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5-Arc transitive cubic Cayley graphs on finite simple groups ☆

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Abstract

In this paper, we determine all connected 5-arc transitive cubic Cayley graphs on the alternating group $\mathbf{A}_{47}$; there are only two such graphs (up to isomorphism). By earlier work of the authors, these are the only two non-normal connected cubic arc-transitive Cayley graphs for finite nonabelian simple groups, and so this paper completes the classification of such non-normal Cayley graphs.

1. Introduction

Let G be a group. The subset S of G is called a *Cayley subset* if $1 \notin S$ and $S^{-1} = S$. The *Cayley graph* $\Gamma := \mathrm{Cay}(G, S)$ on G with respect to S is defined by

$$\begin{array}{ll} \text{its vertex set} & V(\Gamma) := G, \text{ and} \\ \text{its edge set} & E(\Gamma) := \{\{g, sg\} \mid g \in G,\ s \in S\}. \end{array}$$

Clearly, its full automorphism group $\mathrm{Aut}(\Gamma)$ acts transitively on the vertex set $V(\Gamma)$ since $\mathrm{Aut}(\Gamma) \geq R(G)$, the right regular representation of G, and hence Γ is vertex-transitive. We always denote $R(G)$ by G for short. It is well-known that Γ is connected if and only if $\langle S\rangle = G$.

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To study the symmetry properties of Cayley graphs, we need more concepts of isomorphisms between Cayley graphs and their full automorphism groups.

Denote the automorphism group of the group G by $\mathrm{Aut}(G)$. A Cayley subset S of G is called a *CI-subset* of G (where CI stands for "Cayley isomorphism"), if for any isomorphism $\mathrm{Cay}(G, S) \cong \mathrm{Cay}(G, T)$ of Cayley graphs there exists an $\alpha \in \mathrm{Aut}(G)$ such that $S^\alpha = T$.

Denote $\mathrm{Aut}(G, S) = \{\alpha \in \mathrm{Aut}(G) \mid S^\alpha = S\}$, and we easily have $\mathrm{Aut}(\Gamma) \geq G \rtimes \mathrm{Aut}(G, S)$. As a matter of fact, $\mathrm{Aut}(\Gamma) = G \rtimes \mathrm{Aut}(G, S)$ is equivalent to $G \trianglelefteq \mathrm{Aut}(\Gamma)$ (see [2]). In this case we call the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ *normal* for G.

Let Γ be a graph, $G \leq \mathrm{Aut}(\Gamma)$ and s a positive integer. Γ is said to be (G, s)-*arc transitive*, if G acts transitively on the set of s-arcs of Γ, where an s-arc is a sequence $(v_0, v_1, \ldots, v_s)$ in $V(\Gamma)$ satisfying $(v_{i-1}, v_i) \in E(\Gamma)$ and $v_{i-1} \neq v_{i+1}$ for all i. In particular, $(\mathrm{Aut}(\Gamma), s)$-arc transitive is called *s-arc transitive*, and 1-arc transitive is simply called *arc transitive*.

Sabidussi gave a construction for all vertex-transitive (not only Cayley) graphs by using a group-theoretic method.

Let G be a finite group and T a subgroup of it. Let D be a union of several double cosets of T satisfying $D^{-1} = D$. He defined a graph Γ with vertex set $V(\Gamma) = [G : T]$, the set of all right cosets of T, and edge set $E(\Gamma) = \{\{Tg, Tdg\} \mid g \in G, d \in D\}$. This graph is called the *Sabidussi cosets graph* of G with respect to T and D, denoted by $\mathrm{Sab}(G, T, D)$.

Obviously, Γ is connected if and only if $\langle D \rangle = G$. It is easy to check that $\mathrm{Sab}(G, T, D)$ is G-arc transitive if and only if $D = TdT$ (a single double coset) for some $d \in G$. We always denote $\mathrm{Sab}(G, T, TdT)$ by $\mathrm{Sab}(G, T, d)$ for short.

In fact, any vertex-transitive graph Γ is the Sabidussi coset graph of its full automorphism group $A := \mathrm{Aut}(\Gamma)$ with respect to $T = A_v$, the stabilizer of any vertex v, and $D := \{\alpha \in A \mid \{v, v^\alpha\} \in E(\Gamma)\}$, which is a union of several double cosets of T.

Let $P(G)$ be the right multiplication action of G on $[G : T]$. Since $\mathrm{Aut}(\mathrm{Sab}(G, T, D)) \geq P(G)$, all Sabidussi coset graphs are vertex transitive. If T is core-free, that is $\cap_{g \in G} T^g = 1$, then $P(G) \cong G$. We always denote $P(G)$ by G.

Regarding connected cubic s-arc transitive graphs, the first important result due to Tutte ([3, Theorem 18.6]) claims that there is no finite s-arc transitive cubic graphs for $s > 5$. Also, it is easy to check that for normal cubic s-arc transitive Cayley graphs, we have $s \leq 2$. So, if a connected cubic Cayley graph is s-arc transitive $s > 2$, then it must be nonnormal.

Much excellent work has dealt with arc-transitive Cayley graphs on finite nonabelian simple groups. For example, in [4, Theorem 7.1.3], Li proved that all connected cubic arc-transitive Cayley graphs are normal except for the following exceptions listed below:

$$\mathbf{A}_5, \mathbf{PSL}_2(11), \mathbf{M}_{11}, \mathbf{A}_{11}, \mathbf{M}_{23}, \mathbf{A}_{23}, \text{ and } \mathbf{A}_{47}.$$

In [1] we proved that the only exception is $\mathbf{A}_{47}$. For all other groups listed above, we proved that their connected cubic arc-transitive Cayley graphs are normal. There we also constructed a connected 5-arc transitive cubic Cayley graph for $\mathbf{A}_{47}$.

The purpose of this paper is to classify all connected 5-arc transitive cubic Cayley graphs on the alternating group $\mathbf{A}_{47}$. By the remarks above, it is also a classification of connected 5-arc transitive cubic Cayley graphs on finite simple groups.

The rest of this paper is organized as follows. After giving some preliminary results in Section 2, we construct all connected 5-arc transitive cubic Cayley graphs on $\mathbf{A}_{47}$ in Section 3, then in the next section we determine the isomorphisms between them, and finally we complete the classification in the last section.

2. Preliminaries

The first lemma is about the relation between Sabidussi coset graphs and Cayley ones.

Lemma 2.1. (1) *Let* $\Gamma := \mathrm{Cay}(G, S)$ *be a Cayley graph, and* $A := \mathrm{Aut}(\Gamma)$*. Then the vertex-stabilizer* A_1 *is a complement of* G *in* A*, where 1 is the identity of* G*, and we have* $\Gamma \cong \overline{\Gamma} := \mathrm{Sab}(A, A_1, A_1 S A_1)$*. In particular, there exists an* $s \in S$ *such that* $A_1 S A_1 = A_1 s A_1$ *when* Γ *is arc transitive.*

(2) *Conversely, let* $\overline{\Gamma} := \mathrm{Sab}(A, T, D)$ *be a Sabidussi coset graph and* G *a complement of* T *in* A*. Denote* $S = G \cap D$*. Then the Cayley graph* $\Gamma := \mathrm{Cay}(G, S)$ *is isomorphic to* $\overline{\Gamma}$*, and hence* $|S| = |D : T|$*. In particular,* S *contains an involution of* G *if the valency of* $\overline{\Gamma}$ *is odd. Also* Γ *is arc transitive if* D *is a single double coset of* T*.*

Proof. (1) Obvious.

(2) Since $A = GT$ and $G \cap T = 1$, each coset in $[A : T]$ has only an element of G as its representative. We define a bijection σ from Γ to $\overline{\Gamma}$ such that $g^\sigma := Tg \in V(\overline{\Gamma}) = [A : T]$ for all $g \in V(\Gamma) = G$. Since

$$\{g, g'\} \in E(\Gamma) \Leftrightarrow g'g^{-1} \in S = G \cap D \Leftrightarrow \{Tg, Tg'\} \in E(\overline{\Gamma})$$

for any $g, g' \in G$, we find $\Gamma \cong \overline{\Gamma}$. □

By results of [1] and [3] (respectively) we easily have

Lemma 2.2. *Let* $G \cong \mathbf{A}_{47}$ *and* $\Gamma := \mathrm{Cay}(G, S)$ *be a connected 5-arc transitive cubic Cayley graph for* G*. Denote* $A = \mathrm{Aut}(\Gamma)$*. Then the following hold.*

(1) $A \cong \mathbf{A}_{48}$*;*

(2) *There exist an involution* s *and a subgroup* T *in* A *which is isomorphic to* $\mathbf{S}_4 \times \mathbb{Z}_2$ *such that the Sabidussi coset graph* $\overline{\Gamma} := \mathrm{Sab}(A, T, s) \cong \Gamma$*. Also we have* $|T : T \cap T^s| = 3$*, and* $\langle T, s\rangle = A$*.*

The next lemma will play a very important role in proving our theorem.

Lemma 2.3. *Suppose that* R *is a regular subgroup on* $\Omega := \{1, 2, \ldots, n\}$ *and* $s \in \mathbf{S}_n$*. The following hold.*

(1) *Let* K *be a subgroup of* R*. Then there are* $|R : K|$ K*-orbits with length* $|K|$*. If* $g \in R$ *normalizes* K*, then* g *induces a permutation action on the set of* K*-orbits. In particular, the action is transitive if* $\langle K, g\rangle = R$*;*

(2) *If* $n = 4$ *and* $R = \langle a\rangle \times \langle b\rangle \cong \mathbb{Z}_2^2$ *such that* $\langle a\rangle^s = \langle a\rangle$ *and* $b^s = ab$*, then* s *is an odd permutation. In particular,* s *is a transposition if* s *is an involution;*

(3) *If* $n = 8$ *and* $R = \langle a\rangle \rtimes \langle b\rangle \cong \mathbf{D}_8$ *such that* $\langle a\rangle^s = \langle a\rangle$ *and* $b^s = ab$*, then* s *is an odd permutation. In particular,* s *is a product of three disjoint transpositions if* s *is an involution.*

Proof. (1) Clearly, K is semiregular on Ω and each K-orbit has the same length $|K|$. Since $|\Omega| = |R|$, then there are $|R : K|$ K-orbits on Ω. Let Δ be a K-orbit. If g normalizes K, then Δ^g is an orbit of $K^g = K$, and hence g may act on the set of k-orbits. Furthermore, if $\langle K, g\rangle = R$, which is transitive on Ω, then $\langle g\rangle$ is also transitive on the set of K-orbits.

(2) As $R = \langle a, b\rangle$ is regular on Ω, we may let $a = (12)(34)$, $b = (13)(24) \in \mathbf{A}_4$. Since s commutes with a but not b, then s is not a 3-cycle on Ω and $s \notin \langle a, b\rangle$, either. But $\langle a, b\rangle$ contains all involutions of $\mathbf{A}_4$, then s is either $(i_1 i_2)$ or $(i_1 i_2 i_3 i_4)$, and hence s is an odd permutation. In particular, $s = (i_1 i_2)$ if its order is 2.

(3) Being semiregular on Ω, $\langle a\rangle$ has two orbits with length 4, denoted by Δ_1, Δ_2. Without loss of generality, we may let $\Delta_1 = \{1, 2, 3, 4\}$, $\Delta_2 = \{5, 6, 7, 8\}$, and $a = (1234)(5678)$.

Since each of b and ab normalizes $\langle a\rangle$, we find $\Delta_1^b = \Delta_1^{ab} = \Delta_2$ by (1). This means that as two permutations with order 2 on Ω, $b = (1\,i_1)(2\,i_2)(3\,i_3)(4\,i_4)$ and $ab = (1\,j_1)(2\,j_2)(3\,j_3)(4\,j_4)$ where $i_r, j_r \in \Delta_2$. Since $a^b = a^{ab} = a^{-1}$, both arrangements $i_1i_2i_3i_4$ and $j_1j_2j_3j_4$ on Δ_2 are in the set $\{8765, 7658, 6587, 5876\}$.

Clearly s also normalizes $\langle a\rangle$, and hence $\Delta_1^s = \Delta_1$ or Δ_2. We deal with these two cases separately.

Case 1: $\Delta_1^s = \Delta_1$. We may let $r^s = k_r$ where $r, k_r \in \Delta_1$. Then $k_1k_2k_3k_4$ is an arrangement on Δ_1. Note that $a^s = ab$ and $b = (1\,i_1)(2\,i_2)(3\,i_3)(4\,i_4)$, only $(r\,i_r)^s = (k_r\,i_r^s)$ is a transposition of ab, and hence $i_r^s = j_{k_r}$. Thus $s = \begin{pmatrix} 1\,2\,3\,4\,i_1\;i_2\;i_3\;i_4 \\ k_1k_2k_3k_4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix}$.

Denote $u := \begin{pmatrix} 1\,2\,3\,4\,5\,6\,7\,8 \\ k_1k_2k_3k_4\,5\;6\;7\;8\end{pmatrix} \in \mathbf{S}_8$, $w := u^{ab}$ and $x := s(uw)^{-1}$. We will finish the proof of case 1 by the following steps:

(i) $w = \begin{pmatrix} 1\,2\,3\,4\,j_1\;j_2\;j_3\;j_4 \\ 1\,2\,3\,4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix}$.

In fact, since $w = u^{ab} = abuab$, then for any $r \in \Delta_1$, $r^w = (r^{ab})^{uab} = j_r^{uab} = j_r^{ab} = r$, and for any $j_r \in \Delta_2$, $j_r^w = (j_r^{ab})^{uab} = r^{uab} = k_r^{ab} = j_{k_r}$.

(ii) $uw = \begin{pmatrix} 1\,2\,3\,4\,j_1\;j_2\;j_3\;j_4 \\ k_1k_2k_3k_4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix}$, and uw commutes with ab.

First, $uw = \begin{pmatrix} 1\,2\,3\,4 \\ k_1k_2k_3k_4\end{pmatrix}\begin{pmatrix} j_1\;j_2\;j_3\;j_4 \\ j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix} = \begin{pmatrix} 1\,2\,3\,4\,j_1\;j_2\;j_3\;j_4 \\ k_1k_2k_3k_4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix}$. Then $(ab)^{uw} = ab$ since $(r, j_r)^{uw} = (k_r, j_{k_r})$ is still a transposition of ab.

(iii) $j_r = i_{r+1}$, where $4 + 1 \equiv 1 (\mathrm{mod}\ 4)$.

In fact, for any $r \in \Delta_1$, $j_r = r^{ab} = (r^a)^b = (r + 1)^b = i_{r+1}$.

(iv) $x = (i_1i_2i_3i_4)$.

In fact, by (ii) and (iii),

$$x = s(uw)^{-1} = \begin{pmatrix} 1\,2\,3\,4\,i_1\;i_2\;i_3\;i_4 \\ k_1k_2k_3k_4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4}\end{pmatrix}\begin{pmatrix} k_1k_2k_3k_4\,j_{k_1}\,j_{k_2}\,j_{k_3}\,j_{k_4} \\ 1\,2\,3\,4\;j_1\;j_2\;j_3\;j_4\end{pmatrix}$$
$$= \begin{pmatrix} 1\,2\,3\,4\,i_1\;i_2\;i_3\;i_4 \\ 1\,2\,3\,4\;j_1\;j_2\;j_3\;j_4\end{pmatrix} = \begin{pmatrix} 1\,2\,3\,4\,i_1\;i_2\;i_3\;i_4 \\ 1\,2\,3\,4\,i_2\;i_3\;i_4\;i_1\end{pmatrix} = (i_1i_2i_3i_4).$$

(v) s is an odd permutation.

In fact, since $uw = u(u^{ab})$ is obviously an even permutation, then by (iv), $s = x(uw) = (i_1i_2i_3i_4)uw$ is an odd permutation.

Case 2: $\Delta_1^s = \Delta_2$. In this case, $\Delta_1^{bs} = \Delta_1$. Then bs, which obviously satisfies the assumption as s does in case 1, is an odd permutation, and so is s.

In particular, as an odd permutation with order 2 of $\mathbf{S}_8$, s is a transposition or a product of three disjoint transpositions. If s is a transposition, then it has 6 fixed points on Ω, and hence $b = (1\,i_1)(2\,i_2)(3\,i_3)(4\,i_4)$ and $ab = (1\,i_2)(2\,i_3)(3\,i_4)(4\,i_1)$ are not conjugate under s. It follows that s is a product of three disjoint transpositions. □

3. How to construct the graphs

In this section, we construct all connected 5-arc transitive cubic Cayley graphs on $\mathbf{A}_{47}$.

Let G be a finite nonabelian simple group and $\Gamma := \mathrm{Cay}(G, S)$ a connected arc-transitive cubic Cayley graph. We know from [1] that Γ is nonnormal for G if and only if $G \cong \mathbf{A}_{47}$, and

$A := \mathrm{Aut}(\Gamma)$ is isomorphic to $\mathbf{A}_{48}$. Recall that this means, (1) Γ is 5-arc transitive; (2) the vertex-stabilizer T of A is isomorphic to $\mathbf{S}_4 \times \mathbb{Z}_2$; (3) there exists an involution s in G such that $|T : T \cap T^s| = 3$, $\langle T, s\rangle = A$, and furthermore the coset graph $\overline{\Gamma} := \mathrm{Sab}(A, T, s)$ is isomorphic to Γ.

To construct all these Cayley graphs, we first let $A \cong \mathbf{A}_{48}$, and $A = GT$, where $G \cong \mathbf{A}_{47}$ and $T \cong \mathbf{S}_4 \times \mathbb{Z}_2$. Secondly, we choose involutions s of G which satisfy that $|T : T \cap T^s| = 3$ and $\langle T, s\rangle = A$. Then we examine the structure of T and its subgroups with index 3.

(A) The structure of T

We will find out generators of T. Without loss of generality, we may let $T = \mathbf{S}_4 \times \mathbb{Z}_2$. Noting that $\mathbf{A}_4 \le \mathbf{S}_4$, we take $K \in \mathrm{Syl}_2(\mathbf{A}_4)$ which is a Klein four-group and $L \in \mathrm{Syl}_3(\mathbf{A}_4)$ which is a cyclic group of order 3 such that $\mathbf{A}_4 = K \rtimes L$. Thus there exist $b \in K$, $t \in L$ such that $K = \{1, b, b^t, b^{t^2}\}$ where $b^{t^2} = bb^t$, and hence $\mathbf{A}_4 = \langle b, t\rangle$. Note that $|\mathbf{S}_4 : \mathbf{A}_4| = 2$, then we may take an element a with order 4 of $\mathbf{S}_4$ such that $\mathbf{S}_4 = \langle \mathbf{A}_4, a\rangle = \langle a, b, t\rangle$.

Consider the relations between a, b and t. Note that $a^2 \in K$, we may let $a^2 = b^{t^2} = b\,b^t$, and accordingly $b^t = a^2 b$, $(a^2)^t = b$. Further, $D := K\langle a\rangle$ is an order 8 Sylow 2-subgroup of $\mathbf{S}_4$, and then $a^b = a^{-1}$. That is $D = \langle a\rangle \rtimes \langle b\rangle \cong \mathbf{D}_8$, and accordingly each Sylow 2-subgroup of T is isomorphic to $\mathbf{D}_8 \times \mathbb{Z}_2$.

By the way, T has 4 Sylow 3-subgroups all of which are in $\mathbf{A}_4$ since $\mathbf{A}_4 \lhd \mathbf{S}_4 \lhd T$ and $|T : \mathbf{A}_4|$ is divisible by 3, and hence all 8 elements of order 3 of T exactly make up the right coset union $Kt \cup Kt^{-1}$.

Represent a^t by a, b and t. First, a^t has order 4 and is not in $\mathbf{A}_4$, and hence in the left coset $a\mathbf{A}_4 = \mathbf{S}_4 \setminus \mathbf{A}_4$. Secondly, $a^t \in Dt \cup Dt^{-1}$ since t does not normalize D. Thus we may let $a^t = a^i b^j t^k$ where $i = \pm 1$, $j = 0$ or 1, $k = \pm 1$. We claim that $i = k = -1$ and $j = 0$, namely $a^t = a^{-1}t^{-1}$. In fact, (i) if $k = 1$, then $t^{-1}at = a^i b^j t$ and $t^{-1} = a^i b^j a^{-1} \in D$, a contradiction. Hence $k = -1$. (ii) If $i = 1$, then $b = (a^2)^t = (a^t)^2 = (ab^j t^{-1})(ab^j t^{-1}) = ab^j a^t t^{-1} b^j t^{-1} = ab^j(ab^j t^{-1})t^{-1}b^j t^2 = a(b^j ab^j)t^{-2}b^j t^2$. In this case, if $j = 0$, then $b = a^2$, a contradiction, and if $j = 1$, $b = aa^{-1}b^{t^2} = a^2$, still a contradiction. Hence $i = -1$. (iii) If $j = 1$, we may let $t' := a^2 t \in Kt$, then t' is still an element of order 3, and $a^{t'} = a^t = a^{-1}bt^{-1} = a^{-1}bt'^{-1}a^2 = a^{-1}b(a^2)^{t'}t'^{-1} = a^{-1}b(a^2)^{a^2 t}t'^{-1} = a^{-1}b(a^2)^t t'^{-1} = a^{-1}bbt'^{-1} = a^{-1}t'^{-1}$. We may replace t by t' such that $a^t = a^{-1}t^{-1}$.

Finally, let $\mathbb{Z}_2 = \langle c\rangle$. Then $T = \langle a, b, c, t \mid a^4 = b^2 = c^2 = t^3 = 1, a^b = a^{-1}, a^c = a, b^c = b, a^t = a^{-1}t^{-1}, b^t = a^2 b, c^t = c\rangle$.

(B) The subgroup of index 3 in T

Let R be a subgroup of T with index 3. Clearly, R is a Sylow 2-subgroup of T, and hence by (A) R is isomorphic to $\mathbf{D}_8 \times \mathbb{Z}_2$. Without loss of generality, we may let $R = \langle a, b, c\rangle$, where a, b, c are the same as in (A). It is easy to check that R has 7 subgroups of order 8 as follows.

Type 1, $\mathbb{Z}_2^3$: $\langle a^2, b, c\rangle$ and $\langle a^2, ab, c\rangle$;

Type 2, $\mathbb{Z}_4 \times \mathbb{Z}_2$: $\langle a, c\rangle$;

Type 3, $\mathbb{Z}_4 \rtimes \mathbb{Z}_2$: $\langle a, b\rangle$, $\langle a, bc\rangle$, $\langle ac, b\rangle$, and $\langle ac, bc\rangle$.

We shall choose an involution s of G such that $|T : T \cap T^s| = 3$ and $\langle T, s\rangle = A$. Clearly, those involutions s which normalize one of Sylow 2-subgroups of T except for T itself must satisfy that $|T : T \cap T^s| = 3$. Note that all three Sylow 2-subgroups of T, including R, are conjugate to each other. Without loss of generality, we may let R be normalized by s, and accordingly $R = T \cap T^s$. However, it is difficult to check if $\langle T, s\rangle = A$. But $\langle T, s\rangle$ should be a simple group, then s does not normalize any nontrivial normal subgroup of T. Then s belongs to the set defined by

$$\Pi_G(R,T) := \{s \in N_G(R) \mid o(s) = 2 \text{ and } \forall\, 1 \neq K \trianglelefteq T,\, K^s \neq K\},$$

where $T = \langle a, b, c, t\rangle$ and $R = \langle a, b, c\rangle$ defined in (A) and (B).

We still denote $\mathbf{S}_4 = \langle a, b, t\rangle$, its subgroup $D = \langle a, b\rangle$ as in (A) and denote $\langle a, c\rangle$ by K.

The next lemma shows us some properties of involutions in $\Pi_G(R,T)$.

Lemma 3.1. *For $s \in \Pi_G(R,T)$, the following hold.*
(1) $(a^2)^s = a^2$ *and* $c^s = a^2c$;
(2) $b^s = ab$;
(3) $a^s = a^{-1}$ *and* $(ac)^s = ac$;
(4) $(c^jD)^s = c^jD$ *and* $(b^jK)^s = b^jK$ *for every* $j \in \{0, 1\}$.

Proof. (1) First, by (B), s normalizes $K = \langle a, c\rangle = \{1, a, a^2, a^3, c, ac, a^2c, a^3c\}$ and hence $\langle a\rangle^s$ equals one of $\langle a\rangle$ and $\langle ac\rangle$. In any case, $(a^2)^s = a^2$. But s normalizes neither $\langle a^2, b, c\rangle$ nor $\langle c\rangle$ since $\langle a^2, b, c\rangle, \langle c\rangle \lhd T$ and $c^s = a^2c$.

(2) Since $\langle a^2, b, c\rangle^s = \{1, a^2, b, c, a^2b, a^2c, bc, a^2bc\}^s = \langle a^2, ab, c\rangle = \{1, a^2, ab, c, a^3b, a^2c, abc, a^3bc\}$, then $b^s = a^{\pm 1}bc^j$ $(j \in \{0, 1\})$. Let $a' = a^{\pm 1}c^j$. Then $R = (\langle a'\rangle \rtimes \langle b\rangle) \times \langle c\rangle$, and $b^s = a'b$. We may replace a by a', and hence (2) holds.

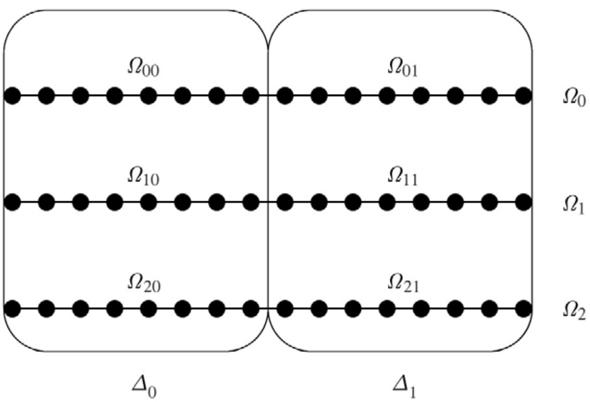
(3) We know from (1) that $\langle a\rangle^s = \langle a\rangle$ or $\langle ac\rangle$. If $\langle a\rangle^s = \langle ac\rangle$, then $a^s = a^{\pm 1}c$ and further $a = a^{s^2} = (a^{\pm 1}c)^s = (a^s)^{\pm 1}c^s = (a^{\pm 1}c)^{\pm 1}(a^2c) = (ac^{\pm 1})(a^2c) = (ac)(a^2c) = a^3 \neq a$. This contradiction shows that $\langle a\rangle^s = \langle a\rangle$ and hence $a^s = a^{\pm 1}$. If $a^s = a$, then $b = b^{s^2} = (ab)^s = a^sb^s = a(ab) = a^2b \neq b$. This contradiction shows that $a^s = a^{-1}$ and consequently $(ac)^s = a^sc^s = a^{-1}a^2c = ac$.

(4) Obvious. □

Of course, we cannot confirm if $\langle T, s\rangle = A$ for $s \in \Pi_G(R,T)$ and so we need some additional assumptions to help us choose s.

With the right multiplication permutation representation of A on $\Omega := [A : G]$ being faithful, we may assume $A = \mathrm{Alt}(\Omega)$. As a complement of G in A, T is a regular subgroup on Ω. Its subgroups R, $\mathbf{S}_4$ and D are semiregular. So there are $|T : R| = 3$ R-orbits denoted by Ω_0, Ω_1, Ω_2, and $|T : \mathbf{S}_4| = 2$ $\mathbf{S}_4$-orbits denoted by Δ_0, Δ_1 in Ω. By Lemma 2.3(1) and $T = \mathbf{S}_4 \rtimes \langle c\rangle$, c interchanges Δ_0 and Δ_1. Furthermore, for all $i \in \{0, 1, 2\}$ and $j \in \{0, 1\}$, $\Omega_{ij} := \Omega_i \cap \Delta_j$ are 6 D-orbits. c interchanges Ω_{i0} and Ω_{i1} for each i.

The R-orbits Ω_i $(i = 0, 1, 2)$, $\mathbf{S}_4$-orbits Δ_j $(j = 0, 1)$ and their intersection orbits Ω_{ij} are depicted in the following figure:



According to the action of $s \in \Pi_G(R, T)$ on $\mathbf{S}_4$-orbits, we say s is of *the first type* if it fixes each Δ_j setwise and s is of *the second type* if it is not of the first type.

Since s normalizes R, s may act on the R-orbit's set $\{\Omega_0, \Omega_1, \Omega_2\}$. As an involution, s must fix one of them. Without loss of generality, we always assume $\Omega_0^s = \Omega_0$. Thus $\Omega_1^s = \Omega_1$ or Ω_2. Analogously, from Lemma 3.1(4) s normalizes D and hence fixes the sets $\{\Omega_{00}, \Omega_{01}\}$ and $\{\Omega_{10}, \Omega_{11}, \Omega_{20}, \Omega_{21}\}$ of D-orbits setwise. But s does not normalize $\mathbf{S}_4$, so the equation $\Delta_0^s = \Delta_1$ may be not true even if s is of the second type.

The next lemma is related to the type of the involutions in $\Pi_G(R, T)$.

Lemma 3.2. *Let $s \in \Pi_G(R, T)$. Then the following hold.*

(1) *The element cs has order 4, and has no fixed point in Ω, and contains no transposition on Ω. In fact, cs is a product of 12 disjoint 4-cycles;*
(2) *If s fixes some Ω_i setwise, then s fixes each of Ω_{i0} and Ω_{i1} setwise, and further s^{Ω_i} is a product of 6 disjoint transpositions on Ω_i. In particular, by our assumption, so is s^{Ω_0};*
(3) *The involution s does not interchange Ω_{i0} and Ω_{i1} for any $i \in \{0, 1, 2\}$;*
(4) *If s fixes Ω_1 or Ω_2 setwise, then s fixes every Ω_{ij} setwise and hence s is of the first type;*
(5) *If s interchanges Ω_{10} and Ω_{20}, then s is of the first type.*

Proof. (1) Since $(cs)^2 = c(scs) = c(a^2c) = a^2 \neq 1$ and $(cs)^4 = (a^2)^2 = 1$, we find $o(cs) = 4$. In particular, $(cs)^2 = a^2$ has no fixed point in Ω so that cs has no fixed point and contains no transposition. Hence cs with order 4 is a product of 12 disjoint 4-cycles.

(2) Since s fixes Ω_i, then $\Omega_{i0}^s = \Omega_{i0}$ or Ω_{i1}. Assume $\Omega_{i0}^s = \Omega_{i1}$. Then $\Omega_{i0}^{cs} = \Omega_{i0}$. As components of Ω_{i0}, $a' := a^{\Omega_{i0}}$, $b' := b^{\Omega_{i0}}$ and $s' := (cs)^{\Omega_{i0}}$ satisfy, by easily checking, the assumption of Lemma 2.3(3), that is, $\langle a' \rangle \rtimes \langle b' \rangle \cong \mathbf{D}_8$, $\langle a' \rangle^{s'} = \langle a' \rangle$ and $(b')^{s'} = a'b'$. So $s' = (cs)^{\Omega_{i0}}$ is an odd permutation on Ω_{i0}. By (1), cs with order 4 has no fixed point and contains no transposition, then $s' = (cs)^{\Omega_{i0}}$ is a product of 2 disjoint 4-cycles on Ω_{i0}, contradicting the oddness of it. Hence s fixes each of Ω_{i0} and Ω_{i1} setwise. In this case, using the way of dealing with $(cs)^{\Omega_{i0}}$ on Ω_{i0} above to deal with $s^{\Omega_{i0}}$ on Ω_{i0} and $s^{\Omega_{i1}}$ on Ω_{i1}, we finally have that $s^{\Omega_i} = s^{\Omega_{i0}} s^{\Omega_{i1}}$ is a product of 6 disjoint transpositions on Ω_i.

(3) and (4) hold by (2).

(5) Since s always fixes Ω_0 setwise, and by (2), we find s also fixes Ω_{00} setwise. It follows that s fixes Δ_0 setwise, and consequently s fixes Δ_1 setwise. So s is of the first type. □

Theorem 3.3. *Let $s \in \Pi_G(R, T)$. Then the following statements are equivalent.*

(1) $\langle T, s \rangle = A$;
(2) *The involution s is of the second type;*
(3) *The involution s interchanges Ω_{10} and Ω_{21}.*

Proof. (1)⇒(2): Since $\langle T, s \rangle = A$ is primitive on Ω but each Δ_j is obviously nonprimitive block of T, then s does not fix Δ_j anymore, namely s is of the second type.

(2)⇒(3): By Lemma 3.2(3), (4) and (5), s must interchange Ω_{10} and Ω_{21}.

(3)⇒(1): See Corollary 4.4 later. □

Theorem 3.3 shows that the Sabidussi coset graph $\overline{\Gamma} := \mathrm{Sab}(A, T, s)$ is a connected 5-arc transitive cubic graph if and only if the involution s is of the second type.

4. Finding the graphs

In this section, we will find out all connected 5-arc transitive cubic Cayley graphs for $\mathbf{A}_{47}$.

We first denote each coset $Ga \in \Omega = [A : G]$ by $\overline{a}$. Then $\Omega = \overline{T} := \{\overline{h} \mid h \in T\}$ and G is the point stabilizer of $\overline{1}$ in A. For any subgroup L of T and its left coset hL, the set $\overline{hL}$ is obviously an L-orbit in Ω. Thus, $\overline{R}$, $\overline{tR}$ and $\overline{t^2R}$ are also R-orbits. But $\overline{R}^s = GRs = GsR = GR = \overline{R}$, then s fixes $\overline{R}$ setwise. So we may let $\overline{R} = \Omega_0$. Without loss of generality, we may assume that $\Omega_i = \overline{t^i R}$ and $\Omega_{ij} = \overline{t^i c^j D}$ for $i \in \{0, 1, 2\}$, $j \in \{0, 1\}$.

By Theorem 3.3, we need only to investigate those $s \in \Pi_G(R, T)$ for which $\Omega_{10}^s = \Omega_{21}$. In this case, $\Omega_1^s = \Omega_2$ and hence $s^{\Omega_1 \cup \Omega_2}$ is a product of 16 disjoint transpositions on $\Omega_1 \cup \Omega_2$. By Lemma 3.2(2), s^{Ω_0} is a product of 6 disjoint transpositions on Ω_0, and then $s = s^{\Omega_0} s^{\Omega_1 \cup \Omega_2}$ is a product of 22 disjoint transpositions on Ω, and s has only 4 fixed-points all belong to $\Omega_0 = \overline{R}$.

To find out all these involutions, we will examine the permutations induced by them on Ω_0 and $\Omega_1 \cup \Omega_2$ respectively.

First, the action by s on Ω_0 is conjugation since for every $\overline{r} \in \overline{R} = \Omega_0$, $\overline{r}^s = Grs = Gss^{-1}rs = Gr^s = \overline{r^s}$. By Lemma 3.1 there is only one choice for s^{Ω_0}.

Secondly, in $\Omega_1 \cup \Omega_2$, since s forces $\overline{t} \in \overline{tD} = \Omega_{10}$ to be in $\Omega_{21} = \overline{t^2cD}$, there exists $d \in D$ such that $\overline{t}^s = \overline{t^2cd}$, or $Gts = Gt^2cd$. Immediately, for each $\overline{tr} \in \Omega_1 = \overline{tR}$, $\overline{tr}^s = Gtrs = Gtss^{-1}rs = (Gts)r^s = (Gt^2cd)r^s = \overline{t^2cdr^s} \in \overline{t^2R} = \Omega_2$. Thus there are 8 choices for $s^{\Omega_1 \cup \Omega_2}$ since $|D| = 8$.

Let $d_0 = 1$, $d_1 = ab$, $d_2 = a$, $d_3 = a^2b$, $d_4 = a^2$, $d_5 = a^3b$, $d_6 = a^3$ and $d_7 = b$ which make up D, then we have 8 involutions $s_0, s_1, \ldots, s_7$ to make $\overline{t}^{s_k} = \overline{t^2cd_k}$ and $\Omega_{10}^s = \Omega_{21}$ ($k = 0, 1, \ldots, 7$).

Accordingly, we have 8 Sabidussi coset graphs:

$$\overline{\Gamma}_k := \mathrm{Sab}(A, T, s_k),$$

where $\mathrm{Val}(\overline{\Gamma}_k) = |Ts_kT : T| = |T : T \cap T^{s_k}| = 3$.

Remark. Each $\overline{\Gamma}_k$ here may not be connected because we do not know if $\langle T, s_k \rangle = A$ yet.

From Theorem 3.3, we immediately have

Corollary 4.1. *Let $s \in \Pi_G(R, T)$. If the Sabidussi coset graph $\overline{\Gamma} := \mathrm{Sab}(A, T, s)$ is connected 5-arc transitive cubic, then $s \in \{s_0, s_1, \ldots, s_7\}$, and further, $\overline{\Gamma} \in \{\overline{\Gamma}_0, \overline{\Gamma}_1, \ldots, \overline{\Gamma}_7\}$.* $\square$

Due to Lemma 2.1(2) we also have 8 Cayley graphs of G:

$$\Gamma_k := \mathrm{Cay}(G, S_k) \cong \overline{\Gamma}_k,$$

where the Cayley subset $S_k = G \cap (Ts_kT)$, and $|S_k| = \mathrm{Val}(\Gamma_k) = \mathrm{Val}(\overline{\Gamma}_k) = 3$.

We will prove soon that S_0 is conjugate to S_1, S_2, S_3 and S_4 is conjugate to S_5, S_6, S_7. Moreover, we will prove that each S_k generates $G \cong \mathbf{A}_{47}$.

Clearly, the three-element set S_k contains s_k. To find the other two elements of S_k, we denote $u_k := t^2cd_ks_kt^2 \in Ts_kT$. Note that $Gts_k = Gt^2cd_k$, then $u_k = t^2cd_ks_kt^2 \in G$ and hence $u_k \in S_k = G \cap (Ts_kT)$. We claim that u_k is not an involution on Ω.

Otherwise, if u_k is an involution, then $1 = u_k^2 = t^2cd_ks_ktcd_ks_kt^2 = t^2d_kcs_kctd_ks_kt^2 = t^2d_ks_kc^{s_k}ctd_ks_kt^2 = t^2d_ks_ka^2td_ks_kt^2 = t^{-1}(d_ks_ka^2td_ks_kt)t = d_ks_k(a^2t)d_ks_kt = d_ks_k(tb)d_ks_kt$, that is, $Gd_k^{-1} = Gd_k^{-1}(d_ks_ktbd_ks_kt) = Gtbd_ks_kt$.

For $k = 0$, $Gtbd_0s_0t = Gtbs_0t = Gts_0abt = (Gt^2cd_0)abt = Gt^2abct \neq G = Gd_0^{-1}$, a contradiction.

For $k = 4$, $Gtbd_4s_4t = Gtba^2s_4t = Gts_4a^3bt = (Gt^2cd_4)a^3bt = Gt^2ca^2a^3bt = Gt^2abct \neq Ga^2 = Gd_4^{-1}$, a contradiction.

Analogously, for $k = 1, 2, 3, 5, 6$ or 7, we also obtain a contradiction (the details are omitted). Thus, u_k is not an involution so that $u_k^{-1} \in S_k \setminus \{s_k, u_k\}$, and hence

$$S_k = \{s_k, u_k, u_k^{-1}\}.$$

Lemma 4.2. *Assume $\sigma \in G$ such that $a^\sigma = a$, $b^\sigma = a^2b$, $c^\sigma = c$, and $t^\sigma = t^2a^2$. Then $S_0^{\sigma^k} = S_k$ and $S_4^{\sigma^k} = S_{k+4}$ for $k \in \{1, 2, 3\}$.*

Proof. We easily have

$$(t^2)^\sigma = tb, \quad t^{\sigma^2} = ta^2b, \quad (t^2)^{\sigma^2} = t^2b, \quad t^{\sigma^3} = t^2a^2b, \quad (t^2)^{\sigma^3} = ta^2, \text{ and } t^{\sigma^4} = t.$$

Consequently, $\sigma^4 \in G$ centralizes T, and hence $\sigma^4 = 1$.

Since $\sigma \in G$, then $(Gt)\sigma = G\sigma t^\sigma = Gt^2a^2$. Analogously, $(Gt^2)\sigma = Gtb$, $(Gt)\sigma^2 = Gta^2b$, $(Gt^2)\sigma^2 = Gt^2b$, $(Gt)\sigma^3 = Gt^2a^2b$ and $(Gt^2)\sigma^3 = Gta^2$.

First, we will prove that for each $k \in \{1, 2, 3\}$, $s_k = s_0^{\sigma^k}$ or $s_ks_0^{\sigma^k} = 1$. Equivalently, we manage to prove $s_ks_0^{\sigma^k}$ fixing $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ pointwise. We deal with these three cases separately.

Case 1: $\overline{r} \in \overline{R} = \Omega_0$.

We claim that $s_ks_0^{\sigma^k}$ commutes with each $r \in R$.

In fact, let $r \in b^jK$ ($j = 0$ or 1). Then $r^\sigma = a^{2j}r$, $r^{\sigma^2} = r$ and $r^{\sigma^3} = a^{2j}r$. That is $r^{\sigma^{\pm k}} = a^{2j(2-k)}r$. Note that $r^{s_k} = r^{s_0}$ and $r^{s_k} \in b^jK$ by Lemma 3.1(4). We have $rs_ks_0^{\sigma^k} = rs_k\sigma^{-k}s_0\sigma^k = s_kr^{s_k}\sigma^{-k}s_0\sigma^k = s_k\sigma^{-k}a^{2j(2-k)}r^{s_k}s_0\sigma^k = s_k\sigma^{-k}a^{2j(2-k)}r^{s_0}s_0\sigma^k = s_k\sigma^{-k}a^{2j(2-k)}s_0r\sigma^k = s_k\sigma^{-k}s_0a^{2j(2-k)}r\sigma^k = s_k\sigma^{-k}s_0\sigma^kr = s_ks_0^{\sigma^k}r$. Thus $s_ks_0^{\sigma^k}$ commutes with each $r \in R$, and hence $s_ks_0^{\sigma^k}$ fixes $\overline{R} = \Omega_0$ pointwise.

Case 2: $\overline{tr} \in \overline{tR} = \Omega_1$.

Let $\overline{h}_k := Gtr(s_k\sigma^{-k}s_0\sigma^k) = Gt(s_k\sigma^{-k}s_0\sigma^k)r = (Gts_k)\sigma^{-k}s_0\sigma^kr = Gt^2cd_k\sigma^{-k}s_0\sigma^kr$, then

(1) $\overline{h}_1 = Gt^2c(ab)\sigma^{-1}s_0\sigma r = Gt^2(abc)\sigma^3s_0\sigma r = Gt^2\sigma^3(a^3bc)s_0\sigma r = (Gta^2)a^3bcs_0\sigma r = Gt(abc)s_0\sigma r = Gts_0(ab)^{s_0}c^{s_0}\sigma r = Gts_0b(a^2c)\sigma r = (Gt^2c)ba^2c\sigma r = Gt^2(ba^2)\sigma r = Gt^2(\sigma b)r = (Gt^2\sigma)br = G(tb)br = Gtr$;

(2) $\overline{h}_2 = Gt^2ca\sigma^2s_0\sigma^2r = Gt^2\sigma^2cas_0\sigma^2r = (Gt^2b)cas_0\sigma^2r = Gt^2c(ba)s_0\sigma^2r = Gt^2c(a^3b)s_0\sigma^2r = (Gt^2c)s_0(a^2b)\sigma^2r = (Gt)(a^2b)\sigma^2r = Gt\sigma^2(a^2b)r = (Gta^2b)(a^2b)r = Gtr$;

(3) $\overline{h}_3 = Gt^2c(a^2b)\sigma s_0\sigma^3r = Gt^2\sigma cbs_0\sigma^3r = (Gtb)cbs_0\sigma^3r = Gtcs_0\sigma^3r = (Gt)s_0(a^2c)\sigma^3r = (Gt^2c)a^2c\sigma^3r = Gt^2\sigma^3a^2r = (Gta^2)a^2r = Gtr$.

Thus $s_ks_0^{\sigma^k}$ fixes $\overline{tR} = \Omega_1$ pointwise.

Case 3: $\overline{t^2r} \in \overline{t^2R} = \Omega_2$.

Let $\overline{h}'_k := Gt^2r(s_k\sigma^{-k}s_0\sigma^k) = (Gt^2cd_k)(cd_k^{-1}r)(s_k\sigma^{-k}s_0\sigma^k) = (Gts_k)(s_k\sigma^{-k}s_0\sigma^k)(cd_k^{-1}r) = (Gt^2cd_ks_k)\sigma^{-k}s_0\sigma^k(cd_k^{-1}r) = Gt\sigma^{-k}s_0\sigma^kcd_k^{-1}r$, then

(1) $\overline{h}'_1 = Gt\sigma^3s_0\sigma(cab)r = (Gt^2a^2b)s_0\sigma(cab)r = Gt^2c(ca^2)bs_0\sigma(cab)r = Gt^2cs_0c(ab)\sigma(cab)r = (Gt^2cs_0)(abc)\sigma(cab)r = (Gt)\sigma(a^3bc)(cab)r = (Gt^2a^2)a^2r = Gt^2r$;

(2) $\overline{h}'_2 = Gt\sigma^2s_0\sigma^2(ca^3)r = (Gta^2b)s_0\sigma^2(ca^3)r = Gts_0a^3b\sigma^2(ca^3)r = (Gt^2c)\sigma^2a^3b(ca^3)r = Gt^2\sigma^2c(bcr) = (Gt^2b)(br) = Gt^2r$;

(3) $\overline{h}'_3 = Gt\sigma s_0\sigma^3(ca^2b)r = (Gt^2a^2)s_0\sigma^3(ca^2b)r = (Gt^2c)(ca^2s_0)\sigma^3(ca^2b)r = (Gts_0)(s_0c)\sigma^3(ca^2b)r = Gt(c\sigma^3)(ca^2b)r = Gt\sigma^3c(ca^2b)r = (Gt^2a^2b)a^2br = Gt^2r$.

Thus $s_k s_0^{\sigma^k}$ fixes $\overline{t^2R} = \Omega_2$ pointwise.

Therefore, $s_k s_0^{\sigma^k} = 1$ or $s_0^{\sigma^k} = s_k$.

Secondly, depending on $u_k = t^2cd_ks_kt^2$, we have

(1) $u_0^{\sigma} = (t^2cd_0s_0t^2)^{\sigma} = (t^2cs_0t^2)^{\sigma} = (tb)cs_1(tb) = t(bcs_1)tb = ts_1(ab)(a^2c)(a^2t) = ts_1(ab)ct = ts_1^{-1}(ab)^{-1}c^{-1}t = (t^2cd_1s_1t^2)^{-1} = u_1^{-1}$;

(2) $u_0^{\sigma^2} = (t^2cs_0t^2)^{\sigma^2} = (t^2b)cs_2(t^2b) = t^2bcs_2(a^2bt^2) = t^2bc(a^3bs_2)t^2 = t^2cas_2t^2 = u_2$;

(3) $u_0^{\sigma^3} = (t^2cs_0t^2)^{\sigma^3} = (ta^2)cs_3(ta^2) = t(a^2cs_3)ta^2 = t(s_3c)(a^2bt) = ts_3(a^2b)ct = (t^2cd_3s_3t^2)^{-1} = u_3^{-1}$.

To sum up, $S_0^{\sigma^k} = S_k$ ($k = 1, 2, 3$).

Similarly, $S_4^{\sigma^k} = S_{k+4}$ (the detailed proof is omitted). □

Theorem 4.3. *With the above notation, we have*

(1) *The graph Γ_0 is isomorphic to Γ_1, Γ_2, Γ_3, and Γ_4 is isomorphic to Γ_5, Γ_6, Γ_7;*

(2) *Each graph Γ_k ($k = 0, 1, \ldots, 7$) is a connected 5-arc transitive cubic graph and its full automorphism group is isomorphic to A;*

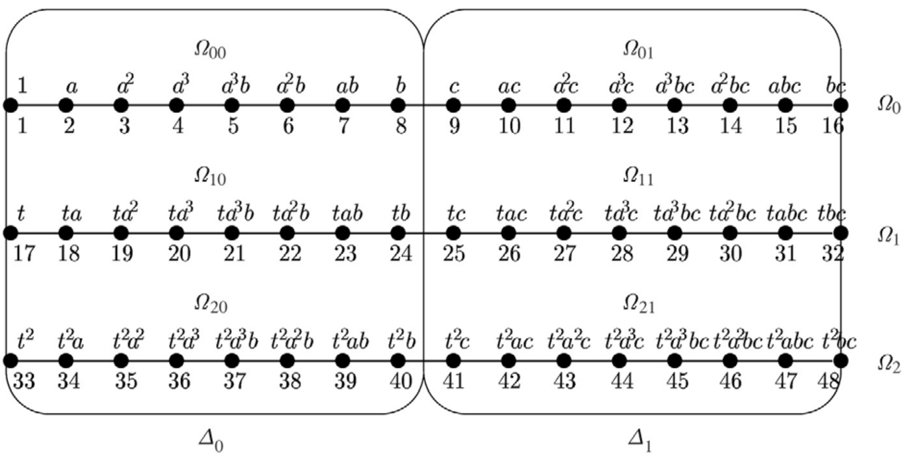
(3) *Each set S_k ($k = 0, 1, \ldots, 7$) is a CI-subset of G;*

(4) *The graphs Γ_0 and Γ_4 are not isomorphic to each other.*

Before our proof we need to represent the permutations of A in another way.

For each element of $\Omega = [A : G]$ which is a right coset of G with the unique representative from T, we will rearrange these representatives. First, we denote all elements of R in turn by $r_1 := 1$, $r_2 = a$, $r_3 := a^2$, $r_4 := a^3$, $r_5 := a^3b$, $r_6 := a^2b$, $r_7 := ab$, $r_8 := b$, $r_9 := c$, $r_{10} := ac$, $r_{11} := a^2c$, $r_{12} := a^3c$, $r_{13} := a^3bc$, $r_{14} := a^2bc$, $r_{15} := abc$, and $r_{16} := bc$.

Then, from $T = R \cup tR \cup t^2R$, we denote all other elements of T by $r_{16i+j} := t^ir_j$ where $i \in \{1, 2\}$ and $j \in \{1, 2, \ldots, 16\}$. Thus we may depict $\Omega = \{\overline{r_1}, \overline{r_2}, \ldots, \overline{r_{48}}\}$ as follows:



Now, acting by its right multiplication, each element of A may simply be denoted as the permutation on $\{1, 2, \ldots, 48\}$, such as

$$a = (1, 2, 3, 4)(5, 6, 7, 8)(9, 10, 11, 12)(13, 14, 15, 16)(17, 18, 19, 20)$$

$$(21, 22, 23, 24)(25, 26, 27, 28)(29, 30, 31, 32)(33, 34, 35, 36)(37, 38, 39, 40)$$
$$(41, 42, 43, 44)(45, 46, 47, 48),$$
$$b = (1, 8)(2, 7)(3, 6)(4, 5)(9, 16)(10, 15)(11, 14)(12, 13)(17, 24)(18, 23)(19, 22)$$
$$(20, 21)(25, 32)(26, 31)(27, 30)(28, 29)(33, 40)(34, 39)(35, 38)(36, 37)(41, 48)$$
$$(42, 47)(43, 46)(44, 45),$$
$$c = (1, 9)(2, 10)(3, 11)(4, 12)(5, 13)(6, 14)(7, 15)(8, 16)(17, 25)(18, 26)(19, 27)$$
$$(20, 28)(21, 29)(22, 30)(23, 31)(24, 32)(33, 41)(34, 42)(35, 43)(36, 44)(37, 45)$$
$$(38, 46)(39, 47)(40, 48).$$

For the t, we easily have that $bt = ta^2b$ and $ct = tc$. In addition, since $a^t = a^{-1}t^{-1}$, then $at^2 = ta^3$, and hence $at = (at^2)t^2 = (ta^3)t^2 = ta^2(at^2) = t^2t^{-1}a^2ta^3 = t^2ba^3 = t^2ab$. So we also obtain the following permutation for t:

$$t = (1, 17, 33)(2, 39, 20)(3, 24, 38)(4, 34, 23)(5, 37, 21)(6, 19, 40)(7, 36, 18)(8, 22, 35)$$
$$(9, 25, 41)(10, 47, 28)(11, 32, 46)(12, 42, 31)(13, 45, 29)(14, 27, 48)(15, 44, 26)$$
$$(16, 30, 43).$$

Proof of Theorem 4.3. (1) From Lemma 4.2 we need only to find out a $\sigma \in G$ such that σ satisfies the assumptions there. Take

$$\sigma := (5, 7)(6, 8)(13, 15)(14, 16)(17, 35, 22, 38)(18, 36, 23, 39)(19, 33, 24, 40)$$
$$(20, 34, 21, 37)(25, 43, 30, 46)(26, 44, 31, 47)(27, 41, 32, 48)(28, 42, 29, 45)$$
$$\in A_1 = G.$$

It is easy to check that $a^\sigma = a$, $b^\sigma = a^2b$, $c^\sigma = c$, $t^\sigma = t^2a^2$, and hence (1) holds.

(2) We will prove that $\langle T, s_l\rangle = A$ for $l \in \{0, 4\}$. This will imply that $\overline{\Gamma}_l$ is a connected 5-arc transitive cubic graph for any l, and so are all other $\overline{\Gamma}_k$ from (1) and also are $\Gamma_k \cong \overline{\Gamma}_k$.

To prove the above assertion, we first determine the permutations of s_0, s_4, u_0 and u_4 on $\{1, 2, \ldots, 48\}$.

For every $\overline{r}$ in Ω_0, since $\overline{r}^{\,s} = (Gr)s = Gr^s = \overline{r^s}$ and $a^s = a^{-1}$, $b^s = ab$, $c^s = a^2c$, we easily have that $s_k^{\Omega_0} = (2, 4)(5, 6)(7, 8)(9, 11)(13, 16)(14, 15)$, being independent of k.

For every $\overline{tr} \in \Omega_1$, since $\overline{tr}^{\,s_k} = (Gtr)s_k = (Gts_k)r^{s_k} = (Gt^2cd_k)r^{s_k} = \overline{t^2cd_kr^{s_k}} \in \Omega_2$, we have the following.

Since $d_0 = 1$,

$$s_0 = (2, 4)(5, 6)(7, 8)(9, 11)(13, 16)(14, 15)(17, 41)(18, 44)(19, 43)(20, 42)(21, 46)$$
$$(22, 45)(23, 48)(24, 47)(25, 35)(26, 34)(27, 33)(28, 36)(29, 40)(30, 39)$$
$$(31, 38)(32, 37);$$

and since $d_4 = a^2c$,

$$s_4 = (2, 4)(5, 6)(7, 8)(9, 11)(13, 16)(14, 15)(17, 43)(18, 42)(19, 41)(20, 44)(21, 48)$$
$$(22, 47)(23, 46)(24, 45)(25, 33)(26, 36)(27, 35)(28, 34)(29, 38)(30, 37)$$
$$(31, 40)(32, 39).$$

Accordingly,

$$u_0 := t^2cs_0t^2 = (2, 7, 4, 24, 41, 25, 33, 22, 29, 11, 42, 20, 3, 37, 43, 13, 32, 38, 5, 19, 26,$$

47, 23, 39, 28, 16, 9, 14, 45, 40, 17, 46, 10, 12, 27, 21, 8, 6, 34, 31, 44, 35)
(18, 36, 48, 30),

$u_4 := t^2ca^2s_4t^2 = (2, 7, 4, 24, 46, 10, 12, 27, 18, 36, 43, 13, 32, 33, 22, 26, 47, 20, 3, 37, 48, 30, 21, 8, 6, 34, 28, 16, 9, 14, 45, 35)(5, 19, 29, 11, 42, 23, 39, 31, 44, 40, 17, 41, 25, 38)$.

Denote $H_l := \langle T, s_l \rangle$ for $l \in \{0, 4\}$. We claim that H_l is 2-transitive on Ω.

In fact, H_l is obviously transitive on Ω. Examining all orbits of $\langle u_l \rangle$ in $\Omega \setminus \{\overline{r_1}\}$ as follows, we easily have that $\langle s_l, u_l \rangle$ is transitive on $\Omega \setminus \{\overline{r_1}\}$ and so is $H_l \cap G$ since $s_l, u_l \in H_l \cap G$.

3 orbits of u_0:

$\{\overline{r_2}, \overline{r_3}, \overline{r_4}, \overline{r_5}, \overline{r_6}, \overline{r_7}, \overline{r_8}, \overline{r_9}, \overline{r_{10}}, \overline{r_{11}}, \overline{r_{12}}, \overline{r_{13}}, \overline{r_{14}}, \overline{r_{16}}, \overline{r_{17}}, \overline{r_{19}}, \overline{r_{20}}, \overline{r_{21}}, \overline{r_{22}}, \overline{r_{23}}, \overline{r_{24}}, \overline{r_{25}}, \overline{r_{26}}, \overline{r_{27}}, \overline{r_{28}}, \overline{r_{29}}, \overline{r_{31}}, \overline{r_{32}}, \overline{r_{33}}, \overline{r_{34}}, \overline{r_{35}}, \overline{r_{37}}, \overline{r_{38}}, \overline{r_{39}}, \overline{r_{40}}, \overline{r_{41}}, \overline{r_{42}}, \overline{r_{43}}, \overline{r_{44}}, \overline{r_{45}}, \overline{r_{46}}, \overline{r_{47}}\}$,

$\{\overline{r_{15}}\}$,

$\{\overline{r_{18}}, \overline{r_{30}}, \overline{r_{36}}, \overline{r_{48}}\}$;

3 orbits of u_4:

$\{\overline{r_2}, \overline{r_3}, \overline{r_4}, \overline{r_6}, \overline{r_7}, \overline{r_8}, \overline{r_9}, \overline{r_{10}}, \overline{r_{12}}, \overline{r_{13}}, \overline{r_{14}}, \overline{r_{16}}, \overline{r_{18}}, \overline{r_{20}}, \overline{r_{21}}, \overline{r_{22}}, \overline{r_{24}}, \overline{r_{26}}, \overline{r_{27}}, \overline{r_{28}}, \overline{r_{30}}, \overline{r_{32}}, \overline{r_{33}}, \overline{r_{34}}, \overline{r_{35}}, \overline{r_{36}}, \overline{r_{37}}, \overline{r_{43}}, \overline{r_{45}}, \overline{r_{46}}, \overline{r_{47}}, \overline{r_{48}}\}$,

$\{\overline{r_5}, \overline{r_{11}}, \overline{r_{17}}, \overline{r_{19}}, \overline{r_{23}}, \overline{r_{25}}, \overline{r_{29}}, \overline{r_{31}}, \overline{r_{38}}, \overline{r_{39}}, \overline{r_{40}}, \overline{r_{41}}, \overline{r_{42}}, \overline{r_{44}}\}$,

$\{\overline{r_{15}}\}$.

But $H_l \cap G$ is exactly the point stabilizer of $\overline{r_1}$ in H_l, then each H_l is 2-transitive on Ω, and hence is primitive. Besides, by direct checking we have

$u_0^{10} = (2, 42, 26, 17, 44, 29, 5, 45, 34, 33, 32, 9, 8, 41, 43, 28, 27, 4, 3, 23, 10)(6, 25, 13, 16, 21, 24, 37, 39, 12, 7, 20, 47, 46, 35, 11, 19, 40, 31, 22, 38, 14)(18, 48)(30, 36)$ with order 42,

$s_1u_0^{10} = (2, 3, 23, 18, 29, 31, 14, 15, 6, 45, 38, 22, 34, 17, 43, 40, 5, 25, 11, 8, 20, 26, 33, 4, 42, 47, 37, 9, 19, 28, 30, 12, 7, 41, 44, 48, 10)(13, 21, 35)(24, 46)(27, 32, 39, 36)$ with order 444,

$(s_0u_0^{10})^{148} = (13, 21, 35) \in H_0$, and

$s_4u_4 = (2, 24, 35, 18, 23, 10, 12, 27)(3, 37, 21, 30, 48, 8, 4, 7, 6, 19, 25, 22, 20, 40, 44)(5, 34, 16, 32, 31, 17, 13, 9, 42, 36, 47, 26, 43, 41, 29)(11, 14, 15, 45, 46, 39, 33, 38)$ with order 120,

$(s_4u_4)^3 = (2, 18, 12, 24, 23, 27, 35, 10)(3, 30, 4, 19, 20)(5, 32, 13, 36, 43)(6, 22, 44, 21, 8)(7, 25, 40, 37, 48)(9, 47, 41, 34, 31)(11, 45, 33, 14, 46, 38, 15, 39)(16, 17, 42, 26, 29)$ with order 40,

$s_4u_4^{-1} = (2, 7, 21, 37, 48, 30, 3, 20, 31, 44, 47, 33, 41, 5, 8)(4, 35, 12, 10, 46, 42, 27, 45)(6, 38, 19, 17, 36, 22, 26, 18, 11, 16, 43, 40, 39, 13, 28)(9, 29, 25, 32, 23, 24, 14, 15)$ with order 120,

$(s_4u_4^{-1})^2 = (2, 21, 48, 3, 31, 47, 41, 8, 7, 37, 30, 20, 44, 33, 5)(4, 12, 46, 27)(6, 19, 36, 26, 11, 43, 39, 28, 38, 17, 22, 18, 16, 40, 13)(9, 25, 23, 14)(10, 42, 45, 35)$

(15, 29, 32, 24) with order 60,

$(s_4u_4)^3(s_4u_4^{-1})^2 = (2, 16, 22, 33, 9, 41, 34, 47, 8, 19, 44, 48, 37, 3, 20, 31, 25, 13, 26, 32, 6, 18, 46, 17, 45, 5, 24, 14, 27, 10, 21, 7, 23, 4, 36, 39, 43)(11, 35, 42)$
$(12, 15, 28, 38, 29, 40, 30)$ with order 777,

$((s_4u_4)^3(s_4u_4^{-1})^2)^{259} = (11, 35, 42) \in H_4.$

It follows from Jordan's theorem that $H_l = \langle T, s_l\rangle = A$ for $l = 0, 4$.

(3) Since $A = \mathrm{Aut}(\Gamma_k) \cong \mathbf{A}_{48}$, then we need only to prove that for any $\sigma \in \mathrm{Sym}(G)$ satisfying $G^\sigma \le A$, G^σ has to be conjugate with G in A. In this case, S_k is a CI-subset by Babai's criterion (see [5]).

In fact, we see that G^σ has at most one fixed point on Ω, since $G^\sigma \cong \mathbf{A}_{47}$ contains a 47-cycle. We claim that G^σ has exactly one fixed point. If not, then G^σ is transitive on Ω, and the point stabilizer is a maximal subgroup since its index in G^σ is 48. Consequently G^σ is primitive on Ω. But G^σ has 43-cycles in Ω, and $|\Omega| - 43 \ge 3$, it follows from Jordan's theorem that $G^\sigma \ge \mathrm{Alt}(\Omega) \cong A$, a contradiction.

Let $\overline{r} \in \Omega$ ($r \in T$) be this fixed point of G^σ. It is easy to check that G^σ is the point stabilizer of $\overline{r}$ in A. Note that G is also the point stabilizer of $\overline{1}$ and hence G^σ is conjugate with G in A.

(4) By direct calculation we have $o(u_0) = o(u_0^{-1}) = 84$ for $u_0 \in S_0$, and $o(u_4) = o(u_4^{-1}) = 224$ for $u_4 \in S_4$. Therefore, for any $\alpha \in \mathrm{Aut}(G)$, $S_0^\alpha \ne S_4$. By (3), Γ_0 is not isomorphic to Γ_4. □

By the proof of Theorem 4.3, we have directly a corollary below and hence complete the proof of Theorem 3.3.

Corollary 4.4. *Let $s \in \Pi_G(R, T)$. Assume that s interchanges Ω_{10} and Ω_{21}. Then the Sabidussi coset graph $\overline{\Gamma} := \mathrm{Sab}(A, T, s)$ is isomorphic to one of Γ_0 and Γ_4. In particular, $\langle T, s\rangle = A$.* □

By the proof of Theorem 4.3(3), we have the following corollary.

Corollary 4.5. *In the alternating group $\mathbf{A}_{48}$, all subgroups which are isomorphic to $\mathbf{A}_{47}$ are conjugate.* □

The following lemma shows that no matter how to choose the involution s, the Sabidussi coset graph $\mathrm{Sab}(A, T, s_k)$ is, up to isomorphism, independent of T.

Lemma 4.6. *Any two mutually isomorphic regular subgroups of $\mathbf{S}_n$ are conjugate in $\mathbf{S}_n$.*

Proof. If X and Y are regular permutation groups on $\Omega = \{1, 2, \ldots, n\}$, and $\alpha : \Omega \to X$ and $\beta : \Omega \to Y$ are bijections with the property that $1^{\alpha(i)} = 1^{\beta(i)}$ for $1 \le i \le n$, and $\theta : X \to Y$ is isomorphism, then $\alpha\theta\beta^{-1} : \Omega \to \Omega$ is a permutation in $\mathbf{S}_n$ that conjugates X to Y. □

5. The main result

Now, we are able to give a complete classification of the nonnormal connected s-arc transitive cubic Cayley graphs of finite nonabelian simple groups.

Theorem 5.1. *Let G be a finite nonabelian simple group and $\Gamma := \mathrm{Cay}(G, S)$ a nonnormal connected s-arc transitive cubic Cayley graph for G. Then Γ is isomorphic to one of Γ_0 and Γ_4.*

Proof. We know from the paper [1] that G here must be isomorphic to $\mathbf{A}_{47}$, and the full automorphism group $A := \mathrm{Aut}(\Gamma)$ of the Cayley graph Γ is isomorphic to $\mathbf{A}_{48}$, and that its point stabilizer A_1 is isomorphic to $\mathbf{S}_4 \times \mathbb{Z}_2$ which is the complement of G in A, that is $A = GA_1$. Thus, by Lemma 2.1(1) there exists an involution s in G such that $\Gamma \cong \overline{\Gamma} := \mathrm{Sab}(A, A_1, s)$ with $|A_1 s A_1 : A_1| = |A_1 : A_1 \cap A_1^s| = 3$ and $\langle A_1, s\rangle = A$.

According to Corollary 4.5, we may assume the pair of simple groups (A, G) to be the same as in Theorem 4.3. In this case, A has two subgroups A_1 and T (T is defined as in Theorem 4.3), both are complements of G and they are regular on $\Omega := [A : G]$. In light of Lemma 4.6, there exists $\sigma \in \mathrm{Sym}(\Omega)$ such that $A_1^\sigma = T$, and then $A = A^\sigma = (GA_1)^\sigma = G^\sigma T$. It follows by Corollary 4.5 that $G^\sigma \in A$ is conjugate to G in A, and we may take $h \in T$ such that $G^{\sigma h} = G$.

Denote $\alpha = \sigma h \in \mathrm{Sym}(\Omega)$. Then $A = A^\alpha = G^\alpha A_1^\alpha = GT$, $|A_1^\alpha s^\alpha A_1^\alpha : A_1^\alpha| = |T s^\alpha T : T| = 3$ and $\langle A_1^\alpha, s^\alpha\rangle = \langle T, s^\alpha\rangle = A$. This implies that $s^\alpha \in \Pi_G(R, T)$. From Corollary 4.1, we have $s^\alpha \in \{s_0, s_1, \ldots, s_7\}$ and $\overline{\Gamma}' := \mathrm{Sab}(A, T, s^\alpha) \in \{\overline{\Gamma}_0, \overline{\Gamma}_1, \ldots, \overline{\Gamma}_7\}$. So we know from Theorem 4.3 that $\overline{\Gamma}'$ is isomorphic to Γ_0 or Γ_4, and hence we complete the proof since $\overline{\Gamma} = \mathrm{Sab}(A, A_1, s)$ is obviously isomorphic to $\overline{\Gamma}' = \mathrm{Sab}(A, T, s^\alpha)$. □

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