On the algebraic theory of pseudo-distance-regularity around a set

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Received 30 March 1999; accepted 13 July 1999

Submitted by R.A. Brualdi

Abstract

Let $\Gamma$ be a connected graph with vertex set $V$, adjacency matrix $A$, positive eigenvector $\nu$ and corresponding eigenvalue $\lambda_0$. A natural generalization of distance-regularity around a vertex subset $C \subset V$, which makes sense even with non-regular graphs, is studied. This new concept is called pseudo-distance-regularity, and its definition is based on giving to each vertex $u \in V$ a weight which equals the corresponding entry $\nu_u$ of $\nu$ and “regularizes” the graph. This approach reveals a kind of central symmetry which, in fact, is an inherent property of all kinds of distance-regularity, because of the distance partition of $V$ they come from. We come across such a concept via an orthogonal sequence of polynomials, constructed from the “local spectrum” of $C$, called the adjacency polynomials because their definition strongly relies on the adjacency matrix $A$. In particular, it is shown that $C$ is “tight” (that is, the corresponding adjacency polynomials attain their maxima at $\lambda_0$) if and only if $\Gamma$ is pseudo-distance-regular around $C$. As an application, some new spectral characterizations of distance-regularity around a set and completely regular codes are given. © 1999 Elsevier Science Inc. All rights reserved.

AMS classification: 05C50; 05C38

Keywords: Distance-regular graph; Local spectrum; Orthogonal polynomials; Completely regular code

* Work supported in part by the Spanish Research Council (Comisión Interministerial de Ciencia y Tecnología, CICYT) under projects TIC 94-0592 and TIC 97-0963.

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1. Introduction

Distance-regularity of graphs is one of the most important concepts in combinatorics, and also has important connections to other subjects, such as design theory, coding theory, geometry and group theory. For a comprehensive treatment of distance-regular graphs, we refer the reader to the textbook of Brouwer et al. [4]. Some other standard references on the subject are also Bannai and Ito [2], Biggs [3], Cvetković and Sachs [9], and Godsil [24]. Thus, because of such a relevance, it is not surprising that a number of generalizations of distance-regularity have been proposed in the literature by different authors. See, for instance, Delsarte [14], Hilano and Nomura [27], Weichsel [41], and Yebra and the authors [18,21,23]. In particular, the concept of distance-regularity around a vertex set (of a regular graph) has much to do with the study of certain codes, which are referred to as “completely-regular codes”. Since their introduction in Delsarte’s thesis [14], these structures have deserved special attention by different authors, such as Courteau and Montpetit [7], Godsil [24], Martin [29], and Neumaier [32], among others. In this work we also study a kind of distance-regularity around a set, but applied to general graphs, not only to those which are regular. The study is mainly based on three facts. The first one is to use the positive eigenvector for the largest eigenvalue to regularize the graph. This approach has been already used by Yebra and the authors to derive bounds on some distance-related parameters of $C$, such as the diameter and the independence number, in terms of its spectrum [15–17,19,22] (following the work of several authors, such as Alon and Milman [1], Chung et. al. [5,6], Van Dam and Haemers [10,11], Delorme et. al. [12,13], Mohar [31], Quenell [34], and Sarnak [35]), and also to study pseudo-distance-regularity around a vertex [18,21]. The second fact is to consider the local spectrum of a vertex subset of the graph, which locally plays a role similar to the standard spectrum of the graph. This shows to be a key concept for developing our algebraic theory of distance-regularity. Finally, the third fact is to employ the properties of some new sequences of orthogonal polynomials (of a discrete variable) to obtain combinatorial properties of pseudo-distance-regularity. One of the main advantages of our approach is that it reveals a kind of central symmetry, which is inherent to all kinds of distance-regularity (because of the linear distribution diagram induced by the distance partition of the vertex set). As a result of the study, some new spectral characterizations of distance-regularity around a set and completely regular codes are obtained.

The plan of the paper is as follows. In Section 2 we introduce the so-called local spectrum of a vertex set, and derive some of their basic properties. As mentioned above, our study also uses some results about orthogonal polynomials of a discrete variable, which is the topic of Section 3. A basic orthogonal system of polynomials is constituted by the so-called $C$-local adjacency polynomials, where $C \subseteq V$. They are defined with respect to a discrete scalar product on the mesh of local eigenvalues of $C$ and measure (or “weight”) function given by the corresponding local multiplicities. These polynomials are studied in detail in Section 4. Special interest is paid, in
Section 5, to the case when such polynomials attain their maximum possible values at $\lambda_0$. This is because such a situation induces a high level of structure to the graph, as shown in Sections 5 and 6. In fact, this structure, introduced in Section 6, shows to be a kind of distance-regularity around the considered vertex set, and generalizes the concept of completely regular code. Section 6 is then devoted to the study of such structure and its relation with the adjacency polynomials. As a main result, a spectral characterization of pseudo-distance regularity around a vertex set is given.

2. The local spectrum of a vertex set

As mentioned in the introduction, the fundamental algebraic concept for our study of a graph “seen” from a vertex set $C$ is the $C$-local spectrum. This section is devoted to introduce such a concept and derive some of the basic results involving it. Before going into the details, we shall first recall some (algebraic) graph theory notation and terminology.

Let $\Gamma = (V, E)$ be a (simple finite connected) graph with $n$ vertices and diameter $D$. Let $A$ be the adjacency matrix of $\Gamma$, with maximum eigenvalue $\lambda_0$, and positive eigenvector $v$, normalized to have smallest entry 1. Otherwise stated, we will suppose that matrices and vectors are indexed by the vertices of $\Gamma$. Thus, for instance, the above condition on $v$ reads $\min_{v \in V} v_a = 1$. We will use the weight function $\rho : \mathcal{P}(V) \rightarrow \mathbb{R}^n$ defined by $\rho U := \sum_{u \in U} v_u e_u$ if $U \neq \emptyset$, where $e_u$ is the $u$th unit vector, and $\rho \emptyset := 0$. (When $\Gamma$ is regular we have $v = j$, the all-1 vector, and then $\rho U$ is just the characteristic vector of $U$.) With this definition, note that

$$\rho V = v, \quad \|\rho U\|^2 + \|\rho (V \setminus U)\|^2 = \|v\|^2$$

and $\|\rho U\|^2 \geq |U|$.

Note also that this corresponds to assigning some weights to the vertices of $\Gamma$, in such a way that it becomes “regularized” since the average weight degree $\delta(\rho(u))$ of each vertex $u$ becomes a constant:

$$\delta(\rho(u)) := \frac{1}{v_u} \sum_{v \in \Gamma(u)} v_v = \lambda_0,$$  \hspace{1cm} (1)

where $\Gamma(u)$ is the set of vertices adjacent to $u$. The spectrum of (the adjacency matrix of) a graph $\Gamma$ is denoted by $\text{sp} \Gamma := \{\lambda_0^m_0, \lambda_1^m_1, \ldots, \lambda_d^m_d\}$, where the superscripts represent multiplicities. Note that, since $\Gamma$ is connected, the matrix $A$ is irreducible and hence $m_0 = 1$. The mesh of distinct eigenvalues of $\Gamma$ will be denoted by $\text{ev} \Gamma := \{\lambda_0 > \lambda_1 > \cdots > \lambda_d\}$. We will assume that $\Gamma$ is not trivial. Hence, $d \geq 1$, $\lambda_0 > 0$, and $-\lambda_0 < \lambda_d < 0$, see for instance Biggs [3]. As usual, we consider $A$ as an endomorphism of $\mathbb{R}^n$. The adjacency (or Bose-Mesner) algebra of $A$, denoted by $\mathcal{A}(A)$, is the algebra of all the matrices which are polynomials in $A$. A polynomial $p$, belonging to the vector space $\mathbb{R}_k[x]$ of real polynomials with degree at most $k$, will operate on $\mathbb{R}^n$ by the rule $pw := p(A)w$, and the matrix is not specified unless some confusion may arise. Moreover, when $w = \rho U$ for some $U \subseteq V$, and since
there is no possible ambiguity, we will simply write $p\rho U := p(A)pU$, instead of $p(\rho U) := p(A)(\rho U)$.

Let $\Gamma = (V, E)$ be a graph as above. Given a non-empty set $C \subset V$, let us consider the spectral decomposition of the unit vector $e_C := \frac{1}{\rho C}\rho C$ (which can be thought of as a weight characteristic vector of $C$):

$$e_C = \sum_{i=0}^{d} E_i e_C,$$

where $E_i$ is the idempotent matrix representing the orthogonal projections onto the eigenspace $E_i$, corresponding to $\lambda_i$. Then, we call

$$m_C(\lambda_i) := \|E_i e_C\|^2 \quad (0 \leq i \leq d)$$

the C-local multiplicity of $\lambda_i$. Now, let $\text{ev} C := \{\mu_0(= \lambda_0) > \mu_1 > \cdots > \mu_{d_C}(= \lambda_{d_C})\}$ be the set of eigenvalues such that the projection onto $E_i$ is non-null, that is $m_C(\mu_i) \neq 0$, $0 \leq i \leq d_C$, and introduce the C-local spectrum of $\Gamma$ or, simply, local spectrum of $C$ as

$$\text{sp } C := \left\{ \lambda_0^{m_C(\lambda_0)}, \mu_1^{m_C(\mu_1)}, \ldots, \mu_{d_C}^{m_C(\mu_{d_C})} \right\}.$$ (2)

Notice that, since $e_C$ is a unit vector, we have $\sum_{i=0}^{d} m_C(\mu_i) = 1$. Moreover, from

$$e_C = \frac{1}{\rho C} \left( \langle \rho C, v \rangle \frac{v}{\|v\|^2} + \sum_{i=1}^{d} E_i \rho C \right) = \frac{1}{\|\rho C\|} \sum_{v \in C} \frac{v_a}{\|v\|^2} v + z$$

where $z := \frac{1}{\|\rho C\|} \sum_{i=1}^{d} E_i \rho C \in v^\perp$, we get that the C-local multiplicity of $\lambda_0$ is

$$m_C(\lambda_0) = \frac{\|\rho C\|^2}{\|v\|^2}.$$ (3)

Hence we have $m_C(\lambda_0) = 1$ iff $C = V$, in which case $d_C = 0$ and $\text{sp } V = \{\lambda_0^1\}$. Consequently, in order to avoid trivial situations we will suppose from now on that $C \neq V$.

The term "local multiplicity" was introduced in [21] when $C$ consists of a single vertex $C = \{u\}$, and it was shown that, in this case, $\sum_{v \in V} m_\mu(\lambda_i) = m_i$. Also, the $u$-local multiplicities are, in fact, the values at $\lambda_i$ of the measure function defined by Terwilliger [40].

The set of vertices at distance not greater than $k$ from (some vertex of) $C$, $N_k(C) := \{v \in V : \delta(v, C) \leq k\}$ will be simply denoted by $N_k$. For instance, $N_0 = C$. The eccentricity of $C$ is defined as $\varepsilon_C := \max \{\delta(v, C) : v \in V\}$, so that $N_{\varepsilon_C} = V$. Notice that this definition generalizes the standard notion of eccentricity of a vertex. Note also that, in coding theory, $\varepsilon_C$ corresponds to the covering radius of the code $C$, and, when $\Gamma$ is regular, $d_C$ coincides with the so-called dual degree of the code. Thus, the
following lemma generalizes for general graphs a result of Delsarte [14] intended for distance-regular graphs (see also [24, Corollary 11.6.3]).

Lemma 2.1. For a given graph \( \Gamma = (V, E) \), let \( C \subset V \) be a set with eccentricity \( e_C \) and \( d_C + 1 \) distinct local eigenvalues. Then, \( e_C \leq d_C \).

Proof. If there exists a vertex \( v \) such that \( d(v, C) > d_C \), then \( (A^k)_{uv} = \langle x^k e_u, e_v \rangle = 0 \), for all \( u \in C \) and \( k = 0, 1, \ldots, d_C \). Therefore, \( \langle p e_v, e_C \rangle = 0 \) for every polynomial \( p \) with \( 0 \leq \text{deg} \, p \leq d_C \). But the polynomial \( \phi = \prod_{l=0}^{d_C} (x - \mu_l) \), with degree \( d_C + 1 \), annihilates \( e_C \). Consequently, \( \langle p e_v, e_C \rangle = 0 \) for every polynomial \( p \in \mathbb{R}[x] \), contradicting that \( \Gamma \) is connected. \( \square \)

Lemma 2.1 suggests to consider the case \( e_C = d_C \), that is when there is some vertex at distance \( d_C \) from \( C \). In this case we will refer to \( C \) as an extremal set of vertices. Notice that \( C \) is extremal if and only if

\[
|C| = |N_0| < |N_1| < \cdots < |N_{d_C-1}| < |N_{d_C}| = |V|. \tag{5}
\]

Given \( C \subset V \), the antipodal set of \( C \), denoted by \( \overline{C} \), is the set of vertices at maximum distance \( e_C \) from \( C \). The extremal character of \( C \) is then equivalent to state that \( d(C, \overline{C}) = d_C \).

Let us consider the following polynomial

\[
H_C := \frac{\|v\|^2}{\pi_0 \|\rho C\|^2} \prod_{l=1}^{d_C} (x - \mu_l) \tag{6}
\]

where \( \pi_0 := \phi'(\lambda_0) = \prod_{l=1}^{d_C} (\lambda_0 - \mu_l) \), and hence \( H_C(\lambda_0) = \|v\|^2 / \|\rho C\|^2 \). We claim that \( H_C \) is the unique polynomial of degree at most \( d_C \) which satisfies

\[
H_C \rho C = v \tag{7}
\]

and so, inspired by Hoffman [28], we call it the Hoffman-like polynomial of \( C \). Indeed, by using (3) we get

\[
H_C e_C = H_C \left( \frac{\|\rho C\|}{\|v\|} v + z \right) = \frac{\|\rho C\|}{\|v\|^2} H_C(\lambda_0) v = \frac{1}{\|\rho C\|} v
\]

and (7) follows. To prove uniqueness, assume that there exist two polynomials \( P, Q \) of degree at most \( d_C \), such that \( P \rho C = Q \rho C \). Thus, \( R \rho C = 0 \) with \( R = P - Q \) and \( \text{deg} \, R \leq d_C \). Then, from \( \|R e_C\|^2 = 0 \) and \( e_C = \sum_{l=0}^{d_C} E_l e_C \) (where now \( E_l \) stands for the projection onto the eigenspace \( \mathcal{E}(\mu_l) \)) we get

\[
\sum_{l=0}^{d_C} R(\mu_l) E_l e_C = \sum_{l=0}^{d_C} R(\mu_l) E_l e_C = \sum_{l=0}^{d_C} R^2(\mu_l) M_C(\mu_l) = 0.
\]
Hence, \( R(\mu_1) = 0 \) for any \( 0 \leq l \leq d_C \) (since \( m_C(\mu_1) > 0 \)) and, therefore, \( R = 0 \).

Notice that Lemma 2.1 can be seen now as a consequence of (7) since, for any vertex \( u \in V, (H_C \rho C)_u = v_u > 0 \) implies \( \epsilon(C, u) \leq \text{dgr} \, H_C = d_C \).

Let \( \Gamma \) be a graph with adjacency matrix \( A \), and let \( C \) be a vertex subset with eccentricity \( \varepsilon \) (from now on we delete the subindex). Consider the distance partition induced by \( C \), namely \( V = C_0 \cup C_1 \cup \cdots \cup C_\varepsilon \), where \( C_0 := C \) and \( C_k := \Gamma_k(C) = N_k \setminus N_{k-1}, 1 \leq k \leq \varepsilon \). Then, the vectors \( \rho C_k, 0 \leq k \leq \varepsilon, \) will be referred to as the (weight) distance(-\( k \)) vectors of \( C \). In this paper we will be interested in the case when such vectors can be expressed in the form \( \rho C_k = p_k \rho C \), where \( p_k \) is some polynomial with \( \text{dgr} \, p_k \leq d_C \). In fact, if such a polynomial exists, it is unique and has degree exactly \( k \), as the next result shows.

**Lemma 2.2.** Let \( C \) be a vertex set with eccentricity \( \varepsilon \). Then, for each \( k = 0, 1, \ldots, \varepsilon \), there is at most one polynomial \( p_k^C \in \mathbb{R}_e[x] \) such that \( p_k^C \rho C = \rho C_k \). Furthermore, \( \text{dgr} \, p_k^C = k \).

**Proof.** The uniqueness of \( p_k^C \), if any, is proved as that of \( H_C \). Moreover, let \( k' = \text{dgr} \, p_k^C \leq \varepsilon \). Then, \( (p_k^C \rho C)_u = (\rho C_k)_u = v_u > 0 \) for every \( u \in C_k \) implies \( k' \geq k \), and if \( k = \varepsilon \) we are done. Moreover, if \( k < \varepsilon \) and \( k' > k \), we would have \( (p_k^C \rho C)_v \neq 0 \) for all \( v \in C_k \), contradicting the definition of the distance vector \( \rho C_k \). ☐

The polynomial \( p_k^C \) will be referred to as the distance(-\( k \)) polynomial of \( C \). The next result provides a sufficient condition, in terms of the existence of all such polynomials, for \( C \) to be extremal.

**Lemma 2.3.** Let \( C \) be a vertex subset of a graph \( \Gamma \), with eccentricity \( \varepsilon \). If all the distance polynomials \( \{p_k^C\}_{0 \leq k \leq \varepsilon} \) exist, then \( C \) is extremal.

**Proof.** Under the hypothesis, we have \( \sum_{k=0}^\varepsilon p_k^C \rho C = \sum_{k=0}^\varepsilon \rho C_k = v \) so that, from the uniqueness of \( H_C \), it must be \( \sum_{k=0}^\varepsilon p_k^C = H_C \), and hence \( \varepsilon = \text{dgr} \, H_C = d_C \). ☐

### 3. Orthogonal polynomials of a discrete variable

In this section we study some general results concerning orthogonal polynomials of a discrete variable, which will be applied in our study of pseudo-distance-regularity around a set.

Let \( \mathcal{M} := \{x_0 > x_1 > \cdots > x_d\} \) be a mesh of real numbers, and consider a weight function \( \rho \) over \( \mathcal{M} \), \( \rho : \mathcal{M} \rightarrow \mathbb{R}_+ \), with \( \rho_l := \rho(x_l), 0 \leq l \leq d \). We will say that \( \rho \) is normalized whenever \( \rho_0 + \rho_1 + \cdots + \rho_d = 1 \). Each real function on \( \mathcal{M} \) can be represented by a unique polynomial of degree at most \( d \). For our study, it will be...
more useful to identify the algebra of real functions defined on \( S \) with the quotient ring \( \mathbb{R}[x]/\mathfrak{I} \), where \( \mathfrak{I} \) is the ideal generated by the polynomial \( \phi = \prod_{i=0}^{d}(x - x_i) \).

As usual, each equivalence class will be represented by its unique element with degree at most \( d \), that is, its canonic representative. We will keep for the classes the notion of degree, which will coincide with the degree of its corresponding canonic representative.

Corresponding to the pair \((S, \rho)\), we can define a scalar product in \( \mathbb{R}[x]/\mathfrak{I}\), as

\[
(f, g) := \sum_{l=0}^{d} p_l f(x_l) g(x_l),
\]

which is clearly well-defined and has, as associated norm, \( \|f\| = \sqrt{(f, f)} \). An orthogonal system, with respect to such a product, is a sequence \( p_0, p_1, \ldots, p_d \), of orthogonal polynomials satisfying \( \deg p_k = k, 0 \leq k \leq d \). It is well-known that the members of every orthogonal system satisfy a three-term recurrence of the form

\[
\begin{align*}
xp_0 &= a_0 p_0 + c_1 p_1, \\
xp_k &= b_{k-1} p_{k-1} + a_k p_k + c_{k+1} p_{k+1} \quad (1 \leq k \leq d-1), \\
xp_d &= b_{d-1} p_{d-1} + a_d p_d,
\end{align*}
\]

initiated with \( p_0 \) any non-zero constant and where \( b_{k-1}, a_k \) and \( c_{k+1} \) are the Fourier coefficients of \( xp_k \) in terms of \( p_{k-1}, p_k \) and \( p_{k+1} \) respectively, so that

\[
\|p_k\|^2 c_k = \|p_k\|^2 \frac{(xp_{k-1}, p_k)}{\|p_k\|^2} = \frac{(p_{k-1}, xp_k)}{\|p_{k-1}\|^2} \|p_{k-1}\|^2 \\
= b_{k-1} \|p_{k-1}\|^2 \quad (1 \leq k \leq d).
\]

(In order to simplify some further computations, we will extend (9) to the range \( 0 \leq k \leq d \) with the convention \( b_{-1} = c_{d+1} = 0 \).) Such coefficients can be arranged as the entries of the following \( 3 \times (d + 1) \) matrix:

\[
M(S, \rho) := \begin{pmatrix}
0 & c_1 & \cdots & c_d-1 & c_d \\
an_0 & a_1 & \cdots & a_{d-1} & a_d \\
b_0 & b_1 & \cdots & b_{d-1} & 0
\end{pmatrix}
\]

which we call the recurrence matrix of the orthogonal system and where, in order to make some expressions uniform, we have introduced the symbols \( c_0 = b_d = 0 \).

Notice that, since \( \deg p_k = k \), the coefficients \( c_1, \ldots, c_d \) are non-null. We also recall that every polynomial of an orthogonal sequence completely factorizes in the open interval \( (x_d, x_0) \).

3.1. The proper polynomials

In this section, and in the following one, we introduce two particular orthogonal systems, whose recurrence matrices differ only by a central symmetry. We will see
later that, within some extremal conditions, the members of such systems turn out to be distance polynomials.

**Proposition 3.1.** Let \(p_0, p_1, \ldots, p_d\) be an orthogonal system with normalized weight function. Then the following assertions are equivalent:

(a) \(a_k + b_k + c_k = x_0\) \((0 \leq k \leq d)\);
(b) \(\|p_k\|^2 = p_k(x_0)p_0(x_0)\) \((0 \leq k \leq d)\).

**Proof.** Assume that (a) holds. Note first that (b) trivially holds for \(p_{-1} = 0\) and \(p_0 = \alpha\) (a constant) since \(\|p_0\|^2 = \sum_{l=0}^{d} \rho_l \alpha^2 = \alpha^2 = p_0^2(x_0)\). By induction, assume that the result is true for \(p_{-1}, p_0, \ldots, p_k\). Then, evaluating (9) at \(x_0\) and multiplying by \(p_0\), we get

\[
 x_0\|p_k\|^2 = b_{k-1}\|p_{k-1}\|^2 + a_k\|p_k\|^2 + c_{k+1}p_{k+1}(x_0)\rho_0
\]

whence, using (10),

\[
p_{k+1}(x_0)p_0 = (x_0 - c_k - a_k)\|p_k\|^2 + \frac{b_k}{c_{k+1}}\|p_k\|^2 = \|p_{k+1}\|^2
\]

and we prove (b). Similarly, if (b) holds, then (12) with \(p_{k+1}(x_0)p_0 = \|p_{k+1}\|^2\) and (10) yield \((x_0 - a_k - b_k - c_k)\|p_k\|^2 = 0\), whence (a) follows. \(\Box\)

Given a scalar product as in (8), it is easy to show (by using an orthonormal system) that there exists a unique orthogonal system \(p_0, p_1, \ldots, p_d\) which satisfies \(\|p_k\|^2 = p_k(x_0)\) for any \(k = 0, 1, \ldots, d\). Hence, if the weight function \(\rho\) is normalized, Proposition 3.1 leads to the following corollary.

**Corollary 3.2.** Let \((\mathcal{H}, \rho)\) be a pair with normalized weight function \(\rho\). Then, there exists a unique orthogonal system \(p_0, p_1, \ldots, p_d\), characterized by any of the following conditions:

(a) \(p_0 = 1\) and \(a_k + b_k + c_k = x_0\) \((0 \leq k \leq d)\);
(b) \(\|p_k\|^2 = p_k(x_0)\) \((0 \leq k \leq d)\).

The unique orthogonal system characterized in Corollary 3.2, and its polynomials, will be called proper. Its recurrence matrix will be denoted by \(M(\mathcal{H}, \rho)\). Moreover, such a system satisfies the two following properties.

**Proposition 3.3.** Let \(\{p_k\}_{0 \leq k \leq d}\) be a proper orthogonal system. Then the following hold:

(a) The product \(\rho_k \phi_k p_d(x_k)\) is constant over \(k\), and so

\[
 \rho_k \phi_k p_d(x_k) = \rho_0 \phi_0 p_d(x_0) \quad (1 \leq k \leq d)
\]

where \(\phi_k := \phi'(x_k) = \prod_{l=0, l \neq k}^{d} (x_k - x_l)\);
(b) The value at \(x_0\) of the highest degree polynomial is
\[ \begin{align*}
p_d(x_0) & = \frac{1}{\rho_0^2 \pi_0^2} \sum_{l=0}^{d} (1/\rho_l \pi_l^2), \\
\pi_l & = (-1)^l \phi_l = \phi'(x_l).
\end{align*} \] (14)

**Proof.** (a) Let us consider the polynomials \( Z_k^* = \prod_{l=1}^{d} (x - x_l) \), \( 1 \leq k \leq d \), so that \( Z_k^*(x_0) = \phi_0/(x_0 - x_k) \) and \( Z_k^*(x_k) = \phi_k/(x_k - x_0) \). Hence, since \( \deg Z_k^* = d - 1 \),
\[ 0 = \langle p_d, Z_k^* \rangle = \rho_0 p_d(x_0) Z_k^*(x_0) + \rho_k p_d(x_k) Z_k^*(x_k) = \rho_0 \frac{p_d(x_0) \phi_0}{x_0 - x_k} - \rho_k \frac{p_d(x_k) \phi_k}{x_0 - x_k} \]
and (13) follows.

(b) Using the property \( p_d(x_0) = \| p_d \|^2 \), and the fact that, from (13), \( p_d(x_k) = \rho_0 \phi_0 p_d(x_0)/\rho_k \phi_k \) for any \( 0 \leq k \leq d \), we get
\[ p_d(x_0) = p_d^2(x_0) \sum_{k=0}^{d} \frac{\rho_0 \phi_0}{\rho_k \phi_k} \left( \frac{\rho_0 \phi_0}{\rho_k \phi_k} \right)^2 = p_d^2(x_0) \sum_{k=0}^{d} \frac{\rho_0^2 \pi_0^2}{\rho_k^2 \pi_k^2} \]
which yields (14). \( \Box \)

Proposition 3.3(a) can also be seen as a consequence of the Darboux–Christoffel formula in the theory of orthogonal polynomials, see for instance [33].

### 3.2. The conjugate polynomials

Let us consider again the proper orthogonal system \([p_k]_{0 \leq k \leq d}\). Notice that \( p_d(x_i) \neq 0 \) for any \( 0 \leq i \leq d \) since, otherwise, backwards application of recurrence (9) from \( p_d(x_i) = 0 \) would give the contradiction \( p_k(x_i) = 0 \) for any \( k \leq d \). In other words, \( p_d \) is relatively prime with the generating polynomial \( \phi \) of the ideal \( \mathcal{I} \) and, therefore, it has inverse in the ring \( \mathbb{R}[x]/\mathcal{I} \), say \( \overline{p_d} := p_d^{-1} \). By using \( \overline{p_d} \) we can now define a new sequence of polynomials \([\overline{p}_k] \), called the **conjugate polynomials** of the \([p_k] \), as
\[ \overline{p}_k := p_d - k \overline{p_d} \quad (0 \leq k \leq d). \] (15)

From its definition, the polynomials \([\overline{p}_k] \) are clearly orthogonal with respect to the scalar product
\[ \langle \overline{f}, \overline{g} \rangle := \sum_{l=0}^{d} \rho_l \frac{p_d^2(x_l)}{p_d(x_l)} \overline{f(x_l)} \overline{g(x_l)} \] (16)
with normalized weight function
\[ \overline{\rho} := \frac{1}{p_d(x_0)} \rho p_d^2 \] (17)
since
\[
\sum_{i=0}^{d} p_i = 1/p_d(x_0) \sum_{i=0}^{d} p_i^2(x_0) = 1/p_d(x_0) \|p_d\|^2 = 1.
\]
Moreover, using the norms associated to (16) and (8), respectively,
\[
\|p_k\|^2 = \frac{\|p_{d-k}\|^2}{p_d(x_0)} = p_{d-k}(x_0)p_d(x_0) = p_k(x_0)
\]
so that the polynomials \(\{p_k\}\) satisfy also Corollary 3.2. To show that they are indeed a proper orthogonal system, it remains to prove that \(\text{dgr} p_k = k\) for every \(0 \leq k \leq d\).
With this aim, we write the recurrence (9), satisfied by the polynomials \(f p_k\), in the form
\[
x p_{d-k} = c_{d-k+1} p_{d-k+1} + a_{d-k} p_{d-k} + b_{d-k-1} p_{d-k-1} \quad (0 \leq k \leq d)
\]
and multiply by \(p_d\) (recall that the computations are carried out in \(\mathbb{R}[x]/\mathcal{I}\)). Then, we get that the polynomials \(\{p_k\}\) satisfy the recurrence
\[
x p_k = p_{k-1} p_{k-1} + a_{k} p_k + c_{k+1} p_{k+1} \quad (0 \leq k \leq d)
\]
starting from \(p_{-1} = 0\) and \(p_0 = 1\), and where we have redenoted the multiplicative constants as \(a_{k} = a_{d-k}\), and \(c_{k+1} = b_{d-k-1}\). Therefore, since \(\text{dgr} p_0 = 0\), we have that \(\text{dgr} p_k = k\) for any \(0 \leq k \leq d\). Notice that the recurrence matrix of the new orthogonal system is now
\[
M(\mathcal{H}, p) = \begin{pmatrix} 0 & b_{d-1} & \cdots & b_1 & b_0 \\ a_d & a_{d-1} & \cdots & a_1 & a_0 \\ c_d & c_{d-1} & \cdots & c_1 & 0 \end{pmatrix}
\]
which corresponds, looking at the “geometry of the matrices”, to going from \(M(\mathcal{H}, \rho)\) to \(M(\mathcal{H}, p)\) by a central symmetry. Note also that, clearly, \(p_k = p_k\).

3.3. The maximal polynomials

Now we introduce another system of orthogonal polynomials, which will be a main tool in our study of graphs. Let us consider the pair \((\mathcal{H}, \rho)\) with normalized \(\rho\) and corresponding scalar product (8). Then we say that the polynomials \(\{Q_k\}\) constitute an \(x_0\)-maximal system when they fulfill the following conditions:
(a) \(Q_k \in \mathbb{R}[x]\) (using the canonic representative);
(b) \(\|Q_k\| = 1\);
(c) \(Q_k(x_0) = \sup\{T(x_0) : T \in \mathbb{R}[x], \|T\| = 1\}\).

**Proposition 3.4.** For any given \((\mathcal{H}, \rho)\) with normalized \(\rho\), there exists a unique \(x_0\)-maximal system characterized by
\[
Q_k = \frac{1}{\sqrt{q_k(x_0)}} q_k \quad (0 \leq k \leq d)
\]
where \( q_k = \sum_{l=0}^{k} p_l \) and \( \{p_k\}_{0 \leq k \leq d} \) is the proper orthogonal system corresponding to \((\mathcal{M}, \rho)\).

**Proof.** For any \( 0 \leq k \leq d \), let \( Q_k = \sum_{j=0}^{k} \alpha_j p_j \). Then the values of \( \alpha_j \) are the solution of the following constrained optimization problem:

- maximize \( f(\alpha_0, \alpha_1, \ldots, \alpha_k) = \sum_{j=0}^{k} p_j(x_0)\alpha_j \)
- subject to \( \sum_{j=0}^{k} p_j(x_0)\alpha_j^2 = \|Q_k\|^2 = 1 \),

whose solution is \( \alpha_j = \left( \sum_{l=0}^{k} p_l(x_0) \right)^{-1/2} \) for any \( j = 0, 1, \ldots, k \). Introducing the polynomials \( q_k = \sum_{l=0}^{k} p_l, 0 \leq k \leq d \), the result follows.

The orthogonality of the polynomials \( \{p_k\} \) imply that the polynomials \( \{q_k\} \) satisfy the same property, namely

\[
\|q_k\|^2 = \sum_{l=0}^{k} p_l(x_0)^2 = q_k(x_0).
\]

The polynomials \( Q_0 \) and \( Q_d \) admit simple explicit expressions. We only need to consider that the maximization of \( Q_d(x_0) \), restricted to \( \sum_{l=0}^{d} p_l Q_d^2(x_l) = 1 \), imply \( Q_d(x_l) = 0 \) for any \( l = 1, \ldots, d \). Then,

\[
Q_0 = 1, \quad Q_d = \frac{1}{\sqrt{\rho_0 \pi_0}} \prod_{l=1}^{d} (x - x_l);
\]

\[
Q_0(x_0) = 1, \quad Q_d(x_0) = \frac{1}{\sqrt{\rho_0}}. \tag{21}
\]

Since, by Proposition 3.4, \( q_k = Q_k(x_0)Q_k \), the above equalities become

\[
q_0 = 1, \quad q_d = \frac{1}{\rho_0 \pi_0} \prod_{l=1}^{d} (x - x_l); \quad q_0(x_0) = 1, \quad q_d(x_0) = \frac{1}{\rho_0}. \tag{22}
\]

4. The adjacency polynomials of a vertex set

Given a graph \( \Gamma = (V, E) \), let us consider a (non-empty) vertex subset \( C \subseteq V \). Let \( \mathcal{M} \) be the mesh of \( C \)-local eigenvalues, \( \text{ev}_C \), with respective local multiplicities \( m_C(\mu_l), 0 \leq l \leq d_C \). Now consider a scalar product with weight function \( \rho_l = m_C(\mu_l) \) which, from the definition of the local multiplicities, is normalized. Moreover, by (4) we now have \( \rho_0 = m_C(\lambda_0) = \|\rho C\|^2/\|w\|^2 \). In what follows, we identify the pair \((\mathcal{M}, \rho)\) of the preceding section with the \( C \)-local spectrum \( \text{sp}_C \) from which such a pair derives. To simplify the notation, we will henceforth delete the sub- or super-script \( C \) from the local multiplicities, polynomials, and some other parameters associated to such a set \( C \). When afterwards we use another set
the corresponding concepts referred to \(\overline{\mathcal{C}}\) will be also overlined. Now the scalar product in \(\mathbb{R}[x]/\mathfrak{I}\), where \(\mathfrak{I}\) is the ideal of \(\mathbb{R}[x]\) generated by the polynomial \(\phi = \prod_{l=0}^{d}(x - \mu_l)\), is

\[
(f, g)_{\mathcal{C}} := \sum_{l=0}^{d} m(\mu_l) f(\mu_l) g(\mu_l)
\]

and its corresponding induced norm will be denoted by \(\|f\|_{\mathcal{C}}\). Note that the relation between this product and the (standard) Euclidean product is

\[
(f e_C, g e_C) = \left( \sum_{l=0}^{d} f(\mu_l) e_l e_C, \sum_{l=0}^{d} g(\mu_l) e_l e_C \right) = \sum_{l=0}^{d} m(\mu_l) f(\mu_l) g(\mu_l) = (f, g)_{\mathcal{C}}. \tag{24}
\]

and, in particular, the relation between the corresponding norms is

\[
\|f e_C\| = \|f\|_{\mathcal{C}} \quad (f \in \mathbb{R}_d(x)). \tag{25}
\]

For each \(k = 0, 1, \ldots, d\), the unit ball \(B^C_k = \{p \in \mathbb{R}_k[x] : \|p\|_{\mathcal{C}} \leq 1\}\) is a compact set in the normed space \(\mathbb{R}_d(x), \|\cdot\|_{\mathcal{C}}\). Then, the linear function \(p \mapsto p(\lambda_0)\) attains a maximum in each \(B^C_k\) at some polynomial \(Q_k^C\), with \(\|Q_k^C\|_{\mathcal{C}} = 1\), that we call the \((\mathcal{C}\text{-local} k\text{-adjacency polynomial})\). When \(\mathcal{C}\) consists of a single vertex \(u\), the \([u]\text{-local}\) adjacency polynomials were studied in [18]. The following results, for general \(\mathcal{C}\), were given there in such a particular case \(\mathcal{C} = [u]\), and they are proved analogously. In particular, note that the construction of the \(\mathcal{C}\text{-local} k\text{-adjacency polynomials}\) immediately follows from our study, in Section 3, of the maximal polynomials with \(\text{sp} \mathcal{C}\). To simplify the notation, from now on we also delete the superscript of \(Q_k^C\).

**Proposition 4.1.** (a) There exists a unique \(\mathcal{C}\text{-local} k\text{-adjacency polynomial} Q_k\), with \(\text{dgr } Q_k = k\), for any \(k = 0, 1, \ldots, d\); 
(b) \([\text{Proposition 3.4}]\) \(Q_k = \frac{1}{\sqrt{Q_0(\lambda_0)}} q_k\), where \(q_k = \sum_{l=0}^{k} p_l\), \(0 \leq k \leq d\), and \(\{p_l\}\) is the proper orthogonal system with respect to the scalar product (23); 
(c) \([\text{Eq. (21)}]\) \(Q_0 = 1\), \(Q_d = \frac{\|\cdot\|}{\text{sp} \mathcal{C}} \prod_{l=1}^{d} (x - \mu_l)\), where \(\phi(\lambda_0) = \prod_{l=1}^{d} (\lambda_l - \mu_l)\); 
(d) \(1 = Q_0(\lambda_0) < Q_1(\lambda_0) < \cdots < Q_d(\lambda_0) = \frac{\|\cdot\|}{\text{sp} \mathcal{C}}\).

Now we prove a striking result about the adjacency polynomials, which is in the basis of the relation between the adjacency polynomials and the concept of pseudo-distance-regularity around a set, studied in the next section.
Theorem 4.2. Let C be a vertex subset of a graph $\Gamma$, with local spectrum $\text{sp}_C = \{\mu_0, \mu_1, \ldots, \mu_d\}$. Let $Q_k$, $0 \leq k \leq d$, be the $C$-local $k$-adjacency polynomial. Then,

$$Q_k(\lambda_0) \leq \frac{\|\rho N_k\|}{\|\rho C\|}$$

and equality is attained if and only if

$$\rho N_k = \frac{\|\rho N_k\|}{\|\rho C\|}Q_k \rho C.$$ (26)

Proof. The result clearly holds for $k = 0, d$, since $Q_0 = 1$ and $Q_d = (\|\rho C\|/\|v\|)H_C$, where $H_C$ is the Hoffman-like polynomial defined in (6). Now suppose that $1 \leq k \leq d - 1$, and consider the spectral decompositions of the vectors $\rho C$ and $\rho(V \setminus N_k)$:

$$\rho C = \frac{\langle \rho C, v \rangle}{\|v\|^2}v + z = \frac{\|\rho C\|^2}{\|v\|^2}v + z; \quad \rho(V \setminus N_k) = \frac{\|\rho(V \setminus N_k)\|^2}{\|v\|^2}v + z'. $$

with $z, z' \in v^\perp$, yielding

$$\|Q_k z\|^2 = \|Q_k \rho C\|^2 - \frac{\|\rho C\|^4}{\|v\|^2}Q_k^2(\lambda_0) = \|\rho C\|^2 \left(1 - \frac{\|\rho C\|^2}{\|v\|^2}Q_k^2(\lambda_0)\right);$$

$$\|z'\|^2 = \|\rho(V \setminus N_k)\|^2 \left(1 - \frac{\|\rho(V \setminus N_k)\|^2}{\|v\|^2}Q_k^2(\lambda_0)\right) = \|\rho(V \setminus N_k)\|^2 \|\rho N_k\|^2 \|\rho C\|^2 \|v\|^2.$$

Then, using these expressions we get

$$0 = \langle Q_k \rho C, \rho(V \setminus N_k) \rangle = Q_k(\lambda_0) \frac{\|\rho C\|^2}{\|v\|^2} \|\rho(V \setminus N_k)\|^2 + \langle Q_k z, z' \rangle$$

$$\geq Q_k(\lambda_0) \frac{\|\rho C\|^2}{\|v\|^2} \|\rho(V \setminus N_k)\|^2 - \|Q_k z\| \|z'\|$$

$$\geq Q_k(\lambda_0) \frac{\|\rho C\|^2}{\|v\|^2} \|\rho(V \setminus N_k)\|^2$$

$$- \|\rho C\| \|\rho(V \setminus N_k)\| \|\rho N_k\| \sqrt{\|v\|^2 - \|\rho C\|^2 Q_k^2(\lambda_0)}$$

and, solving for $Q_k(\lambda_0)$ and using again that $\|\rho(V \setminus N_k)\|^2 + \|\rho N_k\|^2 = \|v\|^2$, the first part of the result follows.

To prove the second part, note that the equality $Q_k(\lambda_0) = \|\rho N_k\|/\|\rho C\|$ holds if and only if all the inequalities above become equalities. Thus, $\cos(Q_k z, z') = -1$, which yields

$$z' = -\frac{\|z'\|}{\|Q_k z\|}Q_k z = -\frac{\|\rho(V \setminus N_k)\| \|\rho N_k\|}{\|\rho C\| \sqrt{\|v\|^2 - \|\rho C\|^2 Q_k^2(\lambda_0)}} Q_k z.$$
\[
\begin{align*}
&= -\frac{\|\rho N_k\|}{\|\rho C\|} \left( Q_k \rho C - \frac{\|\rho C\|^2}{\|v\|^2} Q_k(\lambda_0) v \right) \\
&= -\frac{\|\rho N_k\|}{\|\rho C\|} Q_k \rho C + \frac{\|\rho N_k\|^2}{\|v\|^2} v
\end{align*}
\]

and
\[
\rho(V \setminus N_k) = \frac{\|\rho(V \setminus N_k)\|^2}{\|v\|^2} v + \zeta
\]
\[
= \frac{\|\rho(V \setminus N_k)\|^2 + \|\rho N_k\|^2}{\|v\|^2} v - \frac{\|\rho N_k\|}{\|\rho C\|} Q_k \rho C
\]
\[
= v - \frac{\|\rho N_k\|}{\|\rho C\|} Q_k \rho C.
\]

Consequently,
\[
\rho N_k = v - \rho(V \setminus N_k) = \frac{\|\rho N_k\|}{\|\rho C\|} Q_k \rho C
\]
as claimed. \(\square\)

As a first consequence of the above result we derive the following refinement of Lemma 2.1, which is stated using the language of coding theory.

**Corollary 4.3.** For a given graph \(\Gamma = (V, E)\) with positive eigenvector \(v\), let \(C \subseteq V\) be a code with covering radius \(\epsilon\), and local spectrum \(sp\, C\) as above. Let \(Q_k\), \(0 \leq k \leq d\), be the \(C\)-local \(k\)-adjacency polynomial. Then,
\[
Q_k(\lambda_0) > \frac{1}{\|\rho C\|} \sqrt{\|v\|^2 - 1} \quad \Rightarrow \quad \epsilon \leq k.
\]

**Proof.** From the hypothesis and (26) we get \(\|\rho N_k\|^2 > \|v\|^2 - 1\), and the result follows since \(\lambda_u > 1\) for every \(u \in V\) implies \(\rho N_k = v\). \(\square\)

From (26) and the uniqueness of the adjacency polynomials we obtain the following result.

**Corollary 4.4.** If a polynomial \(p \in \mathbb{R}_k[x]\) has norm \(\|p\|_c \leq 1\) and \(p(\lambda_0) > \|\rho N_k\|/\|\rho C\|\), then \(p\) is the \(C\)-local \(k\)-adjacency polynomial.

It will be useful to consider the version of Proposition 4.1(c,d), and Theorem 4.2, concerning the adjacency polynomials, in terms of the polynomials \(q_k\).

**Corollary 4.5.** The polynomials \(q_k\), \(0 \leq k \leq d\), satisfy the following:

(a) \(q_0 = 1\), \(q_d = \frac{\|v\|^2}{\|\rho C\|^2} \prod_{t=1}^{d} (x - \mu_t) = HC\).
Consequently,

\[ q_0 e_C = \frac{1}{\| \rho C \|_2} \rho N_0 = e_C, \quad q_d e_C = \frac{1}{\| \rho C \|_2} \rho N_d = \frac{1}{\| \rho C \|_2} v. \tag{29} \]

The following result shows that, if \( C \) is extremal, the unique candidate to be the distance-\( d \) polynomial of \( C \) is the highest degree proper polynomial \( p_d \).

**Proposition 4.6.** Let \( C \) be an extremal set of vertices with antipodal set \( \overline{C} \). Let \( \{ p_k \}_{0 \leq k \leq d} \) be the proper orthogonal system associated to \( C \). If there exists a polynomial \( r \in \mathbb{R}_d[x] \) such that \( r \rho C = \overline{\rho C} \) (that is, the distance-\( d \) polynomial), then \( r = p_d \).

**Proof.** Let \( s \) be an arbitrary polynomial of \( \mathbb{R}_{d-1}[x] \). Then, \( \langle r, s \rangle_C = \langle r e_C, s e_C \rangle = \langle r \rho C, s e_C \rangle / \| \rho C \|_2 = 0 \), since the vectors \( \rho C \) and \( s e_C \) have no non-null common component. Thus, the succession of polynomials \( p_0, p_1, \ldots, p_{d-1}, r \), with respective degrees 0, 1, \ldots, \( d-1 \), \( d \), constitutes an orthogonal system for the scalar product \( \langle \cdot, \cdot \rangle_C \). Let us now compute \( r(\lambda_0) \). From the spectral decompositions of \( \rho C \) and \( \overline{\rho C} \), we have

\[ r \rho C = r \left( \frac{\| \rho C \|_2^2}{\| v \|_2^2} v + z \right) = \frac{\| \rho C \|_2^2}{\| v \|_2^2} r(\lambda_0) v + rz; \]

\[ \overline{\rho C} = \frac{\| \rho C \|_2^2}{\| v \|_2^2} v + z'. \]

Hence, equating both coefficients of \( v \), we get \( r(\lambda_0) = \| \overline{\rho C} \|_2^2 / \| \rho C \|_2^2 \). Then, since also

\[ \| r \|_C^2 = \langle r e_C, r e_C \rangle = \frac{1}{\| \rho C \|_2^2} \langle \rho C, \overline{\rho C} \rangle = \frac{\| \overline{\rho C} \|_2^2}{\| \rho C \|_2^2}, \]

we have that \( \| r \|_C^2 = r(\lambda_0) \) and, by Corollary 3.2, \( r = p_d \). \( \square \)

In the preceding study we have seen how to construct the families of polynomials \( \{ p_k \}, \{ q_k \}, \{ Q_k \} \) from the \( C \)-local spectrum (local eigenvalues and multiplicities). Conversely, if we apply Proposition 3.3 (with \( \rho_j = m(\mu_j), 1 \leq j \leq d, p_0 = m(\lambda_0) \) given by Eq. (4), and \( \pi_j = (-1)^j \phi_j \)), we obtain the following result showing that the local multiplicities can be "reconstructed" from the local eigenvalues and the highest degree proper polynomial \( p_d \).
Proposition 4.7. The values of $p_d$ at $\text{ev} \ C$ and the elements of the C-local spectrum are related by the following expressions.

(a) Each C-local multiplicity is given by
\begin{equation}
m(\mu_j) = (-1)^j \frac{\pi_0 p_d(\lambda_0)}{\pi_j p_d(\mu_j)} \frac{\|\rho C\|^2}{\|\nu\|^2}, \quad (0 \leq j \leq d); \tag{30}
\end{equation}

(b) The values of $p_d$ at the local eigenvalues are
\begin{align*}
p_d(\lambda_0) &= \frac{1}{\sum_{i=0}^{d} (1/m(\mu_i)\pi_i^2)}, \\
p_d(\mu_j) &= \frac{(-1)^j m(\lambda_0)\pi_0}{m(\mu_j)\pi_j} (1 \leq j \leq d). \tag{31}
\end{align*}

Note the similarity between (30) and the formula for the multiplicities of the eigenvalues of a distance-regular graph, given in [2].

5. Tight sets

Theorem 4.2 suggests to address our attention to the case in which, for some integer $0 \leq k \leq d$, the vertex set $C$ satisfies
\begin{equation}
Q_k(\lambda_0) = \frac{\|\rho N_k\|}{\|\rho C\|}, \tag{32}
\end{equation}
which, according to such a theorem and Corollary 4.5, is equivalent to any of the following conditions:
\begin{equation}
\frac{\|\rho N_k\|}{\|\rho C\|} Q_k \rho C = \rho N_k, \quad q_k(\lambda_0) = \frac{\|\rho N_k\|^2}{\|\rho C\|^2}, \quad q_k \rho C = \rho N_k. \tag{33}
\end{equation}
In this case we will say that $C$ is $k$-tight. Note that the cases $k = 0$ and $k = d$ are trivial, in the sense that, for these extreme values, any vertex set is 0-tight and $d$-tight, respectively. Indeed, from (29) we have $q_0 \rho C = \rho C = \rho N_0$, and also $q_d \rho C = \rho V = \rho N_d$. Moreover, when $C$ is extremal, the following alternative characterizations of being $(d - 1)$-tight apply.

Proposition 5.1. Let $C$ be an extremal set with $|\text{ev} \ C| = d + 1$, and let $\overline{C}$ denote the antipodal set of $C$. Then the following statements are equivalent:

(a) $C$ is $(d - 1)$-tight;

(b) There exists $r \in \mathbb{R}[x]$ such that $r \rho C = \overline{\rho C}$ (that is, the $d$-distance polynomial);

(c) The proper polynomial $p_d$ satisfies $p_d(\lambda_0) = \frac{\|\rho C\|^2}{\|\rho C\|^2}$.

If this is the case, both polynomials coincide, $r = p_d$. 
Proof. Assume that (a) holds. Then \( q_{d-1} \rho C = \rho N_{d-1} \), and hence \( p_d \rho C = q_d \rho C - q_{d-1} \rho C = \rho N_d - \rho N_{d-1} = \rho C \), which proves (b). Furthermore, if (b) is true, Proposition 4.6 gives \( r = p_d \). Then,
\[
p_d(\lambda_0) = \|p_d\|^2_C = \langle p_d e_C, p_d e_C \rangle = \frac{1}{\|\rho C\|^2} \langle p_d \rho C, p_d \rho C \rangle = \frac{\|\rho C\|^2}{\|\rho C\|^2}.
\]
Finally, assume that (c) holds. Then, using Corollary 4.5,
\[
q_{d-1}(\lambda_0) = q_d(\lambda_0) - p_d(\lambda_0) = \frac{1}{\|\rho C\|^2} (\|\rho V\|^2 - \|\rho C\|^2) = \frac{\|\rho N_{d-1}\|^2}{\|\rho C\|^2}
\]
and \( C \) is \((d-1)\)-tight by (33).

From now on we restrict ourselves to the study of the vertex sets which are \((d-1)\)-tight, called simply tight. As we will see, this extreme condition leads to a high level of structure of the graph “around” such sets. Notice that a tight set is necessarily extremal since, by Proposition 4.1, \( \rho N_{d-1}/\|\rho C\| = Q_{d-1}(\lambda_0) < Q_d(\lambda_0) = \|v\|/\|\rho C\| \) implies \( N_{d-1} \neq V \) and hence \( e = d \).

From our previous results it is clear that, for an extremal set \( C \), the knowledge of its local spectrum and of the “weight ratio” \( \|\rho C\|^2/\|\rho C\| \), allow us to decide whether or not \( C \) is tight. In this sense, Proposition 5.1(c) and the value of \( p_d(\lambda_0) \) given in Proposition 4.7(b) lead to the following characterization.

**Proposition 5.2.** Let \( C \) be an extremal set with \( sp C = \{\lambda_0, m_{\mu_1}, \ldots, m_{\mu_d}\} \) and eccentricity \( e = d \). Let \( C \) denote the antipodal set of \( C \). Then \( C \) is tight if and only if the following equality holds.
\[
\frac{\|\rho C\|^2}{\|\rho C\|^2} = \frac{1/m(\lambda_0)^2 \pi_0^2}{\sum_{l=0}^d (1/m(\mu_l)^2 \pi_l^2)}(34)
\]

Since the spectrum of a tight set \( C \) determines, through the construction of the proper orthogonal system \( [p_\mu] \), the antipodal set \( \overline{C} \) by \( p_d \rho C = \rho C \), we must be able, in particular, to compute \( sp \overline{C} \) from \( sp C \). The relation between both local spectra is shown in the next result.

**Proposition 5.3.** Let \( C \) be a tight set with \( sp C = \{\lambda_0, m_{\mu_1}, \ldots, m_{\mu_d}\} \) and proper polynomial \( p_d \). Then, \( sp \overline{C} \) has the same eigenvalues as \( sp C \), and the corresponding local multiplicities are related by
\[
m(\mu_l) = \frac{p^2_d(\mu_l)}{p_d(\lambda_0)} m(\mu_l) \quad (0 \leq l \leq d).
\]

**Proof.** Note first that Proposition 5.1(b) can be formulated as \( p_d e_C = \frac{\rho C}{\|\rho C\|} e_C \).

Thus, projecting onto \( e(\mu_l) \), we get \( p_d(\mu_l) e_C = \frac{\rho C}{\|\rho C\|} E_l e_C \). Taking square norms,
Consequently, the spectra $\text{sp}_C$ and $\overline{\text{sp}}_C$ are constituted by the same eigenvalues since, as it was seen, $p_d(\mu_l) \neq 0$, $0 \leq l \leq d$. Then, (36) together with Proposition 5.1(c) gives the result. □

Comparing (35) with the weight function $\overline{\pi}$ in (17), we get the following result, showing that the polynomials $\{\overline{\pi}_k\}$, associated to $\overline{C}$ through $\overline{\text{sp}}_C$, are just the conjugate polynomials of the $\{p_k\}$, as suggested by the notation; and that tight sets come in pairs.

**Proposition 5.4.** Let $C$ be a tight set of vertices with proper orthogonal system $\{p_k\}_{0 \leq k \leq d}$, and let $\{\overline{\pi}_k\}_{0 \leq k \leq d}$ be the proper orthogonal system associated to its antipodal set $\overline{C}$. Then,

(a) $p_d^{-1} = \overline{\pi}_d$;

(b) $\overline{\pi}_k = \overline{\pi}_d p_{d-k}$ \quad $(0 \leq k \leq d)$;

(c) $\overline{C}$ is also tight.

**Proof.** As commented, (a) and (b) follow from the relation between $\text{sp}_C$ and $\overline{\text{sp}}_C$, the definition of the conjugate polynomials, and the uniqueness of the proper orthogonal systems.

In the way of proving (c), we have that $\overline{\pi}_d \rho C = \overline{\pi}_d p_d \rho C = \rho C$, by (a). Then, taking into account Proposition 5.1(b), it only remains to show that $C$ is the antipodal set of $\overline{C}$. Clearly, $C \subset \overline{C}$. To prove equality, consider the polynomials $\{\overline{\pi}_k\}_{0 \leq k \leq d}$ associated to $\overline{C}$ as in Proposition 4.1(b). Then, applying $\overline{\pi}_d = \overline{\pi}_d + \overline{\pi}_{d-1}$ to $\rho C$, we get $\rho V = \rho C + \overline{\pi}_{d-1} \rho C$ which, compared with $\rho V = \rho \overline{C} + \rho N_{d-1}$, yields $\rho C = \rho \overline{C}$ and $\overline{\pi}_{d-1} \rho C = \rho N_{d-1}$.

Multiplying the result of Proposition 5.4(b), written as $\overline{\pi}_{d-k} = \overline{\pi}_d p_{k}$, $0 \leq k \leq d$, by $p_d$, and using Proposition 5.4(a), (or, simply, interchanging the roles of $C$ and $\overline{C}$) we get

$$p_k = p_d \overline{\pi}_{d-k} \quad (0 \leq k \leq d).$$

Note that, if we compute the right-hand expression of (34) using the spectrum of $\overline{C}$ given in Proposition 5.3 we get

$$\frac{1}{\sum_{l=0}^{d} \overline{\pi}_l(x_0)^2 \pi_0^2} = \frac{1}{\rho_d(\lambda_0) m(\mu_l)} \frac{p_d(\lambda_0) \rho_l m(\mu_l)}{p_d(\lambda_0) m(\mu_l) \pi_0^2} \frac{1}{\sum_{l=0}^{d} p_d(\lambda_0) m(\mu_l)} = \frac{\|\rho C\|^2}{\|\rho \overline{C}\|^2}$$
(where we have used that $p_d^2(\mu l)m(\mu l)\pi_l^2 = p_d^2(\lambda_l)m(\lambda_l)\pi_d^2$ for any $0 \leq l \leq d$), as expected.

As a direct consequence of the above results, note that, if the family $\mathcal{F}$ of tight vertex sets of a graph $\Gamma$ is not empty, then the application which maps every set to its antipodal fixes $\mathcal{F}$ and is involutive.

Now we are ready to give a result which is of central importance to our characterization of tight sets and completely-regular codes studied in the next section. Namely, we will see that the tight character of a set $C$—or, equivalently, the existence of the distance polynomial $p_d^C$—leads to the existence of all the distance polynomials with respect to both sets $C$ and $\overline{C}$, and that they are just the members of their associated proper orthogonal systems. What is more, for every $0 \leq k \leq d$, the action of the polynomial $p_k$ on $\rho C$ coincides with the action of $\overline{\rho}_{d-k}$ on $\rho \overline{C}$, so revealing the symmetry between the roles of $C$ and $\overline{C}$. (See Fig. 1.)

**Proposition 5.5.** Let $C$ be a tight set, with antipodal set $\overline{C}$, and let $\{p_k\}_{0 \leq k \leq d}$, $\{\overline{p}_k\}_{0 \leq k \leq d}$ be the corresponding proper orthogonal systems. Then,

$$p_k \rho C = \rho C_k = \rho \overline{C}_{d-k} = \overline{p}_{d-k} \rho \overline{C} \quad (0 \leq k \leq d).$$

**Proof.** Applying $p_k = \overline{p}_{d-k} p_d$ to the set $\rho C$, we get $p_k \rho C = \overline{p}_{d-k} \rho \overline{C}$. In terms of the involved vertices,

$$p_k \rho C = \sum_{u \in N_k} \xi_u e_u = \overline{p}_{d-k} \rho \overline{C} \quad \text{(38)}$$

If $\hat{d}(u, C) \leq k - 1$, then $\hat{d}(u, \overline{C}) \geq d - k + 1$ and hence $\xi_u = 0$. Thus, $p_k \rho C = \sum_{u \in C_k} \xi_u e_u$ for any $k = 0, 1, \ldots, d$. Then,

$$\sum_{v \in V} v_v e_v = v = \rho V = q_d \rho C = \sum_{k=0}^{d} \sum_{u \in C_k} \xi_u e_u$$

which implies $\xi_u = v_v$, for any $v \in V$. Consequently, $p_k \rho C = \sum_{u \in C_k} v_v e_u = \rho C_k$.

---

**Fig. 1.** Antipodal tight sets and their distance partition.
The same result for the antipodal set of $C$, $\overline{C}$, which is also tight and extremal, gives $\overline{P}_{d-k} \rho \overline{C} = \rho \overline{C}_{d-k}$ for any $0 \leq k \leq d$. Then using Eq. (38) we complete the proof.

When $C$ is tight, Proposition 5.1(c) illustrates the role of $p_k(\lambda_0)$ as the weight ratio (or, if the graph is regular, the order ratio) between $\overline{C} \equiv C_d$ and $C$. This generalizes to the other polynomials $p_k$ with $0 \leq k \leq d$.

**Corollary 5.6.** Let $C$ be a tight set. Then,
\[
\frac{\|\rho C_k\|^2}{\|\rho C\|^2} = p_k(\lambda_0) \quad (0 \leq k \leq d).
\]

**Proof.** This is proved by a simple computation
\[
\frac{\|\rho C_k\|^2}{\|\rho C\|^2} = (\rho C_k, \rho C_k) = (p_k \rho C, p_k \rho C) = \|\rho C\|^2 (p_k e_C, p_k e_C) = \|\rho C\|^2 \|p_k\|^2 = p_k(\lambda_0) \|\rho C\|^2.
\]

Notice also that, by Proposition 5.5, we infer that for a tight set $C$, the polynomials $q_k$ act on $\rho C$ in the form
\[
q_k \rho C = \sum_{h=0}^{k} p_h \rho C = \sum_{h=0}^{k} \rho C_h = \rho N_k \quad (0 \leq k \leq d).
\]

Consequently, from (33) we conclude that if a vertex set $C$ is tight then it is also $k$-tight for every $k = 0, 1, \ldots, d$.

6. **Pseudo-distance-regularity around a set**

In this section we introduce and characterize some combinatorial structures which come out in our study. We begin by introducing a generalization of the so-called equitable or regular partitions.

6.1. **Pseudo-equitable partitions**

For a given graph $\Gamma = (V, E)$, let us consider a given partition $\mathcal{P}$ of its vertex set $V = V_0 \cup V_1 \cup \cdots \cup V_m$. For every vertex $u \in V_k$, consider the numbers
\[
b_{hk}(u) := \sum_{v \in \Gamma(u) \cap V_h} \frac{v_u}{v_h} \quad (0 \leq h, k \leq m).
\]

Then, we say that $\mathcal{P}$ is a pseudo-equitable partition, whenever $b_{hk}(u)$ depends only on the values of $k$ and $h$, (but not on the chosen vertex $u$). In such a case, we denote these intersection numbers by $b_{hk}$, and we can consider the so-called collapsed
or quotient \((m+1) \times (m+1)\) matrix \(B = (b_{hk})\), denoted by \(A(\Gamma / \mathcal{P})\) (that is, the adjacency matrix of the quotient arc-weighted digraph).

As the chosen name suggests, and we will see below, the above concept is a generalization of the so-called equitable partitions, a name due to Schwenk [36]. Equitable partitions (called regular in Brouwer et al. [4]) were studied in some detail in McKay [30] and Godsil [24]. Roughly speaking, the definition of equitable partition is the same as that of pseudo-equitable partition, but now all the vertices have constant weight 1. More precisely, a partition \(V = V_0 \cup V_1 \cup \cdots \cup V_m\), of the vertex set of a graph \(\Gamma = (V, E)\), is equitable if the numbers \(b^*_{hk}(u) := |\Gamma(u) \cap V_h|\), where \(u \in V_k\) and \(0 \leq h, k \leq m\), only depend on the values of \(k\) and \(h\). Most of the results about equitable partitions can be generalized for pseudo-equitable partitions. For instance, it can be proved that the characteristic polynomial of the quotient matrix \(B\) divides the characteristic polynomial of \(A\) (see [23]).

For an equitable partition, it is known (see [24]) that all the entries of the positive eigenvector corresponding to the vertices in each set of the partition \(V_k\), \(0 \leq k \leq m\), have a common value, say \(\nu_k = \nu^*_k / |V_k|\), where \(\nu^*_k\) is the positive eigenvector of the collapsed matrix \(B^* = (b^*_{hk})\), normalized in such a way that \(\min\{\nu^*_k / |V_k| : 0 \leq k \leq m\} = 1\).

Lemma 6.1. Let \(\mathcal{P}\) be an equitable partition of a graph \(\Gamma\), with intersection numbers \(b^*_{hk}\), \(0 \leq h, k \leq m\). Let \(\Gamma\) have positive eigenvector \(\nu\) with entries denoted as above. Then \(\mathcal{P}\) is also a pseudo-equitable partition of \(\Gamma\) with new intersection numbers

\[
b_{hk} = \frac{\nu^*_h}{\nu^*_k} b^*_{hk} \quad (0 \leq h, k \leq m).
\]  

(39)  

Proof. Let \(u \in V_k\). Then,

\[
b_{hk}(u) = \frac{1}{\nu_k} \sum_{v \in \Gamma(u) \cap V_h} \nu_v = \frac{1}{\nu_k} \nu^*_k b^*_{hk}, \quad 0 \leq h, k \leq m. \quad \square
\]

Conversely, when the eigenvector \(\nu\) of a pseudo-equitable partition \(\mathcal{P}\) bears the above mentioned regularity, then \(\mathcal{P}\) is also an equitable partition, and the relation between the corresponding intersection numbers is still given by (39). To show, however, that this is not always the case, let us consider the following example of pseudo-equitable partition which is not equitable. Take the binary tree \(T\) of depth two, with vertices \(*\) (father), \(*0\), \(*1\) (sons), and \(*00\), \(*01\), \(*10\), \(*11\) (grandsons), radius \(r = \varepsilon_k = 2\), maximum eigenvalue \(\lambda_0 = 2\), and positive eigenvector \(\nu^*\) with entries \(v_0 = v_{00} = v_{01} = v_{10} = v_{11} = 1/2\). Then, by using known results about the spectrum and eigenvectors of the cartesian product of graphs (see Cvetković [8]), it is shown that the graph \(\Gamma = T \times \cdots \times T\) (\(t\) factors) has radius \(r' = \varepsilon_{(k, \ldots, k)} = 2t\), maximum eigenvalue \(\lambda_0' = 2t\), and eigenvector \(\nu'\) with \(v_{u_1} v_{u_2} \cdots v_{u_t}\) as the component associated to the vertex \((u_1, u_2, \ldots, u_t)\), \(u_i \in V(T)\). By using
these data, an easy computation shows that the distance partition induced in \( G \) by the central vertex \((*,*,\ldots,*)\) is indeed pseudo-equitable (but not equitable), and its non-null intersection numbers are \( b_{k-1,k} = b_{r-1,k}, 1 \leq k \leq r \). In fact, this is an example of “pseudo-distance-regularity around a vertex,” see [21], which, in turn, is a special case of pseudo-distance-regularity around a vertex set. In what follows, we concentrate on this last concept.

6.2. Locally pseudo-distance-regular graphs

Let \( C \subseteq V \) be a vertex set with eccentricity \( \varepsilon \). Then, we say that \( G \) is pseudo-distance-regular around \( C \) if the distance partition \( V = C_0 \cup C_1 \cup \cdots \cup C_{\varepsilon} \) is pseudo-equitable. Note that, in this case, we only need to consider the numbers

\[
C_k := \frac{1}{v_u} \sum_{v \in F(u) \cap C_{k-1}} v, \quad A_k := \frac{1}{v_u} \sum_{v \in F(u) \cap C_k} v, \quad b_k := \frac{1}{v_u} \sum_{v \in F(u) \cap C_{k+1}} v,
\]

where \( u \in C_k, 0 \leq k \leq \varepsilon \), and, by convention, \( c_0 = a_0 := 0 \). These numbers correspond to the above \( b_{k-1}, b_k, \) and \( b_{k+1} \), respectively (the other intersection numbers being zero), and they are referred to as the \((C-)local intersection numbers\) of \( C \).

Notice that, from (1), they clearly satisfy

\[
C_k + A_k + b_k = \lambda_0, \quad 0 \leq k \leq \varepsilon.
\]

The matrix

\[
M_C := \begin{pmatrix}
0 & c_1 & \cdots & c_{\varepsilon-1} & c_\varepsilon \\
0 & a_0 & a_1 & \cdots & a_{\varepsilon-1} & a_\varepsilon \\
0 & b_0 & b_1 & \cdots & b_{\varepsilon-1} & b_\varepsilon
\end{pmatrix}
\]

is called the intersection matrix around \( C \).

The concept of “pseudo-distance-regularity around a set” is a generalization of what is known, in the language of coding theory, as “completely regular codes” (first defined by Delsarte [14] in the setting of association schemes). This is a consequence of Lemma 6.1, since, in the sense of Neumaier [32], a completely regular code \( C \) in a graph \( G \) is just a vertex subset which induces an equitable distance partition (when \( G \) is distance-regular this definition is equivalent to Delsarte’s original one). Thus, using an alternative terminology, we are here interested in “pseudo-completely regular codes”.

The next result gives a characterization of pseudo-distance-regularity around a set in terms of the existence of its distance polynomials.

**Theorem 6.2.** A graph \( G = (V, E) \) is pseudo-distance-regular around a set \( C \subseteq V \), with eccentricity \( \varepsilon \), if and only if there exist a sequence of polynomials \( p_0, p_1, \ldots, p_{\varepsilon} \), with \( \deg p_k = k \), such that \( \rho C_k = p_k \rho C \) for any \( 0 \leq k \leq \varepsilon \).

**Proof.** Suppose first that \( G \) is pseudo-distance-regular around \( C \). Then, we will show first that the distance vectors \( \rho C_0, \rho C_1, \ldots, \rho C_{\varepsilon} \) satisfy the following recurrence relations:

\[
A \rho C_k = b_{k-1} \rho C_{k-1} + a_k \rho C_k + c_{k+1} \rho C_{k+1} \quad (0 \leq k \leq \varepsilon)
\]

(41)
(with \( C_{-1} = C_{r+1} = \emptyset \)). With this aim, let \( u \in C_{k+1} \). Then,
\[
(A \rho C_k)_u = (A \rho C_k, e_u) = (\rho C_k, A e_u) = \sum_{v \in G(u) \cap C_k} v_u = v_u c_{k+1} = c_{k+1}(\rho C_{k+1})_u.
\]

Similarly, if \( u \in C_k \) we get \((A \rho C_k)_u = a_k(\rho C_k)_u\), and if \( u \in C_{k-1} \), then \((A \rho C_k)_u = b_{k-1}(\rho C_{k-1})_u\). Since, clearly, \( \rho C_0 = p_0 \rho C \) with \( p_0 = 1 \), the recurrence relations imply the existence (and uniqueness) of the \( C \)-local distance polynomials \( p_k \) (of degree \( k \)), \( k = 0, 1, \ldots, \varepsilon \), as claimed.

Conversely, assume that such a sequence of polynomials \( \{p_k\} \) exist. Then, by Lemma 2.3, \( C \) is extremal, \( \varepsilon = d \). Let \( \lambda_0 > \lambda_1 > \cdots > \lambda_d \) be its local eigenvalues. As these polynomials satisfy a three-term recurrence like (9), the theory of orthogonal polynomials assures their orthogonality on a given mesh of \( d + 1 \) points. As expected, these points turn out to be the corresponding local eigenvalues, whereas their (local) multiplicities stand for the weights of the scalar product. Indeed, from \( \langle \rho C_k, \rho C_h \rangle = \langle p_k, p_h \rangle_C = 0, k \neq h \), we get \( \langle p_k, p_h \rangle_C = \langle p_k \epsilon_C, p_h \epsilon_C \rangle = 0 \), whereas for \( k = h \) we have \( \| p_k \|^2_C = \| p_k \|^2_C \) (compare with Corollary 5.6).

Let \( u \in C_k \) for some \( 1 \leq k \leq d \). Then, the polynomial \( x p_{k-1} \) can be written in terms of the orthogonal system \( p_0, p_1, \ldots, p_k \), so giving
\[
(x p_{k-1} \rho C)_u = \sum_{h=0}^{k} \gamma_h(p_h \rho C)_u = \sum_{h=0}^{k} \gamma_h(\rho C_h)_u = \gamma_k v_u,
\]
where \( \gamma_h := \langle x p_{k-1}, p_h \rangle_C / \| p_h \|^2_C \) is the corresponding Fourier coefficient. But, reasoning as before,
\[
(x p_{k-1} \rho C)_u = (A(x p_{k-1} \rho C))_u = \sum_{v \in G(u) \cap C_{k-1}} v_v.
\]

Consequently, \( \gamma_k \), a constant independent of the chosen vertex \( u \), is
\[
\gamma_k = \frac{1}{v_u} \sum_{v \in G(u) \cap C_{k-1}} v_v = c_k.
\]

The independence of \( a_k \) and \( b_k \), with respect to \( u \), is proved similarly by considering the polynomials \( x p_k \) and \( x p_{k+1} \), respectively.

When the set consists of a single vertex, \( C = \{u\} \), the above result was already given in [21]. In this case, using the language and notation of the subconstituent or Terwilliger algebra \( T = T(u) \) generated by \( A, E^*_1, E^*_1, \ldots, E^*_\varepsilon \) (with \( E^*_k, 0 \leq k \leq \varepsilon \), denoting the \( n \times n \) diagonal matrix with \( (E^*_k)_{vv} = 1 \) if \( \delta(u, v) = k \), and \( (E^*_k)_{vv} = 0 \) otherwise), Theorem 6.2 characterizes the situation in which the trivial \( T(u) \)-module is thin. That is, when the unique irreducible \( T(u) \)-module containing \( \rho u \), namely \( W := T \nu \), satisfies \( \dim E^*_k W \leq 1 \) for any \( 0 \leq k \leq \varepsilon \). In other words, the graph \( T \) is pseudo-distance-regular around vertex \( u \) if and only if the space spanned by \( E^*_0 \nu \),
$E_1^*v, \ldots, E_k^*v$ is invariant under $A$. See [37–40] for more details. For a general vertex subset $C$ we can give a similar characterization by simply changing $u$ by $C$ in the above.

In the case where $C$ is a completely regular code, the existence of a sequence of orthogonal polynomials giving the distance vectors (the necessary condition of the above theorem) was also noted by Martin [29].

As a consequence of Theorem 6.2 and the results of the previous section, we obtain the following characterization of pseudo-distance-regularity around a set (or of pseudo-completely regular codes).

**Theorem 6.3.** Let $C$ be a subset of vertices of a given graph $G$. Then $G$ is pseudo-distance-regular around $C$ if and only if $C$ is tight. Moreover, its intersection matrix coincides with the recurrence matrix of the proper polynomials $\{p_k\}_{0 \leq k \leq d}$ associated to $sp C$, that is $M_C = M(sp C)$.

**Proof.** Let $C$ have eccentricity $\varepsilon$, and $|ev C| = d + 1$. If $C$ is tight, then $\varepsilon = d$ and, by Proposition 5.5, the proper polynomials $\{p_k\}$ coincide with the distance polynomials, and Theorem 6.2 applies. Conversely, if $G$ is pseudo-distance-regular around $C$, there exist its distance polynomials, say $\{r_k\}_{0 \leq k \leq d}$, and hence $C$ is extremal, $\varepsilon = d$, by Lemma 2.3. Moreover, $r_d C = \rho C = \rho C$, so that $C$ is tight (Proposition 5.1). Therefore, from Proposition 5.5, $\{r_k\}$ is the proper orthogonal system. The coincidence between the two intersection matrices is clear from the proof of Theorem 6.2. □

From this theorem and Proposition 5.2, we can also give the following characterization of pseudo-distance-regularity around a set $C$, in terms of its local spectrum $sp C$.

**Theorem 6.4.** A graph $G = (V, E)$ is pseudo-distance-regular around a set $C \subset V$, with local spectrum $sp C = \{\lambda_0, \mu_1, \ldots, \mu_d\}$, if and only if
\[
\frac{\|\rho C\|^2}{\|\rho C\|} = \frac{1/m(\lambda_0)^2 \pi_0^2}{\sum_{d=0}^d (1/m(\mu_i) \pi_i^2)}.
\]

**Proof.** If (42) holds, then it must be $\|\rho C\| = \|\rho C\|$ for $C \neq \emptyset$, and hence $C$ is extremal by Lemma 2.1. $C = \overline{C}$. Thus, by Proposition 5.2, $C$ is also tight and Theorem 6.3 applies. Using the same results, the converse is clear. □

Using Lemma 6.1, and the comments which precede and follow it, the above theorem leads to the following characterization of completely regular codes.

**Theorem 6.5.** Let $G = (V, E)$ be a graph with positive eigenvector $v$. A vertex subset $C \subset V$, with local spectrum $sp C = \{\lambda_0, \mu_1, \ldots, \mu_d\}$, is a com-
pletely regular code if and only if the entries of \( \mathbf{v} \) are constant, say \( 0, v_1, \ldots, v_d \), on each set of the partition \( \mathcal{V} = C_0 \cup C_1 \cup \cdots \cup C_d \), and

\[
\frac{n_d}{n_0} = \frac{(v_0/m(\lambda_0)\pi_0 v_d)^2}{\sum_{i=0}^{d}(1/m(\mu_i)\pi_i^2)}
\tag{43}
\]

where \( n_0 := |C_0| = |C| \) and \( n_d := |C_d| \).

**Proof.** Use the above-mentioned results and \( \|\rho C_d\|^2 \|\rho C_0\|^2 = \frac{n_d v_d^2}{n_0 v_0^2} \). \( \square \)

Let us consider a graph \( \Gamma \), which is pseudo-distance-regular around a set \( C \subset \mathcal{V} \), with eccentricity \( \epsilon = d \). Since the distance partitions induced by \( C \) and its antipodal set \( \overline{C} = C_d \) are clearly the same, it is trivial that \( \Gamma \) is also pseudo-distance-regular around \( \overline{C} \) and the respective intersection numbers are related by

\[
\overline{c}_k = c_{d-k}, \quad \overline{a}_k = a_{d-k}, \quad \overline{b}_k = b_{d-k} \quad (0 \leq k \leq d),
\]

which gives \( M(\rho C_d) \) by applying a central symmetry to \( M(\rho C) \), just as \( M(\rho, \overline{\rho}) \) derived from \( M(\rho, \frac{\rho}{\rho}) \). This proves again, through Theorem 6.3, that \( C \subset \mathcal{V} \) is tight if and only if \( \overline{C} \) is. Moreover, looking at the recurrences satisfied by the proper orthogonal systems \( \{p_k\} \) and \( \{\overline{p}_k\} \), we have the following expressions for the intersection numbers associated to \( C \):

\[
c_k = \frac{a_{k-1}}{a_k}, \quad b_k = \overline{b}_{d-k} = \frac{\overline{a}_{d-k-1}}{\overline{a}_{d-k}}, \quad a_k = \lambda_0 - b_k - c_k \quad (0 \leq k \leq d),
\tag{44}
\]

where \( a_k \) and \( \overline{a}_k \), \( 0 \leq k \leq d \), stand for the leading coefficients of \( p_k \) and \( \overline{p}_k \), respectively.

In the case where \( \Gamma \) is regular (\( \mathbf{v} = \mathbf{j} \)) and the set \( C \) consists of a single vertex \( u \) (\( n_0 = 1 \)) we simply speak about distance-regularity around \( u \). Then \( \Gamma \) is called distance-regular if it is distance-regular around each of its vertices and with the same intersection matrix, see [3,4,24]. In such a case, it is known that the \( u \)-local multiplicity of any eigenvalue \( \lambda_i \in \text{ev} \Gamma \) is just \( m_u(\lambda_i) = m_i/n \), \( 0 \leq i \leq d \) (recall that \( m_i \) is the standard multiplicity of \( \lambda_i \) and \( n = |V| \)), see [25]. Substituting these values into (43) we get that, in such a graph, the number of vertices at distance \( d \) from each vertex must be

\[
n_d = \frac{n}{\pi_0^2 \sum_{i=0}^{d}(1/m_i \pi_i^2)}
\tag{45}
\]

In fact, in [18] the authors managed to prove that this condition is also sufficient for a regular graph with spectrum \( \text{sp} \Gamma = \{\lambda_0, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\} \) to be distance-regular. (The case \( d = 3 \) had been already proved by Haemers and Van Dam [26,11].)

**References**

[40] P. Terwilliger, Algebraic Graph Theory, forthcoming.