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Original Article

Categories isomorphic to (L,M)-DFTOP

A.A. Ramadan [a], **A.A. Abd El-latif** [b,*]

[a] *Department of Mathematics, Faculty of Science, Beni-suef University, Egypt*
[b] *Department of Mathematics, Faculty of Science and Arts at Balgarn, P.O. Box 60, Sabt Al-Alaya 61985, Bisha University, Saudi Arabia*

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Abstract The aim of this paper is to introduce the notion of (L,M)-double fuzzy quasi-coincident neighborhood system, and investigate the relationships between (L,M)-double fuzzy quasi-coincident neighborhood spaces and (L,M)-double fuzzy topological spaces in the category aspect. Also, we give a characterization of (L,M)-DFTOP, which is called compatible antichain L-double topologies and consider the categorical connections between them. Finally, to better understand (L,M)-**DFTOP**, several categories were introduced.

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1. Introduction and preliminaries

Kubiak [1] and Šostak [2] introduced the notion of (L-)fuzzy topological space as a generalization of L-topological spaces (originally called (L-)fuzzy topological spaces by Chang [3] and Goguen [4]). It is the grade of openness of an L-fuzzy set. A general approach to the study of topological-type structures on fuzzy powersets was developed in [5–8].

As a generalization of fuzzy sets, the notion of intuitionistic fuzzy sets was introduced by Atanassov [9,10]. Recently, Çoker and his colleagues [11,12] introduced the notion of intuitionistic fuzzy topological space using intuitionistic fuzzy sets. Samanta and Mondal [13,14], introduced the notion of intuitionistic gradation of openness as a generalization of intuitionistic fuzzy topological spaces [12] and L-fuzzy topological spaces.

Working under the name "intuitionistic" did not continue because doubts were thrown about the suitability of this term, especially when working in the case of complete lattice L. These doubts were quickly ended in 2005 by Garcia and Rodabaugh [15]. They proved that this term is unsuitable in mathematics and applications. They concluded that they work under the name "double". Under this name, many works have been launched [16–19].

Q-neighborhood system that was introduced by Pu and Liu [20] generalized the classical theory of neighborhood system. Since then, the Q-neighborhood system played an important role in L-topology. Later, Šostak [21,22] introduced the fuzzy Q-neighborhood system of fuzzy points in I-fuzzy

* Corresponding author. Fax: +966 7630225/222.
E-mail address: ahmeda73@yahoo.com (A.A. Abd El-latif).
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topological spaces as an extension of Q-neighborhood system. Moreover, the relations between I-fuzzy topological spaces and its Q-neighborhood systems were discussed in [22]. However, as pointed out by Demirci [23], there were some errors in Šostak's results, so he proposed some properties of this kind of fuzzy Q-neighborhood system to correct these errors. Furthermore, in the case $L = [0,1]$, Fang [24] introduced I-fuzzy quasi-coincident neighborhood system independently. Later, in case of completely distributive De Morgan algebra [25], Fang [26] considered L-fuzzy quasi-coincident neighborhood system with respect to L-fuzzy topology, and established the relationship between them in category-theoretical sense. Neighborhood system was usually used as a kind of tools, but we can also regard it as an independent structure and find the relations between this independent structure and other structures.

In this paper, we introduce the notion of (L,M)-double fuzzy quasi-coincident neighborhood system, and construct the category of (L,M)-double fuzzy quasi-coincident neighborhood spaces and their continuous maps, denoted (L,M)-**DPrFQN** which contains the category of topological (L,M)-double fuzzy quasi-coincident neighborhood spaces and their continuous maps, denoted (L,M)-**DFQN** as a full subcategory. Also, we can prove that the category (L,M)-**DFQN** is isomorphic with (L,M)-**DFTOP** and (L,M)-**DPrFQN** is a topological category over **SET**. Finally, we introduce several categories which can be used to better understand (L,M)-**DFTOP**.

Let L and M be fuzzy lattices, i.e., a completely distributive lattices with an order-reversing involution $'$, if not otherwise stated. Let 0_L (1_L) and 0_M (1_M) are the smallest (largest) elements of L and M respectively. Let a,b be elements in a complete lattice M. An element $a \in M$ is said to be coprime if $a \le b \vee c$ implies that $a \le b$ or $a \le c$. The set of all coprimes of M is denoted by $c(M)$. Note that we do not regard $0_M \in M$ as a coprime in this paper and $a \in c(M)$ if and only if a' is prime. We say a is a way below (wedge below) b, in symbols, $a \ll b$ ($a \lll b$) or $b \gg a$ ($b \rhd a$), if for every directed (arbitrary) subset $D \subseteq M$, $\bigvee D \ge b$ implies $a \le d$ for some $d \in D$. If a is a coprime, we have $a \ll b$ if and only if $a \lll b$. The lattice M is called continuous (completely distributive) if every element $a \in M$ is the supremum of all elements way below (wedge below) it. Let X be a non-empty set. The family of all L-fuzzy sets on X will be denoted by L^X. The smallest element and the largest one of L^X will be denoted by 0_X and 1_X respectively. A fuzzy point, denoted x_t $(x \in X, t \in c(L))$, is an L-fuzzy set from X to L such that $x_t(x) = t \neq 0_L$ and otherwise $= 0_L$; the set of all fuzzy points is denoted by $Pt(L^X)$. Let $x_t \in Pt(L^X)$ and $\lambda \in L^X$. We say that x_t is quasi-coincident with λ, denoted $x_t q \lambda$ [27], if $t \nleq \lambda'(x)$, where $\lambda'(x) = \lambda(x)'$. The relation that " is not quasi-coincident with " is denoted by $/q$. The reader should note that $Pt(L^X)$ should not be confused with the notion for points of a complete lattice—the carrier set used in Stone duality—in [28, Chapter 2]. Note that for every $x_t \in Pt(L^X)$, $\overrightarrow{x_t}$ denotes the collection $\{\nu \in L^X : x_t q \nu\}$, and for all λ with $x_t q \lambda$, $\overrightarrow{x_t} \mid \lambda$ denotes the collection $\{\nu \in L^X : \nu \le \lambda$ and $x_t q \nu\}$. Let $f : X \to Y$ be a map. Then the Zadeh image and preimage operators of f are defined by: $f^{\to}(\lambda)(y) = \bigvee\{\lambda(x) : f(x) = y\}$, $f^{\leftarrow}(\mu) = \mu \circ f$, for all $\lambda \in L^X, \mu \in L^Y, x \in X$ and $y \in Y$. An L-topology on a set X is a subset δ of L^X closed under finite meets and arbitrary joins. An L-topology on a set is always in Chang-Goguens's sense. The pair (δ,δ^*) is called an L-double topology on X if δ and δ^* are L-topologies on X and $\delta \subseteq \delta^*$. The triplet (X,δ,δ^*) is called L-double topological space. Let (X,δ_1,δ_1^*) and (Y,δ_2,δ_2^*) be L-double topological spaces. Then a map $f : (X,\delta_1,\delta_1^*) \to (Y,\delta_2,\delta_2^*)$ is continuous if for each $\nu \in \delta_2$ (resp. $\nu \in \delta_2^*$), $f^{\leftarrow}(\nu) \in \delta_1$ (resp. $f^{\leftarrow}(\nu) \in \delta_1^*$). The following definitions and results will be used frequently in the sequel.

Proposition 1.1 ([29]). *Let M be a complete lattice. The following conditions are equivalent:*

(i) *M ia a completely distributive;*
(ii) *M is a distributive continuous lattice with enough coprimes;*
(iii) *M is a distributive and both M and M^{op} are continuous.*

It is well-known that both the way below relation in a continuous lattice and the wedge below relation in a completely distributive lattice have the interpolation property, hence if $a \ll b$ in a completely distributive lattice M and a is a coprime there is some coprime $c \in M$ such that $a \ll c \ll b$. The way below relation on a complete lattice M is said to be multiplicative if $a \ll b$ and $a \ll c$ implies $a \ll b \wedge c$ for all $a,b,c \in M$. The way below relation on a completely distributive lattice M is called locally multiplicative if for every coprime $a \in c(M)$, $a \lll b$ and $a \lll c$ implies $a \lll b \wedge c$ for all $b,c \in M$. Clearly, if the way below relation on a completely distributive lattice is multiplicative, then it is locally multiplicative [30].

Definition 1.2. For a given nonempty set X, a double quasi-coincident system on X is a family of $(\mathcal{Q},\mathcal{Q}^*) = \{(\mathcal{Q}_{x_t},\mathcal{Q}^*_{x_t}) : \mathcal{Q}_{x_t},\mathcal{Q}^*_{x_t} \subseteq L^X, x_t \in Pt(L^X)\}$ fulfilling the following conditions:

(Q1) $\mathcal{Q}_{x_t} \subseteq \mathcal{Q}^*_{x_t}$;
(Q2) $\lambda \in \mathcal{Q}_{x_t}, \lambda \in \mathcal{Q}^*_{x_t}$ implies $x_t q \lambda$;
(Q3) $\forall\ \lambda,\nu \in L^X$, if $\nu \in \mathcal{Q}_{x_t}$ (resp. $\nu \in \mathcal{Q}^*_{x_t}$) and $\lambda \ge \nu$ then, $\lambda \in \mathcal{Q}_{x_t}$ (resp. $\lambda \in \mathcal{Q}^*_{x_t}$).
(Q4) $\forall\ \lambda,\nu \in L^X$, if $\nu \in \mathcal{Q}_{x_t}$ (resp. $\nu \in \mathcal{Q}^*_{x_t}$) and $\lambda \in \mathcal{Q}_{x_t}$ (resp. $\lambda \in \mathcal{Q}^*_{x_t}$) then, $\lambda \wedge \nu \in \mathcal{Q}_{x_t}$ (resp. $\lambda \wedge \nu \in \mathcal{Q}^*_{x_t}$).
$(\mathcal{Q},\mathcal{Q}^*) = \{(\mathcal{Q}_{x_t},\mathcal{Q}^*_{x_t}) : x_t \in Pt(L^X)\}$ is the double quasi-coincident neighborhood system of some L-double topology on X if and only if it satisfies:
(Q5) $\forall\ \lambda \in \mathcal{Q}_{x_t}$ (resp. $\lambda \in \mathcal{Q}^*_{x_t}$) there is $\nu \le \lambda$ such that $x_t q \nu$ and $\nu \in \mathcal{Q}_{y_s}$ (resp. $\nu \in \mathcal{Q}^*_{y_s}$) $\forall\ y_s q \nu$.

A triplet $(X,\mathcal{Q},\mathcal{Q}^*)$ is called double quasi-coincident neighborhood space. Let $f : (X,\mathcal{P},\mathcal{P}^*) \to (Y,\mathcal{Q},\mathcal{Q}^*)$ be a map from a double quasi-coincident neighborhood space $(X,\mathcal{P},\mathcal{P}^*)$ to another double quasi-coincident neighborhood space $(Y,\mathcal{Q},\mathcal{Q}^*)$. f is continuous if $\forall\ x_t \in Pt(L^X)$, $f^{\leftarrow}(\lambda) \in \mathcal{P}_{x_t}$ (resp. $f^{\leftarrow}(\lambda) \in \mathcal{P}^*_{x_t}$) for each $\lambda \in \mathcal{Q}_{f^{\to}(x_t)}$ (resp. $\lambda \in \mathcal{Q}^*_{f^{\to}(x_t)}$).

Let **DQN**(L^X) denote the complete lattice of all double quasi-coincident neighborhood systems on X. Trivially the elements in **DQN**(L^X) corresponds bijective to the L-double topologies on X. If we denote the category by ***L*** -**DQN**, of which objects are triplet $(X,\mathcal{Q},\mathcal{Q}^*)$ and morphisms are continuous maps as usual, the corresponding relation can be restated as ***L*-DQN** is isomorphic to ***L*-DTOP** in category terminology.

Definition 1.3 ([31] **).** The pair $(\mathcal{T},\mathcal{T}^*)$ of maps $\mathcal{T},\mathcal{T}^*: L^X \to M$ is called an (L,M)-double fuzzy topology on X if it satisfies the following conditions:

(DFT1) $\mathcal{T}(\lambda) \leq (\mathcal{T}^*(\lambda))'$, for each $\lambda \in L^X$,
(DFT2) $\mathcal{T}(0_X) = \mathcal{T}(1_X) = 1_M, \mathcal{T}^*(0_X) = \mathcal{T}^*(1_X) = 0_M$,
(DFT3) $\mathcal{T}(\lambda_1 \wedge \lambda_2) \geq \mathcal{T}(\lambda_1) \wedge \mathcal{T}(\lambda_2)$ and $\mathcal{T}^*(\lambda_1 \wedge \lambda_2) \leq \mathcal{T}^*(\lambda_1) \vee \mathcal{T}^*(\lambda_2)$, for any $\lambda_1,\lambda_2 \in L^X$.
(DFT4) $\mathcal{T}(\bigvee_{i\in\Gamma}\lambda_i) \geq \bigwedge_{i\in\Gamma}\mathcal{T}(\lambda_i)$ and $\mathcal{T}^*(\bigvee_{i\in\Gamma}\lambda_i) \leq \bigvee_{i\in\Gamma}\mathcal{T}^*(\lambda_i)$, for any $\{\lambda_i : i \in \Gamma\} \subseteq L^X$.
The triplet $(X,\mathcal{T},\mathcal{T}^*)$ is called an (L,M)-double fuzzy topological space. If $(\mathcal{T}_1,\mathcal{T}_1^*)$ and $(\mathcal{T}_2,\mathcal{T}_2^*)$ are two (L,M)-double fuzzy topologies on X, we say that $(\mathcal{T}_1,\mathcal{T}_1^*)$ is finer than $(\mathcal{T}_2,\mathcal{T}_2^*)$ (or $(\mathcal{T}_2,\mathcal{T}_2^*)$ is coarser than $(\mathcal{T}_1,\mathcal{T}_1^*)$), denoted by $(\mathcal{T}_2,\mathcal{T}_2^*) \leq (\mathcal{T}_1,\mathcal{T}_1^*)$ iff $\mathcal{T}_2(\lambda) \leq \mathcal{T}_1(\lambda)$ and $\mathcal{T}_2^*(\lambda) \geq \mathcal{T}_1^*(\lambda)$, for each $\lambda \in L^X$.

Let $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \to (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ be a map between (L,M)-double fuzzy topological spaces $(X,\mathcal{T}_1,\mathcal{T}_1^*)$ and $(Y,\mathcal{T}_2,\mathcal{T}_2^*)$. Then f is said to be continuous if for each $\mu \in L^Y$, $\mathcal{T}_2(\mu) \leq \mathcal{T}_1(f^{\leftarrow}(\mu))$ and $\mathcal{T}_2^*(\mu) \geq \mathcal{T}_1^*(f^{\leftarrow}(\mu))$.

Thus we have the category **(L,M)-DFTOP** where the objects are (L,M)-double fuzzy topological spaces and the morphisms are continuous maps.

Definition 1.4 ([32] **).**

(1) Category **A** is said to be a subcategory of category **B** provided that the following conditions are satisfied:
  (a) $Ob(\mathbf{A}) \subseteq Ob(\mathbf{B})$;
  (b) for each $A,A' \in Ob(\mathbf{A})$, $hom_{\mathbf{A}}(A,A') \subseteq hom_{\mathbf{B}}(A,A')$;
  (c) for each **A**-object A, the **B**-identity on A is the **A**-identity on A;
  (d) the composition law in **A** is the restriction of the composition law in **B** to the morphisms of **A**.

(2) **A** is called a full subcategory of **B** if, in addition to the above, for each $A,A' \in Ob(\mathbf{A})$, $hom_{\mathbf{A}}(A,A') = hom_{\mathbf{B}}(A,A')$.

Definition 1.5 ([32,33] **).**

(1) A category **C** is called a topological category over **SET** with respect to the usual forgetful functor from **C** to **SET** if it satisfies the following conditions:

(TC1) Existence of initial structure: For any set X, any class J, and family $((X_j,\xi_j))_{j\in J}$ of **C**-object and any family $(f_j : X \to X_j)_{j\in J}$ of maps, there exists the unique **C**-structure ξ on X which is initial with respect to the source $(f_j : X \to (X_j,\xi_j))_{j\in J}$, this means that for a **C**-object (Y,η), a map $g : (Y,\eta) \to (X,\xi)$ is a **C**-morphism iff for all $j \in J, f_j \circ g : (Y,\eta) \to (X_j,\xi_j)$ is a **C**-morphism.
(TC2) Fibre-smallness: For any set X, the **C**-fibre of X, i.e., the class of all **C**-structure on X, which we denote **C** (X), is a set.

(2) Let **B** be a category and E be a class of **B**-bimorphisms. A full subcategory **A** of **B** is called E-reflective (or bireflective) in **B** provided that each **B**-object has an **A**-reflection arrow in E as a bimorphism. This means that, for any **B**-object B, there exists an **A**-reflection (or **A**-reflection bimorphism) $r : B \to A$ from B to an **A**-object A with the following universal property: for any morphism $f : B \to A'$ from B into some **A**-object A', there exists a unique **A**-morphism $f' : A \to A'$ such that $f' \circ r = f$.

For undefined notions about categories and completely distributive notions, we refer to [7,29,32,34,35].

2. Category (L,M)-DFQN isomorphic to (L,M)-DFTOP

Definition 2.1. An (L,M)-double fuzzy quasi-coincident neighborhood system (briefly, (L,M)-dfqn system) on X is defined to be a set $(Q,Q^*) = \{(Q_{x_t},Q^*_{x_t}) : x_t \in Pt(L^X)\}$ of maps $Q_{x_t},Q^*_{x_t} : L^X \to M$ such that $\forall\lambda,\mu \in L^X$,

(DFQ1) $Q_{x_t}(\lambda) \leq (Q^*_{x_t}(\lambda))'$.
(DFQ2) $Q_{x_t}(1_X) = 1_M, Q_{x_t}(0_X) = 0_M, Q^*_{x_t}(1_X) = 0_M$ and $Q^*_{x_t}(0_X) = 1_M$.
(DFQ3) $Q_{x_t}(\lambda) \neq 0_M, Q^*_{x_t}(\lambda) \neq 1_M$ implies $x_t q\lambda$.
(DFQ4) $Q_{x_t}(\lambda \wedge \mu) = Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$ and $Q^*_{x_t}(\lambda \wedge \mu) = Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$.
The triplet (X,Q,Q^*) is called an (L,M)-double fuzzy quasi-coincident neighborhood space (briefly, (L,M)-dfqn space), and it will be called topological if it satisfies moreover, for all $x_t \in Pt(L^X),\lambda \in L^X$,
(DFQ5) $Q_{x_t}(\lambda) = \bigvee_{\mu\in\vec{x_t}|\lambda} \bigwedge_{y_s q\mu} Q_{y_s}(\mu)$, and $Q^*_{x_t}(\lambda) = \bigwedge_{\mu\in\vec{x_t}|\lambda} \bigvee_{y_s q\mu} Q^*_{y_s}(\mu)$.
A continuous map between two (L,M)-dfqn spaces (X,P,P^*) and (Y,Q,Q^*) is a map $f : X \to Y$ such that for each $x_t \in Pt(L^X)$ and $\nu \in L^Y$,
$Q_{f(x)_t}(\nu) \leq P_{x_t}(f^{\leftarrow}(\nu))$, and $Q^*_{f(x)_t}(\nu) \geq P^*_{x_t}(f^{\leftarrow}(\nu))$.
The category of (L,M)-dfqn spaces and their continuous maps is denoted by (L,M)-**DPrFQN**, and (L,M)-**DFQN** the full subcategory of (L,M)-**DPrFQN** consisting of topological (L,M)-dfqn spaces.

Remark 2.2. For all $x_t \in Pt(L^X)$ and $\nu \in L^X, Q_{x_t}(\nu)$ can be thought as the degree of ν being a quasi-coincident neighborhood of x_t and $Q^*_{x_t}(\nu)$ can be thought as the degree of ν being non-quasi-coincident neighborhood of x_t.

Let X be a nonempty set and $(\mathcal{T},\mathcal{T}^*)$ be an (L,M)-double fuzzy topology on X. Define $Q^{\mathcal{T}}_{x_t},Q^{*\mathcal{T}^*}_{x_t} : L^X \to M$ as:

$$Q^{\mathcal{T}}_{x_t}(\lambda) = \begin{cases} \bigvee\{\mathcal{T}(\nu) : \nu \in\vec{x_t}\ |\lambda\}, & \text{if } x_t q\lambda \\ 0_M, & \text{if } x_t \not{q}\lambda \end{cases}$$

$$Q^{*\mathcal{T}^*}_{x_t}(\lambda) = \begin{cases} \bigwedge\{\mathcal{T}^*(\nu) : \nu \in\vec{x_t}\ |\lambda\}, & \text{if } x_t q\lambda \\ 1_M, & \text{if } x_t \not{q}\lambda, \end{cases}$$

for each $x_t \in Pt(L^X)$ and $\lambda \in L^X$. Then we have:

Proposition 2.3. *The set of* $(Q^{\mathcal{T}},Q^{*\mathcal{T}^*}) = \{(Q^{\mathcal{T}}_{x_t},Q^{*\mathcal{T}^*}_{x_t}) : x_t \in Pt(L^X)\}$ *is a topological* (L,M) *-dfqn system on X, called induced topological* (L,M) *-dfqn system from* $(\mathcal{T},\mathcal{T}^*)$.

Proof. (DFQ1)–(DFQ3) are true trivially.

(DFQ4): From the definition of $Q^{\mathcal{T}}_{x_t}$ and $Q^{*\mathcal{T}^*}_{x_t}$, we have: if $\nu,\lambda \in L^X$ such that $\nu \le \lambda$, then $Q^{\mathcal{T}}_{x_t}(\nu) \le Q^{\mathcal{T}}_{x_t}(\lambda)$ and $Q^{*\mathcal{T}^*}_{x_t}(\nu) \ge Q^{*\mathcal{T}^*}_{x_t}(\lambda)$. Thus for each $\lambda,\mu \in L^X$ we have: $Q^{\mathcal{T}}_{x_t}(\lambda\wedge\mu) \le Q^{\mathcal{T}}_{x_t}(\lambda)\wedge Q^{\mathcal{T}}_{x_t}(\mu)$ and $Q^{*\mathcal{T}^*}_{x_t}(\lambda\wedge\mu) \ge Q^{*\mathcal{T}^*}_{x_t}(\lambda)\vee Q^{*\mathcal{T}^*}_{x_t}(\mu)$. On the other hand, for each $\alpha \in c(M)$ such that $\alpha \lhd Q^{\mathcal{T}}_{x_t}(\lambda)\wedge Q^{\mathcal{T}}_{x_t}(\mu)$, we have $\alpha\lhd Q^{\mathcal{T}}_{x_t}(\lambda)$ and $\alpha \lhd Q^{\mathcal{T}}_{x_t}(\mu)$. Thus there exist $\lambda_1,\mu_1 \in L^X$ such that $x_t q \lambda_1 \le \lambda, \alpha \le \mathcal{T}(\lambda_1)$ and $x_t q\mu_1 \le \mu, \alpha \le \mathcal{T}(\mu_1)$, respectively. Therefore $\alpha \le \mathcal{T}(\lambda_1)\wedge\mathcal{T}(\mu_1) \le \mathcal{T}(\lambda_1\wedge\mu_1)$. It is clear that $x_t q(\lambda_1\wedge\mu_1) \le \lambda\wedge\mu$. Then, by the definition of $Q^{\mathcal{T}}_{x_t}$, $\alpha \le \mathcal{T}(\lambda_1\wedge\mu_1) \le Q^{\mathcal{T}}_{x_t}(\lambda\wedge\mu)$. Hence, $Q^{\mathcal{T}}_{x_t}(\lambda\wedge\mu) \ge Q^{\mathcal{T}}_{x_t}(\lambda)\wedge Q^{\mathcal{T}}_{x_t}(\mu)$. It remains to prove that: $Q^{*\mathcal{T}^*}_{x_t}(\lambda\wedge\mu) \le Q^{*\mathcal{T}^*}_{x_t}(\lambda)\vee Q^{*\mathcal{T}^*}_{x_t}(\mu)$. So let $\beta \lhd Q^{*\mathcal{T}^*}_{x_t}(\lambda\wedge\mu), \beta \in c(M)$. Then, there exists $\nu \in L^X$ such that $x_t q\nu \le \lambda\wedge\mu$ and $\beta \le \mathcal{T}^*(\nu)$. This implies that $x_t q\nu \le \lambda, \beta \le \mathcal{T}^*(\nu)$ and $x_t q\nu \le \mu, \beta \le \mathcal{T}^*(\nu)$. Then $\beta \le Q^{*\mathcal{T}^*}_{x_t}(\lambda)$ and $\beta \le Q^{*\mathcal{T}^*}_{x_t}(\mu)$. Thus $\beta \le Q^{*\mathcal{T}^*}_{x_t}(\lambda)\vee Q^{*\mathcal{T}^*}_{x_t}(\mu)$. Hence, $Q^{*\mathcal{T}^*}_{x_t}(\lambda\wedge\mu) \le Q^{*\mathcal{T}^*}_{x_t}(\lambda)\vee Q^{*\mathcal{T}^*}_{x_t}(\mu)$.

(DFQ5) $\forall\ \mu \in \vec{x_t}|\lambda$, we have $\mathcal{T}(\mu) \le \bigwedge_{y_s q\mu} Q^{\mathcal{T}}_{y_s}(\mu) \le Q^{\mathcal{T}}_{x_t}(\mu) \le Q^{\mathcal{T}}_{x_t}(\lambda)$, and $\mathcal{T}^*(\mu) \ge \bigvee_{y_s q\mu} Q^{*\mathcal{T}^*}_{y_s}(\mu) \ge Q^{*\mathcal{T}^*}_{x_t}(\mu) \ge Q^{*\mathcal{T}^*}_{x_t}(\lambda)$. Therefore,

$$Q^{\mathcal{T}}_{x_t}(\lambda) = \bigvee_{\mu\in\vec{x_t}|\lambda} \mathcal{T}(\mu) \le \bigvee_{\mu\in\vec{x_t}|\lambda}\bigwedge_{y_s q\mu} Q^{\mathcal{T}}_{y_s}(\mu) \le Q^{\mathcal{T}}_{x_t}(\lambda),$$

and

$$Q^{*\mathcal{T}^*}_{x_t}(\lambda) = \bigwedge_{\mu\in\vec{x_t}|\lambda} \mathcal{T}^*(\mu) \ge \bigwedge_{\mu\in\vec{x_t}|\lambda}\bigvee_{y_s q\mu} Q^{*\mathcal{T}^*}_{y_s}(\mu) \ge Q^{*\mathcal{T}^*}_{x_t}(\lambda).$$

This means, $Q^{\mathcal{T}}_{x_t}(\lambda) = \bigvee_{\mu\in\vec{x_t}|\lambda}\bigwedge_{y_s q\mu} Q^{\mathcal{T}}_{y_s}(\mu)$ and $Q^{*\mathcal{T}^*}_{x_t}(\lambda) = \bigwedge_{\mu\in\vec{x_t}|\lambda}\bigvee_{y_s q\mu} Q^{*\mathcal{T}^*}_{y_s}(\mu)$. □

Lemma 2.4. $\forall\ \lambda \in L^X, \mathcal{T}(\lambda) = \bigwedge_{x_t q\lambda} Q^{\mathcal{T}}_{x_t}(\lambda)$ *and* $\mathcal{T}^*(\lambda) = \bigvee_{x_t q\lambda} Q^{*\mathcal{T}^*}_{x_t}(\lambda)$.

Proposition 2.5.

(i) *If* $(\mathcal{T}_1,\mathcal{T}_1^*)$ *and* $(\mathcal{T}_2,\mathcal{T}_2^*)$ *are two* (L,M) *-double fuzzy topologies on X which determine the same topological* (L,M) *-dfqn system, then* $(\mathcal{T}_1,\mathcal{T}_1^*) = (\mathcal{T}_2,\mathcal{T}_2^*)$.

(ii) *Suppose that* $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \to (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ *is a continuous map between* (L,M)*-double fuzzy topological spaces. Then,* $f : (X,Q^{\mathcal{T}_1},Q^{*\mathcal{T}_1^*}) \to (Y,Q^{\mathcal{T}_2},Q^{*\mathcal{T}_2^*})$ *is also continuous with respect to induced topological* (L,M)*-dfqn systems.*

Proof.

(i) Hold by Lemma 2.4.

(ii) Since $f : (X,\mathcal{T}_1,\mathcal{T}_1^*) \to (Y,\mathcal{T}_2,\mathcal{T}_2^*)$ is continuous, then for each $\mu \in L^Y$ we have, $\mathcal{T}_2(\mu) \le \mathcal{T}_1(f^{\leftarrow}(\mu))$ and $\mathcal{T}_2^*(\mu) \ge \mathcal{T}_1^*(f^{\leftarrow}(\mu))$. Notice that, $Q^{\mathcal{T}}_{x_t}(\mu) = \bigvee_{\nu\in\vec{x_t}|\mu}\mathcal{T}(\nu)$ and $Q^{*\mathcal{T}^*}_{x_t}(\mu) = \bigwedge_{\nu\in\vec{x_t}|\mu}\mathcal{T}^*(\nu)$. It follows that:

$$\begin{aligned} Q^{\mathcal{T}_2}_{f(x)_t}(\mu) &= \bigvee_{\nu\in\vec{f(x)_t}|\mu}\mathcal{T}_2(\nu) \le \bigvee_{f^{\leftarrow}(\nu)\in\vec{x_t}|f^{\leftarrow}(\mu)}\mathcal{T}_2(\nu) \\ &\le \bigvee_{f^{\leftarrow}(\nu)\in\vec{x_t}|f^{\leftarrow}(\mu)}\mathcal{T}_1(f^{\leftarrow}(\nu)) \le Q^{\mathcal{T}_1}_{x_t}(f^{\leftarrow}(\mu)),\end{aligned}$$

and

$$\begin{aligned} Q^{*\mathcal{T}_2^*}_{f(x)_t}(\mu) &= \bigwedge_{\nu\in\vec{f(x)_t}|\mu}\mathcal{T}_2^*(\nu) \ge \bigwedge_{f^{\leftarrow}(\nu)\in\vec{x_t}|f^{\leftarrow}(\mu)}\mathcal{T}_2^*(\nu) \\ &\ge \bigwedge_{f^{\leftarrow}(\nu)\in\vec{x_t}|f^{\leftarrow}(\mu)}\mathcal{T}_1^*(f^{\leftarrow}(\nu)) \ge Q^{*\mathcal{T}_1^*}_{x_t}(f^{\leftarrow}(\mu)).\end{aligned}$$

□

From the above proposition, we have obtained a functor from (L,M)-**DFTOP** to (L,M)-**DFQN** which is injective on objects.

Let $(Q,Q^*) = \{(Q_{x_t},Q^*_{x_t}) : x_t \in Pt(L^X)\}$ be a topological (L,M)-dfqn system on X. Define a pair $(\mathcal{T}^Q,\mathcal{T}^{*Q^*})$ of maps $\mathcal{T}^Q,\mathcal{T}^{*Q^*} : L^X \to M$ as:

$$\mathcal{T}^Q(\lambda) = \begin{cases} \bigwedge_{x_t q\lambda} Q_{x_t}(\lambda), & \text{if} \quad \lambda \ne 0_X \\ 1_M, & \text{if} \quad \lambda = 0_X, \end{cases}$$

$$\mathcal{T}^{*Q^*}(\lambda) = \begin{cases} \bigvee_{x_t q\lambda} Q^*_{x_t}(\lambda), & \text{if} \quad \lambda \ne 0_X \\ 0_M, & \text{if} \quad \lambda = 0_X. \end{cases}$$

Then we have:

Proposition 2.6.

(i) $(\mathcal{T}^Q,\mathcal{T}^{*Q^*})$ *defined above is an* (L,M) *-double fuzzy topology on X, called induced* (L,M) *-double fuzzy topology from* (Q,Q^*) *. Moreover, if* (P,P^*) *and* (Q,Q^*) *are two topological* (L,M) *-dfqn systems on X which determine the same* (L,M) *-double fuzzy topology, then* $(P,P^*) = (Q,Q^*)$.

(ii) *If a map* $f : (X,P,P^*) \to (Y,Q,Q^*)$ *is a continuous map between topological* (L,M) *-dfqn spaces, then f is continuous with respect to induced* (L,M) *-double fuzzy topologies.*

Proof.

(i) (DFT1)–(DFT3) are easily proved.
(DFT4) For each $\{\mu_i : i\in\Gamma\} \subseteq L^X$, we have:

$$\begin{aligned}\mathcal{T}^Q\left(\bigvee_{i\in\Gamma}\mu_i\right) &= \bigwedge_{x_t q(\bigvee_{i\in\Gamma}\mu_i)} Q_{x_t}\left(\bigvee_{i\in\Gamma}\mu_i\right) = \bigwedge_{i\in\Gamma}\bigwedge_{x_t q\mu_i} Q_{x_t}\left(\bigvee_{i\in\Gamma}\mu_i\right) \\ &\ge \bigwedge_{i\in\Gamma}\bigwedge_{x_t q\mu_i} Q_{x_t}(\mu_i) = \bigwedge_{i\in\Gamma}\mathcal{T}^Q(\mu_i),\end{aligned}$$

and

$$\begin{aligned}\mathcal{T}^{*Q^*}\left(\bigvee_{i\in\Gamma}\mu_i\right) &= \bigvee_{x_t q(\bigvee_{i\in\Gamma}\mu_i)} Q^*_{x_t}\left(\bigvee_{i\in\Gamma}\mu_i\right) = \bigvee_{i\in\Gamma}\bigvee_{x_t q\mu_i} Q^*_{x_t}\left(\bigvee_{i\in\Gamma}\mu_i\right) \\ &\le \bigvee_{i\in\Gamma}\bigvee_{x_t q\mu_i} Q^*_{x_t}(\mu_i) = \bigvee_{i\in\Gamma}\mathcal{T}^{*Q^*}(\mu_i).\end{aligned}$$

Hence, $(\mathcal{T}^Q,\mathcal{T}^{*Q^*})$ is an (L,M)-double fuzzy topology. In addition, suppose that $(\mathcal{T},\mathcal{T}^*)$ is the same (L,M)-double fuzzy topology. Then, for each $x_t \in Pt(L^X)$ and $\lambda \in L^X$,

$$P_{x_t}(\lambda)=\bigvee_{\mu\in \overrightarrow{x_t}|\lambda}\bigwedge_{y_s q\mu}P_{y_s}(\mu)=\bigvee_{\mu\in \overrightarrow{x_t}|\lambda}\mathcal{T}(\mu)=\bigvee_{\mu\in \overrightarrow{x_t}|\lambda}\bigwedge_{y_s q\mu}Q_{y_s}(\mu)=Q_{x_t}(\lambda),$$

and

$$P^*_{x_t}(\lambda)=\bigwedge_{\mu\in \overrightarrow{x_t}|\lambda}\bigvee_{y_s q\mu}P^*_{y_s}(\mu)=\bigwedge_{\mu\in \overrightarrow{x_t}|\lambda}\mathcal{T}^*(\mu)=\bigwedge_{\mu\in \overrightarrow{x_t}|\lambda}\bigvee_{y_s q\mu}Q^*_{y_s}(\mu)=Q^*_{x_t}(\lambda).$$

Hence, $(P,P^*)=(Q,Q^*)$.

(ii) Since $x_t q f^{\leftarrow}(\mu)$ if and only if $f^{\rightarrow}(x_t)=f(x)_t q\mu, \forall\ \mu \in L^X$, and

$$\{y_t \in Pt(L^Y): y_t q\mu\} \supseteq \{f(x)_t \in Pt(L^Y): x_t \in Pt(L^X), \text{ and } f(x)_t q\mu\}$$

we can get $f:(X,\mathcal{T}^P,\mathcal{T}^{*P^*}) \to (Y,\mathcal{T}^Q,\mathcal{T}^{*Q^*}))$ is continuous.

The next results follow from Propositions 2.3, 2.5 and 2.6. □

Theorem 2.7. *(L,M)-**DFQN** is isomorphic to (L,M)-**DFTOP**.*

Corollary 2.8. *(L,M)-**DFTOP** is concretely bireflective in (L,M)-**DPrFQN**.*

Theorem 2.9. *(L,M) - **DPrFQN** is a topological category over **SET** with respect to the usual forgetful functor.*

Proof. By Definition 1.5, we need to check the conditions of fibre-smallness and existence of initial structures for this category. The fibre-smallness condition is trivial. We need to prove that it fulfills the existence of initial structures. Let $\{f_j: X \to (X^j,Q^j,Q^{*^j})\}_{j\in J}$ be a source in (L,M)-**DPrFQN** and $(Q,Q^*)=\{(Q_{x_t},Q^*_{x_t}): Q_{x_t},Q^*_{x_t}: L^X \to M, x_t \in Pt(L^X)\}$ defined by: $\forall x_t \in Pt(L^X), \forall\lambda \in L^X$,

$$Q_{x_t}(\lambda)=\bigvee_{F\in J^{<w}}\left\{\bigwedge_{j\in F}Q^j_{f_j^{\rightarrow}(x_t)}(\lambda_j):\lambda_j\in L^{X^j},\bigwedge_{j\in F}\ f_j^{\leftarrow}(\lambda_j)\le\lambda\right\},$$

and

$$Q^*_{x_t}(\lambda)=\bigwedge_{F\in J^{<w}}\left\{\bigvee_{j\in F}Q^{*^j}_{f_j^{\rightarrow}(x_t)}(\lambda_j):\lambda_j\in L^{X^j},\bigwedge_{j\in F}\ f_j^{\leftarrow}(\lambda_j)\le\lambda\right\}.$$

We will show that (Q,Q^*) is the unique (L,M)-**DPrFQN**-structure on X which is initial with respect to the source $\{f_j: X \to (X^j,Q^j,Q^{*^j})\}_{j\in J}$.

Step 1 (V-lift): (Q,Q^*) is (L,M)-**DPrFQN**-structure on X, i.e., (Q,Q^*) is (L,M)-dfqn system on X and (Q,Q^*) makes f_j continuous for each $j \in J$.

(DFQ1) and (DFQ2) are trivial, (DFQ3) is routine.

(DFQ4) From the definition of Q_{x_t} and $Q^*_{x_t}$, we know that $Q_{x_t}(\nu)\le Q_{x_t}(\lambda)$ and $Q^*_{x_t}(\nu)\ge Q^*_{x_t}(\lambda)$, when $\nu\le\lambda$. Therefore, for each $\lambda,\mu \in L^X$, we have:

$$Q_{x_t}(\lambda\wedge\mu)\le Q_{x_t}(\lambda)\wedge Q_{x_t}(\mu) \text{ and } Q^*_{x_t}(\lambda\wedge\mu)\ge Q^*_{x_t}(\lambda)\vee Q^*_{x_t}(\mu).$$

On the other hand, let $\alpha \in c(M)$ such that $\alpha\lhd Q_{x_t}(\lambda)\wedge Q_{x_t}(\mu)$. Then, $\alpha\lhd Q_{x_t}(\lambda)$ and $\alpha\lhd Q_{x_t}(\mu)$. By the definition of Q_{x_t}, there exist $\{\lambda_j\}_{j\in F_1}$ with $\bigwedge_{j\in F_1}f_j^{\leftarrow}(\lambda_j)\le\lambda$, and there exist $\{\mu_j\}_{j\in F_2}$ with $\bigwedge_{j\in F_2}f_j^{\leftarrow}(\mu_j)\le\mu$, such that $\alpha\le\bigwedge_{j\in F_1}Q^j_{f_j^{\rightarrow}(x_t)}(\lambda_j)$ and $\alpha\le\bigwedge_{j\in F_2}Q^j_{f_j^{\rightarrow}(x_t)}(\mu_j)$, respectively, where F_1 and F_2 are two finite subsets of J. Let $F=F_1\cup F_2$ and

$$\nu_j=\begin{cases}\lambda_j, & j\in F_1, j\notin F_2\\ \mu_j, & j\notin F_1, j\in F_2\\ \lambda_j\wedge\mu_j, & j\in F_1\cap F_2.\end{cases}$$

Then F is a finite subset of J and we have:

$$\lambda\wedge\mu\ge\bigwedge_{j\in F_1}f_j^{\leftarrow}(\lambda_j)\wedge\bigwedge_{j\in F_2}f_j^{\leftarrow}(\mu_j)$$
$$=\bigwedge_{j\in F}f_j^{\leftarrow}(\nu_j) \text{ and } \alpha\le\bigwedge_{j\in F}Q^j_{f_j^{\rightarrow}(x_t)}(\nu_j).$$

Thus $\alpha\le Q_{x_t}(\lambda\wedge\mu)$. From the arbitrariness of α, we have, $Q_{x_t}(\lambda\wedge\mu)\ge Q_{x_t}(\lambda)\wedge Q_{x_t}(\mu)$. It remains to prove that, $Q^*_{x_t}(\lambda\wedge\mu)\le Q^*_{x_t}(\lambda)\vee Q^*_{x_t}(\mu)$. So, let $\beta\in c(M)$ such that $\beta\lhd Q^*_{x_t}(\lambda\wedge\mu)$. Then, there exists $\{\lambda_j\}_{j\in F_3}$, where F_3 is a finite subset of Γ, with $\bigwedge_{j\in F_3}f_j^{\leftarrow}(\lambda_j)\le\lambda\wedge\mu$ and $\beta\le\bigvee_{j\in F_3}Q^{*^j}_{f_j^{\rightarrow}(x_t)}(\lambda_j)$.

Then, $\bigwedge_{j\in F_3}f_j^{\leftarrow}(\lambda_j)\le\lambda$, $\beta\le\bigvee_{j\in F_3}Q^{*^j}_{f_j^{\rightarrow}(x_t)}(\lambda_j)$, and $\bigwedge_{j\in F_3}f_j^{\leftarrow}(\lambda_j)\le\mu$, $\beta\le\bigvee_{j\in F_3}Q^{*^j}_{f_j^{\rightarrow}(x_t)}(\lambda_j)$. Thus, $\beta\le Q^*_{x_t}(\lambda)$ and $\beta\le Q^*_{x_t}(\mu)$, this implies that $\beta\le Q^*_{x_t}(\lambda)\vee Q^*_{x_t}(\mu)$.

Since β is arbitrary, $Q^*_{x_t}(\lambda\wedge\mu)\le Q^*_{x_t}(\lambda)\vee Q^*_{x_t}(\mu)$. Furthermore, from the definition of Q_{x_t} and $Q^*_{x_t}$ we have: $Q_{x_t}(f_j^{\leftarrow}(\lambda_j))\ge Q^j_{f_j^{\rightarrow}(x_t)}(\lambda_j)$, and $Q^*_{x_t}(f_j^{\leftarrow}(\lambda_j))\le Q^{*^j}_{f_j^{\rightarrow}(x_t)}(\lambda_j)$, for each $\lambda_j\in L^{X^j}$. Then, $f_j:(X,Q,Q^*)\to(X^j,Q^j,Q^{*^j})$ is continuous for each $j\in J$.

Step 2 (Initial V-lift): It is easy to show that (Q,Q^*) is initial V-lift, i.e., for an (L,M)-**DPrFQN**-object (Y,S,S^*) a map $f:(Y,S,S^*)\to(X,Q,Q^*)$ is continuous if and only if $f_j\circ f:(Y,S,S^*)\to(X^j,Q^j,Q^{*^j})$ is continuous, for each $j\in J$.

Step 3 (Unique initial V-lift): Suppose that (X,P,P^*) is another initial V-lift with respect to the source $\{f_j: X\to(X^j,Q^j,Q^{*^j})\}_{j\in J}$. Let $id_X:(X,P,P^*)\to(X,Q,Q^*)$. Since, (X,Q,Q^*) is initial V-lift and $f_j\circ id_X=f_j$ is continuous for all $j\in J$, then id_X is continuous. Hence, $P_{x_t}(\lambda)\ge Q_{x_t}(\lambda)$ and $P^*_{x_t}(\lambda)\le Q^*_{x_t}(\lambda)$, for each $\lambda\in L^X$. Then, $(P,P^*)\ge(Q,Q^*)$. Using the similar argument, we have $(P,P^*)\le(Q,Q^*)$. Therefore, $(P,P^*)=(Q,Q^*)$. Hence, (L,M)-**DPrFQN** is a topological category over **SET**. □

Theorem 2.10. *(L,M) - **DFQN** is bireflective in (L,M) - **DPrFQN** ; hence (L,M) - **DFQN** is a topological category over **SET** with respect to the usual forgetful functor.*

Proof. Assume that (X,Q,Q^*) is an (L,M)-dfqn space, we assert that its (L,M)-**DFQN**-reflection is defined by $id_X:(X,Q,Q^*)\to(X,Q^H,Q^{*^{H^*}})$ where, $(Q^H,Q^{*^{H^*}})=\{(Q^H_{x_t},Q^{*^{H^*}}_{x_t}): Q^H_{x_t},Q^{*^{H^*}}_{x_t}: L^X\to M, x_t\in Pt(L^X)\}$, and for each $x_t\in Pt(L^X),\lambda\in L^X$, $Q^H_{x_t}(\lambda)=\bigvee_{\nu\in\overrightarrow{x_t}|\lambda}\bigwedge_{y_s q\nu}Q_{y_s}(\nu)$ and $Q^{*^{H^*}}_{x_t}(\lambda)=\bigwedge_{\nu\in\overrightarrow{x_t}|\lambda}\bigvee_{y_s q\nu}Q^*_{y_s}(\nu)$. Then, $(X,Q^H,Q^{*^{H^*}})$ is a topological

(L,M)-dfqn space, i.e., $(X,Q^H,Q^{*^{H^*}})$ is (L,M)-**TDFQN**-object, $id_X : (X,Q,Q^*) \to (X,Q^H,Q^{*^{H^*}})$ is continuous and it is bimorphism in (L,M)-**DPrFQN**, and for each topological (L,M)-dfqn space (Y,P,P^*) and each map $f : X \to Y$, the continuity of $f : (X,Q,Q^*) \to (Y,P,P^*)$ implies the continuity of $f : (X,Q^H,Q^{*^{H^*}}) \to (Y,P,P^*)$. □

3. Category DLaTQN isomorphic to (L,M)-DFTOP

In this section, through a categorical aspect, we establish a natural relation between (L,M)-double fuzzy quasi-coincident neighborhood systems in (L,M)-double fuzzy topology and quasi-coincident neighborhood systems in L-double topology. Note that M is fuzzy lattice with locally multiplicative property in this section.

Definition 3.1. The category **DLaQN** contains objects in the form (X,g,g^*) where X is a set, $g,g^* : c(M) \to \mathbf{DQN}(L^X)$ is a map such that for each $\alpha \in c(M)$, the source $\{id_X^{\to} : (X,g(\alpha),g^*(\alpha)) \to (X,g(\alpha),g^*(\alpha))\}$ is initial. The morphism in **DLaQN** is a map $f : (X,g,g^*) \to (Y,h,h^*)$ such that $f : (X,g(\alpha),g^*(\alpha)) \to (Y,h(\alpha),h^*(\alpha))$ is continuous for all $\alpha \in c(M)$. **DLaTQN** denotes the full subcategory of **DLaQN** consisting of those objects (X,g,g^*) such that for each $\alpha \in c(M)$, $(g(\alpha),g^*(\alpha))$ is double quasi-coincident neighborhood system of some L-double topology on L^X.

Proposition 3.2. *Suppose that (X,Q,Q^*) is an (L,M) -dfqn space. Define $g,g^* : c(M) \to \mathbf{DQN}(L^X)$ by: $g(\alpha)(x_t) = \{\lambda \in L^X : \alpha \ll Q_{x_t}(\lambda)\}$ and $g^*(\alpha)(x_t) = \{\lambda \in L^X : Q^*_{x_t}(\lambda) \ll \alpha'\}$, for each $x_t \in Pt(L^X)$ and each $\alpha \in c(M)$. Then we have:*

(i) *(X,g,g^*) is an object in* **DLaQN** *;*
(ii) *(X,g,g^*) is an object in* **DLaTQN** *if (X,Q,Q^*) is topological (L,M) -dfqn space;*
(iii) *If a map $f : (X,P,P^*) \to (Y,Q,Q^*)$ is continuous, then $f : (X,g,g^*) \to (Y,h,h^*)$ is continuous, where g and h (resp. g^* and h^*) are induced from P and Q (resp. P^* and Q^*).*

Proof.

(i) It suffices to show that $(g(\alpha)(x_t),g^*(\alpha)(x_t))$ satisfies (Q1)–(Q4) of Definition 1.2, for each $x_t \in Pt(L^X)$ and each $\alpha \in c(M)$.
(Q1)–(Q3) are easy proved.
(Q4) $\forall\ \lambda,\nu \in L^X$, if $\lambda \in g(\alpha)(x_t)$ and $\nu \in g(\alpha)(x_t)$, then $\alpha \ll Q_{x_t}(\lambda)$ and $\alpha \ll Q_{x_t}(\nu)$. From the local multiplicative of the way below relation on M, we have $\alpha \ll Q_{x_t}(\lambda) \wedge Q_{x_t}(\nu) = Q_{x_t}(\lambda \wedge \nu)$. Then, $\lambda \wedge \nu \in g(\alpha)(x_t)$. Also, if $\lambda \in g^*(\alpha)(x_t)$ and $\nu \in g^*(\alpha)(x_t)$, then $Q^*_{x_t}(\lambda) \ll \alpha'$ and $Q^*_{x_t}(\nu) \ll \alpha'$. Thus $Q^*_{x_t}(\lambda \wedge \nu) = Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\nu) \ll \alpha'$. Then, $\lambda \wedge \nu \in g^*(\alpha)(x_t)$.
(ii) It suffices to show that $(g(\alpha)(x_t),g^*(\alpha)(x_t))$ satisfies (Q5) of Definition 1.2, for each $x_t \in Pt(L^X)$, and each $\alpha \in c(M)$. When $\lambda \in g(\alpha)(x_t)$, we have $\alpha \ll Q_{x_t}(\lambda)$. Since $Q_{x_t}(\lambda) = \bigvee_{\nu \in \vec{x_t}|\lambda} \bigwedge_{y_s q\nu} Q_{y_s}(\nu)$, by the coprimality of α, there is some $\nu \in \vec{x_t}\ |\lambda$ such that $\alpha \ll \bigwedge_{y_s q\nu} Q_{y_s}(\nu)$. Then, $\alpha \ll Q_{y_s}(\nu)$, for each $y_s q\nu$. This implies that, $x_t q\nu \le \lambda$ and $\nu \in g(\alpha)(y_s)$, for each $y_s q\nu$. When $\lambda \in g^*(\alpha)(x_t)$, we have $Q^*_{x_t}(\lambda) \ll \alpha'$. Since $Q^*_{x_t}(\lambda) = \bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu)$, then for each $\nu \in \vec{x_t}\ |\lambda$, $\bigvee_{y_s q\nu} Q^*_{y_s}(\nu) \ll \alpha'$. Then, $Q^*_{y_s}(\nu) \ll \alpha'$, for each $y_s q\nu$. This implies that $x_t q\nu \le \lambda$ and $\nu \in g^*(\alpha)(y_s)$, for each $y_s q\nu$.
(iii) It is clear. □

By the above proposition, we get a functor from (L,M)-**DPrFQN** to **DLaQN** and this functor maps the subcategory of (L,M)-**DPrFQN** consisting of those objects fulfilling (DFQ5) into **DLaTQN**. Conversely, given an object (X,h,h^*) in **DLaQN**, let us define $(Q,Q^*) = \{(Q_{x_t},Q^*_{x_t}) : x_t \in Pt(L^X)\}$ so that, $Q_{x_t}(\lambda) = \bigvee\{\alpha \in c(M) : \lambda \in h(\alpha)(x_t)\}$ and $Q^*_{x_t}(\lambda) = \bigwedge\{\alpha' \in c(M) : \lambda \in h^*(\alpha)(x_t)\}$, for each $x_t \in Pt(L^X)$ and each $\lambda \in L^X$. Then we have:

Proposition 3.3.

(i) *(X,Q,Q^*) is an (L,M) -dfqn space;*
(ii) *(X,Q,Q^*) is topological if (X,h,h^*) is an object in* **DLaTQN** *;*
(iii) *If a map $f : (X,g,g^*) \to (Y,h,h^*)$ is continuous, then $f : (X,P,P^*) \to (Y,Q,Q^*)$ is also continuous, where P and Q (resp. P^* and Q^*) are induced from g and h (resp. g^* and h^*).*

Proof. (DFQ1)–(DFQ3) are easily checked.

(DFQ4) It is easy to show that $Q_{x_t}(\lambda \wedge \mu) \le Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$ and $Q^*_{x_t}(\lambda \wedge \mu) \ge Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$, for each $x_t \in Pt(L^X)$ and $\lambda,\mu \in L^X$. So it remains to prove that, $Q_{x_t}(\lambda \wedge \mu) \ge Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$ and $Q^*_{x_t}(\lambda \wedge \mu) \le Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$, for each $x_t \in Pt(L^X)$ and $\lambda,\mu \in L^X$. Let $k \in c(M)$ with $k \ll Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$. Take coprimes $\beta,\gamma \in c(M)$ such that: $k \ll \beta \ll \gamma \ll Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$, by the interpolation property of the way below relation. Then $\gamma \ll Q_{x_t}(\lambda)$ and $\gamma \ll Q_{x_t}(\mu)$. Since $Q_{x_t}(\lambda) = \bigvee\{\alpha \in c(M) : \lambda \in h(\alpha)(x_t)\}$ and γ is coprime, there exists some $\alpha_\lambda \in c(M)$ such that $\gamma \le \alpha_\lambda$ and $\lambda \in h(\alpha_\lambda)(x_t)$. Similarly, there exists some $\alpha_\mu \in c(M)$ such that $\gamma \le \alpha_\mu$ and $\mu \in h(\alpha_\mu)(x_t)$. Hence, $\beta \ll \gamma \le \alpha_\lambda \wedge \alpha_\mu$. Therefore, $\lambda \wedge \mu \in h(\beta)(x_t)$, which implies, $k \le Q_{x_t}(\lambda \wedge \mu)$. From the arbitrariness of k we have, $Q_{x_t}(\lambda \wedge \mu) \ge Q_{x_t}(\lambda) \wedge Q_{x_t}(\mu)$. Now, suppose that there is $a \in c(M)$ and $\lambda,\mu \in L^X$ such that, $Q^*_{x_t}(\lambda \wedge \mu) > a \ge Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$. Take a coprime $k \in c(M)$ such that, $Q^*_{x_t}(\lambda \wedge \mu) > a \ge k \gg Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$. Then $Q^*_{x_t}(\lambda) \ll k$ and $Q^*_{x_t}(\mu) \ll k$. Thus $\lambda \in h^*(k')(x_t)$ and $\mu \in h^*(k')(x_t)$. By (Q4) of Definition 1.2, $\lambda \wedge \mu \in h^*(k')(x_t)$. Thus $Q^*_{x_t}(\lambda \wedge \mu) \le a$. It is a contradiction. Then $Q^*_{x_t}(\lambda \wedge \mu) \le Q^*_{x_t}(\lambda) \vee Q^*_{x_t}(\mu)$ for each $\lambda,\mu \in L^X$.

(ii) We need to prove that, $Q_{x_t}(\lambda) = \bigvee_{\mu \in \vec{x_t}|\lambda} \bigwedge_{y_s q\mu} Q_{y_s}(\mu)$, and $Q^*_{x_t}(\lambda) = \bigwedge_{\mu \in \vec{x_t}|\lambda} \bigvee_{y_s q\mu} Q^*_{y_s}(\mu)$, for each $x_t \in Pt(L^X)$ and $\lambda \in L^X$. On the one hand, $Q_{x_t}(\lambda) \ge \bigvee_{\mu \in \vec{x_t}|\lambda} \bigwedge_{y_s q\mu} Q_{y_s}(\mu)$ and $Q^*_{x_t}(\lambda) \le \bigwedge_{\mu \in \vec{x_t}|\lambda} \bigvee_{y_s q\mu} Q^*_{y_s}(\mu)$, are trivial. On the other hand, suppose that $a \in c(M)$ with $\alpha \ll Q_{x_t}(\lambda)$. Since $Q_{x_t}(\lambda) = \bigvee\{\alpha \in c(M) : \lambda \in h(\alpha)(x_t)\}$, then there exists some $\beta \in c(M)$ such that $\alpha \ll \beta$ with $\lambda \in h(\beta)(x_t)$. Then, by (Q5) of Definition 1.2, there exists $\nu \leq \lambda$ such that $x_t q\nu$ and $\nu \in h(\beta)(y_s)$ for all $y_s q\nu$. Thus, $\alpha \ll \beta \leq \bigwedge_{y_s q\nu} Q_{y_s}(\nu)$ therefore, $\alpha \leq \bigvee_{\nu \in \vec{x_t}|\lambda} \bigwedge_{y_s q\nu} Q_{y_s}(\nu)$. Consequently, $Q_{x_t}(\lambda) \leq \bigvee_{\nu \in \vec{x_t}|\lambda} \bigwedge_{y_s q\nu} Q_{y_s}(\nu)$. It remains to prove that, $Q^*_{x_t}(\lambda) \geq \bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu)$. suppose that there exists $a \in c(M)$ and $\lambda \in L^X$ such that, $Q^*_{x_t}(\lambda) \leq a < \bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu)$. Take a coprime $k \in c(M)$ such that: $Q^*_{x_t}(\lambda) \ll k \leq a < \bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu)$. Since $Q^*_{x_t}(\lambda) \ll k$, then $\bigwedge\{\alpha' : \lambda \in h^*(\alpha)(x_t)\} \ll k$. Thus there exists some $\alpha_k \in c(M)$ such that $\lambda \in h^*(\alpha_k)(x_t)$ and $\alpha'_k \ll k$. By (Q5) of Definition 1.2, there exists $\nu \in L^X$ such that $x_t q\nu \leq \lambda$ and $\nu \in h^*(\alpha_k)(y_s)$ for all $y_s q\nu$. This implies, $\bigvee_{y_s q\nu} Q^*_{y_s}(\nu) \leq \alpha'_k$. Thus $\bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu)) \leq \alpha'_k \leq a$, a contradiction. Thus $Q^*_{x_t}(\lambda) \geq \bigwedge_{\nu \in \vec{x_t}|\lambda} \bigvee_{y_s q\nu} Q^*_{y_s}(\nu), \forall \lambda \in L^X$.

(iii) Obvious. □

By the above proposition, we get a functor from **DLaQN** to (L,M)-**DPrFQN**, which maps **DLaTQN** into (L,M)-**DFQN**. It can be easily showed that it is inverse to the functor defined in Proposition 3.2. Therefore, we have:

Theorem 3.4. (L,M) - **DPrFQN** *is concretely isomorphic to* **DLaQN** *, and* (L,M) - **DFQN** *to* **DLaTQN** .

4. Category L-AIDTOP isomorphic to (L,M)-DFTOP

In this section, we will construct the category *L* -**AIDTOP** induced from indexed families of L-double topologies, which is isomorphic to (L,M)-**DFTOP**. Thus, the close relation between (L,M)-double fuzzy topology and L-double topology is established, as desired. In the following, **LDTO(** X **)** denotes all L-double topologies on a nonempty set X, and note that **LDTO(** X **)** is complete lattice.

Given an (L,M)-double fuzzy topology $(\mathcal{T},\mathcal{T}^*)$ on X, we can obtain a collection of L-double topologies $\{(\mathcal{T}_\alpha,\mathcal{T}^*_\alpha) : \alpha \in c(M)\}$ on X where, $\mathcal{T}_\alpha = \{\lambda \in L^X : \mathcal{T}(\lambda) \gg \alpha\}, \mathcal{T}^*_\alpha = \{\lambda \in L^X : \mathcal{T}^*(\lambda) \ll \alpha'\}$. Moreover, if we let $h^{\mathcal{T}}(\alpha) = \bigvee_{\beta \gg \alpha} \mathcal{T}_\beta$ and $h^{*\mathcal{T}^*}(\alpha) = \bigvee_{\beta \gg \alpha} \mathcal{T}^*_\beta$, where $\alpha \in c(M)$, then $\{(h^{\mathcal{T}}(\alpha), h^{*\mathcal{T}^*}(\alpha)) : \alpha \in c(M)\}$ is compatible antichain of L-double topologies in the sense that $\alpha \leq \beta \Rightarrow h^{\mathcal{T}}(\alpha) \supseteq h^{\mathcal{T}}(\beta)$ and $h^{*\mathcal{T}^*}(\alpha) \supseteq h^{*\mathcal{T}^*}(\beta)$. It is clear that, $h^{\mathcal{T}}(\alpha) = \bigvee_{\beta \gg \alpha} h^{\mathcal{T}}(\beta)$, and $h^{*\mathcal{T}^*}(\alpha) = \bigvee_{\beta \gg \alpha} h^{*\mathcal{T}^*}(\beta)$.

Lemma 4.1. *Let* $(\mathcal{T},\mathcal{T}^*)$ *be an* (L,M)*-double fuzzy topology. Then* $\mathcal{T}(\lambda) \geq \alpha$*, for all* $\lambda \in h^{\mathcal{T}}(\alpha)$ *and* $\mathcal{T}^*(\lambda) \leq \alpha'$*, for all* $\lambda \in h^{*\mathcal{T}^*}(\alpha)$.

Proof. It is trivial. □

Definition 4.2. An object of the category *L* -**AIDTOP** is a triplet (X,h,h^*), where X is a nonempty set and, $h,h^* : c(M) \to$ **LDTO**(X) such that $\forall\ \alpha \in c(M)$, $h(\alpha) = \bigvee_{\beta \gg \alpha} h(\beta), h(1_M) = \{0_X,1_X\}$, and $h^*(\alpha) = \bigvee_{\beta \gg \alpha} h^*(\beta), h^*(1_M) = \{0_X,1_X\}$. A morphism $f : (X,h,h^*) \to (Y,g,g^*)$ in *L*-**AIDTOP** is a map $f : X \to Y$ such that $\forall\ \alpha \in c(M)$, $f : (X,h(\alpha),h^*(\alpha)) \to (Y,g(\alpha),g^*(\alpha))$ is continuous. An object (X,h,h^*) of *L* -**AIDTOP** is called a compatible antichain L-double topological space and (h,h^*) is said to be a compatible antichain of L-double topology on X. From the definition of $h^{\mathcal{T}}, h^{*\mathcal{T}^*}$ above, we know that $(X,h^{\mathcal{T}},h^{*\mathcal{T}^*})$ is an object of *L* -**AIDTOP**.

Proposition 4.3.

(i) *If two* (L,M) *-double fuzzy topologies on X determine the same object in* *L* - **AIDTOP** *, then they are equal.*
(ii) *If a map* $f : (X,\mathcal{S},\mathcal{S}^*) \to (Y,\mathcal{T},\mathcal{T}^*)$ *is continuous between two* (L,M) *-double fuzzy topological spaces, then* $f : (X,h^{\mathcal{S}},h^{*\mathcal{S}^*}) \to (Y,h^{\mathcal{T}},h^{*\mathcal{T}^*})$ *is continuous.*

Proof.

(i) Let $(\mathcal{T},\mathcal{T}^*)$ and $(\mathcal{S},\mathcal{S}^*)$ be two (L,M)-double fuzzy topologies on X satisfying $h^{\mathcal{T}} = h^{\mathcal{S}}$ and $h^{*\mathcal{T}^*} = h^{*\mathcal{S}^*}$. We want to show that $\mathcal{T} = \mathcal{S}$ and $\mathcal{T}^* = \mathcal{S}^*$. On the one hand, suppose that $\lambda \in L^X$ and $\alpha \in c(M)$ with $\alpha \ll \mathcal{T}(\lambda)$. Fix a coprime $\beta \in c(M)$ such that, $\alpha \ll \beta \ll \mathcal{T}(\lambda)$. Then, $\lambda \in \mathcal{T}_\beta \subseteq h^{\mathcal{T}}(\alpha) = h^{\mathcal{S}}(\alpha)$. By Lemma 4.1, we get $\mathcal{S}(\lambda) \geq \alpha$. From the arbitrariness of α , we have $\mathcal{T}(\lambda) \leq \mathcal{S}(\lambda)$. Suppose that, there exist $\lambda \in L^X$ and $\alpha \in c(M)$ such that $\mathcal{T}^*(\lambda) \leq \alpha < \mathcal{S}^*(\lambda)$. take $\beta \in c(M)$ such that $\mathcal{T}^*(\lambda) \ll \beta \leq \alpha < \mathcal{S}^*(\lambda)$. Then, $\lambda \in \mathcal{T}^*_{\beta'} \subseteq \bigvee_{\beta' \geq \alpha'} \mathcal{T}^*_{\beta'} = h^{*\mathcal{T}^*}(\alpha') = h^{*\mathcal{S}^*}(\alpha')$. By Lemma 4.1, we get $\mathcal{S}^*(\lambda) \leq \alpha$, a contradiction. Then, $\mathcal{T}^*(\lambda) \geq \mathcal{S}^*(\lambda), \forall\ \lambda \in L^X$. On the other hand, by the similar way, we can show that $\mathcal{T}(\lambda) \geq \mathcal{S}(\lambda)$ and $\mathcal{T}^*(\lambda) \leq \mathcal{S}^*(\lambda), \forall\ \lambda \in L^X$. Hence, $\mathcal{T} = \mathcal{S}$ and $\mathcal{T}^* = \mathcal{S}^*$.
(ii) Suppose that a map $f : (X,\mathcal{S},\mathcal{S}^*) \to (Y,\mathcal{T},\mathcal{T}^*)$ is continuous. To show that $f : (X,h^{\mathcal{S}},h^{*\mathcal{S}^*}) \to (Y,h^{\mathcal{T}},h^{*\mathcal{T}^*})$ is continuous, we have to show that: for each $\alpha \in c(M), f^{\leftarrow}(\lambda) \in h^{\mathcal{S}}(\alpha)$ for each $\lambda \in h^{\mathcal{T}}(\alpha)$ and $f^{\leftarrow}(\lambda) \in h^{*\mathcal{S}^*}(\alpha)$ for each $\lambda \in h^{*\mathcal{T}^*}(\alpha)$. Let $\lambda \in h^{\mathcal{T}}(\alpha) = \bigvee_{\beta \gg \alpha} h^{\mathcal{T}}(\beta)$, we know that λ has a form of $\lambda = \bigvee_{j \in J} \bigwedge_{k \in K_j} \lambda_{jk}$, where K_j is finite indexed set for all $j \in J$ and $\lambda_{jk} \in h^{\mathcal{T}}(\beta_{jk})$ $(\beta_{jk} \gg \alpha)$. By Lemma 4.1, $\mathcal{T}(\lambda_{jk}) \geq \beta_{jk} \gg \alpha$. Thus, $\mathcal{S}(f^{\leftarrow}(\lambda_{jk})) \geq \mathcal{T}(\lambda_{jk})) \gg \alpha$. Moreover, $f^{\leftarrow}(\lambda_{jk}) \in h^{\mathcal{S}}(\alpha)$. Therefore, $f^{\leftarrow}(\lambda) = \bigvee_{j \in J} \bigwedge_{k \in K_j} f^{\leftarrow}(\lambda_{jk}) \in h^{\mathcal{S}}(\alpha)$. Also, we can put λ in the form $\lambda = \bigvee_{j \in J} \bigwedge_{k \in K_j} \nu_{jk}$, where K_j is finite indexed set for all $j \in J$ and $\nu_{jk} \in h^{*\mathcal{T}^*}(\beta_{jk})$ $(\beta_{jk} \gg \alpha)$. By Lemma 4.1, $\mathcal{T}^*(\nu_{jk}) \leq \beta'_{jk} \ll \alpha'$. Thus, $\mathcal{S}^*(f^{\leftarrow}(\nu_{jk})) \leq \mathcal{T}^*(\nu_{jk})) \ll \alpha'$. Moreover, $f^{\leftarrow}(\nu_{jk}) \in h^{*\mathcal{S}^*}(\alpha)$. Therefore, $f^{\leftarrow}(\lambda) = \bigvee_{j \in J} \bigwedge_{k \in K_j} f^{\leftarrow}(\nu_{jk}) \in h^{*\mathcal{S}^*}(\alpha)$. □

By the above proposition, we have a functor from (L,M)-**DFTOP** to *L*-**AIDTOP** which is injective on objects. Conversely, given a compatible antichain L-double topology (h,h^*) on X, then we can construct an (L,M)-double fuzzy topology $(\mathcal{T}_h,\mathcal{T}^*_{h^*})$ on X such that $\forall\ \lambda \in L^X$, $\mathcal{T}_h(\lambda) =$

$\bigvee\{\alpha \in c(M) : \lambda \in h(\alpha)\}$ and $\mathcal{T}^*_{h^*}(\lambda) = \bigwedge\{\alpha' \in c(M) : \lambda \in h^*(\alpha)\}$. Then we have,

Proposition 4.4. *$(\mathcal{T}_h,\mathcal{T}^*_{h^*})$ is an (L,M)-double fuzzy topology on X, called induced (L,M)-double fuzzy topology from (h,h^*).*

Proof. We need to check the axioms of (DFT1)–(DFT4).

(DFT1) and (DFT2) are clear and omitted.

(DFT3) Let a be a coprime element ($a \in c(M)$) such that $a \ll \mathcal{T}_h(\lambda) \wedge \mathcal{T}_h(\mu)$. Take $b \in c(M)$ such that $a \ll b \ll \mathcal{T}_h(\lambda) \wedge \mathcal{T}_h(\mu)$. Then $b \ll \mathcal{T}_h(\lambda)$ and $b \ll \mathcal{T}_h(\mu)$. By the definition of $\mathcal{T}_h(\lambda)$, there exists some $\alpha_\lambda \in c(M)$ such that $b \leq \alpha_\lambda$ and $\lambda \in h(\alpha_\lambda)$. Similarly, by the definition of $\mathcal{T}_h(\mu)$ there exists some $\alpha_\mu \in c(M)$ such that $b \leq \alpha_\mu$ and $\mu \in h(\alpha_\mu)$. Then $a \ll b \leq \alpha_\lambda \wedge \alpha_\mu$. Then, $\lambda \in h(a)$ and $\mu \in h(a)$. It implies that $\lambda \wedge \mu \in h(a)$, since $h(a)$ is an L-topology. That is, $\mathcal{T}_h(\lambda \wedge \mu) \geq a$. From the arbitrariness of a, we obtain $\mathcal{T}_h(\lambda \wedge \mu) \geq \mathcal{T}_h(\lambda) \wedge \mathcal{T}_h(\mu)$. Suppose that there exists $\lambda,\mu \in L^X$ and $\alpha \in c(M)$ such that, $\mathcal{T}^*_{h^*}(\lambda \wedge \mu) > \alpha \geq \mathcal{T}^*_{h^*}(\lambda) \vee \mathcal{T}^*_{h^*}(\mu)$. Take a coprime $b \in c(M)$ such that $\mathcal{T}^*_{h^*}(\lambda \wedge \mu) > \alpha \geq b \gg \mathcal{T}^*_{h^*}(\lambda) \vee \mathcal{T}^*_{h^*}(\mu)$. Then, $b \gg \mathcal{T}^*_{h^*}(\lambda)$ and $b \gg \mathcal{T}^*_{h^*}(\mu)$. Then, there exists $b_\lambda \in c(M)$ such that $b \geq b'_\lambda$ and $\lambda \in h^*(b_\lambda)$. Also, there exists $b_\mu \in c(M)$ such that $b \geq b'_\mu$ and $\mu \in h^*(b_\mu)$. Then, $\alpha' \leq b' \leq b'_\lambda \wedge b'_\mu$. Then, $\lambda \in h^*(\alpha')$ and $\mu \in h^*(\alpha')$. Since $h^*(\alpha')$ is L-topology, $\lambda \wedge \mu \in h^*(\alpha')$. Then $\mathcal{T}^*_{h^*}(\lambda \wedge \mu) \leq (\alpha')' = \alpha$, a contradiction. Then, $\mathcal{T}^*_{h^*}(\lambda \wedge \mu) \leq \mathcal{T}^*_{h^*}(\lambda) \vee \mathcal{T}^*_{h^*}(\mu), \forall\, \lambda,\mu \in L^X$.

(DFT4) Let $a \in c(M)$ and a family of $\{\nu_j : j \in J\} \subseteq L^X$ such that $a \ll \bigwedge_{j\in J}\mathcal{T}_h(\nu_j)$. Take $b \in c(M)$ such that $a \ll b \ll \bigwedge_{j\in J}\mathcal{T}_h(\nu_j)$. Thus $\mathcal{T}_h(\nu_j) \gg b, \forall\, j \in J$. By the definition of $\mathcal{T}_h$, there exists $\alpha_j \geq b \gg a$ such that $\nu_j \in h(\alpha_j)$ for all $j \in J$. Hence we obtain, $\bigvee_{j\in J}\nu_j \in \bigvee_{\beta\gg a}h(\beta) = h(a)$. Thus $a \leq \mathcal{T}_h(\bigvee_{j\in J}\nu_j)$. Therefore, $\mathcal{T}_h(\bigvee_{j\in J}\nu_j) \geq \bigwedge_{j\in J}\mathcal{T}_h(\nu_j)$ from the arbitrariness of a. Suppose that there is a family of $\{\nu_j : j \in J\} \subseteq L^X$ and $\alpha \in c(M)$ such tat, $\mathcal{T}^*_{h^*}(\bigvee_{j\in J}\nu_j) > \alpha \geq \bigvee_{j\in J}\mathcal{T}^*_{h^*}(\nu_j)$. Take $b \in c(M)$ such that: $\mathcal{T}^*_{h^*}(\bigvee_{j\in J}\nu_j) > \alpha \geq b \gg \bigvee_{j\in J}\mathcal{T}^*_{h^*}(\nu_j)$. Then, $b \gg \mathcal{T}^*_{h^*}(\nu_j), \forall\, j \in J$. By the definition of $\mathcal{T}^*_{h^*}$, there exists $\alpha_j \in c(M)$ such that $b \geq \alpha'_j$ and $\nu_j \in h^*(\alpha_j)$. Then, $\alpha' \leq b' \leq \alpha_j$. This implies that $\nu_j \in h^*(\alpha_j) \subseteq h^*(\alpha'), \forall\, i \in J$. Since $h^*(\alpha')$ is L-topology, $\bigvee_{j\in J}\nu_j \in h^*(\alpha')$. Then $\mathcal{T}^*_{h^*}(\bigvee_{j\in J}\nu_j) \leq (\alpha')' = \alpha$, a contradiction. Then, $\mathcal{T}^*_{h^*}(\bigvee_{j\in J}\nu_j) \leq \bigvee_{j\in J}\mathcal{T}^*_{h^*}(\nu_j)$, for each $\{\nu_j : j \in J\} \subseteq L^X$. □

Proposition 4.5.

(i) *If (X,g,g^*) and (Y,h,h^*) are two objects in **L-AIDTOP** and they determine the same (L,M)-double fuzzy topology on X, then they are equal.*

(ii) *If a map $f : (X,g,g^*) \to (Y,h,h^*)$ is continuous, then f is continuous with respect to induced (L,M) -double fuzzy topologies.*

Proof.

(i) Let $(\mathcal{T},\mathcal{T}^*)$ be the same (L,M)-double fuzzy topology induced by (h,h^*) and (g,g^*). We want to show $g = h$ and $g^* = h^*$ i.e., $g(\alpha) = h(\alpha)$ and $g^*(\alpha) = h^*(\alpha), \forall\, \alpha \in c(M)$. Let $\nu \in h(\alpha)$. Then, there exists $\beta \in c(M)$ such that $\nu \in h(\beta)$ and $\beta \gg \alpha$. Then, $\mathcal{T}_g(\nu) = \mathcal{T}_h(\nu) \geq \beta \gg \alpha$. Since $\mathcal{T}_g(\nu) \geq \alpha$, there exists $\gamma \in c(M)$ such that $\gamma \gg \alpha$ and $\nu \in g(\gamma)$. Then, $\nu \in g(\alpha)$. Thus $h(\alpha) \subseteq g(\alpha)$ $\forall\, \alpha \in c(M)$. By the same manner we can prove that, $g(\alpha) \subseteq h(\alpha)$ $\forall\, \alpha \in c(M)$. Hence, $h = g$. It remains to show that $h^* = g^*$. Let $\nu \in h^*(\alpha), \alpha \in c(M)$, then there exists $\beta \in c(M)$ such that $\beta \gg \alpha$ and $\nu \in h^*(\beta)$. Then, $\mathcal{T}^*_{g^*}(\nu) = \mathcal{T}^*_{h^*}(\nu) \leq \beta' \ll \alpha'$. Since $\mathcal{T}^*_{g^*}(\nu) \leq \alpha'$, there exists $\gamma \in c(M)$ such that $\gamma' \leq \alpha'$ and $\nu \in g^*(\gamma)$. Then, $\nu \in g^*(\alpha)$. Thus $h^*(\alpha) \subseteq g^*(\alpha), \forall\, \alpha \in c(M)$. In the same manner, we can show $h^*(\alpha) \supseteq g^*(\alpha), \forall\, \alpha \in c(M)$. Then, $h^* = g^*$.

(ii) It is clear. □

From Propositions 4.4, and 4.5, we have the following theorem:

Theorem 4.6. *(L,M) - **DFTOP** is isomorphic to **L** - **AIDTOP**.*

At the end of this paper, we summarize the results as follows:

Theorem 4.7.

(i) *If L and M are completely distributive lattice, then the categories (L,M) - **DFTOP**, (L,M) - **DFQN** and **L** - **AIDTOP** are isomorphic to each other.*

(ii) *If M is a lattice with locally multiplicative property, then these categories (L,M) - **DFTOP**, (L,M) - **DFQN**, **DLaTQN** and **L** - **AIDTOP** are isomorphic to each other.*

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