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Some complexity results on transition systems and elementary net systems

Kunihiko Hiraishi

School of Information Science, Japan Advanced Institute of Science and Technology, 15 Asahi-dai Tatsunokuchi Nomi-gun, Ishikawa 923-12, Japan

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Abstract

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There is a strong relationship between transition systems and elementary net systems. We consider the problem of finding an elementary net system corresponding to a given transition system, where the correspondence is defined as an isomorphism between a transition system and the state transition diagram of an EN system. The problem is decomposed into two related problems, and we show that these problems are NP-complete. However, this result does not mean that the original problem is NP-complete. The problem to construct a labeled EN system corresponding to a given transition system is also considered.

1. Introduction

Transition systems are models for representing global behavior of discrete event systems. However, Petri nets are suitable for representing local behavior. Places (S-elements) in a Petri net represent local states in a system, and global states are represented as markings, that is, configuration of tokens in each place. Each occurrence of transitions depends only on the connected places, and does not affect other places directly.

Elementary net systems (EN systems) as a primitive class of Petri nets. In EN systems, each place contains at most one token. There is strong relationship between

Correspondence to: K. Hiraishi, School of Information Science, Japan Advanced Institute of Science and Technology, 15 Asahi-dai Tatsunokuchi Nomi-gun, Ishikawa 923-12, Japan. Email: hira@jaist.ac.jp.

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transition systems and elementary net systems. Nielsen, Rosenberg et al. found a class of transition systems corresponding to elementary net systems [1, 2, 5]. This class is called elementary transition systems. This correspondence is defined as an isomorphism between a transition system and the state transition diagram of an EN system. They showed a necessary and sufficient condition for a transition system to be in this class. In this paper, we will show some complexity results on the problem to check this condition. This problem can be considered as localization of global states. Especially when the system allows concurrent behavior, we can describe the state space with fewer elements than in transition systems. (The converse of this problem is known as "state explosion" [6].)

In Section 3, we will consider this problem in a different setting, and after that we will discuss relationship between this setting and that of [5]. The problem is decomposed into two related problems, and we will show that these problems are NP-complete. However, this result does not mean that the original problem is NP-complete.

Not every transition system has a corresponding EN system. In Section 4, we will consider labeled EN systems. The behavior of a transition system can always be simulated by some labeled EN system. We will show a simple algorithm to construct a labeled EN system whose transition diagram is isomorphic to a given transition system.

2. Preliminaries

2.1. Transition systems

Definition 2.1. *A transition system* is a quadruple $TS=(S, E, \tau, s_0)$, where S is a nonempty set of *states*, E is a set of *events*, $\tau \subset S \times E \times S$ is *the transition relation*, and $s_0 \in S$ is the initial state.

Let $TS=(S, E, \tau, s_0)$ be a transition system. TS is called *finite* if S is a finite set. TS is called *deterministic* if $\forall (s, e_1, s_1), (s, e_2, s_2) \in \tau$: $[e_1 = e_2 \Rightarrow s_1 = s_2]$. We will write $(s, e\rangle \in \tau$ to indicate $\exists s' \in S$: $(s, e, s') \in \tau$.

We assume that the following for transition systems which we will consider in this paper.

Assumption. Every transition system $TS=(S, E, \tau, s_0)$ satisfies the following axioms:

(A1) $\forall e \in E\ \exists (s, e, s') \in \tau$;

(A2) $\forall s \in S - \{s_0\}\ \exists e^0, e^1, \ldots e^{n-1} \in E$ and $\exists s^1, s^2, \ldots, s^n \in S$ such that $s^1 = s_0$, $s^n = s$ and $(s^i, e^i, s^{i+1}) \in \tau$ for $0 \leqslant i < n-1$.

We define the following morphism between transition systems, which preserves global behavior of the system.

Definition 2.2. Let $TS_i=(S_i, E, \tau_i, s_0^i)$ $(i=1,2)$ be transition systems that have the same set of events.

(1) *An L-morphism* from TS_1 to TS_2 is a mapping $g: S_1 \to S_2$ that satisfies the following:

(i) $g(s_0^1)=s_0^2$;

(ii) $\forall (s,e,s') \in \tau_1: (g(s), e, g(s')) \in \tau_2$.

(2) *An L-isomorphism* is a bijection $g: S_1 \to S_2$ such that (i) g is an L-morphism from TS_1 to TS_2 and (ii) g^{-1} is an L-morphism from TS_2 to TS_1.

If there exists an L-isomorphism between TS_1 and TS_2, then the language generated by TS_1 is the same as that by TS_2, i.e., each L-isomorphism preserves their languages.

2.2. *Elementary net systems*

Definition 2.3. *A net* is a triple $N=(B, E, F)$, where B is a set of *conditions*, E is a set of events $(B \cap E = \emptyset)$, and $F \subset (B \times E) \cup (E \times B)$ is *the flow relation*.

As usual for each $x \in B \cup E$, let ${}^\bullet x = \{y \mid (y,x) \in F\}$ and $x^\bullet = \{y \mid (x,y) \in F\}$. A net N is called *simple* iff $\forall x, y \in B \cup E$: $[{}^\bullet x = {}^\bullet y \wedge x^\bullet = y^\bullet \Rightarrow x = y]$. An element $x \in B \cup E$ is called *isolated* if there is no $y \in B \cup E$ such that $(x,y) \in F \vee (y,x) \in F$.

Definition 2.4. *An EN system* is a quadruple $M=(B, E, F; c_0)$, where $N=(B, E, F)$ is a net called *underlying net* of M, and $c_0 \subset B$.[1]

Let $M=(B,E,F;c_0)$ be an EN system and let $N=(B,E,F)$ be its underlying net. Then $\tau_N \subset 2^B \times E \times 2^B$ is the transition relation given by

$$\tau_N = \{(c,e,c') \mid c - c' = {}^\bullet e \wedge c' - c = e^\bullet\}.$$

The state space of M is represented by a transition system $H(M)=(C_M, E_M, \tau_M, c_0)$, where

(i) C_M is the smallest subset of 2^B containing c_0 such that $[c \in C_M \wedge (c,e,c') \in \tau_N] \Rightarrow c' \in C_M$;

(ii) $\tau_M = \{(c,e,c') \in C_M \times E \times C_M \mid (c,e,c') \in \tau_N\}$;

(iii) $E_M = \{e \mid \exists (c,e,c') \in \tau_M\}$.

We remark that events on a self-loop can never occur in $H(M)$. (i) means that C_M is the set of states reachable from the initial state c_0, and (ii) means that every element of E_M appears in the transition relation at least once. Hence, $H(M)$ satisfies the assumptions (A1) and (A2). Moreover, $H(M)$ is a deterministic transition system.

[1] EN systems defined in [5] requires that N is simple and has no isolated elements. We do not use this requirement here.

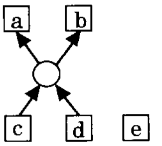


Fig. 1. Atomic systems.

3. Transition systems and corresponding EN systems

In this section, we consider the problem to find an EN system corresponding to a given transition system. An EN system can be decomposed into atomic EN systems, where an EN system is called atomic if it contains exactly one place. We will first study the properties to be satisfied by atomic EN systems so that there is an isomorphism between a given transition system and the composed EN system. We will prove that the problems to check these properties are NP-complete. After that, we will compare these properties with the condition obtained in [5].

3.1. Net atoms

Let E be a set of events. An element x in $2^E \times 2^E$ $(x \neq (\emptyset, \emptyset))$ is called *an atom* over E. Let Atom_E denote the set of all atoms over E. Each $x = (E_1, E_2) \in \mathrm{Atom}_E$ specifies atomic EN systems $A_x = (\{b\}, E, F, \emptyset)$ and $\underline{A}_x = (\{b\}, E, F, \{b\})$ such that $(e, b) \in F$ iff $e \in E_1$, and $(b, e) \in F$ iff $e \in E_2$. For example, Fig. 1 shows A_x for $E = \{a, b, c, d, e\}$ and $x = (\{c, d\}, \{a, b\})$. Each atom specifies dependency among occurring events. An atom $(E_1, E_2) \in \mathrm{Atom}_E$ requires that events in E_1 and E_2 occur alternately. Trace languages were proposed based on such dependency [4], and relationships between trace languages and EN systems were discussed in [3]. We should mention the difference between out atomic EN systems and net atoms used in [4]. In [4], an atom of an EN-system $M = (B, E, F; c_0)$ (determined by a condition b) is defined by

$$N_b = (\{b\}, {}^\bullet b \cup b^\bullet, F_b, c_0 \cap \{b\}),$$

$$F_b = \{(e, b) \mid e \in {}^\bullet b\} \cup \{(b, e) \mid e \in b^\bullet\}.$$

Each atom N_b has ${}^\bullet b \cup b^\bullet$ as the set of event, while in our definition each atomic EN system has E itself as the set of events. This difference is important when we consider an L-morphism between transition system TS and $H(A_x)$.

Definition 3.1. Let $TS = (S, E, \tau, s_0)$ be a transition system. An atom $x \in \mathrm{Atom}_E$ is called *consistent* with TS if there exists an L-morphism from TS to $H(A_x)$. Let $\mathrm{Atom}_E(TS)$ denote the set of all atom $x \in \mathrm{Atom}_E$ consistent with TS, and let $\mathrm{Atom}^{-1}{}_E(TS) = \{(E_2, E_1) \mid (E_1, E_2) \in \mathrm{Atom}_E(TS)\}$.

The following is obvious.

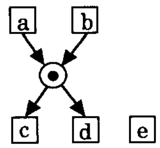


Fig. 2. $\underline{A}y$.

Property 3.2. *Let* $TS=(S,E,\tau,s_0)$ be a transition system and let $x=(E_1,E_2)$ and $y=(E_2,E_1)$ be in Atom_E.

(i) $H(A_x)$ and $H(\underline{A}_y)$ are L-isomorphic to each other.

(ii) Let g_x be an L-morphism from TS to $H(A_x)$, and let g_y be an L-morphism from TS to $H(\underline{A}_y)$. Then for each $s\in S, g_x(s)=\emptyset$ iff $g_y(s)\neq\emptyset$.

Fig. 2 shows the atomic EN system $\underline{A}_y$ for $E=\{a,b,c,d,e\}$ and $y=(\{a,b\}, \{c,d\})$. We can observe that the behavior of A_x (Fig. 1) and $\underline{A}_y$ is equivalent. We remark that Property 3.2 (ii) relies on the assumptions (A1) and (A2), i.e., this property is not always true without these assumptions.

Let $M_i=(B_i,E,F_i;c_i)$ $(i=1,2), B_1\cap B_2=\emptyset$, be EN systems having the same set of events. Then the composition of M_1 and M_2 and M_2 is defined by $M_1\oplus M_2=(B_1\cup B_2,E,F_1\cup F_2;c_1\cup c_2)$. Using $H(M_1)$ and $H(M_2)$, $H(M_1\oplus M_2)$ is obtained as follows:

(i) $C_{M_1\oplus M_2}$ is the smallest subset of $2^{B_1\cup B_2}$ containing $c_1\cup c_2$ such that $[c\in C_{M_1\oplus M_2}\wedge(c\cap B_1,c'\cap B_1)\in\tau_{M_1}\wedge(c\cap B_2,e,c'\cap B_2)\in\tau_{M_2}]\Rightarrow c'\in C_{M_1\oplus M_2}$;

(ii) $\tau_{M_1\oplus M_2}=\{(c,e,c')\in C_{M_1\oplus M_2}\times E\times C_{M_1\oplus M_2}\mid(c\cap B_1,e,c'\cap B_1)\in\tau_{M_1}\wedge(c\cap B_2, e,c'\cap B_2)\in\tau_{M_2}\}$;

(ii) $E_{M_1\oplus M_2}=\{e\mid\exists(c,e,c')\in\tau_{M_1\oplus M_2}\}$.

Such composition has been studied often, eg., [4]. We define the EN systems constructed from a set of atoms. Let W be a subset of Atom_E. When $W\neq\emptyset$, let $M(W)=\bigoplus_{x\in W}A_x$ and let $\underline{M}(W)=\bigoplus_{x\in W}\underline{A}_x$. When $W=\emptyset$, let $M(\emptyset)=\underline{M}(\emptyset)=(\emptyset,E,\emptyset;\emptyset)$ (the EN system that has no conditions). We have the following proposition.

Proposition 3.3. *Let* $TS=(S,E,\tau,s_0)$ *be a transition system. Then* $H(M(\mathrm{Atom}_E(TS)))$ *and* $H(\underline{M}(\mathrm{Atom}^{-1}{}_E(TS)))$ *are L-isomorphic to each other.*

Proof. $\mathrm{Atom}_E(TS)$ contains an element $x=(E_1,E_2)$ if and only if $\mathrm{Atom}^{-1}{}_E(TS)$ contains $y=(E_2,E_1)$. Moreover, by Property 3.2 (i), $H(A_x)$ and $H(\underline{A}_y)$ are L-isomorphic to each other. Considering the above construction of $H(M_1\oplus M_2)$, we can conclude that $H(M(\mathrm{Atom}_E(TS)))$ and $H(\underline{M}(\mathrm{Atom}^{-1}{}_E(TS)))$ are L-isomorphic to each other. $\square$

3.2. Net construction problems from consistent atoms

Now we show a condition satisfied by a set W of atoms such that $H(M(W))$ and a given transition system are L-isomorphic.

Proposition 3.4. *Let $TS=(S,E,\tau,s_0)$ be a transition system and let $W\subset \mathrm{Atom}_E$. Then TS and $H(M(W))$ are L-isomorphic if and only if the following hold:*

(i) *For each $x\in W$, there exists an L-morphism g_x from TS to $H(A_x)$.*

(ii) *For each pair $s_1,s_2\in S$ $(s_1\neq s_2)$, there exists an atom $x\in W$ such that $g_x(s_1)\neq g_x(s_2)$. (Such an atom x is called a state separation atom for s_1 and s_2.)*

(iii) *Let X_{TS} be defined by $X_{TS}=\{(s,e)\in S\times E\,|\,(s,e\rangle\notin\tau\}$. For each $(s,e)\in X_{TS}$, there exists an atom $x\in W$ such that $(g_x(s),e\rangle\notin\tau_{A_x}$. (Such an atom x is called an inhibitor atom for (s,e).)*

Proof. The "Only if" part is clear. We will prove the "If" part. From (i), there is an L-morphism g from TS to $H(M(W))$. From (ii), g is an injection. From (iii), if $(c,e,c')\in\tau_{M(W)}$ and $g(s)=c$, then there exists s' such that $(s,e,s')\in\tau$. Since $H(M(W))$ is deterministic, we have $g(s')=c'$. On the one hand, this implies $g(S)=C_{M(W)}$, since $H(M(W))$ satisfies (A2). On the other hand, it implies for the inverse mapping $g^{-1}:C_{M(W)}\to S$ that for each $(c,e,c')\in\tau_{M(W)}$ we have $(g^{-1}(c),e,g^{-1}(c'))\in\tau$. $\square$

Let us consider the problem to find a state separation atom for each pair of states. Let $TS=(S,E,\tau,s_0)$ be a finite transition system, let $x=(E_1,E_2)\in\mathrm{Atom}_E(TS)$ and let g_x be an L-morphism from TS to $H(A_x)$. Suppose that x is a state separation atom for $s_1,s_2\in S(s_1\neq s_2)$, i.e., $g_x(s_1)\neq g_x(s_2)$. We define mappings f and m as follows. Let $f:E\to\{-1,0,1\}$ be a mapping such that $f(e)=-1$ if $e\in E_2$. $f(e)=1$ if $e\in E_1$, and $f(e)=0$ otherwise. Let $m:S\to\{0,1\}$ be a mapping such that $m(s)=1$ if $g_x(s)\neq\emptyset$, $m(s)=0$ if $g_x(s)=\emptyset$. Then the following holds:

$$m(s)+f(e)=m(s')\quad\text{for each }(s,e,s')\in\tau;\qquad m(s_1)\neq m(s_2).$$

Conversely, if there are such mappings m and f that satisfy the above equations, then $(\{e\in E\,|\,f(e)=1\},\ \{e\in E\,|\,f(e)=-1\})$ is a state separation atom for s_1,s_2. $m(s_1)\neq m(s_2)$ means that $m(s_1)=1\wedge m(s_2)=0$ or $m(s_1)=0\wedge m(s_2)=1$. Let us consider $y=(E_2,E_1)\in\mathrm{Atom}^{-1}{}_E(TS)$. By Property 3.2(i), $H(A_x)$ and $H(\underline{A}_y)$ are L-isomorphic to each other, and therefore there exists an L-morphism g_y from TS to $H(\underline{A}_y)$. We define mappings f' and m' as follows. Let $f':E\to\{-1,0,1\}$ be a mapping such that $f'(e)=-1$ if $e\in E_1$, $f'(e)=1$ if $e\in E_2$, and $f'(e)=0$ otherwise. Let $m':S\to\{0,1\}$ be a mapping such that $m'(s)=1$ if $g_y(s)\neq\emptyset$, $m'(s)=0$ if $g_y(s)=\emptyset$. By Property 3.2(ii), it follows that $m(s)=1$ iff $m'(s)=0$ for each state $s\in S$. Therefore, we can conclude that there is a state separation atom for s_1 and s_2 if and only if there are mappings $f:E\to\{-1,0,1\}$ and $m:\{0,1\}$ that satisfies the following:

$$m(s)+f(e)=m(s')\quad\text{for each }(s,e,s')\in\tau;\qquad m(s_1)=1;\ m(s_2)=0 \tag{1}$$

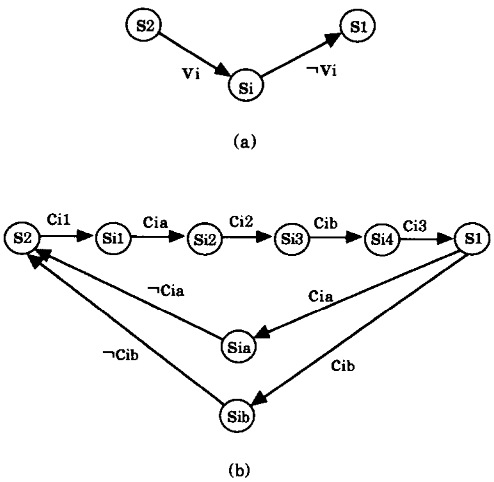


Fig. 3. Construction of a transition system representing 3SAT.

The problem to find the atom x can be written in the following form.

States Separation Atom (SSA)

Instance. A finite transition system $TS=(S,E,\tau,s_0)$ and a pairs of states $s_1,s_2\in S(s_1\neq s_2)$.

Question. Is there a solution of (1)?

Theorem 3.5. SSA *is* NP-*complete.*

Proof. It is easy to see that SSA$\in$NP since we can check in polynomial time whether given m and f satisfies (1). We show that 3SAT below is reducible to SSA in polynomial time.

3-*Satisfiability* (3SAT)

Instance. A collection $C=\{c_1,\ldots,c_m\}$ of clauses on a finite set U of variables such that $|c_i|=3$ for $1\leqslant i\leqslant m$;

Question. Is there a truth assignment for U that satisfies all the clauses in C?

Given a collection C of clauses, a transition system $TS=(S,E,\tau,s_0)$ is constructed as follows (Fig. 3):

(i) $s_1,s_2\in S$.

(ii) For each literals v_i and $\neg v_i$, let

$$s_i\in S,\qquad v_i,\neg v_i\in E,\qquad (s_2,v_i,s_i),(s_i,\neg v_i,s_1)\in\tau.$$

(iii) For each clause $c_i=c_{i1}\vee c_{i2}\vee c_{i3}$, let

$$s_{i1},s_{i2},s_{i3},s_{i4}\in S,\qquad c_{ia},\neg c_{ia},c_{ib},\neg c_{ib}\in E,$$

$$(s_2,c_{i1},s_{i1}),(s_{i1},c_{ia},s_{i2}),(s_{i2},c_{i2},s_{i3}),(s_{i3},c_{ib},s_{i4}),(s_{i4},c_{i3},s_1)\in\tau,$$

$$(s_1,c_{ia},s_{ia}),(s_{ia},\neg c_{ia},s_2),(s_1,c_{ib},s_{ib}),(s_{ib},\neg c_{ib},s_2)\in\tau.$$

(iv) Every element of S, E and τ is defined by the above (i), (ii) and (iii).

(v) Since every state is reachable from s_2, we can choose s_2 as the initial state. (In fact, the selection of the initial state is not important when we find a solution of (1).)

The value $f(v)$ for each literal v is assigned by $f(v)=1$ iff v is true, and $f(v)=0$ iff v is false. Obviously, this instance can be constructed in polynomial time. Suppose that there exists a solution of (1) for TS. Then the following holds.

(i) For each literal v_i and $\neg v_i$, $(f(v_i)=1 \wedge f(\neg v_i)=0) \vee (f(v_i)=0 \wedge f(\neg v_i)=1)$;

(ii) For each clause $c_i = c_{i1} \vee c_{i2} \vee c_{i3}$,

$$(f(c_{ia})=-1 \wedge f(\neg c_{ia})=0) \vee (f(c_{ia})=0 \wedge f(\neg c_{ia})=-1),$$

$$(f(c_{ib})=-1 \wedge f(\neg c_{ib})=0) \vee (f(c_{ib})=0 \wedge f(\neg c_{ib})=-1),$$

$$f(c_{i1})+f(c_{ia})+f(c_{i2})+f(c_{ib})+f(c_{i3})=1.$$

For each clause $c_i = c_{i1} \vee c_{i2} \vee c_{i3}, f(c_{i1})+f(c_{i2})+f(c_{i3}) \geqslant 1$ holds, and therefore at least one of the literals c_{i1}, c_{i2}, c_{i3} must be true. Conversely, if all of the literals in C_i are true, then we have $f(c_{i1})=f(c_{i2})=f(c_{i3})=1$ and $f(c_{ia})=f(c_{ib})=-1$. If two of the literals are true, e.g., $c_{i1}=c_{i2}=$true and $c_{i3}=$false, then we have $f(c_{i1})=f(c_{i2})=1$, $f(c_{i3})=0$ and $(f(c_{ia})=-1 \wedge f(c_{ib})=0) \vee (f(c_{ia})=0 \wedge f(c_{ib})=-1)$. If one of the literals are true, e.g., $c_{i1}=$true and $c_{i2}=c_{i3}=$false, then we have $f(c_{i1})=1$, $f(c_{i2})=f(c_{i3})=0$ and $f(c_{ia})=f(c_{ib})=0$. If no literals are true, then we have no solutions of (1). $\square$

The problem to find an inhibitor atom for each element in X_{TS} can be treated in a similar way. Let $TS=(S,E,\tau,s_0)$ be a finite transition system, let $x \in \mathrm{Atom}_E(TS)$ and let g_x be an L-morphism from TS to $H(A_x)$. Suppose that x is an inhibitor atom for $(s_Z, e_Z) \in X_{TS}$. Then $(g_x(s_z)=\emptyset \wedge f(e_z)=-1) \vee (g_x(s_z)\neq\emptyset \wedge f(e_z)=1)$ holds. Therefore, there is an inhibitor atom for (s_z, e_z) if and only if there are mappings $f: E \to \{-1,0,1\}$ and $m: S \to \{0,1\}$ that satisfies the following:

$$m(s)+f(e)=m(s') \quad \text{for each } (s,e,s') \in \tau; \quad m(s_z)=0; \qquad f(e_z)=-1. \tag{2}$$

The problem to find the atom x can be written as

Inhibitor atom (IA)
Instance. A finite transition system $TS=(S,E,\tau,s_0)$ and $(s_z,e_z) \in X_{TS}$.
Question. Is there a solution of (2)?

Theorem 3.6. IA *is* NP-*complete.*

Proof. It is easy to see that IA$\in$**NP** since we can check in polynomial time whether given m and f satisfies (2). We show that SSA is reducible to IA in polynomial time. We consider an SSA for a transition system $TS=(S,E,\tau,s_0)$ and $s_1, s_2 \in S$ $(s_1 \neq s_2)$. Let $TS'=(S,E',\tau',s_0)$ be a transition system such that $E'=E\cup\{e_z\}$ and $\tau'=\tau\cup\{(s_1,e_z,s_2)\}$. Then the SSA for TS and $s_1, s_2 \in S$ has a solution if and only if an

IA for TS' and $(s_2, e_z) \in X_{TS'}$ has a solution. Clearly, TS' can be constructed in polynomial time. □

Corollary 3.7. *Let $TS = (S, E, \tau, s_0)$ be a transition system. Suppose that there exists* $W \in Atom_E(TS)$ such that TS and $H(M(W))$ *are L-isomorphic to each other. Then there exists such W satisfying* $|W| \leqslant |S|(|S|-1)/2 + |S||E|$.

Proof. W consists of state separation atoms for pairs of states in S and inhibitor atoms for elements in X_{TS}. $|S|(|S|-1)/2$ is the number of pairs of states in S and $|X_{TS}| \leqslant |S||E|$. □

3.3. Regions of elementary transition systems

Elementary transition systems are defined as a subclass of transition systems corresponding to elementary net systems (without isolated elements) [5]. We first give the definitions and results related to elementary transition systems.

Elementary transition systems are based on the notion of regions defined as follows.

Definition 3.8. Let $TS = (S, E, \tau, s_0)$ be a transition system. $r \subset S$ is called *a region* of TS if

(i) $[(s, e, s') \in \tau \wedge s \in r \wedge s' \notin r] \Rightarrow \forall (s_1, e, s_1') \in \tau : [s_1 \in r \wedge s_1' \notin r]$, and

(ii) $[(s, e, s') \in \tau \wedge s \notin r \wedge s' \in r] \Rightarrow \forall (s_1, e, s_1') \in \tau : [s_1 \notin r \wedge s_1' \in r]$.

Let R_{TS} denote the set of non-trivial (neither $\emptyset$ nor S) regions of a transition system $TS = (S, E, \tau, s_0)$.

(i) For each $s \in S$, let $R_s = \{r \mid \exists r \in R_{TS} : s \in r\}$.

(ii) For each $e \in E$, let ${}^\circ e = \{r \mid r \in R_{TS} \wedge \exists (s, e, s') \in \tau : [s \in r \wedge s' \notin r]\}$ and $e^\circ = \{r \mid r \in R_{TS} \wedge \exists (s, e, s') \in \tau : [s \notin r \wedge s' \in r]\}$.

(iii) For each $r \in R_{TS}$, let ${}^\circ r = \{e \in E \mid r \in e^\circ\}$ and $r^\circ = \{e \in E \mid r \in {}^\circ e\}$.

We will use the following property later.

Property 3.9. Let $TS = (S, E, \tau, s_0)$ be a transition system.

(i) r is a region if and only if $S - r$ is a region.

(ii) Suppose that r is a region. Then ${}^\circ r = (S - r)^\circ$ and $r^\circ = {}^\circ(S - r)$.

Using the notion regions, elementary transition systems are defined as a subclass of transition systems.

Definition 3.10. Let $TS = (S, E, \tau, s_0)$ be a transition system (satisfying the assumptions (A1) and (A2)). TS is called *elementary* if the following axioms hold:

(A3) $\forall (s, e, s') \in \tau : s \neq s'$;

(A4) $\forall (s, e_1, s_1), (s, e_2, s_2) \in \tau : [s_1 = s_2 \Rightarrow e_1 = e_2]$;

(A5) $\forall s, s' \in S$: $[R_s = R_{s'} \Rightarrow s = s']$;

(A6) $\forall s \in S\ \forall e \in E : [{}^\circ e \subset R_s \Rightarrow \exists s' \in S : (s, e, s') \in \tau]$.

Note. (A3) directly corresponds to the condition that an EN system has no isolated events, and (A4) corresponds to simple EN systems.

In [5], correspondence between transition systems and EN systems is considered by the following morphism.

Definition 3.11. Let $TS_i = (S_i, E_i, \tau_i, s_0^i)$ $(i = 1, 2)$ be transition systems.

(1) A *G-morphism* from TS_1 to TS_2 is a mapping $g: S_1 \to S_2$ that satisfies the following:

(i) $g(s_0^1) = s_0^2$;

(ii) $\forall (s, e, s') \in \tau_1 : [g(s) = g(s') \vee \exists e_2 \in E_2 : (g(s), e_2, g(s')) \in \tau_2]$;

(iii) $[(s, e, s') \in \tau_1 \wedge (g(s), e_2, g(s')) \in \tau_2] \Rightarrow \forall (s_1, e, s_1') \in \tau_1 : (g(s_1), e_2, g(s_1')) \in \tau_2$.

(2) *A G-isomorphism* is a bijection $g: S_1 \to S_2$ such that (i) g is a G-morphism from TS_1 to TS_2 and (ii) g^{-1} is a G-morphism from TS_2 to TS_1.

Let $TS = (S, E, \tau, s_0)$ be a transition system. Then $J(TS) = (R_{TS}, E, F_{TS}; R_{s0})$ denotes the EN system such that

$$F_{TS} = \{(r, e) \mid r \in R_{TS} \wedge e \in E \wedge r \in {}^\circ e\} \cup \{(e, r) \mid r \in R_{TS} \wedge r \in E \wedge r \in e^\circ\}.$$

The following results was obtained in [5].

Theorem 3.12. *Let $TS = (S, E, \tau, s_0)$ be an elementary transition system. Then the mapping $g: S \to C_{J(TS)}$ given by $g(s) = R_s$ for every $s \in S$ is a G-isomorphism from TS to $H(J(TS))$.*

Theorem 3.13. *A transition system TS is elementary if and only if there exists an EN system M such that* (i) *M is simple,* (ii) *M has no isolated elements, and* (iii) *TS and $H(M)$ are G-isomorphic to each other.*

We first show that there is one-to-one correspondence between regions and consistent atoms.

Proposition 3.14. *Let $TS = (S, E, \tau, s_0)$ be a transition system.*

(i) $\mathrm{Atom}_E(TS) = \{({}^\circ r, r^\circ) \mid r \in R_{TS} - R_{s_0}\}$ *and* $\mathrm{Atom}^{-1}{}_E(TS) = \{({}^\circ r, r^\circ) \mid r \in R_{s_0}\}$.

(ii) *Let r be a non-trivial region, let $x = ({}^\circ r, r^\circ) \in \mathrm{Atom}_E(TS)$ $(\mathrm{Atom}^{-1}{}_E(TS)$, resp.), and let g_x be an L-morphism from TS to $H(A_x)$ $(H(\underline{A}_x)$, resp.). Then $g_x(s) \neq \emptyset$ if and only if $r \in R_s$.*

Proof. (i) Let $r \in R_{TS} - R_{s_0}$ be a non-trivial region, let $x = ({}^\circ r, r^\circ) \in \mathrm{Atom}_E$ and let $A_x = (\{b\}, E, F, \emptyset)$ be an atomic EN system for the atom x. Define a mapping $g: S \to C_{A_x}$ such that for each $s \in S$, $g(s) = \{b\}$ if $s \in r$, and $g(s) = \emptyset$ if $s \notin r$. Then g is an L-morphism from TS to $H(A_x)$. Conversely, suppose that $x = (E_1, E_2) \in \mathrm{Atom}_E(TS)$ and let g_x be an L-morphism from TS to $H(A_x)$. Then $r = \{s \in S \mid g_x(s) = \{b\}\}$ is a non-trivial region such that $E_1 = {}^\circ r$ and $E_2 = r^\circ$. Moreover, $r \notin R_{s_0}$ holds since $g_x(s_0) = \emptyset$. $\mathrm{Atom}^{-1}{}_E(TS) = \{({}^\circ r, r^\circ) \mid r \in R_{s_0}\}$ can similarly be proved.

(ii) We will prove for $x \in \mathrm{Atom}_E(TS)$. The case that $x \in \mathrm{Atom}^{-1}{}_E(TS)$ can be proved in a similar way.

"If". Let $A_x = (\{b\}, E, F, \emptyset)$ be an atomic EN system for the atom x. Assume that $g_x(s) = \emptyset$. Since $s \in r, g_x(s') = \emptyset$ holds for every $s' \in r$. Since r is non-trivial, either ${}^\circ r$ or r° is not empty, i.e., there exist $e \in E, s_1 \in S - r$ and $s_2 \in r$ such that $(e \in {}^\circ r \wedge (s_1, e, e_2) \in \tau) \vee (e \in r^\circ \wedge (s_2, e, s_1) \in \tau)$. Therefore, it follows that $(e \in {}^\bullet b \wedge (g_x(s_1), e, g_x(s_2)) \in \tau_{A_x}) \vee e \in b^\circ \wedge (g_x(s_2), e, g_x(s_1)) \in \tau_{A_x})$. This contradicts $g_x(s_2) = \emptyset$.

"Only if". Assume that $r \notin R_s$. By Property 3.9(i), it follows that $S - r \in R_s$. Let $y = (r^\circ, {}^\circ r) \in \mathrm{Atom}^{-1}{}_E(TS)$ and let g_y be an L-morphism from TS to $H(\underline{A}_y)$. From "If" part, $g_y(s) \neq \emptyset$ holds. By Property 3.2(ii), we obtain $g_x(s) = \emptyset$. □

By the definitions of L-morphism and G-morphism, we have the following lemma.

Lemma 3.15. *Let $TS = (S, E, \tau, s_0)$ be a transition system and let M be an EN system, and let g be an L-isomorphism from TS to $H(M)$. Then g is a G-isomorphism from TS to $H(M)$.*

Using the notion of atoms, we can rewrite Theorem 3.13 as follows.

Corollary 3.16. *A transition system TS is elementary if and only if the following holds:*

(i) *$M(\mathrm{Atom}_E(TS))$ is simple.*

(ii) *$M(\mathrm{Atom}_E(TS))$ contains no isolated elements.*

(iii) *TS and $H(M(\mathrm{Atom}_E(TS)))$ are G-isomorphic to each other.*

Proof. The 'If' part is immediately obtained from Theorem 3.13. By Proposition 3.14, $H(J(TS))$ and $H(M(\mathrm{Atom}_E(TS)) \oplus \underline{M}(\mathrm{Atom}^{-1}{}_E(TS)))$ are L-isomorphic to each other. Since $H(M(\mathrm{Atom}_E(TS)))$ and $H(\underline{M}(\mathrm{Atom}^{-1}{}_E(TS))))$ are L-isomorphic, $H(J(TS))$ and $H(M(\mathrm{Atom}_E(TS)))$ are L-isomorphic, and therefore they are G-isomorphic by Lemma 3.15. Hence, we obtain the "Only if" part by Theorem 3.12. □

Now we consider the problem to check the axiom (A5) and (A6). We can easily obtain the following result from Property 3.14. This proposition shows that these problems correspond to the problems SSA and IA, respectively.

Proposition 3.17. *Let $TS = (S, E, \tau, s_0)$ be a transition system.*

(i) *There exists a state separation atom $x \in \mathrm{Atom}_E(TS)$ for $s_1, s_2 \in S$ $(s_1 \neq s_2)$ if and only if $R_{s_1} \neq R_{s_2}$.*

(ii) *There exists an inhibitor atom $x \in \mathrm{Atom}_E(TS)$ for $(s, e) \in X_{TS}$ if and only if ${}^\circ e - R_s \neq \emptyset$.*

4. Nondeterministic transition systems and labeled EN systems

In this section, we will introduce labeled EN systems, and will consider a problem to find a labeled EN system which is L-isomorphic to a given transitions system. For EN systems, the corresponding transition systems should be deterministic. By using labeled EN systems, the corresponding transition systems can be nondeterministic.

Definition 4.1. Let $M=(B,T,F;c_0)$ be an EN system, let E be a set of labels and let $h:T\to E$ be a mapping. Then $ML=(M;E,h)$ is called *a labeled EN system*, and $H(ML)=(C_M,h(T_M),\tau_{ML},c_0)$ denotes the transition system defined by

$$\tau_{ML}=\{(c,h(t),c')\in C_M\times E\times C_M\,|\,(c,t,c')\in\tau_N\},$$

where $N=(B,T,F)$ is the underlying net of M.

Let $TS=(S,E,\tau,s_0)$ be a transition system. Let $M=(B,T,F;c_0)$ be an EN system and let $ML=(M;E,h)$ be a labeled EN system. ML is called *a trivial representation* of TS if there exists a bijection $\xi:S\cup\tau\to B\cup T$ such that $\xi(S)=B$, $\xi(\tau)=T$, $c_0=\{\xi(s_0)\}$, and for each $v=(s,e,s')\in\tau$, $(\xi(s),\xi(v))\in F$, $(\xi(v),(\xi(s'))\in F$ and $h(\xi(v))=e$. Clearly, TS and $H(ML)$ are L-isomorphic. We need an ML such that $|E|\leqslant|T|<|\tau|$. We can show the following proposition.

Proposition 4.2. *Let $TS=(S,E,\tau,s_0)$ be a transition system, and let $ML=(M(W);E,h)$ be a labeled EN system such that $W\subset$ Atom$_T$ and $h:T\to E$. Then TS and $H(ML)$ are L-isomorphic to each other if and only if the following holds:*

(i) *For each $x\in W$, their exists an L-morphism g_x from TS to $H((A_x;E,h))$.*

(ii) *For each pair $s_1,s_2\in S(s_1\neq s_2)$, there exists an atom $x\in W$ such that $g_x(s_1)\neq g_x(s_2)$. (x is called a state separation pseudo-atom for s_1 and s_2.)*

(iii) *Let $TS^h=(S,T,\tau^h,s_0)$ be the relabeled transition system defined by $(s,t,s')\in\tau^h$ iff $\exists(s,e,s')\in\tau$: $[h(t)=e\wedge\forall x\in W$: $(g_x(x),t,g_x(s'))\in\tau_{A_x}]$. Then for each $(s,t)\in X_{TS^h}$, there exists an atom $x\in W$ such that $(g_x(s),t\rangle\notin\tau_{A_x}$, ($x$ is called an inhibitor pseudoatom for (s,t).)*

Proof. TS^h and $H(M(W))$ are L-isomorphic if and only if TS and $H(ML)$ are L-isomorphic. Therefore, we obtain the proposition by Proposition 3.4. □

By adding state separation pseudo-atoms and inhibitor pseudo-atoms, we can obtain a labeled EN system which is L-isomorphic to a given finite transition system. The following is an algorithm to do this construction.

Algorithm 4.3
Input: a finite transition system $TS=(S,E,\tau,s_0)$.
Output: a labeled EN system ML such that TS and $H(ML)$ are L-isomorphic.
Step 1. (Initialization)
$k:=0$. (k is the iteration counter.)

$T_k = E$. (T_k denote the set of events of the labeled EN system at step k.)
$W_k = \emptyset$. ($W_k \subset \mathrm{Atom}_{T_k}$ denote the set of atoms at step k. Initially, W_0 is empty, i.e., $M(W_0)$ is the EN system which has $T_0 = E$ as the set of events, and has no conditions.)
$u_k : \tau \to T_k$ is defined by $u_k(v) = e$ for each $v = (s, e, s') \in \tau$. (u_k indicates which element of T_k corresponds to each element of τ.)

Let $TS_k = (S, T_k, \tau_k, s_0) := TS$ be a transition system. (TS_k denote the relabeled transition system at step k. Each event of TS is relabeled by corresponding events of $M(W_k)$.) Define a mapping $g_0 : S \to \{\emptyset\}$, which is an L-morphism from TS_0 to $M(W_0)$.
$k := k + 1$.

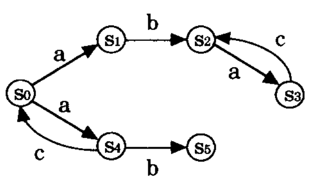


Fig. 4(a). A nondeterministic transition system.

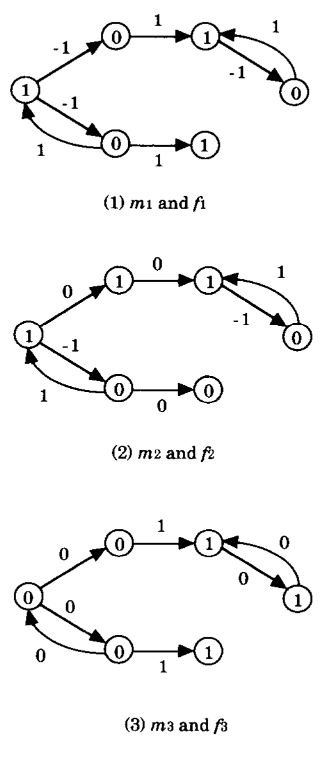


Fig. 4(b). m_i and f_i $(i = 1, 2, 3)$.

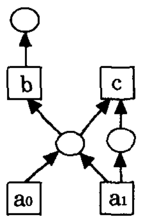


Fig. 4(c). $M(W_3)$.

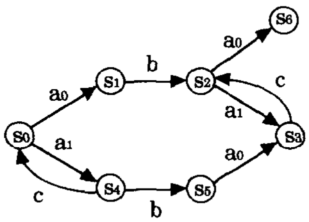
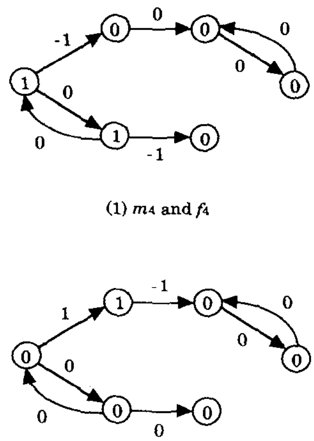


Fig. 4(d). $H(M(W_3))$.



(1) m_4 and f_4

(2) m_5 and f_5

Fig. 4(e). m_i and f_i $(i=4,5)$.

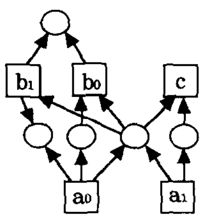


Fig. 4(f). $M(W_5)$.

Step 2. (Finding state separation pseudo-atoms)

2.1. Let g_{k-1} be an L-morphism from TS_{k-1} to $M(W_{k-1})$ and let s_1 and s_2 be states in S such that $s_1 \neq s_2$ and $g_{k-1}(s_1)=g_{k-1}(s_2)$. If there are no such pair of states, then go to Step 3.

2.2. Find mappings $m_k: S \to \{0,1\}$ and $f_k: \tau \to \{-1,0,1\}$ such that

$$m_k(s)+f_k(v)=m_k(s') \quad \text{for each } v=(s,e,s') \in \tau; \qquad m_k(s_1)=1;\, m_k(s_2)=0. \quad (3)$$

We obtain a state separation pseudo-atom from the solution. We should find a solution that minimizes $D_k(t)=|\{f_k(v) \mid u_{k-1}(v)=t\}|$ for each $t \in T_{k-1}$. When $D_k(t)>1$, the transition t will be decomposed into $D_k(t)$ events in T_k.

2.3. Construct an EN system $M(W_k)$ and a relabeled transition system TS_k as follows: $T_k := \{t_q \mid t \in T_{k-1} \wedge f_k(v)=q\}$. (Each event $t \in T_{k-1}$ is decomposed when $D_k(t)>1$.) Let $u_k: \tau \to T_k$ be a mapping defined by $u_k(v)=t_q$ if $u_{k-1}(v)=t \wedge f_k(v)=q$.

Let $W_k := \{x_i \mid i \in \{1, \ldots, k\}\}$ be the new set of atoms over T_k, where

$$x_i := (\{u_k(v) \mid f_i(v) = 1\}, \{u_k(v) \mid f_i(v) = -1\}) \quad \text{if } m(s_0) = 0,$$

$$x_i := (\{u_k(v) \mid f_i(v) = -1\}, \{u_k(v) \mid f_i(v) = 1\}) \quad \text{if } m(s_0) = 1 \quad (i = 1, \ldots, k).$$

Let $TS_k := (S, T_k, \tau_k, s_0)$, where $\tau_k := \{(s, t_q, s') \mid \exists v = (s, e, s') \in \tau : u_k(v) = t_q\}$. Then $W_k \subset \mathrm{Atom}_{T_k}(TS_k)$ holds.

2.4. $k := k+1$ and go to Step 2.

Step 3. (Finding inhibitor pseudo-atoms).

3.1. Let $(s_z, t_z) \in X_{TS_{k-1}}$. If $X_{TS_{k-1}} = \emptyset$, then go to Step 4.

3.2. Find mappings $m_k : S \to \{0, 1\}$ and $f_k : \tau \to \{-1, 0, 1\}$ such that

$$m_k(s) + f_k(v) = m_k(s') \quad \text{for each } v = (s, e, s') \in \tau;$$

$$m_k(s_z) = 0; \quad f_k(v) = -1 \qquad \text{for all } v \in \tau \text{ such that } u_{k-1}(v) = t_z. \tag{4}$$

We obtain an inhibitor pseudo-atom from the solution.

3.3. Construct an EN system $M(W_k)$ and a relabeled transition system TS_k.

3.4. $k := k+1$ and go to Step 3.

Step 4. (Definiting a labeling function). Let $h : T_k \to E$ be a mapping defined by $h(t) = e$ if $\exists v = (s, e, s') \in \tau : u_k(v) = t$.

Output $ML := (M(W_k); E, h)$. □

Remark. We can easily find solutions of (3) and (4). For the equations (3), first assign arbitraly 0 or 1 to each state s other than s_1 and s_2, and let $f_k(v) = s' - s$ for each $v = (s, e, s') \in \tau$. For Eqs. (4), we can obtain a solution as follows:

(i) If $v = (s, e, s') \in \tau$ and $u_{k-1}(v) = t_z$, then let $m_k(s) = 1$ and $m_k(s') = 0$. In this case we can say that there is no $v' = (s', e, s'') \in \tau$ such that $u_{k-1}(v') = t_z$. By processing Step 2, $M(W_{k-1})$ already contains a state separation pseudo-atom for (s, s'), and therefore t_z is not isolated. This implies that the sequence $t_z t_z$ is not enabled at state s. Moreover, s cannot be the state s_z. Hence conflicts do not occur in the above value assignment.

(ii) Assign arbitraly 0 or 1 to other states.

(iii) Let $f_k(v) = s' - s$ for each $v = (s, e, s') \in \tau$ such that $u_{k-1}(v) \neq t_z$.

Each solution of the Eqs. (1) corresponds to a solution of (3) such that $D_k(t) = 1$ for every $t \in T_{k-1}$. This implies that the problem to find a solution minimizing $D_k(t)$ for each $t \in T_{k-1}$ is as hard as the problem SSA. Similarly, for Eq. (4), the problem to find a solution minimizing $D_k(t)$ for each $t \in T_{k-1}$ is as hard as the problem IA.

Example. We will consider a nondeterministic transition system TS shown in Fig. 4(a). For each pair of states (s_0, s_1), (s_1, s_4) and (s_3, s_4), we obtain mappings m_i and f_i $(i = 1, 2, 3)$ shown in Fig. 4(b) (the number in each circle represents m_i and the number on each arrow represents f_i). W_i $(i = 1, 2, 3)$ are obtained as $W_1 = \{(\{a\}, \{b, c\})\}$, $W_2 = \{(\{a_0, a_1\}, \{b, c\}), (\{a_1\}, \{c\})\}$ and $W_3 = \{(\{a_0, a_1\}\{b, c\}), (\{a_1\}, \{c\}), (\{b\}, \emptyset)\}$. $M(W_3)$ is shown in Fig. 4(c) and $H(M(W_3))$ is shown in Fig. 4(d). We can observe that for each pair s_i, s_j of states, $g_3(s_i) = g_3(s_j)$ iff $s_i = s_j$.

Comparing Fig. 4(a) and Fig. 4(d), an inhibitor pseudo-atom $(\{a_0, b_1\}, \emptyset)$ is found for $(s_2, a_0), (s_5, a_0) \in X_{TS_3}$ (Fig. 4(e)). Transition b is decomposed into two transitions b_0 and b_1 at this step. For $(s_4, b_0) \in X_{TS_4}$, $(\{a_0\}, \{b_0\})$ is found. $M(W_5)$ is shown in Fig. 4(f). Defining a labeling function h by $h(a_0) = h(a_1) = a, h(b_0) = h(b_1) = b$, and $h(c) = c$, TS and $ML = (M(W_5); E, h)$ are L-isomorphic to each other.

5. Concluding remarks

We have shown some complexity results on the problem to find an EN system corresponding to a given transition system. Every EN system can be decomposed into a set of atomic EN systems, and there is one-to-one correspondence between atoms and regions. We have considered problems to find an atom which is consistent with the transition system and satisfies the conditions of state separation and inhibition. These two problems correspond to the axioms (A5) and (A6) of elementary transition systems, and are NP-complete. However, this result does not mean that the original problem is NP-complete. In Section 4, we have shown a simple algorithm to construct a labeled EN system from a given transition systems. However, the problem to find a labeled EN system ML that minimizes the number of necessary transitions is as hard as the above problems.

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