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Note

On Kotzig's conjecture for graphs with a regular path-connectedness

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Abstract

Kotzig (see Bondy and Murty (1976)) conjectured that there exists no graph with the property that every pair of vertices is connected by a unique path of length k, $k>2$. Here we prove this conjecture for $k\geqslant 12$.

1. Introduction

We shall consider simple graphs, that is, graphs without loops and multiple edges. All notations and definitions not given here can be found in Bondy and Murty [1].

In 1974 Kotzig stated the following conjecture (see [1. p. 246, Problem 4]).

There exists no graph with the property that every pair of vertices is connected by a unique path of length k ($k>2$).

Let us call a graph with this property a $P(k)$-graph. If a $P(k)$-graph (say G) exists for some $k>2$, then G is connected. Furthermore, because each edge of G belongs to exactly one $(k+1)$-cycle, G is uniquely edge-decomposable into $(k+1)$-cycles, and hence is Eulerian. Kotzig has proved the following lemma [2].

Lemma 1.1. (i) *A $P(k)$-graph G contains a $2n$-cycle with* $3\leqslant n\leqslant k-4$.

(ii) *A $P(k)$-graph G contains no $2n$-cycle with* $n\in\{2, k-3, k-2, k-1, k\}$; *and for* $2<k<9$ *there is no $P(k)$-graph.*

In the following theorem we prove this conjetcure for $k\geqslant 12$.

Theorem 1.2. *There exists no $P(k)$-graph with* $k\geqslant 12$.

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For the proof of this theorem we shall suppose by contradiction that such a graph exists. By Lemma 1.1 a $P(k)$-graph G contains a $2(k-4)$-cycle, thus, in the rest of this paper we assume such a cycle is given and denote it by $C^*=\omega_0\omega_1\omega_2\cdots\omega_{2(k-4)-1}\omega_0$. C^* divides the plane into three disjoint sets called the interior and exterior of C^* and C^* itself. The interior and exterior of C^* are denoted by $\operatorname{int} C^*$ and $\operatorname{ext} C^*$, respectively. We assume that all the vertices and edges of G which are not in C^* are drawn $\operatorname{int} C^*$. If P is a path from ω_i to ω_j such that $V(P)\cap V(C^*)=\{\omega_i,\omega_j\}$, $E(P)\cap E(C^*)=\emptyset$, then we call P a bridge-path. ω_i and ω_j are called the vertices of attachment of P. Two bridge-paths P_1, P_2 are skew if there are four distinct vertices ω_{i_0}, ω_{i_1}, ω_{j_0}, ω_{j_1} such that ω_{i_0} and ω_{i_1} are vertices of attachment of P_1, ω_{j_0} and ω_{j_1} are vertices of attachment of P_2 and the four vertices appear in the cyclic order ω_{i_0}, ω_{j_0}, ω_{i_1}, ω_{j_1} on C^*. Let P be a (u_0,u_1)-path of length l, $P=u_0u_1u_2\cdots u_l$ the segment of P between u_i and u_j is denoted by u_iPu_j if $i<j$ or $u_i\bar{P}u_j$ if $i>j$. We use $\omega_iC^*\omega_j$ and $\omega_i\bar{C}^*\omega_j$ to denote $\omega_i\omega_{i+1}\cdots\omega_{j-1}\omega_j$ and $\omega_i\omega_{i-1}\cdots\omega_{j+1}\omega_j$ (the subscripts are considered mod $2(k-4)$, respectively). A cycle is called an even if the number of its edges is even. A cycle is odd if the number of its edges is odd.

2. Some structural lemmas

Below we shall prove several facts about a $P(k)$-graph.

Lemma 2.1. *If C_{2n} is a $2n$-cycle of a $P(k)$-graph $G(n\leqslant k)$, then there is no (u_0,v_0)-path P of length at least $(k-n)$ such that $V(C_{2n})\cap V(P)=\{u_0\}$. Hence G does not contain a bridge-path P from ω_i to ω_j such that $\omega_iC^*\ \omega_j\bar{P}\omega_i$ is an even cycle.*

Proof. Let $C_{2n}=u_0u_1\cdots u_{n-1}u_nu_{n+1}\cdots u_{2n-1}u_0$. If $P=v_0v_1\cdots v_{k-n-1}u_0$ is a path from v_0 to C_{2n} such that $u_0\in V(C_{2n})$, $v_i\notin V(C_{2a})$, $i=0,1,\cdots,(k-n-1)$, then $P_1=v_0v_1\cdots v_{k-n-1}u_0u_1u_2\cdots u_{n-1}u_n$ and $P_2=v_0v_1\cdots v_{k-n-1}u_0u_{2n-1}\cdots u_{n+1}u_n$ are two paths of length k from v_0 to u_n. This contradicts the assumption of G. □

Lemma 2.2. *For $k\geqslant 12$, a $P(k)$-graph consists of C^* and some bridge-paths of length at most* 4 *and furthermore, the internal vertices of every bridge-path have degree* 2 *and two vertices of attachment of every bridge-path are different.*

Proof. Let G be a $P(k)$-graph for some $k\geqslant 12$. Since G is connected, then for every vertex $u\notin V(C^*)$ we may choose a longest path P from u to some vertex of C^*, say ω_i such that $V(P)\cap V(C^*)=\{\omega_i\}$. Since C^* has length $2(k-4)$ the length l of P is at most three by Lemma 2.1. Now, we distinguish between three cases depending upon l.

Case 1. $l=3$. Let $P=uu_2u_1\omega_i$ be a path from u to C^*, u, u_2, $u_1\notin V(C^*)$. We prove that $d(u_1)=d(u_2)=d(u)=2$.

By Lemma 2.1 $N(u)\subset V(C^*)\cup V(P)$. If $d(u)>2$, then we have $v_1,v_2\in N(u)$, $v_1\neq u_2$, $v_2\neq u_2$. If $u_1\in\{v_1,v_2\}$, say $u_1=v_1$, then $v_2\in V(C^*)$, say $v_2=\omega_i$ and by Lemma 2.1.

$\omega_j u u_2 u_1 \omega_i C^* \omega_j$ is an odd cycle, hence $\omega_j u u_1 \omega_i C^* \omega_j$ is an even cycle, a contradiction. If $v_1, v_2 \in V(C^*)$, say $v_1 = \omega_j$ and $v_2 = \omega_k$, then $\omega_j u u_2 u_1 \omega_i C^* \omega_j$ and $\omega_k u u_2 u_1 \omega_i C^* \omega_k$ are two odd cycles. Consequently $\omega_j u \omega_k C^* \omega_j$ is an even cycle, again a contradiction. It follows that $d(u)=2$.

Suppose now $d(u_2)>2$.

Subcase 1.1. $uu_1 \in B(G)$. In this case we may choose $v \in N(u_2)$, $v \neq u_1$, $v \neq u$. If $v \in V(C^*)$, then $v = \omega_j$ for some j and $\omega_j u_2 u_1 \omega_i C^* \omega_j$ is an odd cycle. Hence $\omega_j u_2 u u_1 \omega_i C^* \omega_j$ is an even cycle. If $v \notin V(C^*)$, then $v u_2 u u_1 \omega_i$ is a path of length 4 from v to C^*. These are impossible. Therefore, Subcase 1.1 fails.

Subcase 1.2. $uu_1 \notin E(G)$. Since $d(u)=2$, some vertex ω_j of C^* is adjacent to u and $\omega_j u u_2 u_1 \omega_i C^* \omega_j$ is an odd cycle. If $d(u_2)>2$, then we may choose $v \in N(u_2)$ such that $v \neq u$, $v \neq u_1$. If $v \in V(C^*)$. If $v' = u_1$, then $v = \omega_k$ and $\omega_k u_2 u_1 \omega_i C^* \omega_k$ is an odd cycle. Hence $\omega_k u_2 u \omega_j C^* \omega_k$ is an even cycle. If $v \notin V(C^*)$, then there exists a vertex v' such that $vv' \in E(G)$, $v' \neq u_2$, $v' \in V(C^*)$ or $v' = u_1$. If $v' = u_1$, then $u u_2 v u_1 \omega_i$ is a path of length 4 from u to C^*. If $v' \in V(C^*)$, say $v' = \omega_k$, then $\omega_k v u_2 u \omega_j C^* \omega_k$ is an even cycle since $\omega_k v u_2 u_1 \omega_i C^* \omega_k$ and $\omega_j u u_2 u_1 \omega_i C^* \omega_j$ are odd cycles. These contradictions imply that Subcase 1.2 fails.

Hence we have $d(u_2)=2$.

Since $d(u)=d(u_2)=2$, if $uu_1 \in E(G)$, then there is no (u,u_2)-path of length $k (k>2)$. Hence $N(u) \cap V(C^*) \neq \emptyset$. Let $u\omega_j \in E(G)$. Thus $u_1 u_2 u \omega_j$ is a path of length 3 from u_1 to C^* and by the proof for $d(u)=2$, we have $d(u_1)=2$.

If $\omega_i = \omega_j$ that is, $\omega_i u_1 u_2 u \omega_j = \omega_i$ is a cycle, then there is no path of length k from u to ω_i. Hence $\omega_i \neq \omega_j$.

Case 2. $l=2$. Let $uu_1\omega_i$ be a path from u to C^* such that $u, u_1 \notin V(C^*)$. Since $l=2$, $N(u) \subset V(C^*) \cup \{u_1\}$ (otherwise, we have $u' \notin V(C^*) \cup \{u_1\}$, $u'u \in E(G)$ then $u'uu_1\omega_i$ is a path from u' to C^*. By Case 1 we have $d(u')=d(u)=d(u_1)=2$). If $d(u)>2$, then we have $\omega_j, \omega_k \in V(C^*)$ such that $u\omega_j, u\omega_k \in E(G)$ and by Lemma 2.1 $\omega_j u u_1 \omega_i C^* \omega_j$ and $\omega_k u u_1 \omega_i C^* \omega_k$ are odd cycles. Thus $\omega_j u \omega_k C^* \omega_j$ is an even cycle, a contradiction. Hence we have $d(u)=2$, that is, there is a vertex $\omega_j \in V(C^*)$ such that $\omega_j u u_1 \omega_i C^* \omega_j$ is an odd cycle, so $u_1 u \omega_j$ is a path from u_1 to C^*, u_1, $u \notin V(C^*)$. We may assume this is a longest path from u_1 to C^* (otherwise by the proof for $l=3$ we have $d(u_1)=2$), and then by using the arguments of the proof for $d(u)=2$ we can easily show that $d(u_1)=2$.

Now $d(u)=d(u_1)=2$, If $u\omega_i \in E(G)$, then there is no (u,ω_i)-path of length k. Hence we have a vertex $\omega_j \neq \omega_i$ such that $\omega_j u \in E(G)$, that is, $P=\omega_i u_1 u \omega_j$ is a bridge-path.

Case 3. $l=1$. Let $u\omega_i \in E(G)$ and $u \notin V(C^*)$. By the assumption of $l=1$ we know $N(u) \in V(C^*)$.

If $d(u)>2$, then there exist two vertices $\omega_j, \omega_k \in N(u) \cap V(C^*)$ such that $\omega_j u \omega_i C^* \omega_j$ and $\omega_k u \omega_i C^* \omega_k$ are odd cycles and therefore $\omega_k = u \omega_j C^* \omega_k$ is an even cycle, a contradiction. So $d(u)=2$ and there is $\omega_j \in V(C^*)$ such that $\omega_i \neq \omega_j$ and $\omega_i u \omega_j$ is a bridge-path. This completes the proof of Lemma 2.2. □

Lemma 2.3. *If* $k \geqslant 12$, *then any two bridge-paths in a* $P(k)$*-graph* G *are not skew. Hence* G *is a planar graph.*

Proof. Let $\omega_{i_0}P_1\omega_{i_1}$, and $\omega_{j_0}P_2\omega_{j_1}$ denote two skew bridge-paths with lengths l and l', respectively. Then C^* can be written as $\omega_{i_0}L_1\omega_{j_0}L_2\omega_{i_1}L_3\omega_{j_1}L_4\omega_{i_0}$. Let l_i denote the length of L_i, $i=1,2,3,4$, and $l_i \geqslant 1$. By Lemma 2.1, $\omega_{i_0}P_1\omega_{i_1}\bar{L}_2\omega_{j_0}\bar{L}_1\omega_{i_0}$ and $\omega_{j_0}P_2\omega_{j_1}\bar{L}_3\omega_{i_1}\bar{L}_2\omega_{j_0}$ are two odd cycles and hence $\omega_{j_0}P_2\omega_{j_1}L_4\omega_{i_0}P_1\omega_{i_1}\bar{L}_2\omega_{j_0}$ and $\omega_{j_0}P_2\omega_{j_1}\bar{L}_3\omega_{i_1}\bar{P}_1\omega_{i_0}L_1\omega_{j_0}$ are two even cycles. Thus, by Lemma 1.1 we have

$$l'+l_4+l+l_2 \leqslant 2(k-4),$$

$$l'+l_3+l+l_1 \leqslant 2(k-4).$$

Since $l_1+l_2+l_3+l_4=2(k-4)$ we have

$$l'+l \leqslant l_1+l_3,$$

$$l'+l \leqslant l_2+l_4.$$

Now we consider the following three possible cases.

Case 1. $l_1 \geqslant l_3$ and $l_2 \geqslant l_4$.

We consider the closed walk:

$$W\colon \omega_{i_0}P_1\omega_{i_1}L_3\omega_{j_1}\bar{P}_2\omega_{j_0}L_2\omega_{i_1}L_3\omega_{j_1}L_4\omega_{i_0}$$

with length $L'=l+l'+l_2+2l_3+l_4$ which is an even number. In this case there exists a vertex $\omega_i \in V(L_2)-\{\omega_{i_1}\}$ such that there are two different (ω_{i_0},ω_i)-paths of length $\frac{1}{2}L'$ from ω_{i_0} to ω_i: $\omega_{i_0}P_1\omega_{i_1}L_3\omega_{j_1}\bar{P}_2\omega_{j_0}L_2\omega_i$ and $\omega_{i_0}\bar{L}_4\omega_{j_1}\bar{L}_3\omega_{i_1}\bar{L}_2\omega_i$. If $\frac{1}{2}L'+l_1 \geqslant k+1$, then there exists a vertex $\omega_j \in V(L_1)-\{\omega_{j_0}\}$ such that there are two different paths of length k from ω_j to ω_i. Hence $\frac{1}{2}L'+l_1 \leqslant k$, that is

$$\tfrac{1}{2}(l'+l+l_3+l_1) \leqslant 4.$$

Similarly, we consider the closed walk

$$W'\colon \omega_{i_1}\bar{P}_1\omega_{i_0}\bar{L}_4\omega_{j_1}\bar{P}_2\omega_{j_0}\bar{L}_1\omega_{i_0}\bar{L}_4\omega_{j_1}\bar{L}_3\omega_{i_1}$$

which has length $L''=l'+2l_4+l+l_1+l_3$, and we can prrove $\frac{1}{2}L''+l_2 \leqslant k$ that is

$$\tfrac{1}{2}(l'+l+l_2+l_4) \leqslant 4.$$

Hence,

$$\tfrac{1}{2}(l'+l+l_1+l_3)+\tfrac{1}{2}(l'+l+l_2+l_4) \leqslant 8,$$

that is,

$$l+l'+(k-4) \leqslant 8.$$

Thus we have $k \leqslant 10$ by l, $l \geqslant 1$. This contradicts the assumption of $k \geqslant 12$.

Case 2. $l_1 < l_3$, and $l_2 \geqslant l_4$.

In this case we consider the closed walks: W' and

$$W''\colon \omega_{j_1}\bar{P}_2\omega_{j_0}\bar{L}_1\omega_{i_0}P_1\omega_{i_1}\bar{L}_2\omega_{j_0}\bar{L}_1\omega_{i_0}\bar{L}_4\omega_{j_1}$$

with length $L'''=l+l'+2l_1+l_2+l_4$ and we can prove $\frac{1}{2}L'''+l_3\leqslant k$, that is,

$$\tfrac{1}{2}(l'+l+l_1+l_3)\leqslant 4.$$

Hence

$$\tfrac{1}{2}(l'+l+l_1+l_3)+\tfrac{1}{2}(l'+l+l_2+l_4)\leqslant 8,$$

that is

$$l'+l+(k-4)\leqslant 8.$$

This is impossible.

Case 3. $l_1<l_2$ and $l_2<l_4$ or $l_1\geqslant l_3$ and $l_2<l_4$.

Similar to Case 1 and Case 2, we can prove that Case 3 is impossible. □

Lemma 2.4. *If $k\geqslant 12$, then any cycle of a $P(k)$-graph G containing exactly two bridge-paths has length at most* 8.

Proof. Let $\omega_{i_0}P_1\omega_{i_1}$ and $\omega_{j_0}P_2\omega_{j_1}$ be two bridge-paths of G and P_1, P_2 have length l and l', respectively. Then $l, l'\leqslant 4$. By Lemma 2.3. P_1 and P_2 are not skew, hence four vertices of attachment of P_1 and P_2 appear in the cyclic order $\omega_{i_0}, \omega_{i_1}, \omega_{j_0}, \omega_{j_1}$ on C^*. Set $C^*=\omega_{i_0}L_1\omega_{i_1}L_2\omega_{j_0}L_3\omega_{j_1}L_4\omega_{i_0}$. Let l_i denote length of L_i, $i=1,2,3,4$. The cycle containing exactly two bridge-paths P_1 and P_2 is unique: $\omega_{i_0}P_1\omega_{i_1}L_2\omega_{j_0}P_2\omega_{j_1}L_4\omega_{i_0}$. If $\omega_{i_0}=\omega_{j_1}$ and $\omega_{j_0}=\omega_{i_1}$, then this cycle has length $l+l'\leqslant 8$. Now, by supposing $\omega_{i_1}\neq\omega_{j_0}$, $l_2\geqslant l_4$, $l_1\geqslant l_3$, we can see that the closed walk

$$W\colon\ \omega_{i_1}\bar{P}_1\omega_{i_0}\bar{L}_4\omega_{j_1}\bar{P}_2\omega_{j_0}L_3\omega_{j_1}L_4\omega_{i_0}L_1\omega_{i_1}$$

has length $L=l+l'+2l_4+l_1+l_3$ which is an even number.

By assumption there is a vertex $\omega_i\in V(L_1)-\{\omega_{i_0}\}$ such that there are two paths from ω_{j_0} to ω_i:

$$\omega_{j_0}L_3\omega_{j_1}L_4\omega_{i_0}L_1\omega_i \quad\text{and}\quad \omega_{j_0}P_2\omega_{j_1}L_4\omega_{i_0}P_1\omega_{i_1}\bar{L}_1\omega_i$$

each having length $\frac{1}{2}L$. If $\frac{1}{2}L+l_2\geqslant k+1$, *then there is a* $\omega_j\in V(L_2)-\{\omega_{i_1}\}$ such that there are two paths of length k from ω_i to ω_j. Hence $\frac{1}{2}L+l_2\leqslant k$, that is,

$$\tfrac{1}{2}(l_1+l_2+l_3+l_4)+\tfrac{1}{2}(l_2+l_4+l+l')\leqslant k$$

and so

$$l+l_2+l'+l_4\leqslant 8. \qquad \square$$

Lemma 2.5. *If $k\geqslant 12$, then any three bridge-paths of a $P(k)$-graph G are not in the same cycle.*

Proof. Assume by contradiction that $\omega_{i_0}P_1\omega_{i_1}$, $\omega_{j_0}P_2\omega_{j_1}$ and $\omega_{k_0}P_3\omega_{k_1}$ are three bridge-paths in the same cycle C, P_1, P_2 and P_3 have lengths l, l' and l'', respectively. Then by Lemma 2.4, 6 vertices of attachment of P_1, P_2 and P_3 appear in the cyclic order ω_{i_0}, ω_{i_1}, ω_{j_0}, ω_{j_1}, ω_{k_0}, ω_{k_1} on C^*. Thus we can denote C^* by $\omega_{i_0}L_1\omega_{i_1}L_2\omega_{j_0}L_3\omega_{j_1}L_4\omega_{k_0}L_5\omega_{k_1}L_6\omega_{i_0}$ and length of L_i by $l_i, i=1,2,\ldots,6$. By Lemma 2.4 we have

$$l+l_2+l'+l_4+l_5+l_6\leqslant 8,$$
$$l+l_2+l_3+l_4+l''+l_6\leqslant 8, \qquad (*)$$
$$l_1+l_2+l'+l_4+l''+l_6\leqslant 8.$$

Hence

$$(l_2+l_2+l_3+l_4+l_5+l_6)+2(l+l''+l''+l_2+l_4+l_6)\leqslant 24.$$

Since $l_1+l_2+l_3+l_4+l_5+l_6=2(k-4)$ we have $(k-4)+(l+l''+l_2+l_4+l_6)\leqslant 12$. Thus if $k\geqslant 14$, this is impossible.

If $k=13$, then $l+l'+l''+l_2+l_4+l_6\leqslant 3$. Only the following case is possible $l_2=l_4=l_6=0$, $l=l'=l''=1$. In this case by $(*)$ we have $l_1=l_3=l_5=6$ and $\omega_{i_0}=\omega_{k_1}$, $\omega_{j_0}=\omega_{i_1}$, $\omega_{j_1}=\omega_{k_0}$. But $C^*\cup P_1\cup P_2\cup P_3$ is not $P(13)$-graph. Hence, there exists a bridge-path different from P_1, P_2 and P_3, say $\omega_i P\omega_j$. Without loss of generality, let $\omega_i\notin V(L_1)-\{\omega_{i_0}\}$. If $\omega_i=\omega_{i_1}$, then ω_j is only ω_{i_0} or ω_{k_0} and $P=\omega_i\omega_j=P_1$ by the above proof. This is impossible. If $\omega_i\neq\omega_{i_0}$, ω_{i_1} and $\omega_j\in V(L_1)$, then P, P_2 and P_3 are in the same cycle. Hence $P=P_1$. If $\omega_i\neq\omega_{i_0}$, ω_{i_1} and $\omega_j\notin V(L_1)$, then $\omega_j\in V(L_3)$ or $\omega_j\in V(L_5)$ thus P and P_1 are skew, this contradicts Lemma 2.3. It follows that P_1, P_2 and P_3 are not in the same cycle.

If $k=12$, then $l_2+l_4+l_6+l+l'+l''\leqslant 4$. We consider the following three cases:

Case 1. $l=l'=l''=1$ and $l_2=l_4=l_6=0$. By $(*)$ we have $l_1, l_3, l_5\leqslant 6$, similar to $k=13$, we can prove that this case fails.

Case 2. $l=l'=l''=1$, $l_2=1$ and $l_4=l_6=0$. By $(*)$ $l_1\leqslant 5$, $l_3\leqslant 5$, $l_5\leqslant 6$, since $\omega_{i_0}P_1\omega_{i_1}\bar{L}_1\omega_{i_0}, \omega_{j_0}P_2\omega_{j_1}\bar{L}_3\omega_{j_0}$ and $\omega_{k_0}P_3\omega_{k_0}\bar{L}_5\omega_{k_0}$ are three odd cycles, hence $l_1\leqslant 4$, $l_3\leqslant 4$, $l_5\leqslant 6$. By $2(k-4)=l_1+l_2+l_3+l_4+l_5+l_6$ we have: $16=2(k-4)=l_1+l_2+l_3+l_4+l_5+l_6=l_1+l_2+l_3+l_5\leqslant 4+1+4+6=15$. Therefore Case 2 is impossible.

Case 3. $l=2, l'=l''=1$ and $l_2=l_4=l_6=0$. By$(*)$ we have $l_1\leqslant 6$, $l_3\leqslant 5$, $l_5\leqslant 5$, similar to Case 2 we have $l_1\leqslant 6$, $l_3\leqslant 4$, $l_5\leqslant 4$.

Thus $16=2(k-4)=l_1+l_3+l_5\leqslant 14$ a contradiction.

This completes the proof of Lemma 2.5. □

3. Proof of Theorem 1.2

If for some $k\geqslant 12$, there is a $P(k)$-graph G, since C^* is not a $P(k)$-graph, then G contains some bridge-paths. Hence by Lemma 2.3 and Lemma 2.5 there exist two bridge-paths, say $\omega_{i_0}P_1\omega_{i_1}$ and $\omega_{j_0}P_2\omega_{j_1}$, such that $C^*=\omega_{i_0}L_1\omega_{i_1}L_2\omega_{j_0}L_3\omega_{j_1}L_4\omega_{i_0}$

and for every vertex $\omega \in V(L_1) - \{\omega_{i_0}, \omega_{i_1}\}$ or $\omega \in V(L_3) - \{\omega_{j_0}, \omega_{j_1}\}$ $d(\omega) = 2$. Since G is uniquely edge-decompossable into $(k+1)$-cycles, each of bridge-paths is contained in some $(k+1)$-cycle. By Lemma 2.4 and Lemma 2.5, the $(k+1)$-cycle containing P_1 is unique $\omega_{i_0} P_1 \omega_{i_1} \bar{L}_1 \omega_{i_0}$ and the $(k+1)$-cycle containing P_2 is unique $\omega_{j_0} P_2 \omega_{j_1} \bar{L}_2 \omega_{j_0}$. Let P_1, P_2 have length l and l', respectively, and let L_i have length l_i, $i = 1, 2, 3, 4$, we have $l_1 + l = k + 1$ and $l_3 + l' = k + 1$, hence $l_1 + l_3 + l + l' = 2(k+1)$. But $l_1 + l_3 \leqslant l_1 + l_2 + l_3 + l_4 = 2(k-4)$ and $l \leqslant 4$, $l' \leqslant 4$ we have $l_1 + l_3 + l + l' \leqslant 2(k-4) + 8 = 2k$ a contradiction. It follows that there is no $P(k)$-graph for $k \geqslant 12$.

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