Action graphs and coverings

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Abstract

An action graph is a combinatorial representation of a group acting on a set. Comparing two group actions by an epimorphism of actions induces a covering projection of the respective graphs. This simple observation generalizes and unifies many well-known results in graph theory, with applications ranging from the theory of maps on surfaces and group presentations to theoretical computer science, among others. Reconstruction of action graphs from smaller ones is considered, some results on lifting and projecting the equivariant group of automorphisms are proved, and a special case of the split-extension structure of lifted groups is studied. Action digraphs in connection with group presentations are also discussed. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

With a group $G$ acting on a set $Z$ we can naturally associate, relative to a subset $S \subset G$, a certain (di)graph called the action (di)graph. Its vertices are the elements of the set $Z$, with adjacencies being induced by the action of the elements of $S$ on $Z$. The definition adopted here is such that a connected action (di)graph corresponds to a Schreier coset (di)graph, with “repeated generators” and semiedges allowed. Note however, that action (di)graphs have a certain advantage over the Schreier coset (di)graphs in that they offer a greater degree of freedom. For similar concepts dealing with (di)graphs and group actions see [1–3,8,12,17,18,21,22,28,45,47]. Some of them, although conceptually different, bear the same name [47], and some of them, quite close to our definition, are referred to by a variety of other names [1,3]. It appears that the term action (di)graph should be attributed to Parsons [45]. For a computer implementation of (a variant of) action graphs see [46].
Group actions are compared by morphisms. The initial observation of this paper is that an epimorphism between two actions invokes a covering projection of the respective action graphs. Surprisingly enough, this simple result does not seem to have been explicitly stated so far, although there are many well-known special cases with numerous applications.

For instance, it is generally known that Schreier coset (di)graphs are actually covering (di)graphs, and that a Schreier coset (di)graph is regularly covered by its corresponding Cayley (di)graph. These facts are commonly used as background results in the theory of group presentations [12–14,28,32,34,50,55], and have recently been applied in the design and analysis of interconnection networks and parallel architectures [1–3,22], among others. Coverings of Cayley graphs are frequently employed to construct new graphs with various types of symmetry and other graph-theoretical properties [9,19,37,49] as well as to prove that a subgroup cannot have genus greater than the group itself [6,19]. As for the maps on surfaces [4,5,10,16,19,20,25–27,31,38,39,42,43,48], one of the many combinatorial approaches to this topic is by means of a Schreier representation [25,42]. Such a representation is actually a certain action graph in disguise, and homomorphisms of maps correspond to covering projections of the respective action graphs. Some important basic facts can be elegantly derived along these lines.

In this paper, we present a unified approach to all these diverse topics, and in addition, we derive certain results which appear to be new or at least bring about some new insights. In Section 2, we state the preliminary concepts. Action graphs are introduced formally in Section 3, and covering projections induced by morphisms of actions in Section 4. Further basic properties of such coverings are discussed in Sections 5 and 6. In Section 7, we briefly consider automorphism groups of action graphs. Section 8 is devoted to lifting and projecting automorphisms, with focus on the equivariant group. In Section 9, we determine the group of covering transformations, and apply some results of [38] to obtain conditions for a natural splitting of a lifted group of map automorphisms (also valid if the map homomorphism is not valency preserving [38,39]). In Section 10, we treat action digraphs in connection with group presentations. The lifting problem along a regular covering (of graphs as well as general topological spaces) is reduced to a question about action digraphs.

2. Preliminaries: group actions, graphs and coverings

An ordered pair \((Z,G)\) denotes a group \(G\) acting on the right on a nonempty set \(Z\). (For convenience, we omit the dot sign indicating the action.) A morphism of actions is an ordered pair \((\phi,\psi):(Z,G)\to(Z',G')\), where \(\phi:Z\to Z'\) is a function and \(\psi:G\to G'\) is a group homomorphism such that \(\phi(u \cdot g) = \phi(u) \cdot \psi(g)\). Morphisms are composed on the left. This defines the category \(\text{Act}_r\) of right actions. Note that \((\phi,\psi)\) is an epimorphism, a monomorphism or an isomorphism, respectively, if and only if \(\phi\) and \(\psi\) are both onto, both into or both 1–1 functions, respectively. An automorphism
is clearly an isomorphism onto itself. Left actions and their morphisms are defined similarly. Morphisms of the form \((\phi,\text{id}): (Z,G) \to (Z',G)\) are called \textit{equivariant}, and morphisms of the form \((\text{id},\psi):(Z,G) \to (Z,G')\) are called \textit{invariant}. Invariant epimorphisms formalize the intuitive notion of ‘groups, acting in the same way on a given set’.

We say that an action \((\tilde{Z},\tilde{G})\) \textit{covers} an action \((Z,G)\) whenever there exists an epimorphism \((\phi,q):(\tilde{Z},\tilde{G}) \to (Z,G)\). Note that the cardinality \(|\phi^{-1}(z)|\) depends just on the orbit of \(G\) to which \(z\in Z\) belongs. A covering of actions \((\phi,q):(\tilde{Z},\tilde{G}) \to (Z,G)\) can be decomposed into an equivariant covering \((\tilde{\phi},\text{id}): (\tilde{Z},\tilde{G}) \to (Z,\tilde{G})\) followed by an invariant covering \((\text{id},q):(Z,\tilde{G}) \to (Z,G)\), where the action of \(\tilde{G}\) on \(Z\) is defined by \(z \cdot \tilde{g} = z \cdot q(\tilde{g})\). It is difficult to give credits for a folklore result such as Proposition 2.1, whose second part can be found in any standard textbook. For a proof see, for instance, [35].

\textbf{Proposition 2.1.} \textit{If \((\tilde{Z},\tilde{G})\) and \((Z,G)\) are transitive actions, then there exists a covering \((\tilde{Z},\tilde{G}) \to (Z,G)\) if and only if there exists, for a fixed chosen \(\tilde{b}\in \tilde{Z}\), a group epimorphism \(q:\tilde{G} \to G\) such that \(q(\tilde{G}_b) \subseteq G_b\) for some \(b\in Z\). The corresponding onto mapping of sets is then given by \(\phi^*_{\tilde{b},b}(\tilde{g}) := b \cdot q(\tilde{g})\). In particular, two transitive actions are isomorphic if and only if there exists an isomorphism between the respective groups mapping a stabilizer onto a stabilizer.}

\textbf{Example 2.2.} \textit{Let \(H \leq H' \leq G\) and \(K\triangleleft G\). The group \(G\) acts by right multiplication on the set of right cosets \(H|G\). Similarly, the quotient group \(G/K\) acts on the set of right cosets \(H'/K|G\). There is an obvious covering of actions \((H|G,G) \to (H'/K|G,G/K)\). In particular, the regular action \((G,G)\) of \(G\) on itself by right multiplication covers any transitive action of \(G/K\).}

\textbf{Example 2.3.} \textit{There is an equivariant isomorphism representing a transitive action of a group \(G\) as an action on the cosets of a stabilizer. Moreover, all transitive and faithful quotient actions of \(G\) can be treated in a similar fashion.}

Indeed, a conjugacy class \(\mathcal{C}\) of subgroups in \(G\) determines the action of \(G/\text{core}(\mathcal{C})\) on the cosets of an element of \(\mathcal{C}\). This action is transitive and faithful. Conversely, let \(q:G \to Q\) be a group epimorphism and let \((Z,Q)\) be transitive and faithful. Define the action \((Z,G)\) such that \((\text{id},q):(Z,G) \to (Z,Q)\) is an invariant covering, and let \(G^Q_b\) be a stabilizer of \((Z,G)\). Let \((c,\text{id}): (Z,G) \to (G^Q_b|G,G)\) be the standard representation of \((Z,G)\). Then \((c,\text{id}): (Z,Q) \to (G^Q_b|G,Q)\) is the standard representation of \((Z,Q)\). It follows that \((Z,Q)\) determines a conjugacy class \(\mathcal{C}^Q\) in \(G\) with \(\text{core}(\mathcal{C}^Q) = \text{Ker} q\). Thus, the isomorphism classes of transitive and faithful actions of quotient groups of \(G\) are in natural correspondence with the conjugacy classes of subgroups in \(G\). Moreover, \((\phi,r):(Z,Q) \to (Z',Q')\) is a covering satisfying \(rq = q'\) if and only if \((\phi,\text{id}): (Z,G) \to (Z',G)\) is a morphism. That is, such a covering exists if and only if \(G^Q_b\) is contained in a conjugate subgroup of \(G^Q_{b'}\). See Examples 4.7 and 4.8 for an application.
We denote the automorphism group of \((Z,G)\) by \(\text{Aut}(Z,G)\). An automorphism \(\psi\) of \(G\) is called \(Z\)-admissible whenever there exists a bijection \(\phi\) on \(Z\) such that \((\phi,\psi) \in \text{Aut}(Z,G)\). The group of \(Z\)-admissible automorphisms is denoted by \(\text{Adm}_Z G\). By \(\text{Aut}(Z)_G\) we denote the equivariant group of the action, formed by all bijections \(\phi\) on \(Z\) for which \((\phi,\text{id}) \in \text{Aut}(Z,G)\). Suppose that \((Z,G)\) is transitive. Then \(\text{Aut}(Z)_G\) can be computed explicitly relative to a point of reference \(b \in Z\) as \(\text{Aut}(Z)_G = \{ \tau \mid \tau(b \cdot g) = b \cdot a g, a \in N(G_b) \text{ mod } G_b \} \cong N(G_b)/G_b\). Also, the left action of \(\text{Aut}(Z)_G\) on \(Z\) is fixed-point free. Moreover, it is transitive if and only if \(G\) acts with a normal stabilizer. Hence if \(G\) is, in addition, faithful, then \(\text{Aut}(Z)_G\) is regular if and only if the action of \(G\) is regular. In this case, \((Z,G)\) is equivariantly isomorphic to the right multiplication \((G,G)_s\), whereas \((\text{Aut}(Z)_G,Z)\) is isomorphic to the left multiplication \((G,G)_l\). For more on group actions see, for instance, [44].

A graph is an ordered 4-tuple \(X = (D,V; \text{beg},\text{inv})\), where \(D\) and \(V\) are disjoint nonempty sets of darts and vertices, respectively, \(\text{beg}:D \rightarrow V\) is a mapping which assigns to each dart \(x\) its initial vertex \(\text{beg}\,x\), and \(\text{inv}:D \rightarrow D\) is an involution which interchanges every dart \(x\) and its inverse \(x^{-1} = \text{inv}\,x\). We use \(\text{beg}\) and \(\text{inv}\) just as symbolic names denoting the actual concrete functions. The terminal vertex \(\text{end}\,x\) of a dart \(x\) is the initial vertex of \(x^{-1}\). The orbits of \(\text{inv}\) are called edges. An edge is called a semiedge if \(x^{-1} = x\), a loop if \(x^{-1} \neq x\) and \(\text{end}\,x = \text{beg}\,x^{-1} = \text{beg}\,x\), and is called a link otherwise. Walks are defined as sequences of darts in the obvious way. By \(\mathcal{W} = \mathcal{W}(X)\) and \(\mathcal{W}^u = \mathcal{W}(X,u)\) we denote the set of all walks and the set of all \(u\)-based closed walks of a graph \(X\), respectively. By recursively deleting all consecutive occurrences of a dart and its inverse in a given walk we obtain its reduction. Two walks with the same reduction are called homotopic. The naturally induced operation in the set of all reduced \(u\)-based closed walks defines the fundamental group \(\pi^u = \pi(X,u)\). A morphism of graphs \(f:(D,V;\text{beg},\text{inv}) \rightarrow (D',V';\text{beg},\text{inv})\) is a function \(f:V \cup D \rightarrow V' \cup D'\) such that \(fV \subseteq V'\), \(fD \subseteq D'\) and \(f\text{beg} = \text{beg}\,f\), \(f\text{inv} = \text{inv}\,f\).

For convenience, we write \(f = f_V + f_D:V \cup D \rightarrow V' \cup D'\), where \(f_V = f|_V:V \rightarrow V'\) and \(f_D = f|_D:D \rightarrow D'\) are the appropriate restrictions. Graph morphisms are composed on the left.

A graph epimorphism \(p: \tilde{X} \rightarrow X\) is called a covering projection if, for every vertex \(\tilde{u} \in \tilde{X}\), the set of darts with \(\tilde{u}\) as the initial vertex is bijectively mapped onto the set of darts with the initial vertex \(p(\tilde{u})\). The graph \(X\) is called the base graph and \(\tilde{X}\) the covering graph. By \(\text{fib}_u = p^{-1}(u)\) and \(\text{fib}_x = p^{-1}(x)\) we denote the fibre over the vertex \(u\) and the dart \(x\) of \(X\), respectively. A morphism of covering projections \(p \rightarrow p'\) is an ordered pair \((f, \tilde{f})\) of graph morphisms \(f:X \rightarrow X'\) and \(\tilde{f}:\tilde{X} \rightarrow \tilde{X}'\) such that \(fp = p'\tilde{f}\). An equivalence of covering projections \(p\) and \(p'\) of the same base graph is a morphism of the form \((\text{id}, \tilde{f})\), where \(\tilde{f}\) is a graph isomorphism. Equivalence of covering projections defined on the same covering graph is defined similarly. An automorphism of \(p: \tilde{X} \rightarrow X\) is of course a pair of automorphisms \((\tilde{f}, f)\) satisfying \(fp = p\tilde{f}\). The automorphism \(\tilde{f}\) is called a lift of \(f\), and \(f\) the projection of \(\tilde{f}\). In particular, all lifts of the identity automorphism form the group \(\text{CT}(p)\) of covering transformations. If the covering graph (and hence the base graph) is connected, then

CT(p) acts semiregularly on vertices and on darts of \( \tilde{X} \). A covering projection of connected graphs is \textit{regular}, whenever CT(p) acts regularly on each fibre.

Let \( p : \tilde{X} \to X \) be an arbitrary covering projection. There exists an \textit{action} of the set of walks \( \mathcal{W} \) of \( X \) on the vertex-set of \( \tilde{X} \) defined by \( \tilde{u} \cdot W = \text{end } \tilde{W} \), where \( \tilde{W} \) is the unique lift of \( W \) such that \( \text{beg } W = \tilde{u} \). In other words, we have \( (\tilde{u} \cdot W_1) \cdot W_2 = \tilde{u} \cdot W_1 W_2 \) and \( \tilde{u} \cdot W W^{-1} = \tilde{u} \). The mapping \( \tilde{u} \mapsto \tilde{u} \cdot W \) defines a bijection \( \text{fib}_{\text{beg } W} \to \text{fib}_{\text{end } W} \).

Homotopic walks have the same action. In particular, \( W^u \) and \( \pi^u \) have the same action on \( \tilde{u} \). The walk-action implies that the coverings (of connected graphs) can be studied from a purely combinatorial point of view [19,38]. A \textit{voltage space} \( (F; \xi, \zeta) \) on a connected graph \( X = (D; V; \text{beg}, \text{inv}) \) is defined by an action of a \textit{voltage group} \( \Gamma \) on a set \( F \), called the \textit{abstract fibre}, and by an assignment \( \xi : D \to \Gamma \) such that \( \xi^{-1} = (\xi x)^{-1} \). This assignment extends to a homomorphism \( \xi : \mathcal{W} \to \Gamma \), with homotopic walks carrying the same voltage. The group \( \text{Loc}^u = \Gamma^u = (\mathcal{W}^u) = (\xi^n) \) is called the \textit{local group} at the vertex \( u \). As the graph is assumed connected, the local groups at distinct vertices are conjugate subgroups, and if any of them is transitive we call such a voltage space \textit{locally transitive}. With every voltage space \( (F, \Gamma; \xi) \) on a connected graph \( X = (D; V; \text{beg}, \text{inv}) \) we can associate a covering \( p_\xi : \text{Cov}(F, \Gamma; \xi) \to X \). The graph \( \text{Cov}(F, \Gamma; \xi) \) has \( \tilde{V} = V \times F \) as the vertex-set and \( \tilde{D} = D \times F \) as the dart-set. The incidence function is \( \text{beg}(x, i) = (\text{beg } x, i) \) and the switching involution \( \text{inv} \) is given by \( (x, i)^{-1} = (x^{-1}, i \cdot \xi x) \). The covering graph is connected if and only if the voltage space is locally transitive. In particular, the \textit{Cayley voltage space} \( (\Gamma, \Gamma; \xi) \), where \( \Gamma \) acts on itself by right multiplication, gives rise to a regular covering. Conversely, each covering of a connected base graph is associated with some voltage space, and each regular covering is associated with a Cayley voltage space \( (\Gamma, \Gamma; \xi) \), where \( \Gamma \cong \text{CT}(p) \) [19,38].

3. Action graphs

Let \( G \) be a (nontrivial) group acting (on the right) on a nonempty set \( Z \) and let \( S \subseteq G \) be a \textit{Cayley set}, that is, \( \emptyset \neq S = S^{-1} \). Any triple \( (Z, G, S) \) is naturally associated with the \textit{action graph} \[
\text{Act}(Z, G; S) = (Z \times S, Z; \text{beg}, \text{inv}),
\]
where \( \text{beg}(z, s) = z \) and \( \text{inv}(z, s) = (z \cdot s, s^{-1}) \). We shall actually need to consider \textit{Cayley multisets}, that is, \( S \) is allowed to have repeated elements (where for each \( s \in S \) the elements \( s \) and \( s^{-1} \) are of the same multiplicity). Our definition of a graph must then be extended accordingly. Some well-known examples of action graphs are listed below.

\textbf{Example 3.1.} A \textit{monopole} is a one-vertex graph with loops and/or semiedges. A monopole without semiedges is a \textit{bouquet of circles} [19]. The action graph of a group \( G \), acting on a one-element set relative to a Cayley (multi)set \( S \), is a monopole which we denote by \( \text{mnp}(S) \). Conversely, any monopole can be viewed as an action graph.
Example 3.2. The action graph \( \text{Act}(H|G,G;S) \) is the Schreier coset graph \( \text{Sch}(G,H;S) \). By taking \( H = 1 \) and \( G \) we get the Cayley graph \( \text{Cay}(G;S) \) and a monopole, respectively. For an account on Schreier coset graphs see [19,54].

Example 3.3. Let \( x_1, x_2, \ldots, x_n \) be permutations of a finite set \( Z \). Then there is a naturally defined action graph for the permutation group \( \langle x_1, x_2, \ldots, x_n \rangle \) relative to the symmetrized generating (multi)set \( \{x_1^{-1}, x_2^{-1}, \ldots, x_n^{-1}\} \). In [45], the digraphs (see Section 10) relative to \( \{x_1, x_2, \ldots, x_n\} \) are referred to as permutation action graphs.

Example 3.4. A finite oriented map \( M \) is a finite graph, cellularly embedded into a closed orientable surface endowed with a global orientation. Let \( D \) be the dart-set of the embedded graph, \( L \) its dart-reversing involution, and \( R \) the local rotation which cyclically permutes the darts in their natural order around vertices, consistently with the global orientation. In studying the combinatorial properties of such a map we only need to consider the permutation group \( \langle R, L \rangle \) on \( D \) (together with the generating set \( \{R, L\} \)), and consequently, its Schreier representation. See Jones and Singerman [25]. Equivalently, we only need to consider the action graph of the group \( \langle R, L \rangle \) acting on \( D \) relative to the Cayley set \( \{R, R^{-1}, L\} \). This graph, here denoted by \( \text{Map}(D; R, L) \) or sometimes by \( \text{Act}(M) \), is also known as the truncation of the map [42].

More generally, maps on all compact surfaces can be viewed combinatorially in terms of certain permutation groups and their generators, as shown by Bryant and Singerman [10]. For closed surfaces, the associated action graphs correspond to graph encoded maps of Lins [31].

A walk \( W \) in the action graph \( \text{Act}(Z; G; S) \) defines a word \( w(W) \in S^* \) over the alphabet \( S \). Conversely, if \( w \in S^* \) is a word, then \( W(z,w) \) denotes the (set of ‘parallel’) walk(s) starting at \( z \in Z \) and determined by \( w \). Note that in the case of repeated generators there is no bijective correspondence between words over \( S \) and walks, rooted at a chosen vertex. Although this fact does not represent a serious difficulty, it has to be well remembered with voltage assignments—because two distinct arcs arising from the same generator need not have the same voltage. The action graph is connected if and only if \( S \) generates a transitive subgroup of \( G \). Without loss of generality we can in most cases assume that the action is transitive and that the Cayley (multi)set generates the group.

4. Morphisms arising from coverings of actions

A mapping \( \phi + \theta : \tilde{Z} + \tilde{Z} \times \tilde{S} \to Z + Z \times S \) is a morphism of action graphs \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(G, Z; S) \) if and only if

\[
\theta(\tilde{z}, \tilde{s}) = (\phi(\tilde{z}), \theta_{\tilde{z}}(\tilde{s})),
\] (1)
where \( \{ \theta z | z \in \tilde{Z} \} \) is a collection of mappings \( \tilde{S} \rightarrow S \) satisfying

\[
\phi(\tilde{z} \cdot \tilde{s}) = \phi(\tilde{z}) \cdot \theta_s(\tilde{s}) \quad \text{and} \quad (\theta_s(\tilde{s}))^{-1} = \theta_{\tilde{s}^{-1}}(\tilde{s}^{-1}). \tag{2}
\]

This follows directly from the definition of a graph morphism. In particular, morphisms arising naturally from coverings of actions can be viewed as graph covering projections (by taking the Cayley (multi)sets to correspond bijectively in a natural way). We state this formally as a theorem, and list some of the well-known special cases and applications.

**Theorem 4.1.** Let \((\phi, q): (\tilde{Z}, \tilde{G}) \rightarrow (Z, G)\) be a covering of actions and let \( \tilde{S} \subset \tilde{G} \) be a Cayley (multi)set. Consider \( S = q(\tilde{S}) \) as a multiset in a bijective correspondence with \( \tilde{S} \). Then the induced graph morphism \( p_{\phi, q} = \phi + \phi \times q|\tilde{S}: \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \rightarrow \text{Act}(Z, G; S) \) is a covering projection of action graphs.

**Proof.** The mapping \( \theta = \phi \times q|\tilde{S} \) obviously satisfies (1) and (2). Hence, \( \phi + \phi \times q|\tilde{S} \) is a graph morphism. Since it is onto with \( q|\tilde{S} \) a bijection, it is a covering projection. □

**Example 4.2.** Let the action be transitive. Then \( p_{c, id}: \text{Act}(Z, G; S) \rightarrow \text{Sch}(G, G_b; S) \), where \( c(b \cdot g) = Gbg \), is an equivariant isomorphism.

**Example 4.3.** Let \( H \leq G \). Then \( p_{\phi, id}: \text{Sch}(G, H; S) \rightarrow \text{Sch}(G, H'; S) \), where \( \phi(Hg) = H'g \), is an equivariant covering projection.

**Example 4.4.** Let \( K \triangleleft G \). Then \( p_{id, q}: \text{Sch}(G, K; S) \rightarrow \text{Cay}(G/K; q(S)) \) is an invariant 1-fold covering projection, and hence an invariant isomorphism [19].

**Example 4.5.** Let \((\text{id}, q):(Z, G) \rightarrow (Z, \tilde{G})\) be an invariant covering of actions. Then \( \text{Act}(Z, G; S) \) is invariantly isomorphic to \( \text{Act}(Z, \tilde{G}; q(S)) \). Thus, in studying action graphs we may restrict ourselves to faithful actions by taking \( q: G \rightarrow \tilde{G} \cong G/G_Z \). See also [51].

**Example 4.6.** A homomorphism \( \tilde{M} \rightarrow M \) of oriented maps is a morphism of the underlying graphs which extends to a mapping between the supporting surfaces. Topologically it corresponds to a branched covering with possible singularities in face-centres, edge-centres and vertices. Combinatorially, we have a mapping of the respective dart-sets \( \phi: \tilde{D} \rightarrow D \) such that \( \phi(\tilde{x} \cdot \tilde{R}) = \phi(\tilde{x}) \cdot R \) and \( \phi(\tilde{x} \cdot \tilde{L}) = \phi(\tilde{x}) \cdot L \). This together with \( \tilde{L} \leftrightarrow L \) and \( \tilde{R} \leftrightarrow R \) defines a covering of actions and consequently, a covering projection of action graphs \( \text{Map}(\tilde{D}; \tilde{R}, \tilde{L}) \rightarrow \text{Map}(D; R, L) \).

**Example 4.7.** Recall from Example 2.3 that the transitive and faithful actions of quotient groups of \( G \) can be ‘modelled by conjugacy classes of subgroups’ in \( G \). It follows that there exists a covering projection \( \phi + \phi \times r: \text{Act}(Z, Q; q(S)) \rightarrow \text{Act}(Z', Q'; q'(S)) \).
arising from actions (where $rq = q'$) if and only if $G'_b$ is contained in a conjugate subgroup of $G'_b$. As a special case, we obtain a result of Širáň and Škoviera [51, Theorem 2]. The situation as described above is encountered in the theory of maps and hypermaps.

**Example 4.8.** Oriented maps and their homomorphisms can be modelled by conjugacy classes within the triangle groups, see Jones and Singerman [25]. The idea extends to all maps [10] and even hypermaps [26].

5. **Structure-preserving morphisms**

Nonisomorphic actions can give rise to isomorphic graphs, as shown by Examples 4.4, 4.5 and by Example 5.1 below. Also, isomorphic actions can have isomorphic graphs with no graph isomorphism arising from an isomorphism of actions, see Example 5.2.

**Example 5.1.** The triangular prism is a Cayley graph for the groups $S_3$ and $Z_6$. It is also an action graph for the group $S_4$. This is less obvious. First, label the vertices of a triangular prism by 1, 2, 3, 4, 5 and 6 so that the involutions $a = (12)(45)(36)$, $b = (23)(56)(14)$ and $c = (13)(46)(25)$ give rise to a natural 1-factorization of the prism. The tricky part consists in showing that $a$, $b$ and $c$ generate a group isomorphic to $S_4$. See Section 10.

**Example 5.2.** Take a Cayley graph $\text{Cay}(G,S)$, where the generating set $S$ is not a CI-set. Then there is a generating Cayley set $T$ with $\text{Cay}(G,S) \cong \text{Cay}(G,T)$ such that no automorphism of $G$ maps $S$ onto $T$. See [30] and the references therein for concrete examples.

In view of these remarks we note the following. When considering an action graph as having a certain structure arising from the action, we are actually considering the induced equivariant covering $\text{Act}(Z,G;S) \rightarrow \text{mp}(S)$. A morphism $\phi + \theta : \text{Act}(\tilde{Z},\tilde{G};\tilde{S}) \rightarrow \text{Act}(Z,G;S)$ is **structure-preserving** if there exists a mapping of the monopoles $\text{mp}(\tilde{S}) \rightarrow \text{mp}(S)$ such that $\phi + \theta$, together with $\text{mp}(\tilde{S}) \rightarrow \text{mp}(S)$, is a morphism of covering projections. In other words, the mapping of darts $\theta_z$ as in (1) does not depend on the vertex $\tilde{z}$. Equivalently, $\theta$ has the form $\theta = \phi \times \psi$, where $\psi : \tilde{S} \rightarrow S$. For example, coverings arising from coverings of actions are structure-preserving, with $\text{mp}(\tilde{S}) \cong \text{mp}(S)$.

**Proposition 5.3.** Let $\phi + \phi \times \psi : \text{Act}(\tilde{Z},\tilde{G};\tilde{S}) \rightarrow \text{Act}(Z,G;S)$ be a structure-preserving covering, where $\tilde{S}$ and $S$ are generating Cayley (multi)sets and $G$ is faithful. Then this covering arises from a covering of actions.
Proof. By induction we have $h_{RS}(\tilde{z} \cdot \tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_n) = h_{RS}(\tilde{z}) \cdot (\psi(\tilde{s}_1) \psi(\tilde{s}_2) \ldots \psi(\tilde{s}_n))$ for each $\tilde{z} \in \tilde{Z}$ and any choice of generators $\tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_n \in \tilde{S}$. Let $\tilde{s}_1 \tilde{s}_2 \ldots \tilde{s}_n = 1$. As $\phi$ is onto and $G$ is faithful, we have $\psi(\tilde{s}_1) \psi(\tilde{s}_2) \ldots \psi(\tilde{s}_n) = 1$. Hence, $\psi$ extends to a homomorphism, as required.

Example 5.4. A covering projection of action graphs $\text{Map}(\tilde{D}; \tilde{R}; \tilde{L}) \to \text{Map}(D; R; L)$ such that $\tilde{R} \mapsto R$ and $\tilde{L} \mapsto L$ arises from a covering of actions $\langle \tilde{R}, \tilde{L} \rangle \to \langle R, L \rangle$, and hence represents a homomorphism of the respective maps.

Although coverings of action graphs, even isomorphisms, in general do not arise from actions nor are at least structure-preserving, we may still ask the following. Let $\text{Act}(Z; G; S) \to X$ and $\tilde{X} \to \text{Act}(Z; G; S)$ be covering projections. Is there an action-structure for the graph $X$ (respectively, $\tilde{X}$) such that these projections arise from coverings of actions?

Theorem 5.5. Let $\phi + \theta : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to X$ be a covering projection. Then there exists an action graph structure for $X$ such that $\phi + \theta$ is equivalent to a covering arising from actions if and only if there exists a covering projection $X \to \text{mnp}(\tilde{S})$ which makes the following diagram commutative. In this case, the action-structure for $X$ can be chosen in such a way that the respective covering is equivariant.

Proof (Sketch). If $\phi + \theta$ is equivalent to a projection arising from a covering of actions, then such a decomposition clearly exists. Conversely, let $Z$ be the vertex-set of $X$. One can easily show that $z \cdot \tilde{g} = \phi(\tilde{z} \cdot \tilde{g})$, $\tilde{z} \in \tilde{Z}$, is a well-defined action of $\tilde{G}$ on $Z$, with $(\phi, \text{id}) : (\tilde{Z}, \tilde{G}) \to (Z, G)$ an equivariant covering of actions. The projection $p_{\phi, \text{id}} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(Z, G; S)$ is equivalent to $\phi + \theta$.

We can interpret Theorem 5.5 by expressing that a quotient of a Schreier coset graph decomposing the natural projection onto a monopole is again a Schreier coset graph of the same group. Stated in this form, the result is due to Širán and Škoviera [52].

We now turn to the second question above. We assume that $(Z, G)$ is transitive and that $S$ is a generating Cayley (multi)set. We may also assume that the covering projection $p = p_{\xi} : \tilde{X} \to \text{Cov}(F, \Gamma; \xi) \to \text{Act}(Z, G; S)$ is given by means of a voltage space $(F, \Gamma; \xi)$ on $\text{Act}(Z, G; S)$.

Theorem 5.6. With the notation and assumptions above there exists an action graph structure $\text{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ for $\text{Cov}(F, \Gamma; \xi)$ such that the corresponding natural projection
The incidence function and the switching involution being, respectively, \( \text{beg}(z; s, i) = (z, i) \) and \( \text{inv}(z; s, i) = (z \cdot s, i \cdot (z, s)^{-1}) \). By the unique walk lifting, the collection of closed walks in \( \text{Act}(Z, G; S) \) representing the orbits of \( s \in S \) lift to a collection of closed walks representing a permutation of \( Z \times F \). This permutation, denoted by \( x_s \), is defined by \( (z, i) \cdot x_s = (z \cdot s, i \cdot (z, s)^{-1}) \). Note that \( s \mapsto x_s \) is a bijection and that \( x_{s^{-1}} = x^{-1}_s \).

Let \( \tilde{G} = \langle x_s \mid s \in S \rangle \) and \( \tilde{S} = \{ x_s \mid s \in S \} \). The action graph \( \text{Act}(Z \times F, \tilde{G}; \tilde{S}) \) has \( Z \times S \) as the vertex-set and \( Z \times S \times F \) as the dart-set. The incidence is given by \( \text{beg}(z; s, i) = (z, i) \) and the switching involution is \( \text{inv}(z, i) \) isomorphism \( \text{Cov}(F, \Gamma; \xi) \rightarrow \text{Act}(Z \times F, \tilde{G}; \tilde{S}) \). This induces a projection \( \text{Act}(Z \times F, \tilde{G}; \tilde{S}) \rightarrow \text{Act}(Z, G; S) \) equivalent to \( p_\xi \), and the natural projection \( \text{Act}(Z \times F, \tilde{G}; \tilde{S}) \rightarrow \text{mn}(\tilde{S}) \) making the required diagram commutative.

The last statement in the theorem follows from Proposition 5.3. \( \square \)

Example 5.7. Let a connected graph \( \text{Cov}(F, \Gamma; \xi) \) cover the action graph \( \text{Map}(D; R, L) \) associated with a map \( M \), and let \( \text{Cov}(F, \Gamma; \xi) \) inherit the action-structure as in Theorem 5.6. Since \( x_L \) is an involution, \( \text{Cov}(F, \Gamma; \xi) \) is the action graph \( \text{Map}(D \times F; x_R, x_L) \) of some map \( \tilde{M} \), and the covering projection essentially arises from the map homomorphism \( \tilde{M} \rightarrow M \), by Example 5.4.

This shows that homomorphisms of oriented maps can be studied just by considering coverings of associated action graphs. Compare this with the discussion on Schreier representations of maps in [43]. We define a map homomorphism \( \tilde{M} \rightarrow M \) to be regular if \( \text{Map}(\tilde{D}; \tilde{R}, \tilde{L}) \rightarrow \text{Map}(D; R, L) \) is a regular covering. Compare the above mentioned with [39].

Finally, let us briefly consider the following question. What is the necessary and sufficient condition for a connected graph \( \text{Cov}(F, \Gamma; \xi) \) as in Theorem 5.6 to be the Cayley graph of the group \( \tilde{G} = \langle x_s \mid s \in S \rangle \) relative to \( \tilde{S} = \{ x_s \mid s \in S \} \)?

First of all, \( x_s, x_{s_2}, \ldots, x_{s_n} \in \tilde{G}(z, i) \) if and only if \( s_1, s_2, \ldots, s_n \in G_G \) and \( \tilde{\xi} w(z, s_1, s_2, \ldots, s_n) \in \Gamma_i \), and hence \( x_{s_1} x_{s_2} \ldots x_{s_n} = 1 \) if and only if \( s_1 s_2 \ldots s_n = 1 \) and \( \tilde{\xi} w(z, s_1, s_2, \ldots, s_n) \in \Gamma_F \) for each \( z \in Z \).

Thus, assuming that \( G \) is faithful and the covering is regular, the answer to the above question is the following: any closed walk \( W \) with \( \tilde{\xi} = 1 \) must correspond to a relation \( s_1 s_2 \ldots s_n = 1 \), and in this case, the voltage of all closed walks corresponding to this word must be trivial. In particular, a regular covering of a Cayley graph as in Theorem 5.6
is the required Cayley graph if and only if the following holds: if the voltage of a closed walk is trivial, then the voltages of all closed walks corresponding to this word must be trivial. This condition is equivalent to saying that the equivariant group lifts, see Section 8 and [38].

**Example 5.8.** Let $\tilde{M} \to M$ be a regular homomorphism of oriented maps, where $M$ is a regular map (see Section 7). Since the action graph $\text{Act}(\tilde{M})$ can be reconstructed from the action graph $\text{Act}(M)$ as in Theorem 5.6 (see Theorem 6.1), it follows that $\tilde{M}$ is a regular map if and only if $\text{Aut} M$ lifts (see Section 7 and Example 8.9, and also [39]).

6. Reconstruction

Let $(\phi, q): (\tilde{Z}, \tilde{G}) \to (Z, G)$ be a covering of transitive actions, let $\tilde{S} \subset \tilde{G}$ be a generating Cayley (multi)set and $S=q(\tilde{S})$ a (multi)set in bijective correspondence with $\tilde{S}$. We would like to reconstruct the action graph $\text{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$ from $\text{Act}(Z, G; S)$ in terms of voltages.

This can be done by means of a *canonical* voltage space $(\tilde{Z}, \tilde{G}; \tilde{\xi})$ (relative to $(\phi, q)$) on $\text{Act}(Z, G; S)$, with voltages on darts defined by the rule $\tilde{\xi}(z, s) = \tilde{s} = q^{-1}(s)$. The derived covering graph $\text{Cov}(\tilde{Z}, \tilde{G}; \tilde{\xi})$ has a vertex-set $\tilde{V} = Z \times Z$ and a dart-set $\tilde{D} = Z \times S \times Z$. The incidence function $\text{beg} : \tilde{D} \to \tilde{V}$ is given by the projection $\text{beg}(z, s, \tilde{u}) = (z, \tilde{u})$, and the switching involution is $\text{inv}(z, s, \tilde{u}) = (z \cdot s, s^{-1}, \tilde{u} \cdot \tilde{s})$, where $\tilde{s} = q^{-1}(s)$. The corresponding local group is $\text{Loc}^b = q^{-1}(G_b)$. Its action on $\tilde{Z}$ is, modulo relabelling, the same as the action of $\Psi^b$ on the vertex fibre over $b$ in $\text{Cov}(\tilde{Z}, \tilde{G}; \tilde{\xi})$. But $\Psi^b$ and $q^{-1}(G_b)$ also act on the fibre $\text{fib}_b$ in $\text{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$. In fact, $(\text{id}, \tilde{\xi}): (\text{fib}_b, \Psi^b) \to (\text{fib}_b, q^{-1}(G_b))$ is an invariant covering of actions.

**Theorem 6.1.** With the notation above, the connected component $C(b, \tilde{b})$ of $\text{Cov}(\tilde{Z}, \tilde{G}; \tilde{\xi})$ containing the vertex $(b, \tilde{b})$, $b = \phi(\tilde{b})$, consists exactly of all vertices of the form $(z, \tilde{z})$, $z = \phi(\tilde{z})$, and the restriction $p_{\tilde{z}} : C(b, \tilde{b}) \to \text{Act}(Z, G; S)$ is equivalent to $p_{\phi, q} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(Z, G; S)$. (If $\tilde{G}$ is a permutation group, then the action graph structure imposed on $C(b, \tilde{b})$ as in Theorem 5.6 coincides with $\text{Act}(\tilde{Z}, \tilde{G}; \tilde{S})$.)

Next, all restrictions of $p_{\tilde{z}} : \text{Cov}(\tilde{Z}, \tilde{G}; \tilde{\xi}) \to \text{Act}(Z, G; S)$ to its connected components are equivalent to $p_{\phi, q} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(Z, G; S)$ if and only if the restrictions of the action of $q^{-1}(G_b)$ on all its orbits have the same conjugacy class of stabilizers. In particular, let the action of $\tilde{G}$ be such that no stabilizer is properly contained in another stabilizer (say, the group is finite). Then all restrictions of $p_{\tilde{z}}$ to its connected components are equivalent to $p_{\phi, q}$ if and only if $q(\tilde{G}_b) \leq G_Z$.

**Proof.** Let $(z, \tilde{u})$ be in the same component as $(b, \tilde{b})$, $b = \phi(\tilde{b})$. Then $\tilde{u} = \tilde{b} \cdot (s_1, s_2, \ldots, s_n)$ and $\phi(\tilde{u}) = b \cdot (s_1, s_2, \ldots, s_n) = z$. So all vertices in this component are of the required form. If $(z, \tilde{z})$ and $(u, \tilde{u})$ have the same label $\tilde{z} = \tilde{u}$, then $z = \phi(\tilde{z}) = \phi(\tilde{u}) = u$. 
Hence, no two vertices are labelled by the same label. Moreover, all labels from \( \tilde{Z} \) actually appear since \( \tilde{S} \) generates \( \tilde{G} \) and \( \tilde{G} \) is transitive on \( \tilde{Z} \). The fibre over \( b \) in \( C(b, \tilde{b}) \) is labelled by the orbit of \( \text{Loc}^b \) on \( \tilde{Z} \) which is precisely \( \text{fib}_b = \phi^{-1}(b) \). It follows that the actions of \( \text{fib}^b \) on fibres over \( b \) in \( C(b, \tilde{b}) \) and \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) are essentially the same. Hence, \( p_{\tilde{z}} : C(b, \tilde{b}) \to \text{Act}(Z, G; S) \) and \( p_{\phi^{-1}(b)} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(Z, G; S) \) are equivalent. The explicit graph isomorphism \( C(b, \tilde{b}) \to \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) which establishes this equivalence is \((z, \tilde{z}) \mapsto \tilde{z} \) on vertices and \((z, s, \tilde{z}) \mapsto (\tilde{z}, \tilde{s}) \) on darts. If \( \tilde{G} \) is faithful, then the action graph structure imposed on \( C(b, \tilde{b}) \) as in Theorem 5.6 obviously coincides with \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \).

Clearly, all restrictions of \( p_{\tilde{z}} \) to the components of \( \text{Cov}(\tilde{Z}, \tilde{G}; \tilde{S}) \) are equivalent if and only if the induced actions of \( \text{Loc}^b = q^{-1}(G_b) \) on its orbits in \( \tilde{Z} \) are equivariantly isomorphic. Hence all of these actions must have the same conjugacy class of stabilizers in \( \text{Loc}^b \).

Observe that \( \text{Loc}^b_{\tilde{z}} = \tilde{G}_{\tilde{z}} \) for \( \tilde{z} \in \phi^{-1}(b) \), and \( \text{Loc}^b_u = \tilde{G}_u \cap \text{Loc}^b \) for \( u \notin \phi^{-1}(b) \). Now, if all stabilizers of \( \tilde{G} \) are contained in \( \text{Loc}^b \), then all the above actions of \( \text{Loc}^b \) do have the same conjugacy class of stabilizers. Conversely, the fact that each \( \text{Loc}^b_{\tilde{z}} = \tilde{G}_{\tilde{z}} \cap \text{Loc}^b \) is also the stabilizer of some point \( \tilde{z} \in \phi^{-1}(b) \) implies \( \tilde{G}_{\tilde{z}} \leq \tilde{G}_u \). But actually we have equality since, by assumption, no stabilizer is properly contained in another stabilizer. Hence, each stabilizer of \( \tilde{G} \) must be contained in \( \text{Loc}^b = q^{-1}(G_b) \). This condition can be further rephrased as follows. Since \( q \) is onto and \( \tilde{G}_z, \tilde{z} \in \tilde{Z} \) are conjugate subgroups, we have \( g^{-1}q(\tilde{G}_z)g \leq G_b \) for all \( g \in G \). Thus, \( q(\tilde{G}_z) \leq G_z \). Conversely, if \( q(\tilde{G}_z) \leq G_z \), then \( q(\tilde{G}_z) = g^{-1}q(\tilde{G}_z)g \leq g^{-1}G_Zg = G_Z \leq G_b \). This completes the proof. \( \square \)

Example 6.2. For instance, if \( \tilde{G} \) (or \( G \)) acts with a normal stabilizer, then the action graph \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) can be reconstructed from \( \text{Act}(Z, G; S) \) by taking any connected component of the derived covering. A special case is encountered when reconstructing Cayley graphs from Cayley graphs of quotient groups [19].

Example 6.3. Consider the dihedral group \( G = D_{18} = \langle R, L \mid R^9 = L^2 = (RL)^2 = 1 \rangle \) with subgroups \( H = \langle 1, L \rangle \cong \mathbb{Z}_2 \) and \( H' = \langle R^3, L \rangle \cong S_3 \), and let \( S = \{ R, R^{-1}, L \} \). Then \( \text{Sch}(G, H; S) \) equivariantly covers \( \text{Sch}(G, H'; S) \). Since not all conjugates of \( H' \) contain \( H \), the graph \( \text{Sch}(G, H; S) \) cannot be reconstructed by taking just any connected component of the derived covering \( \text{Cov}(H|G, G; \tilde{S}) \).

In this particular case, we can also apply the Burnside–Frobenius counting lemma. Namely, the action of \( H' \) on the cosets of \( H \) has two orbits, whereas the derived covering is 9-fold.

7. Automorphisms

We henceforth assume that the considered actions are transitive and that the Cayley (multi)sets generate the groups in question. A bijective mapping \( \tau + \theta \) of \( Z + Z \times S \) onto itself is an automorphism of \( \text{Act}(Z, G; S) \) if and only if \( \theta(z, s) = (\tau(z), \theta_z(s)) \), where
\{\theta_z | z \in \mathbb{Z}\} is a collection of bijections of \(S\) onto itself satisfying \(\tau(z \cdot s) = \tau(z) \cdot \theta_s(s)\) and 
\((\theta_z(s))^{-1} = \theta_{z^{-1}}(s^{-1})\). To study the full automorphism group of \(\text{Act}(Z, G; S)\), even the subgroup of structure-preserving automorphisms, is extremely difficult. More tractable, but still difficult, is a very particular subgroup, namely the subgroup \(\text{Aut}\subseteq G\) of action-automorphisms, \(\tau + \tau \times \psi\), where \(\psi\) extends to (or just is) an automorphism of 
\(G\). Proposition 5.3 implies Proposition 7.1.

**Proposition 7.1.** Let a transitive action \((Z, G)\) be faithful, and let \(S \subseteq G\) be a generating Cayley (multi)set. Then each automorphism of \(\text{Act}(Z, G; S)\) of the form \(\tau + \tau \times \psi\) is an action-automorphism.

Let \(\text{Aut}^S G = \{\psi \in \text{Aut} G | \psi(S) = S\}\) denote the subgroup of automorphisms of \(G\) which preserves \(S\) (as a set), and let \(\text{Adm}^S G = \text{Aut}^S G \cap \text{Adm} G\) denote the subgroup formed by automorphisms of \(G\) which preserves \(S\) as well as the conjugacy class of stabilizers of \(G\) (as a set). Of course, \(\tau + \tau \times \psi\) is an action-automorphism of \(\text{Act}(Z, G; S)\) if and only if \((\tau, \psi) \in \text{Aut}(Z, G)\) with \(\psi \in \text{Adm}^S G\). In particular, we can identify \(\text{Eq}(Z)_G\), the equivariant group of automorphisms of \(\text{Act}(Z, G; S)\), and the equivariant group of the action, \(\text{Aut}(Z)_G\). It is easy to see that the projection \(\text{Aut}^S(Z, G) \rightarrow \text{Adm}^S G\), \((\tau, \psi) \mapsto \psi\) is a group epimorphism with kernel \(\text{Aut}(Z)_G\). We state this folklore observation formally.

**Theorem 7.2.** The group \(\text{Aut}^S(Z, G)\) of action-automorphisms of \(\text{Act}(Z, G; S)\) is isomorphic to an extension of the equivariant group \(\text{Aut}(Z)_G\) by \(\text{Adm}^S G\).

While \(\text{Aut}(Z)_G\) is isomorphic to \(N_G(G_b)/G_b\), the identification of \(\text{Adm}^S G\) and the extension itself might not be easy. Here is a simple and well-known example.

**Example 7.3.** In view of Proposition 7.1 there are two kinds of automorphisms of a Cayley graph \(\text{Cay}(G; S)\). Those for which the mapping of darts \(\theta_g : S \rightarrow S\) depends on the vertex \(g \in G\), and those that are action-automorphisms. Let \(\tau + \tau \times \psi\) be an action-automorphism. Then \(\tau(g) = \tau_{\alpha \cdot \psi}(g) = a \psi(g)\), for some \(a \in G\) and \(\psi \in \text{Adm}^S G = \text{Aut}^S G\). Moreover, the assignment \((a, \psi) \mapsto (\tau_{a \cdot \psi}, \psi)\) is an isomorphism \(G \rtimes \text{id} \text{Aut}^S G \rightarrow \text{Aut}^S(G, G)\). Hence, the group of action-automorphisms \(\text{Aut}^S(G, G)\) of \(\text{Cay}(G; S)\) is isomorphic to a subgroup in the holomorph of \(G\). The group \(\text{Aut}^S(G, G)\), can as well be characterized as the normalizer of the left regular representation \(\{\tau_{a \cdot \text{id}} | a \in G\}\) of \(G\) within the full automorphism group of \(\text{Cay}(G; S)\) [18].

**Example 7.4.** From the very definition it follows that the automorphism group \(\text{Aut} M\) of an oriented map \(M\) can be identified with the equivariant group \(\text{Aut}(D)_{(R, L)}\) [25].

**Example 7.5.** An oriented map is called regular if its automorphism group acts transitively (and hence regularly) on the dart-set of the underlying graph. Recall that the equivariant group acts regularly if and only if the group itself is regular. Therefore,
a map is regular if and only if Map(D; R, L) is a Cayley graph for the group \langle R, L \rangle \{25,42\}. Since any Map(D; R, L) is equivariantly covered by Cay(\langle R, L \rangle; \{R, R^{-1}, L\}), every map is a (branched) regular quotient of some regular map \{25,27,48\}. The idea extends to maps on bordered surfaces \{4\}.

As our last remark, let us ask as to what are the necessary and sufficient conditions for a transitive and faithful action (Z, G) to extend to a group of automorphisms of Act(Z, G; S). This problem is of interest \{47\} (however, action graphs in \{47\} differ from ours), and difficult in general. But let the requirement be that the action of G extends to a group of action automorphisms (which is equivalent—because the action of G is faithful—to the requirement that the induced group preserves the natural structure of Act(Z, G; S) as a covering of mnp(S)). Under this assumption, the answer is trivial.

**Proposition 7.6.** Let the action (Z, G) be transitive and faithful, and let S \subset G be a generating Cayley (multi)set. Set \tau_g(z) = z \cdot g (g \in G). Then there are group automorphisms \psi_g (g \in G) such that T(G) = \{\tau_g + \psi_g | g \in G\} is a group of action automorphisms of Act(Z, G; S) if and only if S is a union of conjugacy classes. In particular, T(G) acts as a subgroup of the equivariant group if and only if the action graph is a Cayley graph Cay(G; S) and G is abelian.

Other types of symmetries of action graphs will not be discussed here. Arc-transitivity of Cayley digraphs is considered, for instance, in \{2,22\}.

8. Lifting and projecting automorphisms

Let \phi: Act(\tilde{Z}, \tilde{G}; \tilde{S}) \rightarrow Act(Z, G; S) be a covering arising from transitive actions, where \tilde{S} and S = q(\tilde{S}) are generating Cayley (multi)sets. The problem of lifting automorphisms has recently received considerable attention \{4,5,7,9,15,23,24,29,36,38,39,49,52,53\} in various contexts. From the general theory \{40, Theorem 5.1, pp. 156\} we infer that the lifting condition, expressed in terms of the canonical voltages valued in \tilde{G}, reads: an automorphism \tau + \theta lifts along \phi if and only if there exists \tilde{z} \in \text{fib}_z such that, for each \text{W} \in \mathcal{W}^h,

\[ \xi_w \in \tilde{G}_b \text{ if and only if } \xi_{0w} \in \tilde{G}_z. \]

However, we explore the possibility here to express the lifting condition without the use of an explicit reference to mappings of closed walks and their voltages, but rather to be expressed in terms of certain subgroups of G and \tilde{G}. To this end we have to restrict our considerations either to a very special class of action-automorphisms (the equivariant group), or else to action-automorphisms with additional requirements imposed on the covering of actions (Example 8.2). We also note that the lifts of action-automorphisms, although structure-preserving, need not be action-automorphisms.
Example 8.1. Let $\tilde{G}$ be faithful. Then, a lift of an action-automorphism of $\text{Act}(Z,G;S)$ is an action-automorphism of $\text{Act}(\tilde{Z},\tilde{G};\tilde{S})$, by Proposition 7.1.

Example 8.2. Suppose that the covering of actions is equivariant. Then the conclusion from the previous example holds, too. Moreover, from (3) we easily derive that an action-automorphism $\tau+\tau \times \psi$ lifts if and only if there exists $g \in G$ such that $\tau(b)=b \cdot g$ and $\psi(G_b)=g^{-1}G_{\tilde{g}}g$. (An alternative direct proof avoiding (3) is readily at hand and is left to the reader.) In particular, action-automorphisms do lift along the equivariant covering $\text{Cay}(G;S) \to \text{Act}(Z,G;S)$.

Now, let us focus on the equivariant group $\text{Eq}(Z)_G$ of $\text{Aut}(Z,G;S)$. It is easy to derive a lifting condition for $\text{Eq}(Z)_G$ in terms of subgroups of $\tilde{G}$ and $G$. This is not surprising because with equivariant automorphisms we could as well consider just coverings of actions.

Theorem 8.3. The equivariant group $\text{Eq}(Z)_G$ lifts along $p_{\phi,\eta} : \text{Act}(\tilde{Z},\tilde{G};\tilde{S}) \to \text{Act}(Z,G;S)$ if and only if $q(N_G(\tilde{G}_b))$ intersects every coset of $G_b$ within $N_G(G_b)$, in which case the lifted group is a subgroup of the equivariant group $\text{Eq}(Z)_G$. In particular, if the covering projection is regular, then $\text{Eq}(Z)_G$ lifts if and only if $N_G(G_b) \leq q(N_G(\tilde{G}_b))$ (equivalently, $q^{-1}(N_G(G_b)) \leq N_G(\tilde{G}_b)$).

Proof. We present a proof which avoids the reference to (3). Clearly, a lift of an equivariant automorphism is equivariant. Consider a pair of equivariant automorphisms of the covering graph and of the base graph, respectively. Their action on vertices is given, relative to $\tilde{b}$ and $b = \phi(\tilde{b})$, by $\tilde{\tau}(\tilde{b} \cdot \tilde{g}) = \tilde{b} \cdot \tilde{c} \tilde{g}$ where $\tilde{c} \in N_G(\tilde{G}_b)$ mod $\tilde{G}_b$, and $\tau_c(b \cdot g) = b \cdot cg$, where $c \in N_G(G_b)$ mod $G_b$. Let $\tilde{\tau}_c + \tilde{\tau}_c \times \text{id}$ be a lift of $\tau_c + \tau_c \times \text{id}$, that is, let $\phi(\tilde{\tau}_c) = \tau_c \phi$. We easily get $q(\tilde{c}) \in G_{bc}$. Conversely, if $c$ and $\tilde{c}$ satisfy this condition, then $\tilde{\tau}_c + \tilde{\tau}_c \times \text{id}$ is a lift of $\tau_c + \tau_c \times \text{id}$. The lifting condition can now be expressed as: for each $c \in N_G(G_b)$ there exists $\tilde{c} \in N_G(\tilde{G}_b)$ such that $q(\tilde{c}) \in G_{bc}$. The claim follows.

The covering projection $p_{\phi,\eta}$ is regular if and only if $q^{-1}(G_b) \leq N_G(\tilde{G}_b)$, by Theorem 9.1 below. This implies $G_b \leq q(N_G(\tilde{G}_b))$, and hence the lifting condition now obviously reduces to $N_G(G_b) \leq q(N_G(\tilde{G}_b))$. The alternative form follows because $N_G(\tilde{G}_b)$ contains $\text{Ker}q$. \( \square \)

Example 8.4. Let the group $\tilde{G}$ act with a normal stabilizer. Then the equivariant group lifts. In particular, $\text{Eq}(Z)_G$ lifts along $p_{\phi,\eta} : \text{Cay}(\tilde{G};\tilde{S}) \to \text{Act}(Z,G;S)$.

Example 8.5. Let $\tilde{M} \rightarrow M$ be a homomorphism of oriented maps. If $\tilde{M}$ is a regular map, then $\text{Aut}M$ lifts [39].

Example 8.6. The lift of the equivariant group $\text{Eq}(Z)_G$ along a regular covering projection $p_{\phi,\eta}$ is isomorphic to $q^{-1}(N_G(G_b))/\tilde{G}_b$. 


Proposition 8.7. Let the covering \( p_{\phi,q} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Cay}(G; S) \) be regular. If the equivariant group of \( \text{Cay}(G; S) \) lifts, then \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) is isomorphic to the Cayley graph \( \text{Cay}(\tilde{G}/\tilde{G}_2; \tilde{S}) \).

Proof. By Theorem 8.3 we have \( q(N_{\tilde{G}}(\tilde{G}_b)) = G \). Thus \( N_{\tilde{G}}(\tilde{G}_b) \) intersects every coset of \( \text{Ker } q \). Since \( \text{Ker } q \leq q^{-1}(G_b) \) and \( q^{-1}(G_b) \leq N_{\tilde{G}}(\tilde{G}_b) \) (see Theorem 9.1 below), \( N_{\tilde{G}}(\tilde{G}_b) \) contains \( \text{Ker } q \). Hence, \( N_{\tilde{G}}(\tilde{G}_b) = \tilde{G} \), and the proof follows.

Example 8.8. Consider a regular covering \( p_{\phi,q} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Cay}(G; S) \), where \( \tilde{G} \) is faithful. Then \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) is isomorphic to the Cayley graph \( \text{Cay}(\tilde{G}; \tilde{S}) \) if and only if the equivariant group of \( \text{Cay}(G; S) \) lifts, by Example 8.4 and Proposition 8.7. In view of the lifting condition (3) we may rephrase this as in Section 5.

Example 8.9. Let \( \tilde{M} \to M \) be a regular homomorphism, where \( M \) is a regular map. If \( \text{Aut } M \) lifts, then \( \tilde{M} \) is also a regular map. In view of Example 8.5 we obtain the if and only if statement of Example 8.5. See also [20,39].

Let us now consider projecting automorphisms of \( \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \) along \( p_{\phi,q} \). An automorphism \( \tilde{\tau} + \theta \) projects whenever each vertex-fibre is mapped onto some vertex-fibre and each dart-fibre is mapped onto some dart-fibre. If the covering is regular, then an automorphism projects if and only if it normalizes the group of covering transformations. This is actually a theorem of Macbeath [33] which holds for general topological coverings as well as our combinatorial context.

Example 8.10. The automorphisms of \( \text{Cay}(G; S) \) which project along \( \text{Cay}(G; S) \to \text{mnp}(S) \) constitute, in view of Example 7.3, precisely the group \( \text{Aut}^\delta(G,G) \) of action automorphisms. A similar result (which can be interpreted along these lines) about vertex stabilizers of Cayley digraphs is proved in [41].

Note that the projections of action-automorphisms are structure-preserving but need not be action-automorphisms. One particular instance when such projections are indeed action-automorphisms is when the covering is equivariant; the case when \( G \) acts faithfully is another example. In general, the following holds.

Proposition 8.11. Let an action-automorphism \( \tilde{\tau} + \tau \times \tilde{\psi} \) project along \( p_{\phi,q} : \text{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \text{Act}(Z, G; S) \). Then the projected automorphism is an action-automorphism of \( \text{Act}(Z, G; S) \) if and only if \( \text{Ker } q \) is invariant under \( \tilde{\psi} \).

Proof. Let \( \tau + \tau \times \psi \) be the projected automorphism. We know that \( \psi \) is defined as \( \psi = q\tilde{\psi}q^{-1} \) on \( S \). Now \( \psi \) extends to an automorphism of \( G \) if and only if \( \psi(s_1) \ldots \psi(s_n) = 1 \) whenever \( s_1 \ldots s_n = 1 \). Equivalently, we must have \( q\tilde{\psi}(\tilde{s}_1 \ldots \tilde{s}_n) = 1 \) whenever \( (\tilde{s}_1 \ldots \tilde{s}_n) \in \text{Ker } q \), that is, \( \tilde{\psi}(\tilde{s}_1 \ldots \tilde{s}_n) \in \text{Ker } q \) whenever \( (\tilde{s}_1 \ldots \tilde{s}_n) \in \text{Ker } q \).
Theorem 8.12. An action-automorphism \( \bar{\tau} + \bar{\tau} \times \bar{\psi} \) projects along \( p_{\phi,q} : \text{Act}(\bar{Z}, \bar{G}; \bar{S}) \to \text{Act}(Z, G; S) \) if and only if there exists \( \bar{g} \in \bar{G} \) such that
\[
\bar{\tau} \bar{b} = \bar{b} \cdot \bar{g} \quad \text{and} \quad \bar{\psi}(q^{-1}(G_b)) = \bar{g}^{-1}(q^{-1}(G_b)) \bar{g}.
\]

Proof. First of all, if an action-automorphism is vertex-fibre preserving, then it is also dart-fibre preserving. Moreover, it is enough to have a requirement that just one vertex-fibre is mapped to a fibre. The proof of this fact is left to the reader. It follows that \( \bar{\tau} + \bar{\tau} \times \bar{\psi} \) projects if and only if \( \bar{\psi} \) maps \( q^{-1}(G_b) \) onto \( q^{-1}(G_{\bar{b}}) \), where \( \bar{b} \in \text{fib}_b \).

Writing \( \bar{\tau} \bar{b} = \bar{b} \cdot \bar{g} \) and taking into account that the stabilizers are conjugate subgroups, we obtain the desired result. \( \square \)

Example 8.13. An action-automorphism \( \bar{\tau} + \bar{\tau} \times \bar{\psi} \) projects along \( p_{\phi,q} : \text{Act}(\bar{Z}, \bar{G}; \bar{S}) \to \text{Cay}(G; S) \) if and only if \( \bar{\psi}(\text{Ker } q) = \text{Ker } q \), and the projection is necessarily an action-automorphism.

Theorem 8.14. The group \( \text{Eq}(\bar{Z})_\bar{G} \) projects along \( p_{\phi,q} : \text{Act}(\bar{Z}, \bar{G}; \bar{S}) \to \text{Act}(Z, G; S) \) if and only if \( N_{\bar{G}}(\bar{G}_b) \leq N_{\bar{G}}(q^{-1}(G_b)) \) (equivalently, \( q(N_{\bar{G}}(\bar{G}_b)) \leq N_{\bar{G}}(G_b) \)). The projected group is a subgroup of \( \text{Eq}(Z)_G \). For regular coverings, the condition simplifies to \( q^{-1}(G_b) \triangleleft N_{\bar{G}}(\bar{G}_b) \).

Proof. Clearly, the projection of an equivariant automorphism is equivariant. By Theorem 8.12, an automorphism \( \bar{\tau} + \bar{\tau} \times \bar{\psi} \) projects if and only if \( \bar{\psi}(\text{Ker } q) = \text{Ker } q \), and the projection is necessarily an action-automorphism. Theorem 8.12, an automorphism \( \bar{\tau} + \bar{\tau} \times \bar{\psi} \) projects if and only if \( \bar{\psi}(\text{Ker } q) = \text{Ker } q \), and the projection is necessarily an action-automorphism.

Example 8.15. Let \( G \) act with a normal stabilizer. Then the group \( \text{Eq}(\bar{Z})_\bar{G} \) projects along \( p_{\phi,q} : \text{Act}(\bar{Z}, \bar{G}; \bar{S}) \to \text{Act}(Z, G; S) \). In particular, it projects along \( p_{\phi,q} : \text{Act}(\bar{Z}, \bar{G}; \bar{S}) \to \text{Cay}(G; S) \).

Example 8.16. Let \( \tilde{M} \to M \) be a homomorphism of oriented maps, where \( M \) is a regular map. Then \( \text{Aut} \tilde{M} \) projects [39].

Corollary 8.17. Let the group \( \text{Eq}(\bar{G})_\bar{G} \) project along \( p_{\phi,q} : \text{Cay}(\bar{G}; \bar{S}) \to \text{Act}(Z, G; S) \). Then \( \text{Act}(Z, G; S) \) is isomorphic to the Cayley graph \( \text{Cay}(G/G_2; S) \).

Proof. By Theorem 8.14 we have \( q^{-1}(G_b) \triangleleft \bar{G} \). Hence \( G_b \triangleleft G \), and the proof follows. \( \square \)
Example 8.18. Let $\tilde{M} \to M$ be a homomorphism of oriented maps, where $\tilde{M}$ is a regular map. In view of Example 8.16 and Corollary 8.17, the group $\operatorname{Aut}(\tilde{M})$ projects if and only if $M$ is also a regular map. (The homomorphism itself must then be regular, see Example 9.3.)

Corollary 8.19. Let the covering projection $p_{\phi,q}: \operatorname{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \operatorname{Act}(Z, G; S)$ be regular. Then the equivariant group $\operatorname{Eq}(Z)_G$ lifts and the equivariant group $\operatorname{Eq}(\tilde{Z})_{\tilde{G}}$ projects if and only if $q(N_G(\tilde{G})) = N_G(G_b)$. In this case, $\operatorname{Eq}(Z)_G$ lifts to $\operatorname{Eq}(\tilde{Z})_{\tilde{G}}$ and $\operatorname{Eq}(\tilde{Z})_{\tilde{G}}$ projects onto $\operatorname{Eq}(Z)_G$.

Proof. By Theorems 8.3 and 8.14 we must have $N_G(G_b) \leq q(N_G(\tilde{G}))$ and $q(N_G(\tilde{G})) = N_G(G_b)$, and the claim follows. The last statement is also evident. □

Example 8.20. The projection $p_{\phi,q}: \operatorname{Cay}(\tilde{G}; \tilde{S}) \to \operatorname{Cay}(G; S)$ is regular (see Example 9.3). The left regular representation of $G$ lifts to the left regular representation of $\tilde{G}$, and hence the latter projects onto the former. In particular, if $\tilde{M} \to M$ is a homomorphism of regular maps, then $\operatorname{Aut}(\tilde{M})$ lifts to $\operatorname{Aut}(M)$ and $\operatorname{Aut}(\tilde{M})$ projects onto $\operatorname{Aut}(M)$ [39].

9. The structure of lifted groups

Theorem 9.1. Consider the covering projection $p = p_{\phi,q}: \operatorname{Act}(\tilde{Z}, \tilde{G}; \tilde{S}) \to \operatorname{Act}(Z, G; S)$ arising from a covering of transitive actions, where $\tilde{S}$ and $S = q(\tilde{S})$ are generating Cayley (multi)sets. Choose $\tilde{b} \in \tilde{Z}$ and $b = \phi(\tilde{b})$ as base-points. Then:

(a) $\operatorname{CT}(p) = \{\tilde{\tau} + \tau \times \operatorname{id} \mid \tilde{\tau} \in \operatorname{Aut}(\tilde{Z})_{\tilde{G}}, \phi(\tilde{\tau}) = \phi\}$ is a subgroup of $\operatorname{Eq}(\tilde{Z})_{\tilde{G}}$.
(b) $\operatorname{CT}(p) = \{\tilde{\tau} + \tau \times \operatorname{id} \mid \tilde{\tau} \tilde{b} \cdot \tilde{g} = \tilde{b} \cdot \tilde{g}, \tilde{a} \in (q^{-1}(G_b) \cap N(\tilde{G}_b)) \mod \tilde{G}_b\}$ is isomorphic to $(q^{-1}(G_b) \cap N(\tilde{G}_b))/\tilde{G}_b$.
(c) The covering projection $p = [q^{-1}(G_b) : \tilde{G}_b]$-fold, and is regular if and only if $\tilde{G}_b \leq q^{-1}(G_b)$ (equivalently, $q^{-1}(G_b) \leq N_G(\tilde{G}_b)$).

Proof. The statement (a) is obvious. Since $\tilde{G}$ is transitive on $\tilde{Z}$ we can explicitly calculate the elements of $\operatorname{Aut}(\tilde{Z})_{\tilde{G}}$ relative to $\tilde{b}$ as $\tilde{\tau} \tilde{b} \cdot \tilde{g} = \tilde{b} \cdot \tilde{g}$, where $\tilde{c} \in N_G(\tilde{G}_b) \mod \tilde{G}_b$.

Example 9.2. Let $H \leq H' \leq G$, and let $p = p_{\phi,\operatorname{id}}: \operatorname{Sch}(G, H; S) \to \operatorname{Sch}(G, H'; S)$ be the corresponding covering projection, where $\phi(H_g) = H'g$. Then $\operatorname{CT}(p) = \{\tilde{\tau} + \tau \times \}$
id | \tilde{h}(Hg) = Hag, a \in (H' \cap N(H)) \mod H \} is isomorphic to \((H' \cap N(H))/H\). The covering projection is \([H':H]-fold, and is regular if and only if \(H < H'\).

**Example 9.3.** A covering \(p_{\phi,q} : \text{Cay}(\tilde{G}; \tilde{S}) \rightarrow \text{Act}(Z;G;S)\) is always regular. Hence a homomorphism \(\tilde{M} \rightarrow M\) of oriented maps, where \(\tilde{M}\) is a regular map, must be a regular homomorphism. In particular, homomorphisms between regular maps are regular [39].

Let \(p : \tilde{X} \rightarrow X\) be a covering projection of connected graphs. Then the lifted group \(\tilde{A} \leq \text{Aut} \tilde{X}\) of \(A \leq \text{Aut} X\) along \(p\) is an extension of \(\text{CT}(p)\) by \(A\). This extension is difficult to analyse in general [9,20,36,38,39,53]. But suppose that the extension splits and that the group \(A\) acts without fixed points. Then each orbit of an arbitrary complement to \(\text{CT}(p)\) within \(\tilde{A}\) intersects each fibre in at most one point (thus forming an *invariant transversal*). This is equivalent [36] with the requirement that it is possible to reconstruct \(p\) by means of a voltage space for which the distribution of voltages is well behaved relative to the action of \(A\). The claim takes a particularly good form whenever the covering projection is regular [36,38]. A straightforward application of these considerations to regular homomorphisms of oriented maps gives Theorem 9.4. A direct proof in terms of voltages associated with angles of the map can be found in [39]. By \(\mathcal{W}^\Omega\) we denote the set of all walks with end vertices in a subset of vertices \(\Omega\).

**Theorem 9.4.** Let \(\tilde{M} \rightarrow M\) be a regular homomorphism of oriented maps, and let \(\Omega\) be an orbit (or a union of orbits) of a dart in \(M\) relative to \(\text{Aut} M\). Then \(\text{Aut} M\) lifts as a split extension of \(\text{CT}\) (the lift of the identity automorphism) if and only if the covering projection of action graphs \(\text{Act}(\tilde{M}) \rightarrow \text{Act}(M)\) can be reconstructed by means of a Cayley voltage space \((\text{CT},\mathcal{W},\xi)\) such that the set of walks \(\{W \in \mathcal{W}^\Omega | \xi_W = 1\}\) in \(\text{Act}(M)\) is invariant under the action of \(\text{Aut} M\). Moreover, the extension is a direct product if and only if \(\text{Act}(\tilde{M}) \rightarrow \text{Act}(M)\) can be reconstructed by a Cayley voltage space \((\text{CT},\mathcal{W},\xi)\) such that each of the sets \(\{W \in \mathcal{W}^\Omega | \xi_W = v\}\) \((v \in \text{CT})\) is invariant under the action of \(\text{Aut} M\).

**Proof.** The map automorphism group corresponds to the equivariant group in the associated action graph, and the lift of the identity automorphism group corresponds to the group of covering transformations for the regular covering projection of the respective action graphs. As the equivariant group acts without fixed points, the theorem follows by applying Theorems 9.1 and 9.3, and Corollaries 9.7 and 9.8 of [38].

10. Generators and relations

Let \(\tilde{S} \subset G\) denote a nonempty antisymmetric subset, that is, \(\tilde{S} \cap \tilde{S}^{-1}\) is either empty or otherwise all of its elements are of order at most 2. With \((Z,G)\) and \(\tilde{S}\) we associate the *action digraph* \(\text{Act}(Z,G;\tilde{S})\) with the vertex-set \(Z\) and the arc-set \(Z \times \tilde{S}\), where \(\text{beg}(z,s) = z\) and \(\text{end}(z,s) = z \cdot s\). Like the case of action graphs, it is sometimes
necessary to consider the set \( \tilde{S} \) as a multiset. The underlying graph of \( \text{Act}(Z; G; \tilde{S}) \) is the action graph \( \text{act}(Z; G; \tilde{S}) = \text{Act}(Z; G; \tilde{S} \cup \tilde{S}^{-1}) \). (Note that the trivial and the involutory loops collapse to semiedges.) Omitting formal basic definitions we only mention that epimorphisms of actions give rise to covering projections of action digraphs.

The study of group presentations involves a variety of techniques, see [13,32,34,50] and the references therein. Action digraphs can often provide much insight into a formal algorithmic approach, and are (in disguise) at least partially present in the original works of Reidemeister and Schreier.

Choose a spanning tree in \( \text{Act}(Z; G; \tilde{S}) \), where \((Z, G)\) is transitive and \( \tilde{S} \) generates \( G \). Each cotree arc gives rise to a unique fundamental closed walk based at \( z \in Z \), and the set of all such walks generates the set of all closed walks at \( z \), up to reduction. Thus, if \( \tilde{C} \) is a set of labels bijectively associated with all the cotree arcs, which evaluate the words in \((\tilde{S} \cup \tilde{S}^{-1})^*\) defined by the fundamental closed walks rooted at \( z \in Z \), then each element of \( G_z \), expressed as a word in \((\tilde{S} \cup \tilde{S}^{-1})^*\), can be written as a word in \((\tilde{C} \cup \tilde{C}^{-1})^*\). This is done by trailing the closed walk associated with a given word in \((\tilde{S} \cup \tilde{S}^{-1})^*\), and simultaneously keeping track of the labels in \( \tilde{C} \cup \tilde{C}^{-1} \) when traversing a cotree arc. The process is known as the rewriting process relative to \( z \in Z \). A variant of the Schreier–Reidemeister theorem now states the following. Let \( G = \langle \tilde{S}; \mathcal{R} \rangle \) be a presentation of \( G \), and let \( \text{Rew} \mathcal{R} \) be the set of (reduced) words in \((\tilde{C} \cup \tilde{C}^{-1})^*\), obtained from all the relators in \( \mathcal{R} \) by a rewriting process relative to all \( z \in Z \). Then the stabilizer \( G_b \) has the presentation \( \langle \tilde{C}; \text{Rew} \mathcal{R} \rangle \). Many of the generators and relators obtained by this method can be redundant. However, sophisticated techniques for simplifying the presentation do exist in certain cases [12,13,34,50,55].

An action digraph \( \text{Act}(Z; G; \tilde{S}) \), where \( \tilde{S} \) generates \( G \), obviously determines the group \( G/G_Z \) up to isomorphism (even if \( G \) is not transitive). Suppose that the action digraph is finite. Denote by \( \tilde{S}_1 \) the generators of the stabilizer \( G_{b_1} \), expressed as words in \((\tilde{S} \cup \tilde{S}^{-1})^*\) associated with fundamental closed walks at \( b_1 \) relative to a spanning tree in the appropriate component of \( \text{Act}(Z; G; \tilde{S}) \). By repeating this process on \( \text{Act}(Z \setminus \{b_1\}, G_{b_1}; \tilde{S}_1) \), \( \text{Act}(Z \setminus \{b_1, b_2\}, G_{b_1, b_2}; \tilde{S}_2) \) and so on, we can recursively construct a generating set for the pointwise stabilizer \( G_Z \). Hence, if \( G \) is faithful, we can find a presentation of \( G \). If \( |\tilde{S}| = n \) and \( |Z| = m \), then the number of generators of \( G_Z \) obtained in this way can amount up to \((n - 1)! + 1\). Thus, the method is not practical unless one can detect sufficiently many redundant generators at each step, or has sufficient control over the recursive construction of the generators. As for the improvements which allow effective computer implementation we refer to [11,12,50] and the references therein.

Despite the remarks above, \( \text{Act}(Z; G; \tilde{S}) \) proves useful in gathering at least partial information about the defining relations, particularly when its underlying graph is highly asymmetric with special structure. The idea is to use graph-theoretical properties of \( \text{Act}(Z; G; \tilde{S}) \) to derive such information.

Example 10.1. Consider the alternating group \( A_n \), where \( n \geq 11 \) is odd, and the generators \( a = (1,2,3,4,5,6,7,8,\ldots,n) \) and \( b = (3,6,1,4,5,7,8,\ldots,n) \). A careful
analysis of the action digraph $\text{Act}(\{1,2,\ldots,n\},A_n;\{a,b\})$ shows that in the Cayley graph $\text{Cay}(A_n;\{a,a^{-1},b,b^{-1}\})$, the cycles of girth-length, which is 6, arise essentially from the relation $(ab^{-1})^3 = 1$, and that cycles of length $n$ arise essentially from the obvious relations $a^n = b^n = 1$. This information is crucial in proving that the above Cayley graph is $\frac{1}{2}$-transitive [37].

What we have discussed so far can be applied to a problem encountered with lifts of automorphisms. Let $p: \text{Cov}(F,\Gamma;\xi) \to X$ be a covering projection of connected graphs (or even more general topological spaces, see [35,36]), given by means of a voltage space $(F,\Gamma;\xi)$, where $\Gamma$ acts faithfully on $F$. A necessary condition (also sufficient if the covering is regular) for an automorphism to lift is that the set of all closed paths with trivial voltage should be invariant under its action [36,38]. In order to test this effectively (assuming of course, that the covering has finite number of folds and that the fundamental group of $X$ is finitely generated) we only need the generators of the kernel of $\xi: \pi^b \to \Gamma$, expressed in terms of a generating set $\tilde{S}$ of $\pi^b$, save for those cases where ad hoc techniques apply.

One possibility is to consider the auxiliary regular covering $\text{Cov} = \text{Cov}(\Gamma,\Gamma;\xi) \to X$. (It is always possible to replace the original voltage space by one in which the local group is equal to the whole group. Thus, $\text{Cov}$ can be assumed connected; otherwise, we consider one of its components.) The required generators of $\text{Ker} \xi$ are then obtained by projecting the generators of $\pi(\tilde{b},\text{Cov})$, where $\tilde{b} \in \text{fib}_b$. However, this requires the construction of $\text{Cov}(\Gamma,\Gamma;\xi)$, which is not always convenient.

A better alternative is to consider the fundamental group $\pi^b$ acting on the local group $\Gamma^b$ by the rule $v \cdot W = v \xi_W$. The stabilizer of this action is $\text{Ker} \xi$, and so the required generators can be found by means of a spanning tree in $\text{Act}(\Gamma^b,\pi^b;\tilde{S}) \cong \text{Cay}(\Gamma^b;\tilde{\xi}(\tilde{S}))$. The generators, expressed as words in $(\tilde{S} \cup \tilde{S}^{-1})^*$, are determined by the corresponding fundamental closed walks. Alternatively, we first construct the coset representatives of $\text{Ker} \xi$ within $\pi^b$, that is, for each $v \in \Gamma^b$ we find a closed path (rooted at the base vertex) with voltage $v$. With these representatives we can then apply some variant of the Schreier method to obtain the required generators.

Yet another possibility is to consider the action of $\pi^b$ on the abstract fibre $F$ given by $i \cdot W = i \cdot \xi_W$. The kernel $\text{Ker} \xi$ is then equal to the pointwise stabilizer of this action. Thus, the required generators can be found recursively by considering the action digraph $\text{Act}(F,\pi^b;\tilde{S})$.

A similar problem is encountered with the question whether a group $A \leq \text{Aut} X$ lifts along a regular covering projection $p: \tilde{X} \to X$ as a split extension of $\text{CT}(p)$ with an invariant transversal over an invariant subset $A(\Omega) = \Omega$ (see Section 9 and [36,38]).

This can be tested as follows (we tacitly assume that the fundamental group of $X$ is finitely generated, that $p$ has finite number of folds, that $A$ is finitely generated and that $\Omega$ is finite). Suppose that $p$ is given by means of a regular voltage space (the voltage action is regular). By considering all the possible transversals above $\Omega$ and using the explicit formulas for the lifts of the generators of $A$ [36,38], check whether the chosen transversal is indeed invariant for certain lifts of these generators.
However, there is an alternative method, namely, the group $A$ has the required type of lift if and only if there exists a regular voltage space $(F, \Gamma; \xi)$ which reconstructs the covering and has the following property: the set $\mathcal{W}^\Gamma_1 = \{ W \in \mathcal{W}^\Gamma \mid \xi_W = 1 \}$ of trivial voltage paths with endpoints in $\Omega$ is invariant under the action of $A$ [36,38] (recall Theorem 9.4). In other words, relative to $(F, \Gamma; \xi)$, the invariant transversal above $\Omega$ is labelled by the same label. In general, the number of voltage spaces that have to be checked is equal to the number of all the transversals over $\Omega$. Of course, with each individual voltage space the property needs to be tested just on some generating set of $\mathcal{W}^\Gamma_1$.

Now, let $p: \tilde{X} \to X$ be a covering projection of finite connected graphs (or finite connected CW-complexes) given by some regular voltage space valued in $\Gamma$. We proceed as follows. Introduce a new vertex $B$ not in $X$, connect $B$ with all the vertices in $\Omega$, and extend the voltages valued in $\Gamma$ to the new arcs arbitrarily. Then modify the voltage space to an equivalent one by means of an arbitrary spanning tree so that the new arcs carry the trivial voltage [19]. The required generating set of walks is obtained as before by considering the extended graph (or CW-complex) with $B$ as the base point (and then omitting the artificially added arcs). If the test of the above property is positive, then the group $A$ lifts as the required split extension along $p$ (equivalently, viewed as the stabilizer of $B$ in the extended graph (CW-complex), it lifts along the extended covering projection). The invariant transversal is determined by the voltages assigned to the artificially added arcs at $B$. If the test is negative, we repeat the process using another voltage assignment for the arcs at $B$. Thus, in principle we have to consider $|\mathcal{W}^\Gamma_1|$ different voltage spaces.

Part of the preceding discussion is summarized in the following theorem.

**Theorem 10.2.** With notation and assumptions above, the problem—whether a given group $A$ of automorphisms of a graph $X$ (or a more general topological space) lifts along a regular covering projection given by a regular voltage space $(F, \Gamma; \xi)$—can be tested in time proportional to the number of generators of $A$, multiplied by the time required for the construction of $\text{Cay}(\Gamma; \xi(\tilde{S}))$ and its spanning tree together with checking whether the voltages of the mapped fundamental walks are trivial.

Let $X$ be a graph (or a finite CW-complex) and $(F, \Gamma; \xi)$ a regular voltage space on $X$. Then the problem—whether a group $A \subseteq \text{Aut}X$ lifts along the derived regular covering as a split extension with an invariant transversal over an invariant subset of vertices $\Omega = A(\Omega)$—can be tested in time proportional to the time required to solve the first problem above, multiplied by $|\Gamma|^{1/2}$, and multiplied by the time required to modify an existing voltage space to an equivalent one employing a spanning tree.

**References**

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[53] A. Venkatesh, Graph coverings and group liftings, submitted for publication.
