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The Making of Geometry

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Abstract

Geometry has been a source of inspiration in the design of the manmade world for millennia; it also provides representational means enabling development of a concept into a built object. In the past three decades computing methodologies have provided the designer with unprecedented tools to explore highly complex forms, create digital models and fabricate them. This paper describes a computational methodology for the transition of forms from abstract geometric configurations to physical objects: a parametric design process assists from the initial ideation to the final prototyping with 3D printing technologies. The five regular polyhedra are used as a case study; this paper explores how parametric based procedures develop these geometric shapes into digital models of structures to be fabricated in different sizes and materials.

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1. Introduction

The role of mathematics in the understanding and interpretation of the physical world is at the foundation of science and perhaps can be summarized in Galileo Galilei's poetic definition of the universe "written in the language of mathematics, and its characters are triangles, circles, and other geometrical figures" [1]. Mathematics underlies laws of nature and it has been used to measure and understand the physical world; nature itself presents geometric arrangements in the majority of forms, found in physics and biology.

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Mathematics and geometry have had a major role in architecture and any manmade fabrication, providing artists, architects and designers with sources of inspiration, representational means and structural verification. Proportions in architecture, symmetry in paintings and the harmonic sequences in the sound of music are only a few of the countless examples of the presence of mathematics in human arts endeavors. The statement from the biologist D'Arcy Thompson "form is a diagram of forces" [2] has guided not only structural engineering explorations but has also inspired and provided aesthetic intention for design production at different scales and functional requirements throughout many historical periods and styles [3].

Despite the entanglement between geometry and nature, somehow the world of geometry is abstracted from the constraints of the physical world ruled by forces and their interactions. Points, lines and surfaces do not have size limitations and are not ruled by the gravity, heat and other physical constraints of the physical products of design. Recent computational digital methodologies can assist in the translation from ideal geometric shapes to digital and physical models of design objects which can be constructed in different materials and sizes.

This paper outlines a computational methodology which can translate geometric entities into digital and physical models using a set of operations in conjunction with user defined parameters. The five regular solids are explored as a case study; the methodology is software independent. Some of the models have been fabricated as stand-alone using 3D printing technologies while others integrate 3D printed parts with off-the shelf components to create a cost effective product.

2. The Platonic Solids in Geometry

The choice of the regular convex polyhedra as a case study derives from their relevance not only in geometry but also in art, architecture and design [4]. Each of these solids is composed of equal and regular faces, that is, by regular convex polygons. A regular polyhedron can be identified by the Schläfli symbol {p, q} where p indicates the number of sides of the regular polygon — defining the faces of the polyhedron— and q is the number of polygons meeting at each vertex [5]. For example, according to the notation of the Schläfli symbol, p =3 denotes a triangle, p=4 a square, p=5 a pentagon. In each polyhedron a set of faces meets in a vertex defining a solid angle: all the solid angles of a regular polyhedron are equal. Only five regular convex polyhedra exist, due to the property of their faces and that there is only a limited number of the same regular polygons meeting in one solid angle with sum of the planar angles less than 2p [6].

Another characterizing element for each polyhedron is the dihedral angle —that is the angle between two adjacent faces. Table 1 summarizes the geometric properties for the five solids, which characterize each polyhedron and will be used in later sections to define computational construction rules.

Table 1. The Five Regular Polyhedra as Geometric Entities

Polyhedron Name	Number of Faces	Number of Edges	Number of Vertices	Schläfli Symbol	Dihedral Angle
Tetrahedron	4	6	4	{3, 3}	70.53°
Hexahedron (Cube)	6	12	8	{4, 3}	90°
Octahedron	8	12	6	{3, 4}	109.47°
Dodecahedron	12	30	20	{5, 3}	116.57°
Icosahedron	20	30	12	{3, 5}	138.19°

The five polyhedra are also called Platonic solids, from Plato who discussed them from his cosmological approach in the dialogue *Timaeus*. Plato assigned the form of each solid to one of the natural elements which were believed to be the fundamental matter of the universe: the tetrahedron was associated to fire, the cube to earth, the octahedron to air, the icosahedron to water, and the dodecahedron was assigned to the ether [7].

The regular solids are characterized by spherical symmetry: from a point on a sphere only five sets of symmetrical points can be individuated, which originate from that point and holding the same distance from the center. The symmetry related properties make the polyhedra of great interest in architectural and product design.

Space frame structures are often based on tetrahedra and octahedra; the icosahedron is instead the originating geometry for geodesic domes [8].

2.1. Representing and Making the Polyhedra

Leonardo da Vinci was perhaps the first to represent the polyhedra as constructible geometry in his illustrations for the *De Divina Proportione* [9]. The five polyhedra are shown as solids, or as hollow structures bounded by edges. Sculptures, modelmaking and structures based on polyhedra have a long standing tradition in both the digital and analog world. The most basic representations of polyhedra are based only on their geometric characteristics —a portion of space bounded by polygonal faces sharing edges and faces (figure 1).

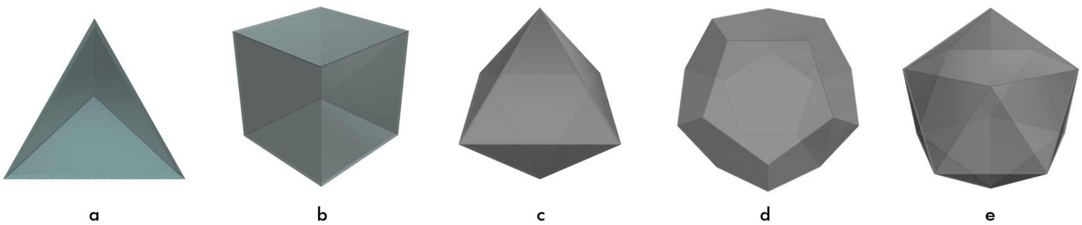

Fig. 1. (a) tetrahedron; (b) hexahedron; (c) octahedron; (d) dodecahedron; (e) icosahedron

The next representation, visually stronger, uses spheres for vertices and cylinders for edges while transparent planes identify faces (figure 2), similarly to ball-and-stick models in chemistry.

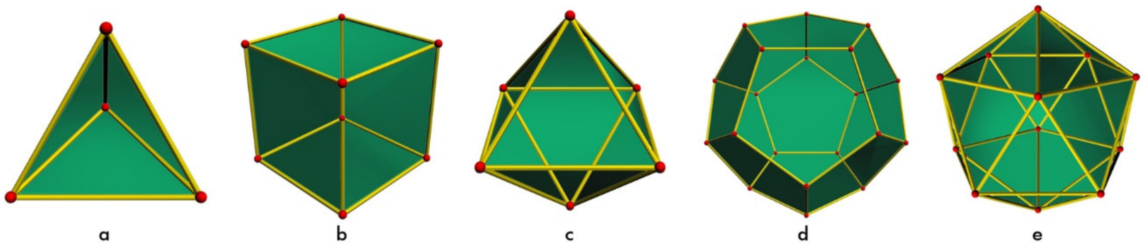

Fig. 2. (a) tetrahedron; (b) hexahedron; (c) octahedron; (d) dodecahedron; (e) icosahedron

3. Articulating complexity from simpler systems

Parametricism is identified by Patrick Schumacher as a style in architectural and urban theory. It is based on differentiation and articulation of complexity from sub-systems and involves many different facets and scales of design —from urban and architectural to interior, furniture and product design [10]. Previous work in art, design and natural sciences has also identified sets of rules to generate form complexity from interaction of simpler systems; such work included shape grammars [11], generative algorithms [12], Lindenmayer systems [13].

The commonality between these design approaches is based on the notion of substitution of a system with another system respecting initial generative rules, e.g. spatial relationships or geometric transformation. The process is often defined with an algorithm or script utilizing computational methodologies; parameters —which can be also be thought as variables— are used to generate a multitude of variations from initial concepts. Three-dimensional models are usually generated by such process which, when used to generate a spatial configuration, can be generalized and summarized as below:

- Definition of a parameters for a sub-system

- Definition of three-dimensional coordinate systems
- Definition of a sub-system position in three-dimensional space according to a previously specified coordinate system
- Use of geometric transformations or other functions to articulate the parameters of systems of higher complexity

4. Parametricing the Polyhedra

The digital models of the regular polyhedra shown in the following examples have been developed using GenerativeComponents (GC), a parametric associative modeling software by Bentley Systems [14]. Parametric based computational procedures are employed for both the geometric definition of a polyhedron as well as the digital model used for the polyhedron fabrication. The translation of a geometric shape into a physical object requires parameters in the form of numerical variables, based on a user defined unit system and related to the size of the material properties. Each polyhedron can be considered a system comprised of three different sub-systems: vertices, edges and faces.

Scripts written in C# programming language generate a 3D model for each polyhedron, as it has been shown in previous work [15]. At this stage the parameter/variable is size —provided by the distance SR of a vertex from the center of the polyhedron. In the model generated by such a script the geometric primitives characterizing the polyhedron are still present: vertices are points, edges are segments of straight lines and faces are portions of planes.

4.1. Evolution of a CAD Model

The polyhedron model generated by a GC script is imported into a CAD software, where the initial sub-systems —comprised of vertices, edges, faces— are substituted with sub-systems of higher geometric complexity [16]. In the CAD model each sub-system is an instance replicated according to the symmetry relationships characteristic of each regular polyhedron. Morphological evolutions develop the regular solids into manifolds presenting different geometrical and topological properties. Figure 4 shows an example of substitution of geometric primitives, which generate the models described in the following paragraphs.

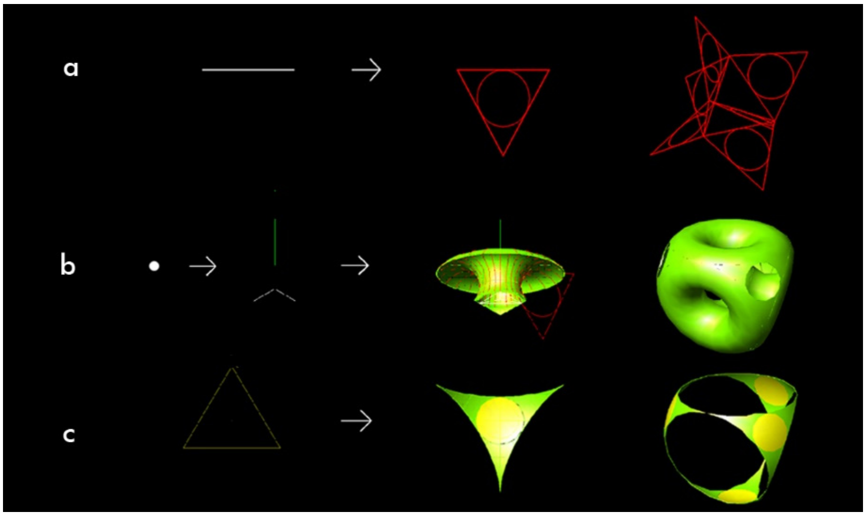

Fig. 4. Substitution of the geometric primitives of a polyhedron

4.1.1. Inside-Out Forms

The digital models of figure 5 show regular solids which are transformed into manifolds. The initial polyhedra turn inside-out by substitution rules, and the initial geometric primitives are substitute by more complex geometries, often with different topological properties. Each vertex in the initial polyhedron is replaced by a revolution surface, defining a hole symmetric about the axis connecting the polyhedron centre and the vertex. This revolution surface is generated by a profile, a 180 degrees arc, revolving 360 degrees around an axis defined by the centre of the polyhedron and a vertex. The arc belongs to the plane defined by the polyhedron centre and one of the edges with the vertex as endpoint and becomes a replacement for the edge. Faces are substituted by surfaces connecting the revolution surfaces which replace the vertices. The resulting manifold is a shape with a number of holes equal to the vertices of the initial polyhedron; the symmetry relationships are maintained. The profile, generating the revolution surface replacing the vertex, lays in the plane identified by one of the edges meeting in the vertex and by the centre of the polyhedron.

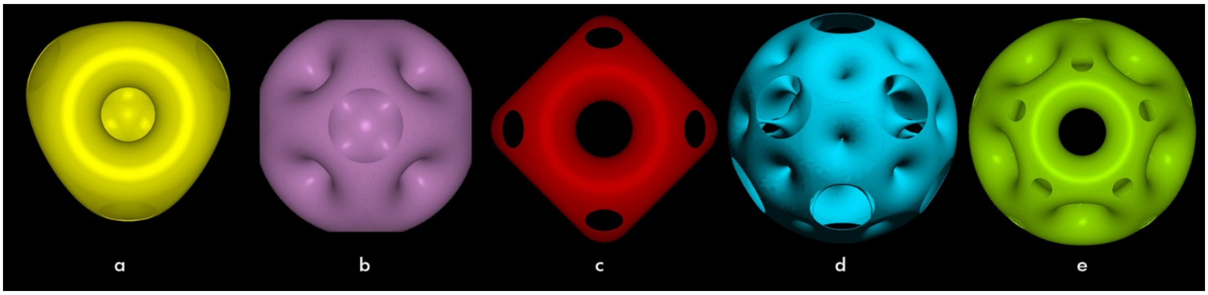

Fig. 5. Polyhedra transformation into manifolds: (a) tetrahedron; (b) hexahedron; (c) octahedron; (d) dodecahedron; (e) icosahedron

The photo of figure 6 shows a manifold evolved from the dodecahedron; it is shown as wireframe rendering of the digital model and as fabricated in plaster using 3D printing technologies. The digital model has been realized using the procedure previously described; two parameters in the initial development of dodecahedron are assigned in the GC script, according to the constraint of the 3D printer used for fabrication (ZPrinter model 650). The first parameter is the radius of the sphere inscribing the dodecahedron, limited by bounding box of the 3D printer. The other parameter is the wall thickness defined by the material, object size and 3D printer specifications.

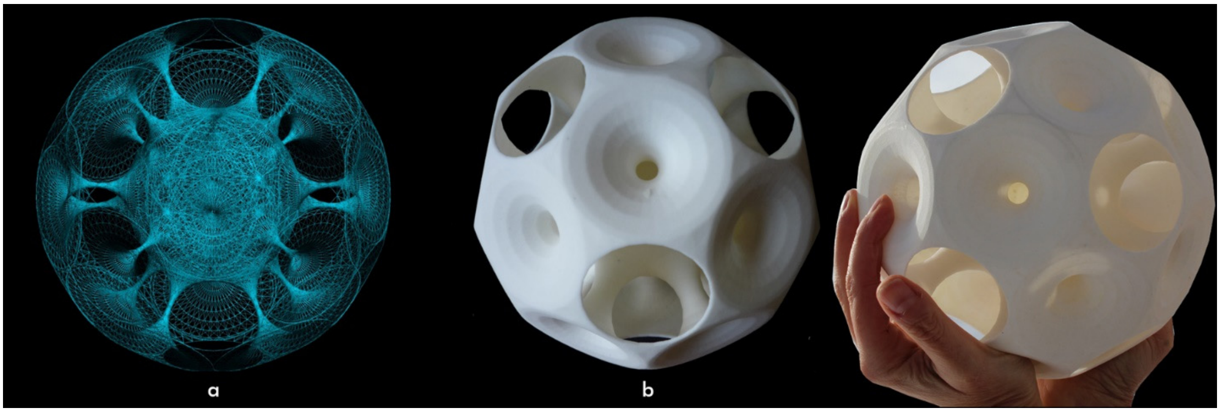

Fig. 6. A 3D printed manifold from the dodecahedron. Printing courtesy of Z Corporation

4.1.2. Polyhedra and Organic Forms

The biologist and artists Ernst Haeckel illustrated hundreds of forms found in nature, mainly microorganism: several of these natural forms present an underlying symmetry, often based on the five regular polyhedra [17]. D'Arcy Thompson also associated some of the organic shapes illustrated by Haeckel to minimal surfaces [18] emphasizing the relationship between the geometric form and the surface-tension due to molecular forces. Following the parametric approach described above, models have been developed by replacing each edge of a polyhedron with a surface bounded by a spherical arc and the segments connecting the vertices endpoints of the edge with the center of the polyhedron. A radial grid further evolves from the membranes generating forms similar to the radiolaria illustrated by Haeckel. Additional parameters are provided by grid density in terms of concentric arcs and angular divisions. Figure 7 shows digital models and 3D prints —from Selective Laser Sintering techniques— created by such formal evolutions.

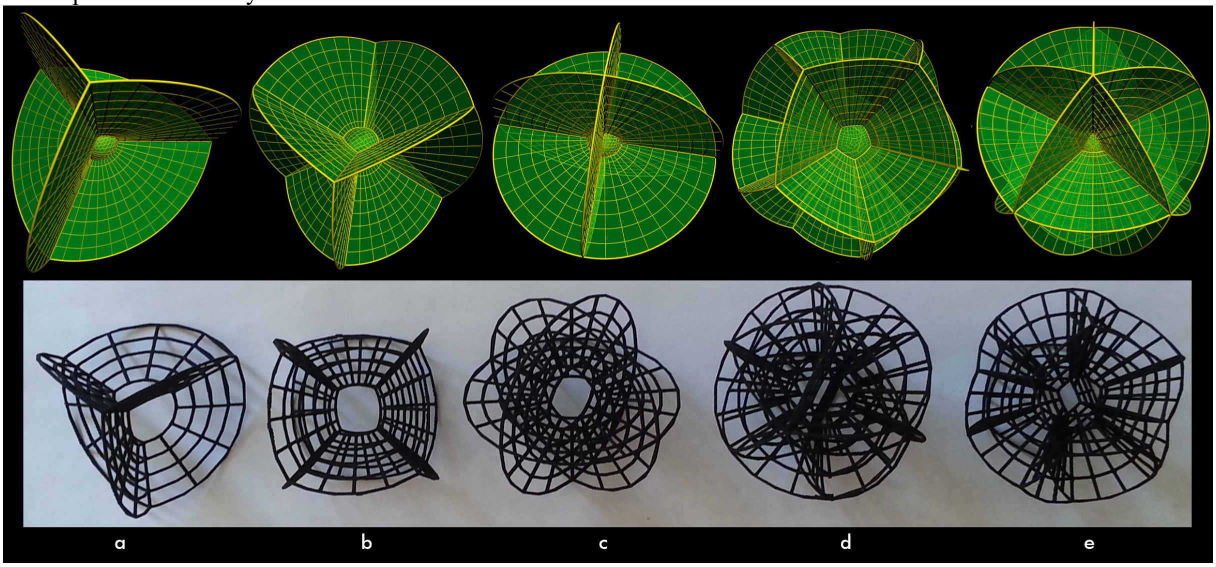

Fig. 7. Evolution based on minimal surfaces, digital models (above) and 3D prints for: (a) tetrahedron; (b) hexahedron; (c) octahedron; (d) dodecahedron; (e) icosahedron

The tetrahedron model, resembling the radiolarian Callimitra has been developed further as a light fixture (figure 8c), and has been 3D printed in plaster.

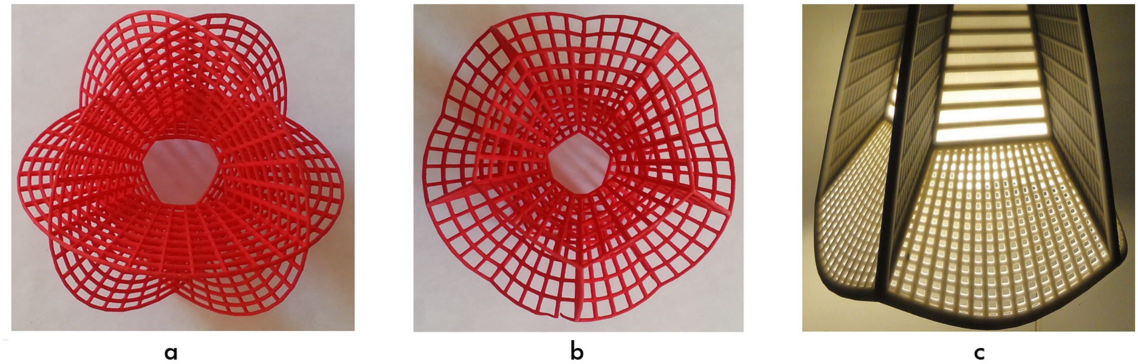

Fig. 7. Changing the grid parameters for: (a) octahedron; (b) dodecahedrob; (c) tetrahedron

5. Conclusion

As design revolves around form and function, the association between geometrical configurations associated to material parameters can offer an optimal design solution form in terms of structural and functional efficiency. This paper has shown how parameters based on geometric properties is one of the means used for understanding and making constructible forms from concepts and sketches to three-dimensional digital and physical models. 3D printing has been used to build complex manifolds evolving from the five regular solids.

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