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Perfect bases for differential equations

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Abstract

We present a method for the construction of solutions of certain systems of partial differential equations with polynomial and power series coefficients. For this purpose we introduce the concept of perfect differential operators. Within this framework we formulate division theorems for polynomials and power series. They in turn yield existence theorems for solutions of systems of linear partial differential equations and algorithms to explicitly construct solutions.

Keywords: Differential operators; Division theorem; Perfect bases

We investigate the construction of solutions of differential equations with polynomial and power series coefficients based on the concept of *perfect* differential operators. This concept is similar to the notion of Gröbner or standard bases for polynomial and power series ideals or left ideals of differential operators. Our definition refers to the *action* of the operators on polynomials and power series – instead of the left module generated by the operators in the Weyl algebra.

If $D = \sum_{\alpha\beta} c_{\alpha\beta} x^\alpha \partial^\beta$ is such an operator, its initial form with respect to a weight vector $\lambda \in \mathbb{R}_+^n$ is defined as the operator $\mathrm{in}(D) = \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ with τ the maximum (in the polynomial case) or the minimum (in the power series case) of the products $\lambda(\alpha - \beta)$ over all α, β with $c_{\alpha\beta} \neq 0$. Several operators $D_1, \dots, D_p$ are called *perfect* with respect to λ if the evaluations $\mathrm{in}(D_i)x^\gamma$ of their initial forms on monomials x^γ span the vector space of all initial monomials of the evaluations $\sum_i D_i a_i$ of the D_i on polynomials, or power series $a_1, \dots, a_p$.

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The definition thus differs from the notion of Gröbner and standard bases for differential operators considered by Castro, Granger, Saito, Sturmfels, Takayama and other authors. Though we know of no algorithm for checking whether given differential operators are perfect, this property is very useful for constructing the associated solutions of the homogeneous or inhomogeneous differential equations. In the case of convergent power series, we indicate additional assumptions which are sufficient to ensure also the convergence of the solutions.

Our first result, the Division Theorem, establishes the division of a polynomial or power series e by differential operators $D_1, \ldots, D_p$, say $e = \sum D_i a_i + b$ with quotients a_i and remainder b. If the D_i are perfect, the remainder b, which is subject to certain support conditions, is unique. This allows us to check, at least theoretically, whether an inhomogeneous differential equation has a polynomial or power series solution: the remainder of the division of the inhomogeneous term by the D_i must be zero.

The second result, the Monomialization Theorem, shows that perfect differential operators behave in specific situations similarly to their initial form, i.e., as a monomial operator. In particular, we show how the solution spaces of the associated homogeneous differential equation can be computed from the solution space of the initial form (which is particularly simple since it is spanned by monomials).

It seems that both results have been known and used in some form – at least implicitly or in special cases – by people working in the algebraic theory of differential equations; see for instance the work of Janet (1920), Abramov (1995) and Petkovšek (1992), or the book of Saito et al. (2000). In a certain sense our approach conceptualizes the construction of solutions through an ansatz with unknown coefficients.

Definitions

We consider linear differential operators $D = \sum_{\alpha\beta} c_{\alpha\beta} x^\alpha \partial^\beta$ with polynomial, or formal or convergent power series coefficients $c_\beta(x) = \sum_\alpha c_{\alpha\beta} x^\alpha$ in n variables $x_1, \ldots, x_n$ over a field $\mathbb{K}$ (assumed complete valued in the convergent setting). Here, ∂ denotes the vector of partial derivatives $\partial_i = \partial_{x_i}$. We denote by $\mathcal{P}_n$ the rings $\mathbb{K}[x]$, $\mathbb{K}\{x\}$ and $\mathbb{K}[[x]]$ of polynomials, convergent and formal power series, respectively, in n variables over $\mathbb{K}$, with associated algebra $\mathcal{P}_n[\partial]$ of linear differential operators.

The *support* $\mathrm{supp}(D)$ of a differential operator D is the set of exponents $(\alpha, \beta) \in \mathbb{N}^{2n}$ with $c_{\alpha\beta} \neq 0$. We set

$$\mathrm{supp}_\partial(D) = \{\beta \in \mathbb{N}^n,\ \text{there is an } \alpha \in \mathbb{N}^n \text{ with } (\alpha, \beta) \in \mathrm{supp}(D)\}.$$

The differences $\tau = \alpha - \beta \in \mathbb{Z}^n$ are called the *shifts* of D, forming the set $S_D = \mathrm{shift}(D) \subseteq \mathbb{Z}^n$. There exists a $\delta \in \mathbb{Z}^n$ such that $\mathrm{shift}(D) \subset \delta + \mathbb{N}^n$. The filtration of $\mathcal{P}_n[\partial]$ via shifts is known as the F-filtration.

A *monomial differential operator* is of the form $D = \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ for some shift $\tau \in \mathbb{Z}^n$. Each D decomposes into a (possibly infinite) sum $D = \sum_{\tau \in S_D} \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ of monomial operators. A differential operator D is a *pure monomial* if it has just one summand $D = c_{\alpha\beta} x^\alpha \partial^\beta$ for some pair $(\alpha, \beta) \in \mathbb{N}^{2n}$ and a coefficient $c_{\alpha\beta} \in \mathbb{K}$. A subvector space M of $\mathcal{P}_n$ is a *monomial subspace* if it contains with each polynomial/series

$\sum_\alpha c_\alpha x^\alpha$ all its summands $c_\alpha x^\alpha$, or, equivalently, if there exists a subset $\Sigma \subseteq \mathbb{N}^n$ such that M consists of all polynomials/series with support in Σ. We shall sometimes write $M = \mathcal{P}_n^\Sigma$. The *canonical monomial direct complement* of a monomial subspace M of $\mathcal{P}_n$ is the subvector space N of $\mathcal{P}_n$ of polynomials/series with support in the complement $\Sigma' = \mathbb{N}^n \setminus \Sigma$ of Σ, namely $N = \mathcal{P}_n^{\Sigma'}$.

Differential operators $D \in \mathcal{P}_n[\partial]$ will be identified with the induced linear operator $D : \mathcal{P}_n \to \mathcal{P}_n$ given by $a \to Da$. For convenience, we shall mostly refer to the convergent or formal power series case. The polynomial case may require small modifications of the definitions and reasoning which will be omitted if obvious.

For $k, m \in \mathbb{N}$ we denote by $k^{\underline{m}}$ the falling factorial $k^{\underline{m}} = k!/(k-m)! = k(k-1)\ldots(k-m+1)$, and similarly $\gamma^{\underline{\beta}} = \Pi_i \gamma_i^{\underline{\beta_i}}$ for n-tuples $\beta, \gamma \in \mathbb{N}^n$. By definition, $k^{\underline{0}} = 1$ for $k > 1$, $0^{\underline{m}} = 0$ for $m > 1$ and $0^{\underline{0}} = 1$. Observe that $(x^\alpha \partial^\beta)x^\gamma = \gamma^{\underline{\beta}} x^{\tau+\gamma}$ with $\tau = \alpha - \beta$. We associate with n-tuples $\tau \in \mathbb{Z}^n$ and polynomial functions $\kappa_\tau : \mathbb{Z}^n \to \mathbb{K}$ with $\kappa_\tau(\gamma) = 0$ if $\gamma + \tau \notin \mathbb{N}^n$ a linear operator $\kappa_\tau \xi^\tau : \mathcal{P}_n \to \mathcal{P}_n$, by

$$(\kappa_\tau \xi^\tau)x^\gamma = \kappa_\tau(\gamma)x^{\gamma+\tau}.$$

We call $\kappa_\tau \xi^\tau$ the *monomial operator* on $\mathcal{P}_n$ with *coefficient function* κ_τ and shift τ. Clearly, monomial differential operators are monomial operators. As the falling factorials form a basis of the polynomial ring, the converse is also true, so the two notions coincide. Kernels and images of monomial operators are monomial subspaces of $\mathcal{P}_n$.

Let now $D_1, \ldots, D_p \in \mathcal{P}_n[\partial]$ be differential operators with induced map

$$D : \mathcal{P}_n^p \to \mathcal{P}_n,\ a = (a_1, \ldots, a_p) \to \textstyle\sum_i D_i a_i.$$

Our main interest will be the solutions $a \in \mathcal{P}_n^p$ of the differential equation $\sum_i D_i a_i = e$ for some polynomial or series $e \in \mathcal{P}_n$. As a preliminary stage of their construction, we shall “divide” e by the D_i – say $e = \sum_i D_i a_i + b$ with a well defined remainder $b \in \mathcal{P}_n$ and quotients $a_i \in \mathcal{P}_n$. In the case where the remainder of the division can be made unique, $\sum_i D_i a_i = e$ will have a solution in $\mathcal{P}_n^p$ if and only if the remainder b of e is zero. In this way, division is a more general procedure than the actual construction of a solution.

In our context, the uniqueness of the remainder will not always be ensured – but it does hold if the D_i are perfect.

We fix throughout a weight vector $\lambda \in \mathbb{R}^n$ with positive and $\mathbb{Q}$-linearly independent components $\lambda_1, \ldots, \lambda_n$. It induces a total order $<_\lambda$ on $\mathbb{Z}^n$ via $\alpha <_\lambda \beta$ if $\lambda\alpha < \lambda\beta$. Consequently, each differential operator D decomposes into a sum of monomial operators according to decreasing or increasing shifts τ. Let $\tau \in \mathbb{Z}^n$ be the maximal (in the case of polynomials) or minimal (in case of power series) shift of D with respect to λ, $\tau = \max/\min\{\alpha - \beta,\ (\alpha, \beta) \in \mathrm{supp}(D)\}$. We write

$$D = D^\circ - \overline{D}$$

where $D^\circ = \mathrm{in}_\lambda(D) = \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ denotes the *initial form* of D with respect to λ, and $\overline{D} = -\sum_{\alpha-\beta<_\lambda\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ (for polynomials) or $\overline{D} = -\sum_{\alpha-\beta>_\lambda\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ (for power series) is the *queue* of D (cf. e.g. Assi et al. (1996), Saito et al. (2000), Oaku et al. (2001), and the relation of the initial form to the indicial polynomial). Notice the difference between the initial form D° and the symbol $\sigma(D) = \sum_{|\beta|=\max} c_{\alpha\beta} x^\alpha \partial^\beta$ of D.

By definition, D° is a monomial operator with shift τ, $D^\circ = \kappa_\tau \xi^\tau$ and coefficient function $\kappa_\tau(\gamma) = \sum_{\alpha-\beta=\tau} c_{\alpha\beta}\gamma^{\underline{\beta}}$. Setting $\Delta = \{\gamma \in \mathbb{N}^n,\ \kappa_\tau(\gamma) \neq 0\}$, the image of D° in $\mathcal{P}_n$ consists of polynomials or series with support in $\Delta + \tau$.

In the case where no derivatives occur, $D = h \in \mathcal{P}_n$ a polynomial or power series, the initial form is a monomial in x, the *initial monomial* of h with respect to $<_\lambda$.

We call differential operators $D_1, \ldots, D_p$ *perfect* with respect to $\lambda \in \mathbb{R}^n_+$ if the initial monomials of the images of $D : \mathcal{P}_n^p \to \mathcal{P}_n$, $a \to \sum_i D_i a_i$ are images of $D^\circ : \mathcal{P}_n^p \to \mathcal{P}_n$, $a \to \sum_i D_i^\circ a_i = \sum_i \mathrm{in}(D_i)a_i$. More explicitly, if for each $a \in \mathcal{P}_n^p$ there exist a $\gamma \in \mathbb{N}^n$ and a $j \in \{1, \ldots, p\}$ such that $\mathrm{in}(\sum D_i a_i) = D_j^\circ x^\gamma = \mathrm{in}(D_j)x^\gamma$.

In a similar but slightly different vein, one could also consider the action $\tilde{D} : \mathcal{P}_n[\partial]^p \to \mathcal{P}_n[\partial]$, $E \to \sum_i D_i \cdot E_i$ of $D_1, \ldots, D_p$ on $\mathcal{P}_n[\partial]^p$ and require that the right ideal of $\mathcal{P}_n[\partial]$ generated by $D_1, \ldots, D_p$ is generated by the initial forms $D_1^\circ, \ldots, D_p^\circ$. We shall not pursue this variation of the concept of perfect operators in the present paper.

Our notion of perfect operators is different from the definition of Gröbner and standard bases used in Castro-Jiménez (1984, 1987) and Saito et al. (2000). However, in the case where no derivatives occur, $D_i = h_i \in \mathcal{P}_n$, we recover the classical notions of Gröbner bases for polynomials and standard bases for power series. For vectors of differential operators $D_i \in \mathcal{P}_n[\partial]^q$, the definition extends in a natural way by ordering the components of vectors in $\mathcal{P}_n^q$ so as to dispose of the notion of their *initial monomial vector* as a vector with one monomial entry, the other entries being zero.

Even in the case where $p = 1$, a single operator D need not be perfect; just take $D = y\partial_x - \partial_y$ with $D^\circ = y\partial_x$. Here we have $I^\circ = \mathrm{Im}\, D^\circ = \mathcal{P}_2 \cdot y$ and $\mathrm{in}(\mathrm{Im}\, D) = \mathcal{P}_2$. This occurrence was to be expected since any $\mathcal{P}_n$-linear map $h : \mathcal{P}_n^p \to \mathcal{P}_n$, $a \to \sum_i h_i a_i$ can be interpreted as a differential operator $D_h : \mathcal{P}_{n+p} \to \mathcal{P}_{n+p}$ identifying $a = (a_1, \ldots, a_p) \in \mathcal{P}_n^p$ with $\sum_i a_i t_i$ and setting $D_h = \sum_i h_i \partial_{t_i}$, for new variables $t_1, \ldots, t_p$. If $h_1, \ldots, h_p$ do not form a standard basis in the usual sense of polynomials or power series, D_h will not be perfect.

In contrast to the polynomial or power series case $D_i = h_i \in \mathcal{P}_n$, it may be very intricate to check whether given differential operators $D_i \in \mathcal{P}_n[\partial]$ are perfect. In a first step, one will restrict the operator to the kernel of its initial form and check whether the images thereof produce new initial monomials; cf. the examples below. But this is not sufficient – there are other ways in which initial monomials can occur. No general method like the Buchberger criterion for polynomials and power series seems to be known for checking whether differential operators are perfect.

The subtlety of the concept is revealed already in simple examples: the operator $D = x^2\partial_x^2 - x\partial_x - x^3$ is perfect, whereas $D = x^2\partial_x^2 - x\partial_x - x^2$ and $D = x^2\partial_x^2 - x\partial_x - x$ are not (both for slightly different reasons); cf. the examples later on.

In the case of convergent power series, we say that the initial form $D^\circ = \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ *dominates* D if there is a constant $C > 0$ such that for all $\beta' \in \mathrm{supp}_\partial(\overline{D})$ and all γ with $\kappa_\tau(\gamma) \neq 0$, we have

$$\gamma^{\underline{\beta'}} \leq C \cdot |\sum_{\alpha-\beta=\tau} c_{\alpha\beta}\gamma^{\underline{\beta}}|.$$

The inequality implies that for each $j = 1, \ldots, n$ and each $\beta' \in \mathrm{supp}_\partial(\overline{D})$ there is at least one $\beta \in \mathrm{supp}_\partial(D^\circ)$ whose j-th component is larger than or equal to the j-th component of β', say $\beta'_j \leq \beta_j$. This is not sufficient for dominance, as shown by the example in two variables $D = x\partial_x - y\partial_y + x^2\partial_x$ with $D^\circ = x\partial_x - y\partial_y$, $\kappa_\tau(\gamma) = \gamma_1 - \gamma_2$, $\mathrm{supp}_\partial(\overline{D}) = \{(1,0)\}$ and unbounded quotient $\gamma_1/(\gamma_1 - \gamma_2)$ on the line $\gamma_1 = \gamma_2 + 1$ in $\mathbb{N}^2$.

Dominance is a condition which ensures that the initial form of an operator controls sufficiently the division properties of the operator, i.e., that the queue $\overline{D}$ is a negligible perturbation of D°. This holds trivially if no derivatives appear at all. In the one-variable case $n = 1$, the choice of the weight vector is redundant and the initial form D° is unique. It dominates D if there appears in D° a derivative ∂^β of highest order, i.e., if the order of the operator equals the order of its initial form. This implies that in the one-variable case dominance is equivalent to 0 being a regular singular point; cf. Komatsu (1971) and Malgrange (1974). For several variables, the situation is much more involved. A sufficient though not necessary condition for dominance is that D° is a pure monomial $c_{\alpha\beta}x^\alpha\partial^\beta$ with a ∂-exponent β which is componentwise the maximum of all ∂-exponents β' of D, say $\beta'_j \leq \beta_j$ for all $\beta' \in \mathrm{supp}_\partial(\overline{D})$ and all $j = 1, \ldots, n$.

In the polynomial case, we equip $\mathcal{P}_n = \mathbb{K}[x]$ with the topology of coefficientwise convergence. In the case of formal power series, $\mathcal{P}_n = \mathbb{K}[[x]]$ comes with the m-adic topology given by the basis of zero-neighborhoods $(x_1, \ldots, x_n)^k$. Note here that also the ideals P_k generated by the monomials x^γ with $\lambda\gamma \geq k$ form such a basis. In the convergent case, $\mathcal{P}_n = \mathbb{K}\{x\}$ has the inductive limit topology given by the filtration $\mathcal{P}_n = \cup_{s>0}\mathcal{P}_n(s)$, where $\mathcal{P}_n(s)$ denotes the Banach space of series $a = \sum_\gamma a_\gamma x^\gamma$ with norm $|a|_s = \sum_\gamma |a_\gamma| s^{\lambda\gamma} < \infty$.

A $\mathbb{K}$-linear map $w : \mathcal{P}_n \to \mathcal{P}_n$ is *contractive* with respect to $\lambda \in \mathbb{R}^n_+$ if there exists an $\varepsilon > 0$ such that for all $\gamma \in \mathbb{N}^n$ and all monomials x^δ of the expansion of $w(x^\gamma)$ one has $\lambda\delta \leq \lambda\gamma - \varepsilon$ in the polynomial case and $\lambda\delta \geq \lambda\gamma + \varepsilon$ in the power series case. If w is contractive and $a \in \mathcal{P}_n$, the geometric series $\sum_{k=0}^\infty w^k(a)$ is finite in the polynomial case and converges in the formal power series case with respect to the m-adic topology on $\mathbb{K}[[x]]$. For convergent power series, one requires in addition that there are constants $s_0 > 0$ and $0 < C < 1$ such that the restrictions $w_s : \mathcal{P}_n(s) \to \mathcal{P}_n(s)$ are well defined for $0 < s < s_0$ and have operator norm $|w_s| \leq C$. In all cases the geometric series gives rise to a well defined linear map $\sum_{k=0}^\infty w^k : \mathcal{P}_n \to \mathcal{P}_n$.

Division of polynomials and series by differential operators

Let now $D_1, \ldots, D_p \in \mathcal{P}_n[\partial]$. We describe in this section how polynomials or power series e of $\mathcal{P}_n$ will be divided by $D_1, \ldots, D_p$, say $e = \sum D_i a_i + b$ with a_i and b in $\mathcal{P}_n$ specified by support conditions. This division can be extended to vectors $D_1, \ldots, D_p$ in $\mathcal{P}_n[\partial]^q$.

Expand $D_i = \sum_{\alpha\beta} c_{\alpha\beta i}x^\alpha\partial^\beta$ and set again $D_i = D_i^\circ - \overline{D}_i$ with initial form $D_i^\circ = \sum_{\alpha-\beta=\tau_i} c_{\alpha\beta i}x^\alpha\partial^\beta = \kappa_{\tau_i}\xi^{\tau_i}$ with respect to λ as before. We denote by $V_i = V(\kappa_{\tau_i})$ the zero-set of κ_{τ_i} in $\mathbb{N}^n$ and set $\Delta_i = \mathbb{N}^n \setminus V_i$. Then $K_i^\circ = \mathrm{Ker}\, D_i^\circ$ and $I_i^\circ = \mathrm{Im}\, D_i^\circ$ are monomial subspaces of $\mathcal{P}_n$ with support V_i and $\Delta_i + \tau_i$ respectively. Let $\Xi = \cup_i \Delta_i + \tau_i$ and choose a partition $\Xi = \dot\bigcup \Gamma_i + \tau_i$ with $\Gamma_i \subset \Delta_i$. Let $L^\circ \subset \mathcal{P}_n^p$ be the monomial subvector space of vectors $a = (a_1, \ldots, a_p)$ with a_i having support in $\Gamma_i - \tau_i$. It is then clear that L° is a direct complement of $K^\circ = \mathrm{Ker}\, D^\circ$ in $\mathcal{P}_n^p$, say $K^\circ \oplus L^\circ = \mathcal{P}_n^p$, where

$$D^\circ : \mathcal{P}_n^p \to \mathcal{P}_n,\ a = (a_1, \ldots, a_p) \to \textstyle\sum_i D_i^\circ a_i.$$

Similarly, $I^\circ = \operatorname{Im} D^\circ$ consists of series with support in Ξ and has direct monomial complement J° consisting of series with support in the complement $\Xi' = \mathbb{N}^n \setminus \Xi$. Of course, all these objects depend on the choice of λ.

Division Theorem. *Let $D_1, \ldots, D_p$ be differential operators in $\mathcal{P}_n[\partial]$ and let $\lambda \in \mathbb{R}^n_+$ be a weight vector with $\mathbb{Q}$-linearly independent components. Let D_i° be the initial form of D_i with respect to λ and let D and D° be the induced maps $\mathcal{P}_n^p \to \mathcal{P}_n$. Set $I^\circ = \operatorname{Im} D^\circ$, $K^\circ = \operatorname{Ker} D^\circ$ and choose the direct monomial complements L° of K° in $\mathcal{P}_n^p$ and J° of I° in $\mathcal{P}_n$ as before. In the convergent case, assume in addition that each D_i is dominated by its initial form D_i° with respect to λ. Then the map*

$$u : L^\circ \times J^\circ \to \mathcal{P}_n : (a, b) \to \sum D_i a_i + b$$

is a topological isomorphism. In particular, $\operatorname{Im} D + J^\circ = \mathcal{P}_n$ *and* $\operatorname{Ker} D \cap L^\circ = 0$. *If the $D_1, \ldots, D_p$ are perfect, one even has the direct sum decompositions* $\operatorname{Im} D \oplus J^\circ = \mathcal{P}_n$, *and* $\operatorname{Ker} D \oplus L^\circ = \mathcal{P}_n^p$.

Remarks. (a) Stated differently, for each $e \in \mathcal{P}_n$ there are unique series a_i and b with support in $\Gamma_i - \tau_i$ and Ξ' such that $e = \sum D_i a_i + b$. This implies that $\operatorname{Ker} D \cap L^\circ = 0$ and $\operatorname{Im} D + J^\circ = \mathcal{P}_n$. Moreover, the proof of the theorem provides an algorithm for computing a_i and b. The algorithm follows from the construction of the inverse of the operator u. It is given by the geometric series $(\sum_{k=0}^{\infty}(wv^{-1})^k)v$ where $v(a, b) = \sum_i D_i^\circ a_i + b$ and $w = v - u$. The partial sums $(\sum_{k=0}^{r}(wv^{-1})^k)v$ of the expansion of u^{-1} allow one to compute the quotients a_i and the remainder b up to arbitrarily high degree. We emphasize that without support conditions on the quotients a_i the remainder b is no longer unique.
(b) In the convergent case, u and u^{-1} respect the filtrations of $\mathcal{P}_n^p$ and $\mathcal{P}_n$ given by $\mathcal{P}_n = \bigcup_{s>0} \mathcal{P}_n(s)$.
(c) It would be interesting (and probably not too hard) to extend the Division Theorem to the case where $D_1, \ldots, D_p$ act on $\mathcal{P}_n[\partial]^p$ from the left via $E = (E_1, \ldots, E_p) \to \sum_i D_i \cdot E_i$ with $D \cdot E$ the multiplication in $\mathcal{P}_n[\partial]$. The image of D is then a right ideal of $\mathcal{P}_n[\partial]$. More generally, the D_i could be vectors in $\mathcal{P}_n[\partial]^q$ generating a right submodule of $\mathcal{P}_n[\partial]^q$. Compare this type of division with the Division Theorem of Castro-Jiménez (1984, 1987) for left ideals in $\mathcal{P}_n[\partial]$ or left submodules of $\mathcal{P}_n[\partial]^q$.
(d) As a variation, one may take in the theorem difference operators instead of differential operators, with a similar proof.
(e) In the case where the D_i are just polynomials or power series $h_i \in \mathcal{P}_n$, i.e., if no derivatives appear, dominance is a void condition. We recover the classical Buchberger and Grauert–Hironaka–Galligo division theorems for ideals in $\mathcal{P}_n$. Our method of proof works for all three cases simultaneously.

Note here that then the image of L° under u is a subvector space of $\mathcal{P}_n$ which, in general, will be strictly contained in the ideal $(h_1, \ldots, h_p)$ generated by the h_i. In the case where the h_i are perfect with respect to λ, the two spaces coincide (because they have the same initial monomials) and the theorem yields $(h_1, \ldots, h_p) \oplus J^\circ = \mathcal{P}_n$, where J° is the canonical direct monomial complement of $I^\circ = (\mathrm{in}(h_1), \ldots, \mathrm{in}(h_i))$.

Similarly, if the differential operators $D_1, \ldots, D_p$ are perfect with respect to λ we have that $\operatorname{Im} D \oplus J^\circ = \mathcal{P}_n$ and $\operatorname{Ker} D \oplus L^\circ = \mathcal{P}_n^p$. In particular, the remainders of the division by

D_i are unique without imposing any support conditions on the quotients. This leads to the following criterion for the existence of particular solutions of inhomogeneous differential equations.

Existence Theorem for inhomogeneous differential equations. *Let $D_1, \dots, D_p$ be differential operators in $\mathcal{P}_n[\partial]$ and let $\lambda \in \mathbb{R}^n_+$ be a weight vector with $\mathbb{Q}$-linearly independent components. Assume that $D_1, \dots, D_p$ are perfect with respect to λ. In the convergent case, assume in addition that each D_i is dominated by its initial form D_i° with respect to λ. Then, for any $e \in \mathcal{P}_n$, the differential equation $\sum_i D_i a_i = e$ has a solution $a = (a_1, \dots, a_p) \in \mathcal{P}_n^p$ if and only if the division of e by $D_1, \dots, D_p$ has remainder equal to zero.*

Despite the fact that there seems to be no known algorithm for checking whether differential operators are perfect and that for power series the actual division requires infinitely many substitution steps, the algorithm can be useful, e.g., for proving that a differential equation has *no* polynomial or formal power series solution: if after some steps of the algorithm there appears a remainder, no solution will exist. Conversely, if after many steps no remainder has shown up, it may become probable that a solution exists, or that such a solution can be guessed from the expansion of the approximated solution computed so far. Of course, the method is very close to a power series ansatz with unknown coefficients.

Variants of the Existence Theorem and different algorithms for computing solutions of differential equations in various circumstances abound. To mention just a few, we recommend Janet's work from 1920 (Janet, 1920), Abramov's (Abramov, 1995) and Petkovšek's (Petkovšek, 1992) algorithms for difference and differential equations in one variable, the book of Saito et al. (2000) on Gröbner deformations, and algorithms developed by Oaku et al. (2001), or by Della Dora et al. (1982).

Proof of the Division Theorem. We follow the technique of the proof of Theorem 5.1 in Hauser and Müller (1994), interpreting u as a perturbation of an operator which is trivially an isomorphism. This method is frequent when using arguments from functional analysis for the study of differential equations; cf. Malgrange (1974, p. 148), or Castro-Jiménez (1984, 1987) and Hauser and Narváez-Macarro (2001). The dominance condition is used to bound the size of the coefficients of the series when applying the inverses of monomial differential operators. In contrast to the division theorems in Castro-Jiménez (1984, 1987) and Hauser and Narváez-Macarro (2001), the filtration of $\mathcal{P}_n[\partial]$ is given in the present theorem by the shifts of the operators.

Decompose $u : L^\circ \times J^\circ \to \mathcal{P}_n$ into $u = v - w$ with $v(a, b) = \sum D_i^\circ a_i + b$ according to $D_i = D_i^\circ - \overline{D}_i$. By definition of L° and J°, v is a topological isomorphism. It is therefore sufficient to show that $uv^{-1} = \mathrm{Id} - wv^{-1} : \mathcal{P}_n \to \mathcal{P}_n$ is also a topological isomorphism. Its inverse is formally given by the infinite sum of operators $\sum_{k=0}^{\infty} (wv^{-1})^k$. One has to prove that this sum does indeed define a continuous linear operator from $\mathcal{P}_n$ to $\mathcal{P}_n$.

In the polynomial and formal power series cases it suffices to show that wv^{-1} is contractive with respect to λ, i.e., that there is an $\varepsilon > 0$ such that for all $\gamma \in \mathbb{N}^n$ the series $wv^{-1}(x^\gamma)$ involves only monomials x^δ with $\lambda\delta \leq \lambda\gamma - \varepsilon$ and $\lambda\delta \geq \lambda\gamma + \varepsilon$, respectively. In the convergent power series case we have to show in addition that there exist constants $s_0 > 0$ and $0 < C < 1$ such that the restrictions $(wv^{-1})_s : \mathcal{P}_n(s) \to \mathcal{P}_n(s)$ are well defined and of norm $\leq C$ for all $0 < s < s_0$ as linear operators between Banach spaces. This signifies that for s sufficiently small and all $e \in \mathcal{P}_n(s)$ one has

$$|wv^{-1}e|_s \leq C \cdot |e|_s.$$

We shall concentrate on the case of convergent power series, which is the hardest one. Moreover it involves the required inequality from the dominance assumption. The polynomial and formal power series cases are easier and can be obtained from the proof below by the obvious modifications.

Write $e = \sum_i e_i + b$ according to the direct sum decomposition of $\mathcal{P}_n$ induced by v and given by the supports $\Gamma_i + \tau_i$ and Ξ'. Then the e_i and b have pairwise disjoint support, so

$$|e|_s = \textstyle\sum_i |e_i|_s + |b|_s.$$

Expand e_i as $e_i = \sum_{\gamma \in \Gamma_i} e_{\gamma i} x^{\gamma + \tau_i}$. Then

$$\begin{aligned} v^{-1}e &= \left(\left(\textstyle\sum_{\alpha-\beta=\tau_i} c_{\alpha\beta i} x^\alpha \partial^\beta\right)^{-1} e_i\right)_i, b) \\ &= \left(\left(\textstyle\sum_{\gamma\in\Gamma_i} e_{\gamma i} (\sum_{\alpha-\beta=\tau_i} c_{\alpha\beta i} \gamma^{\underline{\beta}})^{-1} x^\gamma\right)_i, b\right). \end{aligned}$$

Therefore, setting $\kappa_i(\gamma) = \sum_{\alpha-\beta=\tau_i} c_{\alpha\beta i} \gamma^{\underline{\beta}}$, we have

$$\begin{aligned} wv^{-1}e &= w\left(\left(\textstyle\sum_{\gamma\in\Gamma_i} e_{\gamma i} \kappa_i(\gamma)^{-1} x^\gamma\right)_i, b\right) \\ &= \textstyle\sum_i \sum_{\alpha'-\beta'>\tau_i} c_{\alpha'\beta' i}\, x^{\alpha'} \partial^{\beta'} \left(\textstyle\sum_{\gamma\in\Gamma_i} e_{\gamma i} \kappa_i(\gamma)^{-1} x^\gamma\right) \\ &= \textstyle\sum_i \sum_{\alpha'-\beta'>\tau_i} \sum_{\gamma\in\Gamma_i} c_{\alpha'\beta' i}\, e_{\gamma i} \kappa_i(\gamma)^{-1} \gamma^{\underline{\beta'}} x^{\gamma+\alpha'-\beta'}. \end{aligned}$$

Here, $\alpha' - \beta' > \tau_i$ stands for $\lambda(\alpha' - \beta') > \lambda\tau_i$. As there are only finitely many β', there exists an $\varepsilon > 0$ such that $\lambda(\alpha' - \beta') > \lambda\tau_i + \varepsilon$ for all $i = 1, \ldots, p$ and all $(\alpha', \beta') \in \operatorname{supp} D_i$. This shows in the polynomial and formal power series cases that wv^{-1} is contractive. In the convergent case, we need norm estimates for the series involved.

By dominance we know that there is a constant $C > 0$ such that $\gamma^{\underline{\beta'}} \leq C \cdot \kappa_i(\gamma)$ for all $i = 1, \ldots, p$ and all $\beta' \in \operatorname{supp}_\partial(\overline{D}_i)$, $\gamma \in \Gamma_i$. This implies the inequalities

$$\begin{aligned} \frac{|wv^{-1}e|_s}{|e|_s} &\leq C \cdot \frac{\sum_i \sum_{\alpha'-\beta'>\tau_i} \sum_{\gamma\in\Gamma_i} |c_{\alpha'\beta' i}| \cdot |e_{\gamma i}| \cdot s^{\lambda(\gamma+\alpha'-\beta')}}{\sum_i \sum_{\gamma\in\Gamma_i} |e_{\gamma i}| \cdot s^{\lambda(\gamma+\tau_i)} + |b|_s} \\ &\leq C \cdot \frac{\sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| \cdot s^{\lambda(\alpha'-\beta'-\tau_i)} \sum_{\gamma\in\Gamma_i} |e_{\gamma i}| \cdot s^{\lambda(\gamma+\tau_i)}}{\sum_i \sum_{\gamma\in\Gamma_i} |e_{\gamma i}| \cdot s^{\lambda(\gamma+\tau_i)}} \\ &\leq C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} \frac{|c_{\alpha'\beta' i}| \cdot s^{\lambda(\alpha'-\beta'-\tau_i)} \sum_{\gamma\in\Gamma_i} |e_{\gamma i}| \cdot s^{\lambda(\gamma+\tau_i)}}{\sum_{\gamma\in\Gamma_i} |e_{\gamma i}| \cdot s^{\lambda(\gamma+\tau_i)}} \\ &\leq C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| \cdot s^{\lambda(\alpha'-\beta'-\tau_i)}. \end{aligned}$$

The convergence of the coefficients $\sum_\alpha c_{\alpha\beta i} x^\alpha$ of ∂^β in D_i implies that this last sum converges for s sufficiently small, say $0 < s < s_0$. We have already seen that there is an $\varepsilon > 0$ such that $\lambda(\alpha' - \beta' - \tau_i) > \varepsilon$ for all i and all α' and β'. Therefore, choosing s_0 sufficiently small, we get

$$C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| \cdot s^{\lambda(\alpha'-\beta'+\tau_i)} < 1$$

for all $0 < s < s_0$. Hence $|wv^{-1}e|_s \leq C' \cdot |e|_s$ for some $C' < 1$ and all $0 < s < s_0$. This proves also in the convergent case that u is a topological isomorphism.

It remains to show that $\operatorname{Im} D \cap J^\circ = 0$ and $\operatorname{Ker} D + L^\circ = \mathcal{P}_n^p$ if $D_1, \ldots, D_p$ are perfect. For the first, an element of the intersection $\operatorname{Im} D \cap J^\circ$ has, by the very definition of perfect operators, its initial monomial in I° and J°, whence $\operatorname{Im} D \cap J^\circ = 0$. For the second, notice that $\operatorname{Im} D \oplus J^\circ = \mathcal{P}_n$ and $D(L^\circ) \oplus J^\circ = \mathcal{P}_n$ implies that $\operatorname{Im} D = D(L^\circ)$. This in turn implies that $\operatorname{Ker} D + L^\circ = \mathcal{P}_n^p$. The proof of the Division Theorem is completed.

Application to inhomogeneous differential equations

We illustrate the division algorithm in simple examples by computing particular power series solutions of inhomogeneous differential equations. Let us take one differential operator $D = \sum_{k=0}^m \sum_{j=0}^\infty c_{jk} x^j \partial^k$ in one variable x of order m. Denote by o_k the order of the coefficient series $c_k(x) = \sum_{j=0}^\infty c_{jk} x^j$ at 0. The choice of a weight vector λ is superfluous. Let τ be the minimal shift of D, $\tau = \min\{o_k - k, k \leq m\}$, let $D^\circ = \sum_{j-k=\tau} c_{jk} x^j \partial^k$ be the initial form of D and set $D = D^\circ - \overline{D}$. Then D^0 is dominant if and only if $o_m - m = \tau$, i.e., if and only if 0 is a regular singular point of the equation $D(a) = 0$.

The kernel K° of D° is spanned by monomials x^l with $\sum_{j-k=\tau} c_{jk} l^{\underline{k}} = 0$ and is hence finite dimensional. The description of its direct monomial complement L° and the image I° of D° with direct monomial complement J° is then immediate.

Example 1. Let us take for instance the operator $D = x^2\partial^2 - x\partial - x^3$ with $D^\circ = x^2\partial^2 - x\partial$ and $\overline{D} = x^3$. Here, 0 is a regular singular point, i.e., D° is dominant, so we may neglect convergence questions and work with formal power series. The kernel K° of D° is spanned by the monomials x^l with $l^{\underline{2}} - l^{\underline{1}} = l(l-2) = 0$, say $K^\circ = \mathbb{K} \oplus \mathbb{K}x^2$. We get $L^\circ = I^\circ = \mathbb{K}x \oplus \mathbb{K}[[x]]x^3$ and $J^\circ = \mathbb{K} \oplus \mathbb{K}x^2$. As $\overline{D}$ sends $\mathbb{K}[[x]]$ to $\mathbb{K}[[x]]x^3$ and thus produces no new initial monomials it follows that D is a perfect operator.

Let us divide a monomial $e = x^l$ by D. For $e = 1$ or $e = x^2$ in J°, no real division occurs, $e = D0 + e$, and e equals the remainder. For $e = x^l$ with $l \neq 0, 2$ the first step of the algorithm produces

$$x^l = D^\circ \tfrac{1}{l(l-2)} x^l = D \tfrac{1}{l(l-2)} x^l + \tfrac{1}{l(l-2)} x^{l+3}.$$

For $l = 1$ or $l \geq 3$ the algorithm repeats infinitely often and yields

$$x^l = D \left(\textstyle\sum_{i\geq 0} \frac{1}{l(l+3)\cdots(l+3i)(l-2)(l+1)\cdots(l+3i-2)} x^{l+3i} \right).$$

We can conclude that the differential equation $x^2a'' - xa' - x^3 = e(x)$ has (convergent or formal) power series solutions $a(x)$ at $x = 0$ if and only if $e(x)$ has no constant term and no monomial x^2. This, of course, could have also been deduced – due to the simplicity of the example – by direct inspection of the equation or a power series ansatz for $a(x)$.

Example 1*bis*. The operator $D = x^2\partial^2 - x\partial - x^2$ with $D^\circ = x^2\partial^2 - x\partial$ and $\overline{D} = x^2$ is not perfect, because $\overline{D}$ sends the kernel $K^\circ = \mathbb{K} + \mathbb{K}x^2$ to $\mathbb{K}x^2 + \mathbb{K}x^4$ with new initial monomial $x^2 \notin I^\circ = \mathbb{K}x + \mathbb{K}[[x]]x^3$. The division of x^2 by D with initial form D° and quotient in $L^\circ = \mathbb{K}x + \mathbb{K}[[x]]x^3$ is trivial with remainder x^2, say $x^2 = D0 + x^2$, though x^2 lies in the image of D via $x^2 = D1$. We see that the remainder of the division is not unique without imposing support conditions on the quotient.

Example 1*ter*. The operator $D = x^2\partial^2 - x\partial - x$ with $D^\circ = x^2\partial^2 - x\partial$ and $\overline{D} = x$ is not perfect either, but the argument is a little trickier. We have as before $K^\circ = \mathbb{K} \oplus \mathbb{K}x^2$, $L^\circ = I^\circ = \mathbb{K}x \oplus \mathbb{K}[[x]]x^3$ and $J^\circ = \mathbb{K} \oplus \mathbb{K}x^2$. For $e = 1$ and $e = x^2$ we get the trivial divisions $e = D0+e$, though $x^2 = D(1-x)+0$ lies in the image of D. So the remainder of the division of x^2 by D is not unique and D is not a perfect operator. And indeed, $x^2 \notin I^\circ$ is a new initial monomial of the image of D. If we divide $e = x$ by D we get after the first substitution step $x = D(-x) - x^2$ and have to redivide x^2 by D as before, producing the same ambiguity of the remainder. When dividing a monomial $e = x^l$ with $l \geq 3$ by D the remainder is unique, with

$$x^l = D\left((l-1)!(l-3)! \sum_{i\geq 0} \frac{1}{(l+i)!(l+i-2)!} x^{l+i}\right).$$

Example 2. Let us now consider in the case of formal power series the operator $D = -y\partial_x + \partial_y$ with shifts $(-1, 1)$ and $(0, -1)$. Here the choice of the weight vector $\lambda = (\lambda_1, \lambda_2)$ becomes relevant. If $-\lambda_1 + \lambda_2 > -\lambda_2$, then $D^\circ = \partial_y$ and D is perfect. As D° has image $\mathbb{K}[[x, y]]$ and hence $J^\circ = 0$, the Division Theorem implies that for any $e \in \mathbb{K}[[x, y]]$, the differential equation $Da = e$ has a formal power series solution $a \in \mathbb{K}[[x, y]]$. For $e = x^k y^l$ we get the polynomial solution

$$a = \sum_{i=0}^{k} \frac{k^{\underline{i}}}{(l+1)(l+3)\dots(l+2i+1)} x^{k-i} y^{l+2i+1}.$$

Observe here that D° is not dominant, though all formal solutions are already convergent.

Example 2*bis*. Take again $D = -y\partial_x + \partial_y$, but with weight vector λ satisfying $-\lambda_1+\lambda_2 < -\lambda_2$. The initial form of D equals $D^\circ = -y\partial_x$ with kernel $K^\circ = \mathbb{K}[[y]]$ and image $I^\circ = \mathbb{K}[[x, y]]y$. We have $L^\circ = \mathbb{K}[[x, y]]x$ and $J^\circ = \mathbb{K}[[x]]$. In this case, D is not perfect, because, for instance, $Dy = 1$ is a monomial in the image of D which does not lie in I°. Dividing $x^k y^l$ by D we get for $l = 0$ the trivial division $x^k = D0 + x^k$ with remainder x^k, and for odd $l \geq 1$ the quotient

$$a = -\sum_{i=0}^{(l-1)/2} \frac{(l-1)(l-3)\cdots(l-2i+1)}{(k+1)(k+2)\cdots(k+i+1)} x^{k+i+1} y^{l-2i-1},$$

with remainder 0, i.e., a is a solution of $Da = x^k y^l$. For even $l \geq 1$ the division has quotient

$$a = -\sum_{i=0}^{l/2-1} \frac{(l-1)(l-3)\cdots(l-2i+1)}{(k+1)(k+2)\cdots(k+i+1)} x^{k+i+1} y^{l-2i-1}$$

and remainder

$$b = \frac{(l-1)(l-3)\cdots 1}{(k+1)(k+2)\cdots(k+l/2+1)} x^{k+l/2}.$$

Monomialization

In the case of perfect differential operators, the Division Theorem describes direct complements of the kernel and image of $D : \mathcal{P}_n^p \to \mathcal{P}_n$. This, in turn, allows us to describe particular solutions of inhomogeneous equations $Da = e$. We are now interested in the operator itself, and in the solution space of the homogeneous equation $Da = 0$. For this we shall show that by linear isomorphisms on the source and target, D can be transformed into its initial form D°.

Monomialization Theorem. *Let $D : \mathcal{P}_n^p \to \mathcal{P}_n$ be given by differential operators $D_1, \ldots, D_p$ which are perfect with respect to a weight vector $\lambda \in \mathbb{R}_+^n$ with $\mathbb{Q}$-linearly independent components. Set $D = D^\circ - \overline{D}$ with initial form D° with respect to λ. In the convergent power series case, assume in addition that the D_i° are dominant for D_i. There then exist topological linear automorphisms u of $\mathcal{P}_n^p$ and v of $\mathcal{P}_n$ such that*

$$vDu^{-1} = D^\circ.$$

In the case where the $D_1, \ldots, D_p$ are not necessarily perfect operators one has $vD_{|L^\circ} = D^\circ u_{|L^\circ}$, where L° denotes a direct monomial complement of Ker D° *in $\mathcal{P}_n^p$.*

Remarks. (a) Again, requiring that the D_i are perfect is quite restrictive and in practice possibly hard to verify. We describe below examples where this works and fails.
(b) A suitable automorphism u can be explicitly described, $u = \mathrm{Id}_{\mathcal{P}_n^p} - S\overline{D}$, where S is a *scission* of D°; see the proof. Then $u^{-1} = \mathrm{Id}_{\mathcal{P}_n^p} + \sum_{k=1}^{\infty}(S\overline{D})^k$, which allows us to compute the solutions of D from the solutions of D° up to arbitrary degree. The automorphism v is given as $v = \mathrm{Id}_{\mathcal{P}_n} - (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1}SD^\circ S$.
(c) In the convergent case, the automorphisms u and v respect the filtration of $\mathcal{P}_n^p$ and $\mathcal{P}_n$ by the Banach spaces $\mathcal{P}_n(s)^p$ and $\mathcal{P}_n(s)$.
(d) The equality $vDu^{-1} = D^\circ$ can be rephrased by saying that D° is a normal form for D. This does not mean the normal form in the usual sense with respect to coordinate changes in the variables $x_1, \ldots, x_n$ but the normal form with respect to linear automorphisms on the source space $\mathcal{P}_n^p$ and the target space $\mathcal{P}_n$ of the operator D.
(e) As with the Division Theorem, it may be worthwhile to try to extend the assertion of the theorem to the case of operators $D : \mathcal{P}_n[\partial]^p \to \mathcal{P}_n[\partial]$.

The Monomialization Theorem applies directly to the characterization of the solution space of the induced homogeneous differential equation.

Existence Theorem for homogeneous differential equations. *With the assumptions and notation of the Monomialization Theorem, a vector of polynomials or power series $a \in \mathcal{P}_n^p$ forms a solution of the differential equation $\sum_i D_i a_i = 0$ if and only if $a = u^{-1}(a^\circ)$ where $a^\circ \in K^\circ$ is a solution of the differential equation $\sum_i D_i^\circ a_i^\circ = 0$.*

Remarks. (a) Solving the differential equation of a single operator $D \in \mathcal{P}_n[\partial]$ yields the monomial differential equation $D^\circ a = 0$. Its solution space is spanned by monomials which can be determined by solving the diophantine equation $\kappa_\tau(\gamma) = 0$ for $\gamma \in \mathbb{N}^n$, where $\kappa_\tau(\gamma) = \sum_{\alpha-\beta=\tau} c_{\alpha\beta}\gamma^{\underline{\beta}}$ is the coefficient function of D°. To do this explicitly for $n \geq 2$ may be arbitrarily difficult.

In general, solving $\sum_i D_i^\circ a_i = 0$ reduces immediately to the "binomial" case $D_1^\circ a_1 = D_2^\circ a_2$ which can be solved by comparing the shifts and the coefficient functions.

(b) The assertion of the theorem extends naturally to the case of systems of differential equations given by vectors $D_1, \dots, D_p \in \mathcal{P}_n[\partial]^q$.

(c) In the holonomic case, the solution space $\operatorname{Ker} D$ is finite dimensional. By the theorem, its dimension then equals the dimension of K°.

Proof of the Monomialization Theorem. We use the notation of the Division Theorem. In particular, we set $I^\circ = \operatorname{Im} D^\circ$, $K^\circ = \operatorname{Ker} D^\circ$ and let L° and J° be the canonical direct monomial complements of K° and I° in $\mathcal{P}_n^p$ and $\mathcal{P}_n$ respectively. Then $D^\circ{}_{|L^\circ} : L^\circ \to I^\circ$ is an isomorphism and $S = (D^\circ{}_{|L^\circ})^{-1}\pi_{I^\circ} : \mathcal{P}_n \to L^\circ$ is a scission of D°, i.e., by definition, $D^\circ S D^\circ = D^\circ$, where $\pi_{I^\circ} : \mathcal{P}_n \to I^\circ$ is the projection given by $\mathcal{P}_n = I^\circ \oplus J^\circ$.

Set $u = \operatorname{Id}_{\mathcal{P}_n^p} - S\overline{D} : \mathcal{P}_n^p \to \mathcal{P}_n^p$. We prove first that u is an isomorphism with inverse $u^{-1} = \operatorname{Id}_{\mathcal{P}_n^p} + \sum_{k=1}^\infty (S\overline{D})^k$. The definition of v and the equality $vDu^{-1} = D^\circ$ will be postponed to after this proof. The polynomial and the formal power series cases are contained in the case of convergent series as in the proof of the Division Theorem and will be omitted.

In the convergent case we show that u is compatible with the Banach space filtrations and that there is an $s_0 > 0$ such that for $0 < s < s_0$ the restriction $(S\overline{D})_s$ to $\mathcal{P}_n(s)^p$ has norm $\leq C$ with a constant $C < 1$ independent of s. This implies the convergence of the geometric series $\sum_{k=0}^\infty (S\overline{D})_s^k = u_s^{-1}$ and shows that u_s is an isomorphism.

Let $e = \sum_\gamma e_\gamma x^\gamma$ be in $\mathcal{P}_n^p$ with $e_\gamma \in \mathbb{K}^p$. Then

$$\begin{aligned}\overline{D}e &= \sum_i \sum_{\alpha'-\beta'>\tau_i} c_{\alpha'\beta' i} x^{\alpha'} \partial^{\beta'} \sum_\gamma e_{\gamma i} x^\gamma \\ &= \sum_i \sum_{\alpha'-\beta'>\tau_i} \sum_\gamma c_{\alpha'\beta' i} e_{\gamma i} \gamma^{\underline{\beta'}} x^{\gamma+\alpha'-\beta'}.\end{aligned}$$

Decompose this series according to $\mathcal{P}_n = I^\circ \oplus J^\circ$. More specifically, write $a \in I^\circ$ as $a = \sum_i \sum_{\delta \in \Gamma_i} a_{\delta+\tau_i} x^{\delta+\tau_i}$ where, as in the proof of the Division Theorem, the sets $\Gamma_i + \tau_i \subset \Delta_i + \tau_i$ form a partition of $\Xi = \bigcup_i \Delta_i + \tau_i$ with $\Delta_i = \{\delta \in \mathbb{N}^n,\ \kappa_i(\delta) \neq 0\}$ and $\kappa_i(\delta) = \sum_{\alpha-\beta=\tau_i} c_{\alpha\beta i}\delta^{\underline{\beta}}$. This allows us to write

$$\pi_{I^\circ}\overline{D}e = \sum_i \sum_{\alpha'-\beta'>\tau_i} \sum_{\gamma \in \Gamma_i+\beta'-\alpha'+\tau_i} c_{\alpha'\beta' i} e_{\gamma i} \gamma^{\underline{\beta'}} x^{\gamma+\alpha'-\beta'-\tau_i} x^{\tau_i}.$$

The i-th component of $S\overline{D}e$ can be expanded into

$$(S\overline{D}e)_i = (\kappa_i \xi^{\tau_i})^{-1} \left(\sum_{\alpha'-\beta'>\tau_i} \sum_{\gamma \in \Gamma_i+\beta'-\alpha'+\tau_i} c_{\alpha'\beta' i} e_{\gamma i} \gamma^{\underline{\beta'}} x^{\gamma+\alpha'-\beta'} \right)$$

$$= \sum_{\alpha'-\beta'>\tau_i} \sum_{\gamma\in\Gamma_i+\beta'-\alpha'+\tau_i} c_{\alpha'\beta' i} e_{\gamma i} \gamma^{\underline{\beta'}} \kappa_i(\gamma+\alpha'-\beta'-\tau_i)^{-1} x^{\gamma+\alpha'-\beta'-\tau_i}$$
$$= \sum_{\alpha'-\beta'>\tau_i} c_{\alpha'\beta' i} x^{\alpha'-\beta'-\tau_i} \sum_{\gamma\in\Gamma_i+\beta'-\alpha'+\tau_i} e_{\gamma i} \gamma^{\underline{\beta'}} \kappa_i(\gamma+\alpha'-\beta'-\tau_i)^{-1} x^{\gamma}.$$

We claim that $\gamma^{\underline{\beta'}}\kappa_i(\gamma+\alpha'-\beta'-\tau_i)^{-1}$ remains bounded for all $\gamma \in \Gamma_i+\beta'-\alpha'+\tau_i$ and all $(\alpha',\beta') \in \operatorname{supp}(\overline{D}_i)$. To see this, set $\delta = \gamma+\alpha'-\beta'-\tau_i \in \Gamma_i$. Then notice that there is – since there are only finitely many β' and since $\delta-(\alpha'-\beta'-\tau_i)$ belongs to $\mathbb{N}^n$ – a constant $C' > 0$ with

$$(\delta-(\alpha'-\beta'-\tau_i))^{\underline{\beta'}} \leq C' \cdot \delta^{\underline{\beta'}}$$

for all $\delta \in \Gamma_i$ and all $(\alpha',\beta') \in \operatorname{supp}(\overline{D}_i)$. By dominance, we get $(\delta-(\alpha'-\beta'-\tau_i))^{\underline{\beta'}} \leq C \cdot \kappa_i(\delta)$ for some constant $C > 0$, whence the required boundedness. We may therefore continue with

$$|S\overline{D}e|_s \leq C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| s^{\lambda(\alpha'-\beta'-\tau_i)} \cdot \sum_{\gamma\in\Gamma_i+\beta'-\alpha'+\tau_i} |e_{\gamma i}| s^{\lambda\gamma}$$

for some constant $C > 0$, so

$$\frac{|S\overline{D}e|_s}{|e|_s} \leq C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| s^{\lambda(\alpha'-\beta'-\tau_i)} \cdot \frac{\sum_{\gamma\in\Gamma_i+\beta'-\alpha'+\tau_i} |e_{\gamma i}| s^{\lambda\gamma}}{\sum_\gamma |e_{\gamma i}| s^{\lambda\gamma}}$$
$$\leq C \cdot \sum_i \sum_{\alpha'-\beta'>\tau_i} |c_{\alpha'\beta' i}| s^{\lambda(\alpha'-\beta'-\tau_i)}.$$

Use now again that there is an $\varepsilon > 0$ such that $\lambda(\alpha'-\beta'-\tau_i) > \varepsilon$ for all $i = 1,\ldots,p$ and $(\alpha',\beta') \in \operatorname{supp}(\overline{D}_i)$. It follows that there is an $s_0 > 0$ and a constant $C' < 1$ independent of e such that this last sum is $\leq C'$ for $0 < s < s_0$. This proves that $(S\overline{D}e)_s$ has norm < 1 as required.

We have shown that u is a compatible isomorphism for $0 < s < s_0$. Set $v = \mathrm{Id}_{\mathcal{P}_n} - (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1}SD^\circ S$. It is checked that v is a compatible linear isomorphism of $\mathcal{P}_n$ with inverse $v = \mathrm{Id}_{\mathcal{P}_n} + (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1}SD^\circ S$. We claim that $vDu^{-1} = D^\circ$. For this we need:

Lemma. *The map u restricts to an isomorphism from $K = \operatorname{Ker} D$ to $K^\circ = \operatorname{Ker} D^\circ$.*

Proof. We have

$$\begin{aligned} D^\circ u &= D^\circ(\mathrm{Id}_{\mathcal{P}_n^p} - S\overline{D}) \\ &= D^\circ(\mathrm{Id}_{\mathcal{P}_n^p} - SD + SD^\circ) \\ &= D^\circ - D^\circ SD + D^\circ SD^\circ \\ &= D^\circ SD \\ &= D^\circ(D^\circ_{|L^\circ})^{-1}\pi_{I^\circ} D \\ &= \pi_{I^\circ} D. \end{aligned}$$

Let $a \in \mathcal{P}_n^p$. Then $u(a) \in K^\circ$ if and only if $D^\circ u(a) = \pi_{I^\circ} D(a) = 0$, say $D(a) \in J^\circ$. By the Division Theorem and since the D_i are perfect, we have $\operatorname{Im} D \cap J^\circ = 0$ so that $u(a) \in K^\circ$ if and only if $D(a) = 0$, say $a \in K$. This proves the assertion.

The lemma implies that $Du^{-1}(K^\circ) = 0$, which can be written as $Du^{-1}\pi_{K^\circ} = Du^{-1}(\mathrm{Id}_{\mathcal{P}_n^p} - SD^\circ) = 0$, and hence $Du^{-1} = Du^{-1}SD^\circ$. Because $D^\circ SD = D^\circ S(D^\circ - \overline{D}) = D^\circ SD^\circ - D^\circ S\overline{D} = D^\circ - D^\circ S\overline{D} = D^\circ u$, we obtain

$$\begin{aligned} vDu^{-1} &= Du^{-1} - (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1}SD^\circ SDu^{-1} \\ &= Du^{-1} - (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1}SD^\circ \\ &= Du^{-1} - (\mathrm{Id}_{\mathcal{P}_n} - D^\circ S)Du^{-1} \\ &= D^\circ SDu^{-1} = D^\circ. \end{aligned}$$

This proves the Monomialization Theorem.

Application to homogeneous differential equations

Let us now show how the Monomialization Theorem can be used to compute the solution space of a homogeneous differential equation. We work for simplicity with formal power series.

Example 1. Take again the perfect operator $D = x^2\partial_x^2 - x\partial_x - x^3$ with $D^\circ = x^2\partial_x^2 - x\partial_x$ and $\overline{D} = x^3$. We have $D^\circ x^l = l(l-2)x^l$ and $K^\circ = J^\circ = \mathbb{K} \oplus \mathbb{K}x^2$, $L^\circ = I^\circ = \mathbb{K}x \oplus \mathbb{K}[[x]]x^3$. To compute the isomorphism $u = \mathrm{Id} - S\overline{D}$ and its inverse u^{-1}, observe that $S\overline{D}(x^l) = \frac{1}{(l+3)(l+1)}x^{l+3}$. This gives for the kernel $\operatorname{Ker} D = u^{-1}(K^\circ)$ the $\mathbb{K}$-basis

$$\begin{aligned} &u^{-1}(1) = 1 + \tfrac{1}{3}x^3 + \tfrac{1}{3\cdot 24}x^3 + \tfrac{1}{3\cdot 24\cdot 36}x^3 + \cdots \\ &u^{-1}(x^2) = x^2 + \tfrac{1}{15}x^5 + \tfrac{1}{15\cdot 48}x^8 + \tfrac{1}{15\cdot 48\cdot 99}x^{11} + \cdots. \end{aligned}$$

Example 1bis. For the operator $D = x^2\partial_x^2 - x\partial_x - x$ which is not perfect, we get the same kernel $K^\circ = \mathbb{K} \oplus \mathbb{K}x^2$ of $D^\circ = x^2\partial_x^2 - x\partial_x$ but $u^{-1}(K^\circ) \neq \operatorname{Ker} D$ because $u^{-1}(1) = 1 + x$ does not belong to $\operatorname{Ker} D$. However, $u^{-1}(x^2) = x^2 + \frac{1}{3}x^3 + \frac{1}{3\cdot 8}x^4 + \frac{1}{3\cdot 8\cdot 15}x^5 + \cdots$ is in this kernel.

Example 2. Consider $D = D = \partial_y - y\partial_x$ with initial form $D^\circ = \partial_y$ and queue $\overline{D} = y\partial_x$. It forms a perfect operator. We have $K^\circ = \mathbb{K}[[x]]$, $L^\circ = \mathbb{K}[[x, y]]y$, $I^\circ = \mathbb{K}[[x, y]]$, $J^\circ = 0$, so $S(x^k y^l) = \frac{1}{l+1}x^k y^{l+1}$ and $S\overline{D}(x^k y^l) = \frac{k}{l+2}x^{k-1}y^{l+2}$. We get $\operatorname{Ker} D = u^{-1}(K[[x]])$ with

$$u^{-1}(x^k) = \sum_{i=0}^{k} \frac{k^{\underline{i}}}{2^i \cdot i!} x^{k-i} y^{2i}.$$

Example 2bis. Take instead $D = D = y\partial_x - \partial_y$ with initial form $D^\circ = y\partial_x$ and queue $\overline{D} = \partial_y$. It is not perfect. We have $K^\circ = \mathbb{K}[[y]]$, $L^\circ = \mathbb{K}[[x, y]]x$, $I^\circ = \mathbb{K}[[x, y]]y$, $J^\circ = \mathbb{K}[[x]]$, so $S(x^k) = 0$ and $S(x^k y^l) = \frac{1}{l+1}x^k y^{l+1}$ for $l \geq 1$. Taking e.g. $y \in K^\circ$ we get $S\overline{D}(y) = S(1) = 0$ and $u^{-1}(y) = y$, but $y \notin \operatorname{Ker} D$.

Outlook and open problems

In the perspective of the results of the present paper, there remain several things to be investigated: (1) how to check whether differential operators $D_1, \ldots, D_p$ are already perfect; (2) if they are not, how to compute all initial monomials of their image – if this is done, a refined division algorithm has to be formulated; (3) in the convergent power series case, dominance is a sufficient but not necessary condition to ensure the convergence of the solutions – find other criteria for convergence.

We comment on points (1) and (2). For simplicity, we restrict consideration to a single operator $D = \sum_{\alpha\beta} c_{\alpha\beta} x^\alpha \partial^\beta \in \mathcal{P}_n[\partial]$. Let $D^\circ = \sum_{\alpha-\beta=\tau} c_{\alpha\beta} x^\alpha \partial^\beta$ be its initial form with respect to a chosen weight vector λ, with kernel $K^\circ = \operatorname{Ker} D^\circ$ and direct monomial complement L°. We wish to check whether $I^\circ = \operatorname{Im} D^\circ = D^\circ(L^\circ)$ already contains all initial monomials of the image $I = \operatorname{Im} D$ of D. This is certainly the case if all monomial summands $D^\sigma = \sum_{\alpha-\beta=\sigma} c_{\alpha\beta} x^\alpha \partial^\beta$ of D produce under application to $\mathcal{P}_n$ only initial monomials which lie in I°. Then D will be perfect. This was the case in Examples 1 and 2.

If some D^σ produces initial monomials in the direct monomial complement J° of I°, the next step is to restrict D to L° and to check whether these monomials can really occur as initial monomials of the image of L° under D. This was the case in Example $1^{\textit{bis}}$.

There is a third possibility, illustrated by Examples $1^{\textit{ter}}$. There can be cancellations between the monomials produced by D° and the summands D^σ for $\sigma \neq \tau$. The phenomenon is similar to the cancellation of initial monomials in Buchberger's S-polynomials. The cancellations can be completely controlled if D has just two monomial summands, $D = D^\circ + D^\sigma$, for one $\sigma \in \mathbb{Z}^n$. This will be the subject of forthcoming work. For more summands, the situation can be much more complicated. The objective here will be to describe a finite algorithm which determines all initial monomials of the image of D up to a given degree. To be effective, it must contain a criterion which allows one to check whether all such initial monomials are already found.

As for point (3), the problem is to characterize regular singular points of differential equations through combinatorial criteria which can easily be checked. The methods of proof used in the present paper are still too coarse to capture the intricate phenomena which may appear in more than one variable.

Program for division

We briefly describe a program written for Maple 9.5 which realizes the division by one differential operator as indicated in the Division Theorem. The complete program is available from the authors. There are versions for both the polynomial case and the formal power series case. In order to ensure the feasibility in the latter case, the input has to be polynomial, and the algorithm stops after a prescribed number of steps.

Input:

D a differential operator with polynomial coefficients

D° the initial form of D with respect to a chosen weight vector λ.

$\overline{D}$ the queue $D - D^\circ$ of D

τ the shift of D°

κ_τ the coefficient function of D°

r the maximal number of allowed division steps

e the polynomial to be divided by D

l parameter controlling the log-output

Output:

a, b and ee such that

$$e - ee = Da + b,$$

with $a \in L^\circ$, $b \in J^\circ$ and

$\operatorname{in} e <_\lambda \operatorname{in} e$ or $e = 0$ (in the polynomial case)

$\operatorname{in} e >_\lambda \operatorname{in} e$ or $e = 0$ (in the power series case)

Description of program:

initialize

$ee = e$

$b = 0$

counter $= 0$

while counter $= 0$ **and** $ee \neq 0$ **do**

if $e^\circ = \operatorname{in} ee \notin I^\circ = \operatorname{Im} D^\circ$ **then**

$b = b + e^\circ$

$ee = ee - e^\circ$

else

$a^\circ = (D^\circ)^{-1} e^\circ$

$ee = ee - e^\circ$

$ee = ee - \overline{D} a^\circ$

$a = a + a^\circ$

counter $=$ counter $+1$

Example of Maple-input and log-output:

Example $D = \partial_y - y\partial_x$ with $D^\circ = \partial_y$

```
> DD:= P->collect( diff(P,y) - y*diff(P,x), distributed):
> D0:= P->collect( diff(P,y), distributed):
```

```
> D1:= P->DD(P)-D0(P):
> tau:=[0,-1]:
> kappa := exponent -> exponent[2]:
> r:=5:
> e:=x^2:
> seriesdiffdiv(P,D0,kappa,tau,D1,r,1);

Main: division step 1 of 5
      Main: trying to divide x^2 by D0
      Main: ImageChecker reports x^2 can be divided.
      Main: x^2 generated by x^2*y via D0.
      Main: substitution gives new polynomial 2*y^2*x to check.
      Main: current remainder is 0.

Main: division step 2 of 5
      Main: trying to divide 2*y^2*x by D0
      Main: ImageChecker reports 2*y^2*x can be divided.
      Main: 2*y^2*x generated by 2/3*x*y^3 via D0.
      Main: substitution gives new polynomial 2/3*y^4 to check.
      Main: current remainder is 0.

Main: division step 3 of 5
      Main: trying to divide 2/3*y^4 by D0
      Main: ImageChecker reports 2/3*y^4 can be divided.
      Main: 2/3*y^4 generated by 2/15*y^5 via D0.
      Main: substitution gives new polynomial 0 to check.
      Main: current remainder is 0.

      Main: Found exact representation e = D a + b where

      e = x^2,
      a = x^2*y+2/3*x*y^3+2/15*y^5,
      b = 0.

Main: division finished.
```

Program for monomialization

We now describe a program for Maple 9.5 which realizes the computation of the kernel of one differential operator as the pullback under the map u of the kernel of its initial form as explained in the Monomialization Theorem. There are again versions for both the polynomial case and the formal power series case. Below we restrict to the polynomial case.

Input:

D a differential operator with polynomial coefficients

D° the initial form of D with respect to a chosen weight vector λ.

$\overline{D}$ the queue $D - D^\circ$ of D

S the scission of D°

τ the shift of D°

κ_τ the coefficient function of D°

r the maximal number of iterations

m a monomial x^γ

l parameter controlling the log-output

Output:

p a polynomial such that

$p = u^{-1}(m)$,

Description of program:

initialize

$p = m$

$pp = S\overline{D}(p)$

counter $= 0$

while counter $\leq r$ **and** $pp \neq 0$ **do**

$pp = p + pp$

$pp = S\overline{D}(pp)$

counter $=$ counter $+1$

Example of Maple-input and log-output:

Example $D = \partial_y - y\partial_x$ with $D^\circ = \partial_y$

```
> DD:= P->collect( diff(P,y) - y*diff(P,x), distributed):
> D0:= P->collect( diff(P,y), distributed):
> D1:= P->DD(P)-D0(P):
> tau:=[0,-1]:
> kappa := exponent -> exponent[2]:
> r:=10:
> e:=x^4:
> uinverse(P,D0,kappa,tau,D1,r,1);

Main: found approximation
      u^-1(x^4) = x^4
after step 1.

Main: found approximation
      u^-1(x^4) = x^4 -2*x^3*y^2
after step 2.
```

```
Main: found approximation
      u^-1(x^4) = x^4 -2*x^3*y^2+ 3/2*x^2*y^4
after step 3.
Main: found approximation
      u^-1(x^4) = x^4 -2*x^3*y^2+ 3/2*x^2*y^4- 1/2*x*y^6
after step 4.
Main: found approximation
      u^-1(x^4) = x^4 -2*x^3*y^2+ 3/2*x^2*y^4- 1/2*x*y^6+ 1/16*y^8
after step 5.
Main: Found exact solution
      u^-1(x^4) = x^4 -2*x^3*y^2+ 3/2*x^2*y^4- 1/2*x*y^6+ 1/16*y^8
after 5 steps.
```

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