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Note

On the spectral characterization of the union of complete multipartite graph and some isolated vertices[☆]

Haicheng Ma[a], Haizhen Ren[b,*]

[a] Department of Mathematics, Qinghai Nationalities University, Xining, Qinghai, 810007, PR China

[b] Department of Mathematics, Qinghai Normal University, Xining, Qinghai, 810008, PR China

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ABSTRACT

A graph is said to be determined by its adjacency spectrum (DS for short) if there is no other non-isomorphic graph with the same spectrum. In this paper, we focus our attention on the spectral characterization of the union of complete multipartite graph and some isolated vertices, and all its cospectral graphs are obtained. By the results, some complete multipartite graphs determined by their adjacency spectrum are also given. This extends several previous results of spectral characterization related to the complete multipartite graphs.

1. Introduction

All graphs considered here are finite, undirected and simple (loops and multiple edges are not allowed). Let G be a graph with adjacency matrix $A(G)$. We denote $\det(\lambda I - A)$, the characteristic polynomial of G, by $P(G, \lambda)$. The multiset of eigenvalues of $A(G)$ is called the adjacency spectrum of G, or simply the spectrum of G. Since $A(G)$ is a symmetric matrix, the eigenvalues of G are real. Two graphs G and H are called cospectral, symbolically $G \sim H$, if $P(G, \lambda) = P(H, \lambda)$. It is clear that the relation $\sim$ is an equivalence relation on graphs. We denoted by $[G]$ the cospectral equivalence class determined by G under $\sim$. A graph G is said to be determined by its adjacency spectrum (DS for short) if there is no other non-isomorphic graph with the same spectrum, that is $[G] = \{G\}$. For two graphs G and H, $G \cup H$ denotes the disjoint union of G and H, and mH the disjoint union of m copies of H. Undefined notation and terminology will refer to those in [1,2].

As reported in [2], there are many known results about cospectral but non-isomorphic graphs. It is conjectured that almost all graphs are DS. Whenever considering the question to what extent graphs are determined by their adjacency spectrum, however, we noted that it is not easy to prove that a given graph is DS [3–14]. The recent surveys on DS and cospectral graphs can be found in [2,10], and the references therein. Therefore, it would be interesting to characterize the spectrum equivalent classes of graphs and find more examples of DS graphs. For the positive integers $n_1, n_2, \ldots, n_k$, let $K_{n_1,n_2,\ldots,n_k}$ be a complete k-partite graph and K_1 be an isolated vertex. In this paper, we focus our attention on characterizing the cospectral equivalent class of $K_{n_1,n_2,\ldots,n_k} \cup sK_1$, and all its cospectral graphs are obtained. By the results, some complete multipartite graphs determined by their adjacency spectrum are also given. This extends several previous results of spectral characterization related to the complete multipartite graphs.

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* Corresponding author.
E-mail addresses: qhmymhc@163.com (H. Ma), haizhenr@126.com (H. Ren).

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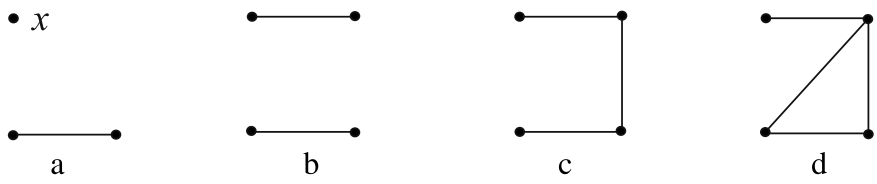

Fig. 1. Four possible modes of induced subgraphs in G.

2. Some lemmas

The following lemmas will be used in what follows.

Lemma 2.1 ([2]). *$P(K_{n_1,n_2,\ldots,n_k},\lambda)=\sum_{i=0}^{k}(1-i)S_i\lambda^{n-i}$, where $n=n_1+n_2+\cdots+n_k$, and S_i $(i=1,2,\ldots,k)$ with order i is the elementary symmetric function of the numbers $n_1,n_2,\ldots,n_k$ and $S_0=1$. Namely,*

$$\begin{cases} S_1=\sum\limits_{1\le i\le k} n_i,\\ S_2=\sum\limits_{1\le i_1<i_2\le k} n_{i_1}n_{i_2},\\ \cdots\cdots\\ S_k=n_1n_2\cdots n_k.\end{cases}$$

As pointed in [2], $P(K_{n_1,n_2,\ldots,n_k},\lambda)$ has exactly one positive root. In fact, let $\lambda_1>0$ be a root of $P(K_{n_1,n_2,\ldots,n_k},\lambda)$. If $\lambda_2>\lambda_1$, without loss of generality, suppose that $\lambda_2=r\lambda_1$, $(r>1)$. Since $P(K_{n_1,n_2,\ldots,n_k},\lambda_1)=0$, i.e. $\lambda_1^n=S_2\lambda_1^{n-2}+2S_3\lambda_1^{n-3}+\cdots+(k-1)S_k\lambda_1^{n-k}$. Then

$$\begin{aligned}\lambda_2^n=r^n\lambda_1^n&=S_2r^n\lambda_1^{n-2}+2S_3r^n\lambda_1^{n-3}+\cdots+(k-1)S_kr^n\lambda_1^{n-k}\\ &>S_2r^{n-2}\lambda_1^{n-2}+2S_3r^{n-3}\lambda_1^{n-3}+\cdots+(k-1)S_kr^{n-k}\lambda_1^{n-k}\\ &=S_2\lambda_2^{n-2}+2S_3\lambda_2^{n-3}+\cdots+(k-1)S_k\lambda_2^{n-k}.\end{aligned}$$

Thus, $P(K_{n_1,n_2,\ldots,n_k},\lambda_2)>0$, which implies that $P(K_{n_1,n_2,\ldots,n_k},\lambda)$ has exactly one positive root.

In [12], J.H. Smith showed that the following Lemma 2.2 holds.

Lemma 2.2 ([12]). *A graph has exactly one positive eigenvalue if and only if its non-isolated vertices form a complete multipartite graph.*

In fact, as illustrated in [12], if we ignore isolated vertices, and if a graph G is not a complete multipartite, then it contains an induced subgraph shown in Fig. 1(a). Since x is not an isolated vertex of G. Thus we conclude that G at least contains one of such graphs (b), (c) and (d) shown in Fig. 1 as an induced subgraph. All of these graphs have two positive eigenvalues, so we know that G has at least two positive eigenvalues.

We noted that Lemmas 2.1 and 2.2 imply the following Lemma 2.3:

Lemma 2.3 ([2,12]). *A graph has exactly one positive eigenvalue if and only if all its non-isolated vertices form a complete k-partite graph with $k=1+\rho_-$, where ρ_- denotes the number of negative eigenvalues of G.*

Lemma 2.4. *Let $n_1,\ldots,n_k(k\ge 2)$ be the integers not less than 2. Then*

$$n_1\cdots n_k\ge n_1+\cdots+n_k.$$

Proof. Let n_1 and n_2 be two integers not less than 2. Since $(n_1-1)(n_2-1)\ge 1$. Then $n_1n_2\ge n_1+n_2$. By the inductive assumption, we can easily generalize the result to the case of k ($\ge$2) integers. This completes the proof. □

Lemma 2.5. *Let $x_1,x_2,\ldots,x_s$ $(s\ge 2)$ be the positive integers, and let $x_1+x_2+\cdots+x_s=N'$. Suppose that $N'=sd+r$, $(0\le r<s)$. Then $x_1x_2\cdots x_s$ is the greatest if and only if*

$$\{x_1,x_2,\ldots,x_s\}=\left\{d,\ldots,d,\overbrace{d+1,\ldots,d+1}^{r}\right\},$$

and the greatest product is $d^{s-r}(d+1)^r$.

Proof. The following Fact 1 clearly holds.

Fact 1. Let x,y be two positive integers. If $x-y\ge 2$, then $(x-1)(y+1)>xy$.

We noted that above Fact 1 implies that for two positive integers a and b, if $a + b$ is constant, the more a being close to b the greater ab. Furthermore, by continuously applying Fact 1 we can easily generalize the result to the case of $s(\geq 2)$ positive integers, i.e., for the positive integers $x_1, x_2, \ldots, x_s$ $(s \geq 2)$, if their sum is constant, the more they being close to each other the greater their product. This proves Lemma 2.5. □

In addition, the dual law of Lemma 2.5 holds, i.e., for the positive integers $x_1, x_2, \ldots, x_s (s \geq 2)$, if their product is constant, the more they being close to each other the smaller their sum, and which can be formulated as the following Lemma 2.6:

Lemma 2.6. *Let $x_1, x_2, \ldots, x_s$ be the positive integers, and let $x_1x_2 \cdots x_s = d^{s-r}(d+1)^r$, where $0 \leq r < s$. Then $x_1+x_2+\cdots+x_s$ is the smallest if and only if*

$$\{x_1, x_2, \ldots, x_s\} = \left\{d, \ldots, d, \overbrace{d+1, \ldots, d+1}^{r}\right\}$$

and the smallest sum is $sd + r$.

3. Main results and proofs

Let $n_1, n_2, \ldots, n_k$ be the positive integers and s be a non-negative integer. For $a \in \{k, k+1, \ldots, n_1 + n_2 + \cdots + n_k + s\}$, if the equation

$$(x - n_1)(x - n_2) \cdots (x - n_k) + (n_1 + n_2 + \cdots + n_k - a)\, x^{k-1} = 0 \tag{1}$$

has only integral roots: $x_1, x_2, \ldots, x_k$, then the sequence $(x_1, x_2, \ldots, x_k, a)$ is called a *suitable solution* of (1). Denoted by $\mathcal{N}$ the set of all suitable solutions of (1).

Theorem 3.1. *Let $n_1, n_2, \ldots, n_k$ be the positive integers and s be a non-negative integer.*

(i)
$$\left[K_{n_1,n_2,\ldots,n_k} \cup sK_1\right] = \left\{K_{x_1,x_2,\ldots,x_k} \cup (n_1 + n_2 + \cdots + n_k + s - a)\, K_1 \mid (x_1, x_2, \ldots, x_k, a) \in \mathcal{N}\right\}$$
where $\mathcal{N}$ denotes the set of all suitable solutions of Eq. (1).

(ii) *$K_{n_1,n_2,\ldots,n_k} \cup sK_1$ is DS if and only if there is no suitable solution of Eq. (1) except $a = n_1 + n_2 + \cdots + n_k$.*

Proof. (i) If $(x_1, x_2, \ldots, x_k, a) \in \mathcal{N}$. Then we have

$$\begin{cases} \sum\limits_{1\leq i\leq k} x_i = a, \\ \sum\limits_{1\leq i_1<i_2\leq k} x_{i_1}x_{i_2} = \sum\limits_{1\leq i_1<i_2\leq k} n_{i_1}n_{i_2}, \\ \sum\limits_{1\leq i_1<i_2<i_3\leq k} x_{i_1}x_{i_2}x_{i_3} = \sum\limits_{1\leq i_1<i_2<i_3\leq k} n_{i_1}n_{i_2}n_{i_3}, \\ \cdots\cdots \\ x_1x_2\cdots x_k = n_1n_2\cdots n_k. \end{cases} \tag{2}$$

By Lemma 2.1, we get

$$K_{x_1,x_2,\ldots,x_k} \cup (n_1 + n_2 + \cdots + n_k + s - a)\, K_1 \sim K_{n_1,n_2,\ldots,n_k} \cup sK_1.$$

Thus

$$K_{x_1,x_2,\ldots,x_k} \cup (n_1 + n_2 + \cdots + n_k + s - a)\, K_1 \in \left[K_{n_1,n_2,\ldots,n_k} \cup sK_1\right].$$

On the contrary, let $H \in [K_{n_1,n_2,\ldots,n_k} \cup sK_1]$. Then H has exactly one positive eigenvalue. By Lemma 2.2, the non-isolated vertices of H form a complete l-partite graph. Further, by Lemma 2.3 and comparing the number of their negative eigenvalues, we have $l - 1 = k - 1$, which implies that the non-isolated vertices of H form a complete k-partite graph. Thus, we may set $H \cong K_{m_1,m_2,\ldots,m_k} \cup tK_1$. Then,

$$P\left(K_{m_1,m_2,\ldots,m_k} \cup tK_1, \lambda\right) = P\left(K_{n_1,n_2,\ldots,n_k} \cup sK_1, \lambda\right).$$

By Lemma 2.1, $(m_1, m_2, \ldots, m_k)$ is a positive integer solution of the Diophantine equations (2) for $a = m_1 + m_2 + \cdots + m_k$, and $k \leq a = m_1 + m_2 + \cdots + m_k \leq m_1 + m_2 + \cdots + m_k + t = n_1 + n_2 + \cdots + n_k + s$. Therefore, $(m_1, m_2, \ldots, m_k, a)$ is a suitable solution of Eq. (1), i.e., $(m_1, m_2, \ldots, m_k, a) \in \mathcal{N}$.

(ii) Suppose that there exist $a \in \{k, k+1, \ldots, n_1 + n_2 + \cdots + n_k + s\}$ and $a \neq n_1 + n_2 + \cdots + n_k$ such that Eq. (1) has suitable solution: $(m_1, m_2, \ldots, m_k, a)$. By the relationship between roots and coefficients, we know that

$$\{m_1, m_2, \ldots, m_k\} \neq \{n_1, n_2, \ldots, n_k\}.$$

So,

$$K_{m_1,m_2,\ldots,m_k} \cup (n_1 + n_2 + \cdots + n_k + s - a)\, K_1 \in \left[K_{n_1,n_2,\ldots,n_k} \cup sK_1\right]$$

and

$$K_{m_1,m_2,\ldots,m_k} \cup (n_1+n_2+\cdots+n_k+s-a)\,K_1 \ncong K_{n_1,n_2,\ldots,n_k} \cup sK_1,$$

which is impossible. This proves the necessity. To prove the sufficiency, note that if $K_{n_1,n_2,\ldots,n_k} \cup sK_1$ is not DS. By (i) we may set $K_{m_1,m_2,\ldots,m_k} \cup (n_1+n_2+\cdots+n_k+s-a)K_1 \in [K_{n_1,n_2,\ldots,n_k} \cup sK_1]$ and $K_{m_1,m_2,\ldots,m_k} \cup (n_1+n_2+\cdots+n_k+s-a)K_1 \ncong K_{n_1,n_2,\ldots,n_k} \cup sK_1$, where $a = m_1+m_2+\cdots+m_k$. Then $(m_1, m_2, \ldots, m_k, a)$ is a suitable solution of Eq. (1), and $a \neq n_1+n_2+\cdots+n_k$ (Otherwise, by Eq. (1) we have $\{m_1, m_2, \ldots, m_k\} = \{n_1, n_2, \ldots, n_k\}$, then $K_{m_1,m_2,\ldots,m_k} \cup (n_1+n_2+\cdots+n_k+s-a)K_1 \cong K_{n_1,n_2,\ldots,n_k} \cup sK_1$. This is contradictory.) This completes the proof. □

Corollary 3.1. *Let s, t be the positive integers. The complete bipartite graph $K_{s,t}$ is DS if and only if the equality $n = st$ is the decomposition of two factors s and t with the smallest sum $s+t$.*

Proof. To prove the sufficiency, suppose that $[K_{s,t}] = \{K_{x_1,x_2} \cup (s+t-a)K_1 \mid (x_1, x_2, a) \in \mathcal{N}\}$. Then by Theorem 3.1, $x_1+x_2 = a \le s+t$ and $x_1x_2 = st = n$. So we have $\{x_1, x_2\} = \{s, t\}$ and $s+t = a$. Thus $[K_{s,t}] = \{K_{s,t}\}$. Now we prove the necessity. If $K_{s,t}$ is DS, we assume that there exist the positive integers s_1 and t_1 such that $n = s_1t_1$ and $s_1+t_1 < s+t$. This implies (by Theorem 3.1) that $K_{s,t} \sim K_{s_1,t_1} \cup (s+t-s_1-t_1)K_1$, which is impossible. □

Theorem 3.2. *Let $p_1, p_2, \ldots, p_s$ be the prime numbers and t be a non-negative integer. Then $K_{\underbrace{1,1,\ldots,1}_{t},p_1,p_2,\ldots,p_s}$ is DS.*

Proof. By Theorem 3.1, let

$$K_{x_1,x_2,\ldots,x_{s+t}} \cup (t+p_1+\cdots+p_s-x_1-\cdots-x_{s+t})\,K_1 \in \left[K_{\underbrace{1,1,\ldots,1}_{t},p_1,p_2,\ldots,p_s}\right].$$

Then $x_1x_2\cdots x_{s+t} = p_1p_2\cdots p_s$. Obviously, the numbers of 1 in $x_1, x_2, \ldots, x_{s+t}$ are at least t. If the numbers of 1 in $x_1, x_2, \ldots, x_{s+t}$ are greater than t, without loss of generality, let $x_1 = x_2 = \cdots = x_a = 1$, $(a > t)$, and let $x_{a+1} = p'_1\cdots p'_{l_1}, \ldots, x_{s+t} = p'_{l_i}\cdots p'_s$, where $p'_1, \ldots, p'_{l_1}, \ldots, p'_{l_i}, \ldots, p'_s$ is a permutation of $p_1, p_2, \ldots, p_s$. By Lemma 2.4, we have

$$x_{a+1} \ge p'_1+\cdots+p'_{l_1}, \ldots, x_{s+t} \ge p'_{l_i}+\cdots+p'_s$$

and

$$\begin{aligned} x_1+x_2+\cdots+x_a+x_{a+1}+\cdots+x_{s+t} &\ge a+p'_1+\cdots+p'_{l_1}+\cdots+p'_{l_i}+\cdots+p'_s \\ &> t+p_1+p_2+\cdots+p_s. \end{aligned}$$

Considering the number of vertices of graphs, this is contradictory. Thus, the numbers of 1 in $x_1, x_2, \ldots, x_{s+t}$ exactly are t. By symmetry and the unique decomposition theorem of integers, we can get $\{x_1, x_2, \ldots, x_{s+t}\} = \{1, \ldots, 1, p_1, \ldots, p_s\}$. Therefore, $K_{1,\ldots,1,p_1,\ldots,p_s}$ is uniquely determined by its adjacency spectrum. This completes the proof. □

By Theorem 3.2, we get the following Corollary 3.2:

Corollary 3.2. *Let $p_1, p_2, \ldots, p_s$ be the prime numbers. Then $K_{p_1,p_2,\ldots,p_s}$ is DS.*

Theorem 3.3. *Let d, s be two positive integers. Then $K_{\underbrace{(d+1),\ldots,(d+1)}_{r},\underbrace{d,\ldots,d}_{s-r}}$ is DS, where $0 \le r < s$.*

Proof. Let

$$K_{x_1,x_2,\ldots,x_s} \cup (sd+r-x_1-\cdots-x_s)\,K_1 \in \left[K_{\underbrace{(d+1),\ldots,(d+1)}_{r},d,\ldots,d}\right].$$

Then $x_1+x_2+\cdots+x_s \le sd+r$ and $x_1x_2\cdots, x_s = d^{s-r}(d+1)^r$ by Theorem 3.1. Thus, by Lemma 2.6 we have

$$\{x_1, x_2, \ldots, x_s\} = \left\{\underbrace{d+1,\ldots,d+1}_{r}, d, \ldots, d\right\}.$$

Therefore, $K_{\underbrace{d+1,\ldots,d+1}_{r},\underbrace{d,\ldots,d}_{s-r}}$ is uniquely determined by its adjacency spectrum, where $0 \le r < s$. This completes the proof. □

By Theorem 3.3, we get $K_{\underbrace{d,\ldots,d}_{t}}$ is DS, where d and t are positive integers.

4. Remarks

We noted that Theorem 3.1 implies that a new method of constructing cospectral graphs. For example, $K_{n(2n-1)^2,(2n-1),(2n-1)}$ is cospectral with $K_{n,(2n-1)^2,(2n-1)^2} \bigcup 4(n-1)^3 K_1$. In addition, we know that the complete multipartite graph is not necessarily DS with respect to its adjacency matrix. Thus, we pose the following problem:

Problem: Characterize all complete multipartite graphs that are DS.

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