Available online at www.sciencedirect.com
ScienceDirect
Discrete Mathematics 309 (2009) 673–685
DISCRETE MATHEMATICS
www.elsevier.com/locate/disc

# Equivalence of operations with respect to discriminator clones

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Received 13 June 2007; received in revised form 3 January 2008; accepted 5 January 2008
Available online 4 March 2008

**Abstract**

For each clone $\mathcal{C}$ on a set $A$ there is an associated equivalence relation, called $\mathcal{C}$-equivalence, on the set of all operations on $A$, which relates two operations iff each one is a substitution instance of the other using operations from $\mathcal{C}$. In this paper we prove that if $\mathcal{C}$ is a discriminator clone on a finite set, then there are only finitely many $\mathcal{C}$-equivalence classes. Moreover, we show that the smallest discriminator clone is minimal with respect to this finiteness property. For discriminator clones of Boolean functions we explicitly describe the associated equivalence relations.


*Keywords:* Clone; Discriminator function; Minor; Subfunction; Boolean function

## 1. Introduction

This paper is a study of how functions on a fixed set can be classified using their substitution instances with inner functions taken from a given set of functions. In the theory of Boolean functions several variants of this idea have been employed. Harrison [5] was interested in the number of equivalence classes when $n$-ary Boolean functions are identified if they are substitution instances of each other with respect to the general linear group $\mathrm{GL}(n, \mathbb{F}_2)$ or the affine general linear group $\mathrm{AGL}(n, \mathbb{F}_2)$ ($\mathbb{F}_2$ is the two-element field). Wang and Williams [14] introduced classification by Boolean minors to prove that the problem of determining the threshold order of a Boolean function is NP-complete. They defined a Boolean function $g$ to be a *minor* of another Boolean function $f$ iff $g$ can be obtained from $f$ by substituting for each variable of $f$ a variable, a negated variable, or one of the constants 0 or 1. Wang [13] characterized various classes of Boolean functions by forbidden minors. A more restrictive variant of Boolean minors, namely when negated variables are not allowed, was used in [4,15] to characterize other classes of Boolean functions by forbidden minors.

In semigroup theory, Green's relation $R$, when applied to transformation semigroups $\mathcal{S}$, is another occurrence of the idea of classifying functions by their substitution instances; namely, two transformations $f, g \in \mathcal{S}$ are $R$-related iff $f(h_1(x)) = g(x)$ and $g(h_2(x)) = f(x)$ for some $h_1, h_2 \in \mathcal{S} \cup \{\mathrm{id}\}$. Henno [6] generalized Green's relations to Menger systems (essentially, abstract clones), and described Green's relations on the clone $\mathcal{O}_A$ of all operations on $A$ for each set $A$.

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0012-365X/$ - see front matter 
doi:10.1016/j.disc.2008.01.003

The notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence where $\mathcal{C}$ is an arbitrary clone provide a common framework for these results. If $\mathcal{C}$ is a fixed clone on a set $A$, and $f, g$ are operations on $A$, then $g$ is a $\mathcal{C}$-minor of $f$ if $g$ can be obtained from $f$ by substituting operations from $\mathcal{C}$ for the variables of $f$, and $g$ is $\mathcal{C}$-equivalent to $f$ if $f$ and $g$ are both $\mathcal{C}$-minors of each other. Thus, for example, the $R$-relation on $\mathcal{O}_A$ described in [6] is nothing else than $\mathcal{O}_A$-equivalence, and the concepts of Boolean minor mentioned in the first paragraph are the special cases of the notion of $\mathcal{C}$-minor where $\mathcal{C}$ is the essentially unary clone of Boolean functions generated by negation and the two constants, or by the two constants only. For the least clone of Boolean functions, the essentially unary clone $\mathcal{P}$ of all projections, the $\mathcal{P}$-minor relation is investigated in [2], and the classes of Boolean functions that are closed under taking $\mathcal{P}$-minors are characterized in [3]. The latter result is extended in [10] to classes of functions on finite sets that are closed under taking $\mathcal{C}$-minors for arbitrary essentially unary clones $\mathcal{C}$. The general notions of $\mathcal{C}$-minor and $\mathcal{C}$-equivalence, as introduced at the beginning of this paragraph, first appeared in print in [7], where the first author studied the $\mathcal{C}$-minor quasiorder for clones $\mathcal{C}$ of monotone and linear operations.

The question this paper will focus on is the following:

**Question.** For which clones $\mathcal{C}$ on a finite set are there only finitely many $\mathcal{C}$-equivalence classes of operations?

The clones that have this property form an order filter (i.e., an upset) $\mathfrak{F}_A$ in the lattice of clones on $A$ (see Proposition 2.3). Henno's result [6] (see Corollary 3.4) implies that $\mathcal{O}_A \in \mathfrak{F}_A$ if and only if $A$ is finite. Thus the order filter $\mathfrak{F}_A$ is nonempty if and only if $A$ is finite. The order filter $\mathfrak{F}_A$ is proper if $|A| > 1$, since the clone $\mathcal{P}_A$ of projections fails to belong to $\mathfrak{F}_A$. The latter statement follows from the fact that $\mathcal{P}_A$-equivalent operations have the same essential arity (i.e., depend on the same number of variables), and on a set with more than one element there exist operations of arbitrarily large essential arity.

In this paper we prove that every discriminator clone on a finite set $A$ belongs to $\mathfrak{F}_A$. Furthermore, we show that if $|A| = 2$, then the members of $\mathfrak{F}_A$ are exactly the discriminator clones; thus in this case $\mathfrak{F}_A$ has a least member, namely the smallest discriminator clone. If $|A| > 2$, then the analogous statements are no longer true, because by a result of the first author in [8], Słupecki's clone belongs to $\mathfrak{F}_A$. Słupecki's clone consists of all operations that are either essentially unary or nonsurjective, therefore it is not a discriminator clone. Thus for finite sets with three or more elements the order filter $\mathfrak{F}_A$ remains largely unknown. However, we show that even in this case the smallest discriminator clone is a minimal member of $\mathfrak{F}_A$.

In the last section of the paper we explicitly describe the $\mathcal{C}$-equivalence and $\mathcal{C}$-minor relations for discriminator clones of Boolean functions.

## 2. Preliminaries

Let $A$ be a fixed nonempty set. If $n$ is a positive integer, then by an $n$-ary operation on $A$ we mean a function $A^n \to A$, and we will refer to $n$ as the *arity* of the operation. The set of all $n$-ary operations on $A$ will be denoted by $\mathcal{O}_A^{(n)}$, and we will write $\mathcal{O}_A$ for the set of all finitary operations on $A$. For $1 \leq i \leq n$ the $i$-th $n$-ary *projection* is the operation $p_i^{(n)}\colon A^n \to A,\ (a_1, \dots, a_n) \mapsto a_i$. Every function $h\colon A^n \to A^m$ is uniquely determined by the $m$-tuple of functions $\mathbf{h} = (h_1, \dots, h_m)$ where $h_i = p_i^{(m)} \circ h\colon A^n \to A$ $(i = 1, \dots, m)$. In particular, $\mathbf{p}^{(n)} = (p_1^{(n)}, \dots, p_n^{(n)})$ corresponds to the identity function $A^n \to A^n$. From now on we will identify each function $h\colon A^n \to A^m$ with the corresponding $m$-tuple $\mathbf{h} = (h_1, \dots, h_m) \in (\mathcal{O}_A^{(n)})^m$ of $n$-ary operations. Using this convention the *composition* of functions $\mathbf{h} = (h_1, \dots, h_m)\colon A^n \to A^m$ and $\mathbf{g} = (g_1, \dots, g_k)\colon A^m \to A^k$ can be written as

$$\mathbf{g} \circ \mathbf{h} = (g_1 \circ \mathbf{h}, \dots, g_k \circ \mathbf{h}) = (g_1(h_1, \dots, h_m), \dots, g_k(h_1, \dots, h_m))$$

where

$$g_i(h_1, \dots, h_m)(\mathbf{a}) = g_i(h_1(\mathbf{a}), \dots, h_m(\mathbf{a})) \qquad \text{for all } \mathbf{a} \in A^n \text{ and for all } i.$$

A *clone* on $A$ is a subset $\mathcal{C}$ of $\mathcal{O}_A$ that contains the projections and is closed under composition; more precisely, this means that for all $m, n$ and $i$ $(1 \leq i \leq n)$, we have $p_i^{(n)} \in \mathcal{C}$ and whenever $g \in \mathcal{C}^{(m)}$ and $\mathbf{h} \in (\mathcal{C}^{(n)})^m$ then $g \circ \mathbf{h} \in \mathcal{C}^{(n)}$. The clones on $A$ form a complete lattice under inclusion. Therefore for each set $F \subseteq \mathcal{O}_A$ of operations there exists a smallest clone that contains $F$, which will be denoted by $\langle F \rangle$ and will be referred to as the *clone generated by* $F$.

Let $\mathcal{C}$ be a fixed clone on $A$. For arbitrary operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$ we say that

- $f$ is a $\mathcal{C}$-*minor* of $g$, in symbols $f \leq_{\mathcal{C}} g$, if $f = g \circ \mathbf{h}$ for some $\mathbf{h} \in (\mathcal{C}^{(n)})^m$;
- $f$ and $g$ are $\mathcal{C}$-*equivalent*, in symbols $f \equiv_{\mathcal{C}} g$, if $f \leq_{\mathcal{C}} g$ and $g \leq_{\mathcal{C}} f$.

Some of the basic properties of the relations $\leq_{\mathcal{C}}$ and $\equiv_{\mathcal{C}}$ are summarized below.

**Proposition 2.1.** *Let $\mathcal{C}$ and $\mathcal{C}'$ be clones on $A$.*

(i) *$\leq_{\mathcal{C}}$ is a quasiorder on $\mathcal{O}_A$.*
(ii) *$\equiv_{\mathcal{C}}$ is an equivalence relation on $\mathcal{O}_A$.*
(iii) *$\leq_{\mathcal{C}} \subseteq \leq_{\mathcal{C}'}$ if and only if $\mathcal{C} \subseteq \mathcal{C}'$.*
(iv) *$\equiv_{\mathcal{C}} \subseteq \equiv_{\mathcal{C}'}$ if $\mathcal{C} \subseteq \mathcal{C}'$.*

**Proof.** $f \leq_{\mathcal{C}} f$ for all $f \in \mathcal{O}_A^{(n)}$ and $n \geq 1$, since $f = f \circ \mathbf{p}^{(n)}$ with $\mathbf{p}^{(n)} \in (\mathcal{C}^{(n)})^n$, as $\mathcal{C}$ contains the projections. If $f \leq_{\mathcal{C}} f' \leq_{\mathcal{C}} f''$ where $f, f', f''$ have arities $k, m, n$, respectively, then by definition, $f = f' \circ \mathbf{h}$ and $f' = f'' \circ \mathbf{h}'$ for some $\mathbf{h} \in (\mathcal{C}^{(k)})^m$ and $\mathbf{h}' \in (\mathcal{C}^{(m)})^n$. Thus $f = (f'' \circ \mathbf{h}') \circ \mathbf{h} = f'' \circ (\mathbf{h}' \circ \mathbf{h})$ with $\mathbf{h}' \circ \mathbf{h} \in (\mathcal{C}^{(k)})^n$ as $\mathcal{C}$ is closed under composition. Hence $f \leq_{\mathcal{C}} f''$. This proves that $\leq_{\mathcal{C}}$ is reflexive and transitive, establishing (i). The claim in (ii) is an immediate consequence of (i).

It follows directly from the definitions that for arbitrary clones $\mathcal{C} \subseteq \mathcal{C}'$ on $A$ we have $\leq_{\mathcal{C}} \subseteq \leq_{\mathcal{C}'}$ and $\equiv_{\mathcal{C}} \subseteq \equiv_{\mathcal{C}'}$. This proves (iv) and the sufficiency in (iii). To prove the necessity in (iii) notice that $\{f \in \mathcal{O}_A : f \leq_{\mathcal{C}} p_1^{(1)}\} = \mathcal{C}$. This equality and the analogous equality for $\mathcal{C}'$ show that $\leq_{\mathcal{C}} \subseteq \leq_{\mathcal{C}'}$ implies $\mathcal{C} \subseteq \mathcal{C}'$. □

By definition, the equivalence relation $\equiv_{\mathcal{C}}$ is the intersection of $\leq_{\mathcal{C}}$ with its converse. Therefore the quasiorder $\leq_{\mathcal{C}}$ induces a partial order on the set $\mathcal{O}_A/{\equiv_{\mathcal{C}}}$ of $\mathcal{C}$-equivalence classes. This partial order will be denoted by $\preceq_{\mathcal{C}}$.

**Corollary 2.2.** *If $\mathcal{C}$ and $\mathcal{C}'$ are clones on $A$ such that $\mathcal{C} \subseteq \mathcal{C}'$, then*

$$\nu_{\mathcal{C}',\mathcal{C}} \colon \mathcal{O}_A/{\equiv_{\mathcal{C}}} \to \mathcal{O}_A/{\equiv_{\mathcal{C}'}}, \qquad f/{\equiv_{\mathcal{C}}} \mapsto f/{\equiv_{\mathcal{C}'}}$$

*is an order preserving mapping of the poset $(\mathcal{O}_A/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$ onto $(\mathcal{O}_A/{\equiv_{\mathcal{C}'}}; \preceq_{\mathcal{C}'})$.*

**Proof.** $\nu_{\mathcal{C}',\mathcal{C}}$ is well defined by Proposition 2.1(iv), and order preserving by Proposition 2.1(iii). The surjectivity of $\nu_{\mathcal{C}',\mathcal{C}}$ is clear from its definition. □

By definition, $\nu_{\mathcal{C}',\mathcal{C}}$ ($\mathcal{C} \subseteq \mathcal{C}'$) maps each $\mathcal{C}$-equivalence class to the $\mathcal{C}'$-equivalence class containing it. Therefore

$$\nu_{\mathcal{C}'',\mathcal{C}} = \nu_{\mathcal{C}'',\mathcal{C}'} \circ \nu_{\mathcal{C}',\mathcal{C}} \quad \text{if } \mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{C}''. \tag{2.1}$$

Now we will assume that $A$ is finite, and will discuss some basic facts on clones $\mathcal{C}$ for which $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$ (that is, the number of $\mathcal{C}$-equivalence classes of operations on $A$ is finite). We will need the following notation. If $\mathcal{C}$ is a clone on $A$ and $B$ is a nonempty subset of $A$ such that every operation in $\mathcal{C}$ preserves $B$, then by restricting all operations in $\mathcal{C}$ to $B$ we get a clone on $B$, which we will denote by $\mathcal{C}|_B$.

**Proposition 2.3.** *Let $\mathcal{C}$ be a clone on a finite set $A$.*

(i) *$\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$ if and only if there exists an integer $d > 0$ such that every operation on $A$ is $\mathcal{C}$-equivalent to a $d$-ary operation on $A$.*
(ii) *If $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$, then $\equiv_{\mathcal{C}'}$ has finite index in $\mathcal{O}_A$ for every clone $\mathcal{C}'$ that contains $\mathcal{C}$.*
(iii) *If $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$ and $B$ is a nonempty subset of $A$ such that every operation in $\mathcal{C}$ preserves $B$, then $\equiv_{\mathcal{C}|_B}$ has finite index in $\mathcal{O}_B$.*

**Proof.** (i) The number of $d$-ary operations on $A$ is finite, since $A$ is finite. Therefore if every operation on $A$ is $\mathcal{C}$-equivalent to a $d$-ary operation on $A$, then $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$. Conversely, assume that $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$, and select a transversal $T$ for the blocks of $\equiv_{\mathcal{C}}$. Since $T$ is finite, there is a $d > 0$ such that every operation in $T$ is at most $d$-ary. Now we will argue that for each operation $f \in T$, the $d$-ary operation $f^*$ obtained by adding fictitious variables to $f$ is $\mathcal{C}$-equivalent to $f$. If $f$ is $k$-ary ($k \leq d$), then $f^* = f \circ (p_1^{(d)}, \dots, p_k^{(d)})$, so $f^* \leq_{\mathcal{C}} f$. Since $(p_1^{(d)}, \dots, p_k^{(d)}) \circ (p_1^{(k)}, \dots, p_k^{(k)}, p_k^{(k)}, \dots, p_k^{(k)}) = \mathbf{p}^{(k)}$, we also get that $f^* \circ (p_1^{(k)}, \dots, p_k^{(k)}, p_k^{(k)}, \dots, p_k^{(k)}) = f \circ \mathbf{p}^{(k)} = f$, so $f \leq_{\mathcal{C}} f^*$. Thus every operation on $A$ is $\mathcal{C}$-equivalent to one of the $d$-ary operations $f^*$, $f \in T$.

(ii) follows immediately from Proposition 2.1(iv).

(iii) Suppose that $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$. By (i) there is an integer $d > 0$ such that every operation on $A$ is $\mathcal{C}$-equivalent to a $d$-ary operation on $A$. Now assuming that $B$ is a nonempty subset of $A$ such that every operation in $\mathcal{C}$ preserves $B$ we will show that every operation on $B$ is $\mathcal{C}|_B$-equivalent to a $d$-ary operation on $B$.

Let $g$ be an $n$-ary operation on $B$. Extend $g$ arbitrarily to an $n$-ary operation $f$ on $A$. Thus $f$ preserves $B$ and $f|_B = g$. By our assumption on $\mathcal{C}$, $f$ is $\mathcal{C}$-equivalent to a $d$-ary operation $f'$ on $A$. Hence there exist $\mathbf{h} \in (\mathcal{C}^{(d)})^n$ and $\mathbf{h}' \in (\mathcal{C}^{(n)})^d$ such that $f' = f \circ \mathbf{h}$ and $f = f' \circ \mathbf{h}'$. Since $f$ preserves $B$ (by construction) and the operations in $\mathcal{C}$ preserve $B$ (by assumption), $f' = f \circ \mathbf{h}$ also preserves $B$. Thus $f'|_B = f|_B \circ \mathbf{h}|_B$ and $f|_B = f'|_B \circ \mathbf{h}'|_B$ where all operations in $\mathbf{h}|_B$ and $\mathbf{h}'|_B$ belong to $\mathcal{C}|_B$. This proves that $g = f|_B$ is $\mathcal{C}|_B$-equivalent to the $d$-ary operation $f'|_B$. □

## 3. The relation $\leq_{\mathcal{C}}$ for discriminator clones $\mathcal{C}$

Let $A$ be an arbitrary set. The *discriminator function* on $A$ is the ternary operation $t$ defined as follows:

$$t(x, y, z) = \begin{cases} z, & \text{if } x = y, \\ x, & \text{otherwise} \end{cases} \qquad (x, y, z \in A).$$

A clone on $A$ will be called a *discriminator clone* if it contains $t$.

Let $\mathcal{C}$ be a clone on $A$. An $n$-ary operation $f$ on $A$ is said to be *locally in* $\mathcal{C}$ if for every finite subset $U$ of $A^n$ there exists an $n$-ary operation $g$ in $\mathcal{C}$ such that $f(u) = g(u)$ for all $u \in U$. The clone $\mathcal{C}$ is called *locally closed* if $f \in \mathcal{C}$ for every operation $f$ that is locally in $\mathcal{C}$. It is easy to see from this definition that if $A$ is finite, then every clone on $A$ is locally closed. Examples of locally closed clones on an infinite set $A$ include the clone of projections and the clone of all operations on $A$.

Throughout this section $\mathcal{C}$ will be a locally closed discriminator clone on a set $A$, and $\mathbf{A}$ will denote the algebra $(A; \mathcal{C})$. An isomorphism between subalgebras of $\mathbf{A}$ is called an *internal isomorphism* of $\mathbf{A}$. We will use the notation $\mathrm{Iso}(\mathbf{A})$ for the family of all internal isomorphisms of $\mathbf{A}$.

$\mathrm{Iso}(\mathbf{A})$ is a set of partial bijections that acts coordinatewise on $A^n$ for all $n \geq 1$ as follows: if $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$, $\iota \in \mathrm{Iso}(\mathbf{A})$, and each $a_i$ is in the domain of $\iota$, then $\iota(\mathbf{a}) = (\iota(a_1), \ldots, \iota(a_n))$; otherwise $\iota(\mathbf{a})$ is undefined. We will follow the convention that when we talk about elements $\iota(\mathbf{a})$ "for some [all] $\iota \in \mathrm{Iso}(\mathbf{A})$" we will always mean "for some [all] $\iota \in \mathrm{Iso}(\mathbf{A})$ for which $\iota(\mathbf{a})$ is defined".

Since $\mathrm{Iso}(\mathbf{A})$ is closed under composition and inverses, the relation $\sim_{\mathcal{C}}$ on $A^n$ defined for all $\mathbf{a}, \mathbf{b} \in A^n$ by

$$\mathbf{a} \sim_{\mathcal{C}} \mathbf{b} \Leftrightarrow \mathbf{b} = \iota(\mathbf{a}) \quad \text{for some } \iota \in \mathrm{Iso}(\mathbf{A})$$

is an equivalence relation whose blocks are the $\mathrm{Iso}(\mathbf{A})$-orbits

$$\mathbf{a}/{\sim_{\mathcal{C}}} = \{\iota(\mathbf{a}) : \iota \in \mathrm{Iso}(\mathbf{A})\}, \quad \mathbf{a} \in A^n.$$

We will choose and fix a transversal $T_n$ for the blocks of $\sim_{\mathcal{C}}$ in $A^n$.

For an $n$-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ let $\mathbf{S}^{\mathcal{C}}_{\mathbf{a}}$ denote the subalgebra of $\mathbf{A}$ generated by the set $\{a_1, \ldots, a_n\}$ of coordinates of $\mathbf{a}$. Now let $\mathbf{a} \in A^n$ and $\mathbf{b} \in A^m$ be such that $\mathbf{S}^{\mathcal{C}}_{\mathbf{b}} \leq \mathbf{S}^{\mathcal{C}}_{\mathbf{a}}$ (i.e., $\mathbf{S}^{\mathcal{C}}_{\mathbf{b}}$ is a subalgebra of $\mathbf{S}^{\mathcal{C}}_{\mathbf{a}}$); in other words, $\mathbf{b} \in (\mathbf{S}^{\mathcal{C}}_{\mathbf{a}})^m$. If $\iota_1, \iota_2 \in \mathrm{Iso}(\mathbf{A})$ are internal isomorphisms of $\mathbf{A}$ such that $\iota_1(\mathbf{a}) = \iota_2(\mathbf{a})$, then $\iota_1, \iota_2$ agree on a generating set of $\mathbf{S}^{\mathcal{C}}_{\mathbf{a}}$. Thus $\iota_1, \iota_2$ are defined and agree on $\mathbf{S}^{\mathcal{C}}_{\mathbf{a}}$, and hence on $\mathbf{S}^{\mathcal{C}}_{\mathbf{b}}$. This implies that $\iota_1(\mathbf{b}) = \iota_2(\mathbf{b})$. Thus

$$\Phi^{\mathcal{C}}_{\mathbf{b},\mathbf{a}}\colon \mathbf{a}/{\sim_{\mathcal{C}}} \to \mathbf{b}/{\sim_{\mathcal{C}}}, \qquad \iota(\mathbf{a}) \mapsto \iota(\mathbf{b}) \quad \text{for all } \iota \in \mathrm{Iso}(\mathbf{A})$$

is a well-defined mapping of the $\sim_{\mathcal{C}}$-block of $\mathbf{a}$ onto the $\sim_{\mathcal{C}}$-block of $\mathbf{b}$. Notice that $\Phi^{\mathcal{C}}_{\mathbf{b},\mathbf{a}}$ is the unique mapping $\mathbf{a}/{\sim_{\mathcal{C}}} \to \mathbf{b}/{\sim_{\mathcal{C}}}$ that sends $\mathbf{a}$ to $\mathbf{b}$ and preserves all internal isomorphisms of $\mathbf{A}$.

**Lemma 3.1.** *Let $\mathcal{C}$ be a locally closed discriminator clone on a set $A$. The following conditions on a function $\mathbf{h}\colon A^n \to A^m$ are equivalent:*

(a) $\mathbf{h}\colon A^n \to A^m$ *belongs to* $(\mathcal{C}^{(n)})^m$.

(b) $\mathbf{h}$ *preserves the internal isomorphisms of* $\mathbf{A}$*; that is,*

$$\mathbf{h}(\iota(\mathbf{a})) = \iota(\mathbf{h}(\mathbf{a})) \quad \textit{for all } \iota \in \mathrm{Iso}(\mathbf{A}).$$

(c) *For each $n$-tuple $\mathbf{c} \in T_n$ there exists an $m$-tuple $\mathbf{d}$ with $\mathbf{S}^{\mathcal{C}}_{\mathbf{d}} \le \mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$ such that the restriction of $\mathbf{h}$ to $\mathbf{c}/{\sim_{\mathcal{C}}}$ is the mapping $\Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}$.*

**Proof.** Since $\mathcal{C}$ is a locally closed clone, therefore $\mathcal{C}$ is the clone of local term operations of the algebra $\mathbf{A} = (A; \mathcal{C})$. The assumption that $t \in \mathcal{C}$, combined with a theorem of Baker and Pixley [1], implies the following well-known claim:

**Claim 3.2.** *An operation $g \in \mathcal{O}_A$ belongs to $\mathcal{C}$ if and only if $g$ preserves all internal isomorphisms of* $\mathbf{A}$.

This implies that an analogous statement holds for $m$-tuples of operations as well. Hence conditions (a) and (b) are equivalent. It remains to show that conditions (b) and (c) are equivalent.

First we will show that (b) $\Rightarrow$ (c). Let $\mathbf{h} \in (\mathcal{C}^{(n)})^m$, and let $\mathbf{c} \in T_n$. Since $\mathbf{h}$ preserves all internal isomorphisms of $\mathbf{A}$, it preserves, in particular, the identity automorphism of each subalgebra of $\mathbf{A}$. Hence $\mathbf{h}$ preserves all subalgebras of $\mathbf{A}$. This implies that the coordinates of the $m$-tuple $\mathbf{d} = \mathbf{h}(\mathbf{c})$ are in $\mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$. Hence $\mathbf{S}^{\mathcal{C}}_{\mathbf{d}} \le \mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$. Moreover,

$$\mathbf{h}(\iota(\mathbf{c})) = \iota(\mathbf{h}(\mathbf{c})) = \iota(\mathbf{d}) = \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}(\iota(\mathbf{c})) \quad \text{for all } \iota \in \mathrm{Iso}(\mathbf{A}).$$

This shows that $\mathbf{h}|_{\mathbf{c}/\sim_{\mathcal{C}}}$ coincides with $\Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}$, as claimed in (c).

To prove the implication (c) $\Rightarrow$ (b) assume that $\mathbf{h}$ satisfies condition (c), and let $\kappa$ be an internal isomorphism of $\mathbf{A}$. We have to show that $\mathbf{h}$ preserves $\kappa$. Let $\mathbf{a}$ be an arbitrary element of $A^n$ such that $\kappa(\mathbf{a})$ is defined, and let $\mathbf{c}$ be the representative of the orbit $\mathbf{a}/{\sim_{\mathcal{C}}}$ in $T_n$. There exists $\iota \in \mathrm{Iso}(\mathbf{A})$ such that $\mathbf{a} = \iota(\mathbf{c})$. Hence $\kappa(\mathbf{a}) = (\kappa \circ \iota)(\mathbf{c})$. Since $\mathbf{h}$ satisfies condition (c), there exists $\mathbf{d} \in A^m$ with $\mathbf{S}^{\mathcal{C}}_{\mathbf{d}} \le \mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$ such that the equality $\mathbf{h}(\lambda(\mathbf{c})) = \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}(\lambda(\mathbf{c}))$ holds for all $\lambda \in \mathrm{Iso}(\mathbf{A})$. Using this equality for $\lambda = \kappa \circ \iota$ and $\lambda = \iota$ (second and sixth equalities below), the definition of $\Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}$ (third and fifth equalities), and the relationship between $\mathbf{a}$ and $\mathbf{c}$ (first and seventh equalities), we get that

$$\begin{aligned}\mathbf{h}(\kappa(\mathbf{a})) &= \mathbf{h}((\kappa \circ \iota)(\mathbf{c})) = \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}((\kappa \circ \iota)(\mathbf{c})) = (\kappa \circ \iota)(\mathbf{d}) \\ &= \kappa(\iota(\mathbf{d})) = \kappa(\Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}(\iota(\mathbf{c}))) = \kappa(\mathbf{h}(\iota(\mathbf{c}))) = \kappa(\mathbf{h}(\mathbf{a})).\end{aligned}$$

This proves that $\mathbf{h}$ preserves $\kappa$, and hence completes the proof of the lemma. □

**Theorem 3.3.** *Let $\mathcal{C}$ be a locally closed discriminator clone on a set $A$. The following conditions on $f \in \mathcal{O}^{(n)}_A$ and $g \in \mathcal{O}^{(m)}_A$ are equivalent:*

(a) $f \le_{\mathcal{C}} g$.

(b) *For each ${\sim_{\mathcal{C}}}$-block $P = \mathbf{c}/{\sim_{\mathcal{C}}}$ $(\mathbf{c} \in T_n)$ in $A^n$ there exists a ${\sim_{\mathcal{C}}}$-block $Q = \mathbf{d}/{\sim_{\mathcal{C}}}$ in $A^m$ such that $\mathbf{S}^{\mathcal{C}}_{\mathbf{d}} \le \mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$ and $f|_P = g|_Q \circ \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}$.*

**Proof.** (a) $\Rightarrow$ (b). If $f \le_{\mathcal{C}} g$, then $f = g \circ \mathbf{h}$ for some $\mathbf{h} \in (\mathcal{C}^{(n)})^m$. Lemma 3.1 shows that for each $\mathbf{c} \in T_n$ there exists $\mathbf{d} \in A^m$ with $\mathbf{S}^{\mathcal{C}}_{\mathbf{d}} \le \mathbf{S}^{\mathcal{C}}_{\mathbf{c}}$ such that by restricting $\mathbf{h}$ to $P = \mathbf{c}/{\sim_{\mathcal{C}}}$ we get the function $\mathbf{h}|_P = \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}} \colon P \to Q = \mathbf{d}/{\sim_{\mathcal{C}}}$. Thus $f|_P = (g \circ \mathbf{h})|_P = g|_Q \circ \mathbf{h}|_P = g|_Q \circ \Phi^{\mathcal{C}}_{\mathbf{d},\mathbf{c}}$.

(b) $\Rightarrow$ (a). Assume that condition (b) holds for $f$ and $g$. For each $\mathbf{c} \in T_n$ fix a tuple $\mathbf{d} = \mathbf{d}_{\mathbf{c}}$ whose existence is postulated in condition (b). Since every ${\sim_{\mathcal{C}}}$-block $P$ in $A^n$ is of the form $P = \mathbf{c}/{\sim_{\mathcal{C}}}$ for a unique $\mathbf{c} \in T_n$, there is a (well-defined) function $\mathbf{h} \colon A^n \to A^m$ such that $\mathbf{h}|_P = \Phi^{\mathcal{C}}_{\mathbf{d}_{\mathbf{c}},\mathbf{c}}$ for all ${\sim_{\mathcal{C}}}$-blocks $P$ in $A^n$. Lemma 3.1 implies that $\mathbf{h} \in (\mathcal{C}^{(n)})^m$. Moreover, we have $f = g \circ \mathbf{h}$, because condition (b) and the construction of $\mathbf{h}$ yield that $f|_P = g|_Q \circ \Phi^{\mathcal{C}}_{\mathbf{d}_{\mathbf{c}},\mathbf{c}} = g|_Q \circ \mathbf{h}|_P = (g \circ \mathbf{h})|_P$ for all ${\sim_{\mathcal{C}}}$-blocks $P$ in $A^n$. Thus $f \le_{\mathcal{C}} g$. □

We conclude this section by applying Theorem 3.3 to the clone $\mathcal{C} = \mathcal{O}_A$, which is clearly a locally closed discriminator clone for every set $A$. If $\mathcal{C} = \mathcal{O}_A$, then the algebra $\mathbf{A} = (A; \mathcal{O}_A)$ has no proper subalgebras and no nonidentity automophisms. Therefore $\mathbf{a}/{\sim_{\mathcal{O}_A}} = \{\mathbf{a}\}$ and $\mathbf{S}^{\mathcal{O}_A}_{\mathbf{a}} = A$ for all $\mathbf{a} \in A^n$, $n \ge 1$. Moreover, $\Phi^{\mathcal{O}_A}_{\mathbf{b},\mathbf{a}}$ is the unique mapping $\{\mathbf{a}\} \to \{\mathbf{b}\}$. Thus condition (b) in Theorem 3.3 for $\mathcal{C} = \mathcal{O}_A$ requires the following: for every block $P = \{\mathbf{c}\}$ in $A^n$, if $f|_P \colon \{\mathbf{c}\} \to \{r\}$, then there exists a block $Q = \{\mathbf{d}\}$ in $A^m$ such that $g|_Q \colon \{\mathbf{d}\} \to \{r\}$; that is, every element $r$ that is in the range $\mathrm{Im}(f)$ of $f$ is also in the range $\mathrm{Im}(g)$ of $g$. Hence Theorem 3.3 yields the following result from [6] (see also [8]).

**Corollary 3.4.** *Let $A$ be a set. For arbitrary operations $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$,*

$$f \leq_{\mathcal{O}_A} g \quad \text{if and only if} \quad \mathrm{Im}(f) \subseteq \mathrm{Im}(g).$$

Further applications of Theorem 3.3 will appear in Sections 4 and 5.

## 4. Finiteness and minimality

Let $A$ be a finite set, and let $\mathfrak{F}_A$ denote the family of all clones $\mathcal{C}$ on $A$ such that there are only finitely many $\mathcal{C}$-equivalence classes of operations on $A$ (that is, $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$). Proposition 2.3(ii) shows that $\mathfrak{F}_A$ is an order filter (i.e., an upset) in the lattice of all clones on $A$. By Corollary 3.4, the clone $\mathcal{O}_A$ belongs to $\mathfrak{F}_A$.

Our goal in this section is to prove that all discriminator clones belong to $\mathfrak{F}_A$. Since $\mathfrak{F}_A$ is an order filter, it will be sufficient to show that the smallest discriminator clone $\mathcal{D} = \langle t \rangle$ belongs to $\mathfrak{F}_A$. We will also prove that $\mathcal{D}$ is a minimal member of $\mathfrak{F}_A$, that is, no proper subclone of $\mathcal{D}$ belongs to $\mathfrak{F}_A$.

Our main result is

**Theorem 4.1.** *Let $A$ be a finite set of cardinality $|A| = k \geq 2$, and let $\mathcal{D}$ be the clone generated by the discriminator function on $A$. For $d = k^k - k^{k-1} + 1$, every operation on $A$ is $\mathcal{D}$-equivalent to a $d$-ary operation on $A$.*

This theorem, combined with Proposition 2.3(i) and (ii), immediately implies the corollary below which states that all discriminator clones belong to $\mathfrak{F}_A$.

**Corollary 4.2.** *For each discriminator clone $\mathcal{C}$ on a finite set $A$ the equivalence relation $\equiv_{\mathcal{C}}$ has finite index in $\mathcal{O}_A$.*

For the proof of Theorem 4.1 we will use the description of $\leq_{\mathcal{D}}$ in Section 3. Recall that since $A$ is finite, all clones on $A$ are locally closed. We will denote the symmetric group on $n$ letters by $S_n$.

**Proof of Theorem 4.1.** Let $\mathbf{A} = (A; \mathcal{D})$. We may assume without loss of generality that $A = \{1, 2, \ldots, k\}$. The discriminator function preserves all bijections between any two subsets of $A$ of the same size. Therefore

(1) All subsets of $A$ are subalgebras of $\mathbf{A}$, and
(2) $\mathrm{Iso}(\mathbf{A})$ is the set of all bijections $B \to C$ such that $B, C \subseteq A$ and $|B| = |C|$.

Hence for each $\mathbf{a} = (a_1, \ldots, a_n) \in A^n$

(3) $\mathbf{S}_{\mathbf{a}}^{\mathcal{D}}$ is the set of coordinates of $\mathbf{a}$, and
(4) the $\mathrm{Iso}(\mathbf{A})$-orbit ($\sim_{\mathcal{D}}$-block) of $\mathbf{a}$ is

$$\mathbf{a}/{\sim_{\mathcal{D}}} = \{(b_1, \ldots, b_n) \in A^n : b_i = b_j \Leftrightarrow a_i = a_j \text{ holds for all } i, j\}.$$

The number of distinct coordinates of $\mathbf{a}$ will be called the *breadth of* $\mathbf{a}$. It follows from (4) that all tuples in a $\sim_{\mathcal{D}}$-block $P = \mathbf{a}/{\sim_{\mathcal{D}}}$ have the same breadth; this number will be called the *breadth of* $P$, and will be denoted by $\nu(P)$. Another consequence of (4) is that

(5) every $\sim_{\mathcal{D}}$-block $P$ of breadth $r$ in $A^n$ can be represented by a tuple $\mathbf{c} = (c_1, \ldots, c_n)$ such that $\{c_1, \ldots, c_n\} = \{1, \ldots, r\}$;
(6) moreover, this representative $\mathbf{c}$ is unique if we require in addition that the first occurrences of $1, \ldots, r$ among $c_1, \ldots, c_n$ appear in increasing order; that is, if the first occurrence of $i$ ($1 \leq i \leq r$) in $(c_1, \ldots, c_n)$ is $c_{j_i}$ for each $i$, then $j_1 < j_2 < \cdots < j_r$.

Thus the $n$-tuples $\mathbf{c}$ that satisfy the conditions described in (5)–(6) form a transversal for the $\sim_{\mathcal{D}}$-blocks of $A^n$. We will select this transversal to be $T_n$.

Let $\mathbf{c} \in T_n$. With the notation used in (5)–(6) we get from (4) that the projection mapping

$$\pi_P \colon P \to P_r, \quad (a_1, \ldots, a_n) \mapsto (a_{j_1}, \ldots, a_{j_r})$$

whose range $P_r$ is the unique $\sim_{\mathcal{D}}$-block of breadth $r$ in $A^r$ is bijective, and maps $\mathbf{c}$ to the $r$-tuple $\vec{r} = (1, \ldots, r) \in T_r$. For a permutation $\sigma \in S_r$ the bijection $P_r \to P_r$, $(x_1, \ldots, x_r) \mapsto (x_{\sigma(1)}, \ldots, x_{\sigma(r)})$ that permutes the coordinates of $P_r$ by $\sigma$ will be denoted by $\sigma^*$.

**Claim 4.3.** *Let $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$. If for every $r\,(1 \le r \le k)$ and for every $\sim_{\mathcal{D}}$-block $P$ of breadth $r$ in $A^n$ there exists a $\sim_{\mathcal{D}}$-block $Q$ of breadth $r$ in $A^m$ such that*

$$f|_P \circ \pi_P^{-1} = (g|_Q \circ \pi_Q^{-1}) \circ \sigma^* \quad \text{for some } \sigma \in S_r, \tag{4.1}$$

*then $f \le_{\mathcal{D}} g$.*

Suppose that the hypotheses of the claim are satisfied. To prove that $f \le_{\mathcal{D}} g$ it suffices to verify that condition (b) in Theorem 3.3 with $\mathcal{C} = \mathcal{D}$ holds. Let $P = \mathbf{c}/{\sim_{\mathcal{D}}}$ ($\mathbf{c} \in T_n$) be an arbitrary $\sim_{\mathcal{D}}$-block of breadth $r$ in $A^n$, and let $Q = \mathbf{c}'/{\sim_{\mathcal{D}}}$ ($\mathbf{c}' \in T_m$) be a $\sim_{\mathcal{D}}$-block in $A^m$ for which (4.1) holds. Furthermore, let $\vec{r} = (1, \dots, r)$, and let $\mathbf{d} = \sigma(\mathbf{c}')$ be the image of $\mathbf{c}'$ under the internal isomorphism $\sigma$ of $\mathbf{A}$.

Notice that each one of the mappings $\pi_P, \pi_Q$, and $\sigma^*$ are bijections between $\sim_{\mathcal{D}}$-blocks, and preserve the internal isomorphisms of $\mathbf{A}$. Therefore the mapping $\pi_Q^{-1} \circ \sigma^* \circ \pi_P \colon P \to Q$ also preserves the internal isomorphisms of $\mathbf{A}$. The image of $\mathbf{c}$ under this mapping is $\mathbf{d}$, as the following calculation shows:

$$\pi_Q^{-1}(\sigma^*(\pi_P(\mathbf{c}))) = \pi_Q^{-1}(\sigma^*(\vec{r})) = \pi_Q^{-1}((\sigma(1), \dots, \sigma(r))) = \pi_Q^{-1}(\sigma(\vec{r})) = \sigma(\mathbf{c}') = \mathbf{d},$$

where the second to last equality holds, because $\pi_Q^{-1}(\vec{r}) = \mathbf{c}'$ and $\sigma$ is an internal isomorphism of $\mathbf{A}$. Since $\Phi_{\mathbf{d},\mathbf{c}}^{\mathcal{D}}$ is the unique mapping $P \to Q$ that preserves the internal isomorphisms of $\mathbf{A}$ and sends $\mathbf{c}$ to $\mathbf{d}$, we get that $\pi_Q^{-1} \circ \sigma^* \circ \pi_P = \Phi_{\mathbf{d},\mathbf{c}}^{\mathcal{D}}$. Thus the equality in (4.1) is equivalent to $f|_P = g|_Q \circ \pi_Q^{-1} \circ \sigma^* \circ \pi_P = g|_Q \circ \Phi_{\mathbf{d},\mathbf{c}}^{\mathcal{D}}$. The $m$-tuple $\mathbf{d} = \sigma(\mathbf{c}')$ clearly satisfies $Q = \mathbf{d}/{\sim_{\mathcal{D}}}$ and $\mathbf{S}_{\mathbf{d}}^{\mathcal{D}} = \{1, \dots, r\} = \mathbf{S}_{\mathbf{c}}^{\mathcal{D}}$ (see statement (3) above). This shows that condition (b) in Theorem 3.3 with $\mathcal{C} = \mathcal{D}$ holds, and hence completes the proof of Claim 4.3.

In Claim 4.3 $f|_P \circ \pi_P^{-1}$ and $g|_Q \circ \pi_Q^{-1}$ are both functions $P_r \to A$, and condition (4.1) says that, up to a permutation of the coordinates of $P_r$, they are the same function. For arbitrary functions $\phi, \psi \colon P_r \to A$ let

$$\phi \approx \psi \Leftrightarrow \phi = \psi \circ \sigma^* \text{ for some } \sigma \in S_r.$$

In other words, $\phi \approx \psi$ iff $\phi$ and $\psi$ are in the same orbit under the action of the symmetric group $S_r$ on the set $A^{P_r}$ of all functions $P_r \to A$ by permuting the coordinates of $P_r$. Hence $\approx$ is an equivalence relation on $A^{P_r}$. With this notation condition (4.1) above can be restated to say that $f|_P \circ \pi_P^{-1}$ and $g|_Q \circ \pi_Q^{-1}$ are in the same $\approx$-block of $A^{P_r}$.

For arbitrary $n$-ary operation $f$ on $A$ ($n \ge 1$) and integer $r$ ($1 \le r \le k$) let $\mathfrak{E}_r(f)$ denote the set of $\approx$-blocks of all functions $f|_P \circ \pi_P^{-1}$ as $P$ runs over all $\sim_{\mathcal{D}}$-blocks of breadth $r$ in $A^n$.

**Claim 4.4.** *Let $f \in \mathcal{O}_A^{(n)}$ and $g \in \mathcal{O}_A^{(m)}$.*

(i) *If $\mathfrak{E}_r(f) \subseteq \mathfrak{E}_r(g)$ for all $r\,(1 \le r \le k)$, then $f \le_{\mathcal{D}} g$.*
(ii) *If $\mathfrak{E}_r(f) = \mathfrak{E}_r(g)$ for all $r\,(1 \le r \le k)$, then $f \equiv_{\mathcal{D}} g$.*

Part (i) is a restatement of Claim 4.3 using the notation introduced after Claim 4.3. Part (ii) is an immediate consequence of (i).

Now let $N(k, r)$ denote the index of $\approx$ (i.e., the number of $\approx$-blocks) in $A^{P_r}$, where $k = |A|$. We will also use the Stirling numbers $S(d, r)$ of the second kind. Since the $\sim_{\mathcal{D}}$-blocks of breadth $r$ in $A^d$ are in one-to-one correspondence with the partitions of $\{1, \dots, d\}$ into $r$ blocks, $S(d, r)$ is the number of $\sim_{\mathcal{D}}$-blocks of breadth $r$ in $A^d$.

**Claim 4.5.** *If $d$ is a positive integer such that*

$$N(k, r) \le S(d, r) \quad \text{for all } r \text{ with } 2 \le r \le k, \tag{4.2}$$

*then every operation $f \in \mathcal{O}_A$ is $\mathcal{D}$-equivalent to a $d$-ary operation.*

Assume that (4.2) holds for $d$, and let $f$ be an arbitrary operation on $A$, say $f$ is $n$-ary. In view of Claim 4.4 (ii) it suffices to show that there exists a $d$-ary operation $g$ on $A$ such that $\mathfrak{E}_r(g) = \mathfrak{E}_r(f)$ for all $r$ ($1 \le r \le k$). Since the $\sim_{\mathcal{D}}$-blocks partition $A^d$, we may define $g$ on each $\sim_{\mathcal{D}}$-block separately.

For the unique $\sim_{\mathcal{D}}$-block $Q = (1, \dots, 1)/{\sim_{\mathcal{D}}}$ of breadth $r = 1$ in $A^d$ we define $g|_Q$ to be $f|_P \circ \pi_P^{-1} \circ \pi_Q$ where $P = (1, \dots, 1)/{\sim_{\mathcal{D}}}$ is the unique $\sim_{\mathcal{D}}$-block of breadth 1 in $A^n$. This will ensure that $\mathfrak{E}_r(f) = \mathfrak{E}_r(g)$ holds for $r = 1$.

If $2 \le r \le k$, then $|\mathfrak{E}_r(f)| \le N(k, r) \le S(d, r)$, where the first inequality follows from the definition of $\mathfrak{E}_r(f)$, while the second equality is our assumption. Let $\phi_1, \dots, \phi_s$ be a transversal for the $\approx$-blocks in $\mathfrak{E}_r(f)$. The inequality

$s = |\mathfrak{E}_r(f)| \leq S(d, r)$ ensures that we can select $s$ distinct $\sim_{\mathcal{D}}$-blocks $Q_j$ $(j = 1, \ldots, s)$ of breadth $r$ in $A^d$. Now for each $\sim_{\mathcal{D}}$-block $Q$ of breadth $r$ in $A^d$ we define $g|_Q = \phi_j \circ \pi_Q$ if $Q = Q_j$ $(j = 1, \ldots, s)$, and $g|_Q = \phi_1 \circ \pi_Q$ otherwise. Clearly, this will imply that $\mathfrak{E}_r(f) = \mathfrak{E}_r(g)$ holds for $r \geq 2$. This completes the proof of the claim.

To finish the proof of Theorem 4.1 it remains to show that (4.2) holds for $d = k^k - k^{k-1} + 1$.

**Claim 4.6.** *Condition (4.2) holds for $d = k^k - k^{k-1} + 1$.*

If $k = 2$, then $d = 3$, and the only value of $r$ to be considered is $r = 2$. It is straightforward to check that in this case $N(k, r) = N(2, 2) = 3$ and $S(d, r) = S(3, 2) = 3$. Therefore (4.2) holds for $k = 2$.

From now on we will assume that $k \geq 3$. Let $2 \leq r \leq k$. We have $d > r$, because $d = k^{k-1}(k - 1) + 1 > (k - 1) + 1 = k$. The number of equivalence relations on $\{1, 2, \ldots, d\}$ with exactly $r$ blocks is at least $r^{d-r}$, since the identity function $\{1, 2, \ldots, r\} \to \{1, 2, \ldots, r\}$ can be extended in $r^{d-r}$ different ways to a function $\{1, 2, \ldots, d\} \to \{1, 2, \ldots, r\}$ and these extensions have distinct kernels, which are equivalence relations on $\{1, 2, \ldots, d\}$ with exactly $r$ blocks. Thus,

$$r^{d-r} \leq S(d, r).$$

The number of functions $P_r \to A$ is $k^{k(k-1)\cdots(k-r+1)}$, therefore

$$N(k, r) \leq k^{k(k-1)\cdots(k-r+1)} \leq k^{k!}.$$

Since $k \geq 3$, we have $k! < k^{k-1}$ and $k \leq 2^{k-1}$. Thus we get the first inequality in

$$k^{k!} \leq (2^{k-1})^{k^{k-1}-1} = 2^{k^k - k^{k-1} + 1 - k} = 2^{d-k} \leq r^{d-r}.$$

The last inequality, $2^{d-k} \leq r^{d-r}$, follows from $2 \leq r \leq k$. Combining the displayed inequalities we get that $N(k, r) \leq k^{k!} \leq r^{d-r} \leq S(d, r)$.

This completes the proof of Theorem 4.1. □

Theorem 4.1 shows that $\mathcal{D}$ belongs to the order filter $\mathfrak{F}_A$ of all clones $\mathcal{C}$ on $A$ for which there are only finitely many $\mathcal{C}$-equivalence classes of operations on $A$. The next theorem will prove that if $|A| = 2$, then all members of $\mathfrak{F}_A$ are discriminator clones. Hence in this case $\mathfrak{F}_A$ is a principal order filter in the lattice of clones on $A$ with least element $\mathcal{D}$.

**Theorem 4.7.** *For a two-element set $A$, if $\mathcal{C}$ is not a discriminator clone on $A$, then $\equiv_{\mathcal{C}}$ has infinite index in $\mathcal{O}_A$.*

**Proof.** We may assume without loss of generality that $A = \{0, 1\}$. The lattice of all clones on $\{0, 1\}$ was described by Post [11]. By inspecting Post's lattice one can see that if $\mathcal{C}$ is not a discriminator clone, then $\mathcal{C}$ is a subclone of one of the following clones:

- the clone $\mathcal{L}$ of linear operations,
- the clone $\mathcal{M}$ of all operations that are monotone with respect to the partial order $0 \leq 1$,
- the clone $\mathcal{R}_0$ of all operations that preserve the binary relation $\rho_0 = \{(0, 0), (0, 1), (1, 0)\}$,
- the clone $\mathcal{R}_1$ of all operations that preserve the binary relation $\rho_1 = \{(1, 1), (1, 0), (0, 1)\}$.

In view of Proposition 2.3(ii), to show that $\equiv_{\mathcal{C}}$ has infinite index in $\mathcal{O}_A$ it suffices to verify that each one of the four equivalence relations $\equiv_{\mathcal{L}}$, $\equiv_{\mathcal{M}}$, and $\equiv_{\mathcal{R}_i}$ $(i = 0, 1)$ has infinite index in $\mathcal{O}_A$. This will be done in the Claims 4.8–4.10 below.

**Claim 4.8.** *The equivalence relation $\equiv_{\mathcal{L}}$ has infinite index in $\mathcal{O}_A$.*

It follows from a result in [7, Proposition 5.9] that if $\mathcal{L}$ is the clone of all linear operations on $A = \{0, 1\}$, then there exists an infinite sequence of operations $g_n \in \mathcal{O}_A$ $(n = 1, 2, \ldots)$ such that $g_m \not\preceq_{\mathcal{L}} g_n$ whenever $m \neq n$. This implies that the equivalence relation $\equiv_{\mathcal{L}}$ has infinite index in $\mathcal{O}_A$.

**Claim 4.9.** *The equivalence relation $\equiv_{\mathcal{M}}$ has infinite index in $\mathcal{O}_A$.*

For $n \geq 1$ let $f_n$ be the $n$-ary linear operation $f_n(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ on $A = \{0, 1\}$. Our claim will follow if we show that the operations $f_n$ are in pairwise distinct $\equiv_{\mathcal{M}}$-blocks. To this end it will be sufficient to verify that $f_m \leq_{\mathcal{M}} f_n$ if and only if $m \leq n$. If $m \leq n$, then $f_m = f_n(p_1^{(m)}, p_2^{(m)}, \dots, p_m^{(m)}, 0, \dots, 0)$, where the projections $p_i^{(m)}$ and the constant function 0 are members of $\mathcal{M}$. Hence $f_m \leq_{\mathcal{M}} f_n$.

Conversely, assume that $f_m \leq_{\mathcal{M}} f_n$. By definition, this means that there exists $\mathbf{h} \in (\mathcal{M}^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. Consider the chain $\mathbf{e}_0 < \mathbf{e}_1 < \cdots < \mathbf{e}_m$ in $(A; \leq)^m$ where $\mathbf{e}_i = (1, \dots, 1, 0, \dots, 0)$ $(0 \leq i \leq m)$ is the $m$-tuple whose first $i$ coordinates are 1 and last $m - i$ coordinates are 0. Since $\mathbf{h} \in (\mathcal{M}^{(m)})^n$, therefore $\mathbf{h}$ is an order preserving mapping of $(A; \leq)^m$ to $(A; \leq)^n$. Thus $\mathbf{h}(\mathbf{e}_0) \leq \mathbf{h}(\mathbf{e}_1) \leq \cdots \leq \mathbf{h}(\mathbf{e}_m)$ holds in $(A; \leq)^n$. Moreover, these elements are pairwise distinct, because the calculation below shows that $f_n(\mathbf{h}(\mathbf{e}_i)) \neq f_n(\mathbf{h}(\mathbf{e}_{i+1}))$ for each $i$ $(0 \leq i < m)$; indeed,

$$f_n(\mathbf{h}(\mathbf{e}_i)) = (f_n \circ \mathbf{h})(\mathbf{e}_i) = f_m(\mathbf{e}_i) \neq f_m(\mathbf{e}_{i+1}) = (f_n \circ \mathbf{h})(\mathbf{e}_{i+1}) = f_n(\mathbf{h}(\mathbf{e}_{i+1})).$$

This proves that $\mathbf{h}(\mathbf{e}_0) < \mathbf{h}(\mathbf{e}_1) < \cdots < \mathbf{h}(\mathbf{e}_m)$ is a chain of length $m$ in $(A; \leq)^n$. In the partially ordered set $(A; \leq)^n$ the longest chain has length $n$, therefore $m \leq n$.

**Claim 4.10.** *The equivalence relation* $\equiv_{\mathcal{R}_\ell}$ $(\ell = 0, 1)$ *has infinite index in* $\mathcal{O}_A$.

The clone $\mathcal{R}_1$ can be obtained from $\mathcal{R}_0$ by switching the role of the two elements of $A = \{0, 1\}$, therefore it suffices to prove the claim for $\ell = 0$. As in the preceding claim, we let $f_n$ $(n \geq 1)$ be the $n$-ary linear operation $f_n(x_1, x_2, \dots, x_n) = x_1 + x_2 + \cdots + x_n$ on $A$, and want to prove that $f_m \leq_{\mathcal{R}_0} f_n$ if and only if $m \leq n$. If $m \leq n$, then $f_m \leq_{\mathcal{R}_0} f_n$ follows the same way as before, since the projections and the constant function 0 are members of $\mathcal{R}_0$.

Now assume that $f_m \leq_{\mathcal{R}_0} f_n$. By definition, there exists $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{R}_0^{(m)})^n$ such that $f_m = f_n \circ \mathbf{h}$. Consider the $m$-tuples $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in A^m$ where the single 1 occurs in the $i$-th coordinate $(1 \leq i \leq m)$. Notice that $\mathbf{e}_i$ and $\mathbf{e}_j$ are $\rho_0$-related coordinatewise for all distinct $i$ and $j$. Since $\mathbf{h} = (h_1, \dots, h_n) \in (\mathcal{R}_0^{(m)})^n$, the operations $h_1, \dots, h_n$ preserve $\rho_0$. Hence the $n$-tuples $\mathbf{h}(\mathbf{e}_i)$ and $\mathbf{h}(\mathbf{e}_j)$ are also $\rho_0$-related coordinatewise for all distinct $i$ and $j$.

We will use the notation $\mathbf{0}$ for tuples (of arbitrary length) whose coordinates are all 0. Since every operation $h_k$ $(1 \leq k \leq n)$ preserves $\rho_0$, and $(0, 0) \in \rho_0$ but $(1, 1) \notin \rho_0$, we get that $h_k(\mathbf{0}) = 0$. Thus $\mathbf{h}(\mathbf{0}) = \mathbf{0}$. The following calculation shows that $f_n(\mathbf{h}(\mathbf{e}_i)) \neq f_n(\mathbf{0})$ for each $i$:

$$f_n(\mathbf{h}(\mathbf{e}_i)) = (f_n \circ \mathbf{h})(\mathbf{e}_i) = f_m(\mathbf{e}_i) \neq f_m(\mathbf{0}) = (f_n \circ \mathbf{h})(\mathbf{0}) = f_n(\mathbf{h}(\mathbf{0})) = f_n(\mathbf{0}).$$

Thus $\mathbf{h}(\mathbf{e}_i) \neq \mathbf{0}$ for each $i$. Let $M$ denote the 0–1 matrix whose rows are the $n$-tuples $\mathbf{h}(\mathbf{e}_i)$ $(1 \leq i \leq m)$. The fact that $\mathbf{h}(\mathbf{e}_i)$ and $\mathbf{h}(\mathbf{e}_j)$ are $\rho_0$-related coordinatewise for all distinct $i$ and $j$ implies that each column of $M$ has at most one occurrence of 1. The fact that each $\mathbf{h}(\mathbf{e}_i)$ is different from $\mathbf{0}$ implies that every row of $M$ has at least one occurrence of 1. Since $M$ is $m \times n$, we get that $m \leq n$.

This completes the proof of Theorem 4.7. □

As we mentioned in the introduction, the statement in Theorem 4.7 fails for clones on a finite set $A$ with more than two elements. For these sets Słupecki's clone is an example of a clone that belongs to the order filter $\mathfrak{F}_A$ (see [8]), but is not a discriminator clone. Therefore in this case the clone $\mathcal{D}$ generated by the discriminator function is not the least element of $\mathfrak{F}_A$. However, we can use Theorem 4.7 to show that $\mathcal{D}$ is a minimal member of $\mathfrak{F}_A$. This will also imply that for finite sets with more than two elements the order filter $\mathfrak{F}_A$ is not principal.

**Theorem 4.11.** *Let $A$ be a finite set of cardinality $|A| > 2$, and let $\mathcal{D}$ be the clone generated by the discriminator function on $A$. If $\mathcal{H}$ is a proper subclone of $\mathcal{D}$, then* $\equiv_{\mathcal{H}}$ *has infinite index in* $\mathcal{O}_A$.

**Proof.** It follows from Proposition 2.3(ii) that the clones $\mathcal{C}$ for which $\equiv_{\mathcal{C}}$ has infinite index in $\mathcal{O}_A$ form an order ideal in the lattice of all clones on $A$. Therefore it suffices to prove the statement when $\mathcal{H}$ is a maximal (proper) subclone of $\mathcal{D}$.

We may assume without loss of generality that $B = \{0, 1\}$ is a subset of $A$. The operations in $\mathcal{D}$ preserve all subsets of $A$, including $B$. Therefore every operation $f \in \mathcal{D}$ can be restricted to $B$ to yield an operation $f|_B$ on $B$. By a result of Marchenkov [9] (see also [12]) $\mathcal{D}$ has two maximal subclones:

- the clone $\mathcal{E}$ consisting of all $f \in \mathcal{D}$ such that $f|_B$ is a linear operation on $B$, and
- the clone $\mathcal{K}$ consisting of all $f \in \mathcal{D}$ such that $f|_B$ is monotone with respect to the order $0 \leq 1$ on $B$.

Thus $\mathcal{E}|_B$ is a subclone of the clone $\mathcal{L}$ of linear operations on $B$, while $\mathcal{K}|_B$ is a subclone of the clone $\mathcal{M}$ of monotone operations on $B$. Hence, by Theorem 4.7, each one of the equivalence relations $\equiv_{\mathcal{E}|_B}$ and $\equiv_{\mathcal{K}|_B}$ has infinite index in $\mathcal{O}_B$. Therefore by Proposition 2.3(iii) each one of $\equiv_{\mathcal{K}}$ and $\equiv_{\mathcal{E}}$ has infinite index in $\mathcal{O}_A$. □

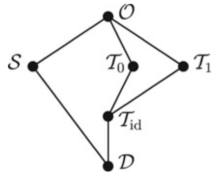


Fig. 1. Discriminator clones of Boolean functions.

## 5. $\mathcal{C}$-equivalence for discriminator clones $\mathcal{C}$ of Boolean functions

Boolean functions are operations on the set $A = \{0, 1\}$. In this section we will explicitly describe the $\mathcal{C}$-equivalence relation for Boolean functions provided $\mathcal{C}$ is a discriminator clone. We will also determine, for each such clone $\mathcal{C}$, the partial order $\preceq_{\mathcal{C}}$ induced on the set of $\mathcal{C}$-equivalence classes by the quasiorder $\leq_{\mathcal{C}}$.

To describe Boolean functions we will use the Boolean algebra operations $\vee$, $\cdot$, and $\bar{\ }$, as well as the Boolean ring operations $+$ and $\cdot$ on $A = \{0, 1\}$. The unary constant operations will be denoted by 0 and 1. If $\mathbf{a} = (a_1, \ldots, a_n)$ is an $n$-tuple in $A^n$, $\overline{\mathbf{a}}$ will denote the $n$-tuple $(\bar{a}_1, \ldots, \bar{a}_n)$. The tuples $(0, \ldots, 0)$ and $(1, \ldots, 1)$ will be denoted by $\mathbf{0}$ and $\mathbf{1}$, respectively.

It is easy to see from Post's lattice (see [11]) or from the characterization of (locally closed) discriminator clones cited in Claim 3.2 that there are six discriminator clones of Boolean functions:

- the clone $\mathcal{O} = \mathcal{O}_A$ of all Boolean functions;
- for each $i = 0, 1$, the clone $\mathcal{T}_i$ of all Boolean functions that fix $i$, that is,

$$\mathcal{T}_0 = \{f \in \mathcal{O} : f(\mathbf{0}) = 0\} \quad \text{and} \quad \mathcal{T}_1 = \{f \in \mathcal{O} : f(\mathbf{1}) = 1\};$$

- the clone $\mathcal{T}_{\mathrm{id}} = \mathcal{T}_0 \cap \mathcal{T}_1$ of all idempotent Boolean functions;
- the clone $\mathcal{S}$ of all self-dual Boolean functions, that is,

$$\mathcal{S} = \{f \in \mathcal{O} : f(\overline{\mathbf{x}}) = \overline{f(\mathbf{x})} \text{ for all } \mathbf{x}\};$$

- the clone $\mathcal{D} = \mathcal{T}_{\mathrm{id}} \cap \mathcal{S}$ of all idempotent self-dual Boolean functions,

and they are ordered by inclusion as shown in Fig. 1.

Our main tool in understanding $\mathcal{C}$-equivalence for these clones $\mathcal{C}$ will be Theorem 3.3. To be able to apply the theorem we will need to know the $\sim_{\mathcal{C}}$-blocks in $A^n$ for each $n \geq 1$, and the subalgebras $\mathbf{S}_{\mathbf{a}}^{\mathcal{C}}$ of $(A; \mathcal{C})$ for all $\mathbf{a} \in A^n$. The descriptions of the six discriminator clones above yield that for each $\mathbf{a} \in A^n$ ($n \geq 1$),

$$\mathbf{a}/{\sim_{\mathcal{C}}} = \begin{cases} \{\mathbf{a}, \overline{\mathbf{a}}\} & \text{if } \mathcal{C} \subseteq \mathcal{S}, \\ \{\mathbf{a}\} & \text{otherwise;} \end{cases} \tag{5.1}$$

and

$$\mathbf{S}_{\mathbf{a}}^{\mathcal{C}} = \begin{cases} \{0\} & \text{if } \mathbf{a} = \mathbf{0} \text{ and } \mathcal{C} \subseteq \mathcal{T}_0, \\ \{1\} & \text{if } \mathbf{a} = \mathbf{1} \text{ and } \mathcal{C} \subseteq \mathcal{T}_1, \\ \{0, 1\} & \text{otherwise.} \end{cases} \tag{5.2}$$

(5.1) implies that each $\sim_{\mathcal{C}}$-block has the same size, which we will denote by $d_{\mathcal{C}}$; namely,

$$d_{\mathcal{C}} = \begin{cases} 2 & \text{if } \mathcal{C} \subseteq \mathcal{S}, \\ 1 & \text{otherwise.} \end{cases}$$

Furthermore,

$$T_n^{\mathcal{C}} = \begin{cases} \{\mathbf{c} = (c_1, \ldots, c_n) \in A^n : c_1 = 0\} & \text{if } \mathcal{C} \subseteq \mathcal{S}, \\ A^n & \text{otherwise} \end{cases}$$

is a transversal for the $\sim_{\mathcal{C}}$-blocks in $A^n$.

For arbitrary Boolean function $f$ let $\mathrm{Im}^{[2]}(f)$ denote the collection of all sets of the form $\{f(\mathbf{a}), f(\overline{\mathbf{a}})\}$ as $\mathbf{a}$ runs over all elements of the domain of $f$, and let $\mathrm{Im}^{[1]}(f)$ denote the collection of all singletons $\{f(\mathbf{a})\}$ as $\mathbf{a}$ runs over all elements of the domain of $f$. Thus $\mathrm{Im}^{[d_\mathcal{C}]}(f)$ consists of the ranges of all functions $f|_P$ as $P$ runs over all $\sim_\mathcal{C}$-blocks in the domain of $f$.

**Theorem 5.1.** *Let $\mathcal{C}$ be a discriminator clone of Boolean functions. The following conditions on $f \in \mathcal{O}^{(n)}$ and $g \in \mathcal{O}^{(m)}$ are equivalent:*

(a) $f \leq_\mathcal{C} g$;
(b) $f(\mathbf{0}) = g(\mathbf{0})$ *if* $\mathcal{C} \subseteq \mathcal{T}_0$, $f(\mathbf{1}) = g(\mathbf{1})$ *if* $\mathcal{C} \subseteq \mathcal{T}_1$, *and* $\mathrm{Im}^{[d_\mathcal{C}]}(f) \subseteq \mathrm{Im}^{[d_\mathcal{C}]}(g)$.

*If* $\mathcal{C} = \mathcal{T}_{\mathrm{id}}, \mathcal{T}_0, \mathcal{T}_1$, *or* $\mathcal{O}$, *then the inclusion* $\mathrm{Im}^{[d_\mathcal{C}]}(f) \subseteq \mathrm{Im}^{[d_\mathcal{C}]}(g)$ *in condition (b) can be replaced by* $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$.

**Proof.** First we will prove the equivalence of conditions (a) and (b). By Theorem 3.3, $f \leq_\mathcal{C} g$ if and only if for all $P = \mathbf{c}/{\sim_\mathcal{C}}$ with $\mathbf{c} \in T_n^\mathcal{C}$,

$$f|_P \in \{g|_Q \circ \Phi^\mathcal{C}_{\mathbf{d},\mathbf{c}} : Q = \mathbf{d}/{\sim_\mathcal{C}},\ \mathbf{d} \in A^m,\ \mathbf{S}^\mathcal{C}_\mathbf{d} \leq \mathbf{S}^\mathcal{C}_\mathbf{c}\}. \tag{5.3}$$

The functions $\Phi^\mathcal{C}_{\mathbf{d},\mathbf{c}} : P \to Q$ here are bijections, since they are surjective by definition, and $|P| = |Q| = d_\mathcal{C}$. If $\mathbf{S}^\mathcal{C}_\mathbf{c} = \{0\}$, then $\mathbf{c} = \mathbf{0}$ and $\mathcal{C} \subseteq \mathcal{T}_0$ by (5.2). Thus $\mathbf{S}^\mathcal{C}_\mathbf{d} \leq \mathbf{S}^\mathcal{C}_\mathbf{c}$ forces $\mathbf{d} = \mathbf{0}$. Similarly, if $\mathbf{S}^\mathcal{C}_\mathbf{c} = \{1\}$, then $\mathbf{c} = \mathbf{1}$, $\mathcal{C} \subseteq \mathcal{T}_1$, and $\mathbf{d} = \mathbf{1}$. In all other cases $\mathbf{S}^\mathcal{C}_\mathbf{c} = \{0, 1\}$, therefore all $\mathbf{d} \in A^m$ satisfy $\mathbf{S}^\mathcal{C}_\mathbf{d} \leq \mathbf{S}^\mathcal{C}_\mathbf{c}$. Since $|P| = d_\mathcal{C} = 1$ or 2, it follows that in this case each bijection of $P$ onto another $\sim_\mathcal{C}$-block $Q$ is of the form $\Phi^\mathcal{C}_{\mathbf{d},\mathbf{c}}$ for an appropriate $\mathbf{d} \in Q$. Consequently, (5.3) is equivalent to the following condition:

(1) $f|_P = g|_Q \circ \phi$ for the unique bijection $\phi : P \to Q$ with $\phi(\mathbf{0}) = \mathbf{0}$, if $P = \mathbf{0}/{\sim_\mathcal{C}}$ and $\mathcal{C} \subseteq \mathcal{T}_0$;
(2) $f|_P = g|_Q \circ \phi$ for the unique bijection $\phi : P \to Q$ with $\phi(\mathbf{1}) = \mathbf{1}$, if $P = \mathbf{1}/{\sim_\mathcal{C}}$ and $\mathcal{C} \subseteq \mathcal{T}_1$;
(3) $f|_P \in \{g|_Q \circ \phi : \phi \text{ is a bijection } P \to Q,\ Q = \mathbf{d}/{\sim_\mathcal{C}},\ \mathbf{d} \in A^m\}$ otherwise.

(1) and (2) require that

(i) $f(\mathbf{0}) = g(\mathbf{0})$ holds if $\mathcal{C} \subseteq \mathcal{T}_0$ and $d_\mathcal{C} = 1$ (that is, if $\mathcal{D} \neq \mathcal{C} \subseteq \mathcal{T}_0$),
(ii) $f(\mathbf{1}) = g(\mathbf{1})$ holds if $\mathcal{C} \subseteq \mathcal{T}_1$ and $d_\mathcal{C} = 1$ (that is, if $\mathcal{D} \neq \mathcal{C} \subseteq \mathcal{T}_1$), and
(iii) both of $f(\mathbf{0}) = g(\mathbf{0})$ and $f(\mathbf{1}) = g(\mathbf{1})$ hold if $\mathcal{C} \subseteq \mathcal{T}_i$ for $i = 0$ or 1 and $d_\mathcal{C} = 2$ (that is, if $\mathcal{C} = \mathcal{D}\ (\subseteq \mathcal{T}_0 \cap \mathcal{T}_1)$).

In (3) the set $\{g|_Q \circ \phi : \phi \text{ is a bijection } P \to Q,\ Q = \mathbf{d}/{\sim_\mathcal{C}},\ \mathbf{d} \in A^m\}$ is equal to the set of functions $P \to A$ whose range is in $\mathrm{Im}^{[d_\mathcal{C}]}(g)$. Therefore condition (3) can be rephrased as follows:

(iv) for all $P = \mathbf{c}/{\sim_\mathcal{C}}$ ($\mathbf{c} \in T_n^\mathcal{C}$) not covered by (i)–(iii) $f|_P$ is a function $P \to A$ whose range is in $\mathrm{Im}^{[d_\mathcal{C}]}(g)$.

It is easy to see now that (i)–(iv) hold for all $f|_P$ ($P = \mathbf{c}/{\sim_\mathcal{C}}$, $\mathbf{c} \in T_n^\mathcal{C}$) if and only if $f$ and $g$ satisfy condition (b). This completes the proof of the equivalence of conditions (a) and (b).

If $\mathcal{C}$ is one of the clones $\mathcal{T}_{\mathrm{id}}, \mathcal{T}_0, \mathcal{T}_1$, or $\mathcal{O}$, then $d_\mathcal{C} = 1$. Hence for each Boolean function $f \in \mathcal{O}$, $\mathrm{Im}^{[d_\mathcal{C}]}(f)$ is the set of singletons $\{r\}$ with $r \in \mathrm{Im}(f)$. Therefore for arbitrary $f, g \in \mathcal{O}$ we have $\mathrm{Im}^{[d_\mathcal{C}]}(f) \subseteq \mathrm{Im}^{[d_\mathcal{C}]}(g)$ if and only if $\mathrm{Im}(f) \subseteq \mathrm{Im}(g)$, proving the last statement of the theorem. □

For each discriminator clone $\mathcal{C}$ of Boolean functions Theorem 5.1 allows us to describe the $\mathcal{C}$-equivalence classes of Boolean functions and also the partial order $\preceq_\mathcal{C}$ induced by $\leq_\mathcal{C}$ on the set $\mathcal{O}/{\equiv_\mathcal{C}}$ of $\mathcal{C}$-equivalence classes.

We will use the following notation: $N$ will denote the set of all nonconstant functions in $\mathcal{O}$, and $[i]$ ($i = 0, 1$) the set of all constant functions with value $i$. For a nonempty subset $R$ of $\{\{0\}, \{1\}, \{0, 1\}\}$, $F_R$ will denote the set of all functions $f \in \mathcal{O}$ such that $\mathrm{Im}^{[2]}(f) = R$. Furthermore, for any ordered pair $(a, b) \in \{0, 1\}^2$ and for any set $U$ of Boolean functions, $U^{ab}$ will denote the set of all functions $f \in U$ such that $f(\mathbf{0}) = a$ and $f(\mathbf{1}) = b$. Notice that with this notation $[i] = F_{\{\{i\}\}} = F^{ii}_{\{\{i\}\}}$ ($i = 0, 1$).

It follows from Theorem 5.1 that $f \equiv_\mathcal{D} g$ if and only if $f(\mathbf{0}) = g(\mathbf{0})$, $f(\mathbf{1}) = g(\mathbf{1})$, and $\mathrm{Im}^{[2]}(f) = \mathrm{Im}^{[2]}(g)$. Therefore the $\mathcal{D}$-equivalence classes are the nonempty sets of the form $F^{ab}_R$ where $\emptyset \neq R \subseteq \{\{0\}, \{1\}, \{0, 1\}\}$ and $(a, b) \in \{0, 1\}^2$. If $F^{ab}_R \neq \emptyset$, then $\{a, b\} \in R$, because $f \in F^{ab}_R$ implies that $R = \mathrm{Im}^{[2]}(f)$ and $\{a, b\} = \{f(\mathbf{0}), f(\mathbf{1})\} \in \mathrm{Im}^{[2]}(f)$. Thus the $\mathcal{D}$-equivalence classes are the nonempty sets among the 16 sets $F^{ab}_R$ with $\{a, b\} \in R \subseteq \{\{0\}, \{1\}, \{0, 1\}\}$. Fig. 2 shows these 16 sets along with representatives for each of them, proving that none of them are empty. Hence there are 16 $\mathcal{D}$-equivalence classes, and according to Theorem 5.1, the ordering $\preceq_\mathcal{D}$ between them is as depicted in Fig. 2. For notational simplicity, in Fig. 2 we omit braces when we list the elements of $R$ in $F^{ab}_R$. For example, we write $F^{10}_{0,01}$ instead of $F^{10}_{\{\{0\},\{0,1\}\}}$.

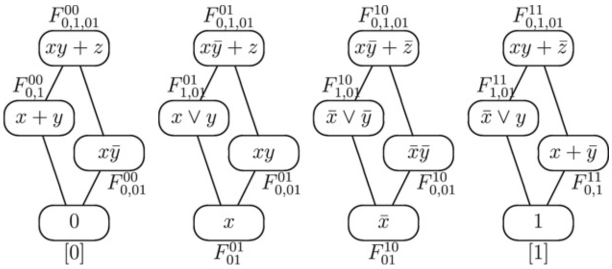


Fig. 2. The poset $(\mathcal{O}/{\equiv_{\mathcal{D}}};\preceq_{\mathcal{D}})$.

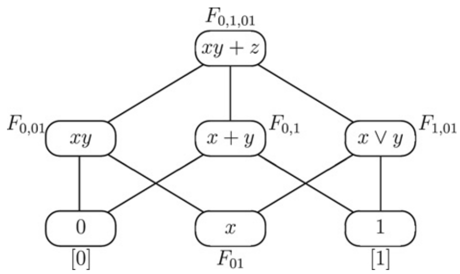


Fig. 3. The poset $(\mathcal{O}/{\equiv_{\mathcal{S}}};\preceq_{\mathcal{S}})$.

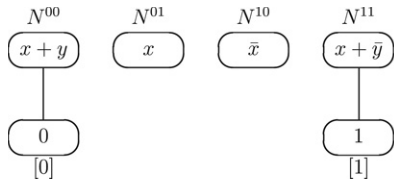


Fig. 4. The poset $(\mathcal{O}/{\equiv_{\mathcal{T}_{\mathrm{id}}}};\preceq_{\mathcal{T}_{\mathrm{id}}})$.

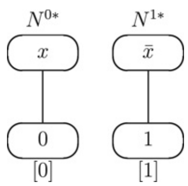


Fig. 5. The poset $(\mathcal{O}/{\equiv_{\mathcal{T}_0}};\preceq_{\mathcal{T}_0})$.

For the clone $\mathcal{S}$ Theorem 5.1 yields that $f \equiv_{\mathcal{S}} g$ if and only if $\mathrm{Im}^{[2]}(f) = \mathrm{Im}^{[2]}(g)$. Thus the $\mathcal{S}$-equivalence classes are the nonempty sets among the 7 sets $F_R$ with $\emptyset \neq R \subseteq \{\{0\},\{1\},\{0,1\}\}$. Fig. 3 shows these sets together with representatives for each of them, hence none of them is empty. Thus there are seven $\mathcal{S}$-equivalence classes, and according to Theorem 5.1, the ordering $\preceq_{\mathcal{S}}$ between them is as indicated in Fig. 3.

Proceeding similarly, for the clone $\mathcal{T}_{\mathrm{id}}$ we get from Theorem 5.1 that $f \equiv_{\mathcal{T}_{\mathrm{id}}} g$ if and only if $f(\mathbf{0}) = g(\mathbf{0})$, $f(\mathbf{1}) = g(\mathbf{1})$, and $\mathrm{Im}(f) = \mathrm{Im}(g)$. Since the range of each nonconstant Boolean function is $\{0,1\}$, we conclude that the $\mathcal{T}_{\mathrm{id}}$-equivalence classes are $[0]$, $[1]$, and $N^{ab}$ with $(a,b) \in \{0,1\}^2$. Fig. 4 shows representatives of these classes and the ordering $\preceq_{\mathcal{T}_{\mathrm{id}}}$ among them according to Theorem 5.1.

Analogously, Theorem 5.1 yields that the $\mathcal{T}_0$-equivalence classes are $[i]$ and $N^{i*} = N^{i0} \cup N^{i1}$ $(i = 0,1)$ with representatives and ordering as shown in Fig. 5. The results for $\mathcal{T}_1$ are similar.

Finally, we obtain either from Theorem 5.1 or from the special case $|A| = 2$ of Corollary 3.4 that the $\mathcal{O}$-equivalence classes are $[i]$ $(i = 0,1)$ and $N$ with representatives and ordering as shown in Fig. 6.

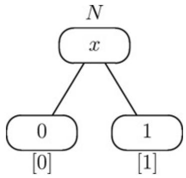


Fig. 6. The poset $(\mathcal{O}/{\equiv_{\mathcal{O}}}; \preceq_{\mathcal{O}})$.

To conclude our discussion of the posets $(\mathcal{O}/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$, recall from Corollary 2.2 that if $\mathcal{C} \subseteq \mathcal{C}'$, then we have a surjective, order-preserving mapping $\nu_{\mathcal{C}',\mathcal{C}}$ from the poset $(\mathcal{O}_A/{\equiv_{\mathcal{C}}}; \preceq_{\mathcal{C}})$ onto $(\mathcal{O}_A/{\equiv_{\mathcal{C}'}}; \preceq_{\mathcal{C}'})$, which assigns to each $\mathcal{C}$-equivalence class the $\mathcal{C}'$-equivalence class containing it. By (2.1) it suffices to look at the mappings $\nu_{\mathcal{C}',\mathcal{C}}$ for covering pairs $\mathcal{C} \subset \mathcal{C}'$.

For each covering pair $\mathcal{C} \subset \mathcal{C}'$ of discriminator clones (see Fig. 1), one can read off of Fig. 2–6 what the corresponding natural mapping $\nu_{\mathcal{C}',\mathcal{C}}$ is. For example, the mapping $\nu_{\mathcal{S},\mathcal{D}}\colon (\mathcal{O}_A/{\equiv_{\mathcal{D}}}; \preceq_{\mathcal{D}}) \to (\mathcal{O}_A/{\equiv_{\mathcal{S}}}; \preceq_{\mathcal{S}})$ preserves the heights of elements, and

- for elements of height 0, it sends $[i]$ to $[i]$ $(i = 0, 1)$, and the other two elements $F^{01}_{01}$, $F^{10}_{01}$ in Fig. 2 to the middle element $F_{01}$ in Fig. 3;
- for elements of height 1, it sends the leftmost and rightmost elements $F^{00}_{0,1}$, $F^{11}_{0,1}$ in Fig. 2 to the middle element $F_{0,1}$ in Fig. 3, and among the remaining six elements in Fig. 2, it sends the three that appear lower to the leftmost element $F_{0,01}$ in Fig. 3, and the three that appear higher to the rightmost element $F_{1,01}$ in Fig. 3;
- finally, it sends all four elements of height 2 in Fig. 2 to the largest element in Fig. 3.

The natural mapping $\nu_{\mathcal{T}_{\mathrm{id}},\mathcal{D}}\colon (\mathcal{O}_A/{\equiv_{\mathcal{D}}}; \preceq_{\mathcal{D}}) \to (\mathcal{O}_A/{\equiv_{\mathcal{T}_{\mathrm{id}}}}; \preceq_{\mathcal{T}_{\mathrm{id}}})$ preserves the four connected components, and

- in the first and last connected components it sends $[i]$ to $[i]$ $(i = 0, 1)$, and the remaining three elements in Fig. 2 to the height one element $N^{ii}$ $(i = 0, 1)$ of the corresponding component in Fig. 4;
- in the second and third connected components it sends all four elements in Fig. 2 to the unique element of the corresponding component in Fig. 4.

## Acknowledgements

This work was initiated while the first author was visiting the University of Waterloo. He is indebted to Professors Ian Goulden and Bruce Richmond for helpful discussions.

This material is based upon work supported by the Hungarian National Foundation for Scientific Research (OTKA) grants no. T 048809 and K60148.

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