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On the Solution of Equations Containing Radicals by the Decomposition Method

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Equations containing radicals are solved using Adomian's decomposition method. © 1985 Academic Press, Inc.

The decomposition method [1, 2] has been used to solve wide classes of equations [2]. Adomian [2] and Adomian and Rach [3] have investigated use of the method when composite nonlinearities of the form $N_0(N_1 x)$ or $N_0(N_1(N_2 x))$, etc., are involved. We now deal specifically with algebraic or differential equations containing radicals using the above referenced works.

We will consider an example in the standard form used in [1, 2] $Ly + Ny = x(t)$, where L is a linear operator and N is a nonlinear operator. First suppose we are considering an algebraic equation with L simply being multiplication by a constant m, with $x(t)$ a constant k, and $Ny = (\alpha y^2 + \beta)^{1/2}$. L^{-1} will now mean division by m and $y = y_0 - L^{-1}Ny$ or

$$y = \sum_{n=0}^{\infty} y_n = y_0 - (1/m)(\alpha y^2 + \beta)^{1/2} \tag{1}$$

where $y_0 = k/m$. We treat the nonlinear or radical term $(\alpha y^2 + \beta)^{1/2}$ as a composite nonlinearity [2, 3] with $N_0(u^0) = \sqrt{u^0} = \sum_{n=0}^{\infty} A_n^0$, where $u^0 = \alpha(N_1 u^1) + \beta$ and $N_1 u^1 = \sum_{n=0}^{\infty} A_n^1$ with $u^1 = y$. Now

$$\sum_{n=0}^{\infty} u_n^0 = \alpha \sum_{n=0}^{\infty} A_n^1 + \beta \tag{2}$$

so that

$$u_0^0 = \beta + \alpha A_0^1 = A_0^0$$

For $n \geqslant 1$ we have

$$u_n^0 = \alpha A_n^1 = A_n^0.$$

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Calculating the A_n^0 polynomials

$$A_0^0 = (u_0^0)^{1/2} = (\beta + \alpha A_0^1)^{1/2}$$

$$A_1^0 = (\tfrac{1}{2})(u_0^0)^{-1/2}(u_1^0) = (\tfrac{1}{2})(\beta + \alpha A_0^1)^{-1/2}(\alpha A_1^1)$$

$$A_2^0 = (\tfrac{1}{2})(u_0^0)^{-1/2}(u_2^0) - (\tfrac{1}{8})(u_0^0)^{-3/2}(u_1^0)^2$$

$$A_3^0 = (\tfrac{1}{2})(u_0^0)^{-1/2}(u_3^0) - (\tfrac{1}{4})(u_0^0)^{-3/2}(u_1^0)(u_2^0) + (\tfrac{1}{16})(u_0^0)^{-5/2}(u_1^0)^3$$

$$A_4^0 = (\tfrac{1}{2})(u_0^0)^{-1/2}(u_4^0) - (\tfrac{1}{4})(u_0^0)^{-3/2}\{(u_2^0)^2/2 + (u_1^0)(u_3^0)\} + (\tfrac{3}{16})(u_0^0)^{-5/2}(u_1^0)^2(u_2^0) - (\tfrac{5}{128})(u_0^0)^{-7/2}(u_1^0)^4$$

$$A_5^0 = (\tfrac{1}{2})(u_0^0)^{-1/2}(u_5^0) - (\tfrac{1}{4})(u_0^0)^{-3/2}\{u_2^0 u_3^0 + u_1^0 u_4^0\} + (\tfrac{3}{16})(u_0^0)^{-5/2}\{(u_1^0)(u_2^0)^2 + (u_1^0)^2(u_3^0)\} - (\tfrac{5}{32})(u_0^0)^{-7/2}(u_1^0)^3(u_2^0) + (\tfrac{7}{256})(u_0^0)^{-9/2}(u_1^0)^5$$

$$\vdots$$

Since $y^2 = \sum_{n=0}^{\infty} A_n^1$, we calculate

$$A_0^1 = y_0^2$$

$$A_1^1 = 2y_0 y_1$$

$$A_2^1 = y_1^2 + 2y_0 y_2$$

$$A_3^1 = 2y_1 y_2 + 2y_0 y_3$$

$$A_4^1 = y_2^2 + 2y_1 y_3 + 2y_0 y_4$$

$$A_5^1 = 2y_2 y_3 + 2y_1 y_4 + 2y_0 y_5$$

$$\vdots$$

Now

$$y_0 = k/m$$

$$y_1 = -(1/m)(\beta + \alpha(k^2/m^2))^{1/2}$$

$$y_2 = -(\alpha/m)(k/m)\{(-1/m)(\beta + (\alpha k^2/m^2))^{1/2}(\beta + \alpha y_0^2)^{-1/2} = \alpha k/m^3.$$

Indicating by ϕ_n the n term approximation $\sum_{\nu=0}^{n-1} y_\nu$, we have the following results if we choose $\alpha = \frac{1}{4}$, $\beta = 1$, $m = 10$, $k = 5$:

$y_0 =$	0.5000000000	$\phi_1 = 0.5000000000$
$y_1 =$	−0.1030776406	$\phi_2 = 0.3969223594$
$y_2 =$	0.0012500000	$\phi_3 = 0.3981723594$

which is already correct to three decimal places (y_3 is 1.364×10^{-4}). (The correct result is 0.4000000 so that the error by ϕ_3 is less than half a percent.) Since this result is obtained with only three terms, and we are solving the *nonlinear* problem, not a linearized version, it is rather remarkable.

Now let us consider a *differential equation in the same standard form.* Let $L=d/dt$, and $x=0$ (for convenience only [2]). L^{-1} is now the definite integral from 0 to t. Let $y(0)=k$ and $Ny=(\alpha y^2+\beta)^{1/2}$ as before. Again,

$$y=y_0-L^{-1}Ny.$$

Letting $y=\sum_{n=0}^{\infty} y_n$, we have $y_0=y(0)=k$ and since we have the same A_n polynomials,

$$\begin{aligned}
y_0 &= k\\
y_1 &= -L^{-1}(\beta+\alpha k^2)^{1/2} = -(\beta+\alpha k^2)^{1/2}\,t\\
y_2 &= -L^{-1}(\beta+\alpha k^2)^{-1/2}\,\alpha y_0 y_1 = \alpha k t^2/2\\
y_3 &= -L^{-1}(1/2)(\beta+\alpha k^2)^{-1/2}\,\alpha(y_1^2+2y_0 y_2)\\
&\quad -(1/8)(\beta+\alpha k^2)^{-3/2}(2\alpha y_0 y_1)^2\\
&= -\alpha(\beta+\alpha k^2)^{1/2}\,t^3/6\\
&\ \vdots\\
y &= k-(\beta+\alpha k^2)^{1/2}\,t+\alpha k t^2/2-\alpha(\beta+\alpha k^2)^{1/2}\,t^3/6...,
\end{aligned}$$

as the solution for

$$\frac{dy}{dt}+\sqrt{\beta+\alpha y^2}=0 \qquad y(0)=k.$$

As shown in [2], cases where L is a higher order operator or $x(t)\neq 0$ are straightforward generalizations. It is interesting to choose some values for the constants. Let $k=5$, $\alpha=\frac{1}{4}$, $\beta=1$. Then

$$\begin{aligned}
y_0 &= 5.0\\
y_1 &= -2.692t\\
y_2 &= 0.625t^2\\
y_3 &= -0.112t^3.
\end{aligned}$$

The solution as a function of t is a rapidly damped oscillation for small t, e.g., at $t=0.1$, the 1-term approximation of ϕ is 5.0, the 2-term approximation ϕ_2 is 4.73, and further changes by going to ϕ_3 or ϕ_4 are in the third decimal place. For $t = 1.0$, $\phi_1 = 5$, $\phi_2 = 2.31$, $\phi_3 = 2.93$, $\phi_4 = 2.82, \ldots,$ so we expect $2.82 < \phi_n < 2.93$ for $n > 4$ and can go to ϕ_6 with the already computed polynomials.

REFERENCES

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