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A NEW COLOR CHANGE TO IMPROVE THE COLORING OF A GRAPH

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For some special classes of graphs an optimal coloring can be obtained by the "Kempe method": color as many vertices as possible with q given colors, and then make one of these colors available for an uncolored vertex x_0 by interchanging the two colors of a "bicolor component". It is well known that in general these interchanges cannot lead to an optimal coloring.

We shall define here another type of color changes which can always lead to an optimal coloring. The basic tool is the concept of an "odd alternating sequence".

We shall prove:

Theorem 1. *Let $G=(X,E)$ be a simple graph with vertex set X, and let $x_0 \in X$. Suppose that we have obtained a q-coloring $\mathcal{S}_q=(S_1,S_2,\ldots,S_q)$ for the subgraph of G induced by $X-\{x_0\}$. Then G is q-colorable if and only if there exists an alternating sequence*

$$\sigma = [x_0(i_0), b_0(j_0), x_1(i_1), b_1(j_1), \ldots, x_{p+1}(i_{p-1})]$$

with the following properties:

(1) *the b_k are distinct vertices of $X-\{x_0\}$, and $b_k \in S_{j_k}$;*

(2) *the x_k are distinct vertices of X, and $x_k \notin S_{i_k}$;*

(3) *for $k=0,1,\ldots,p$, the pair $b_k(j_k)$ is such that some previous element $x_\lambda(i_\lambda)$, $\lambda \le k$, satisfies either one of*

(a) $i_\lambda = j_k$ *and* $[b_k, x_\lambda] \in E$,

(b) $i_\lambda \ne j_k$ *and* $x_\lambda = b_k$;

(4) *for $k=1,2,\ldots,p+1$, the pair $x_k(i_k)$ is such that no $x_\lambda(i_\lambda)$ with $\lambda<k$ satisfies both $[x_k,x_\lambda]\in E$ and $i_k=i_\lambda$;*

(5) *for $k=1,2,\ldots,p+1$, the pair $x_k(i_k)$ is such that some previous element $b_\lambda(j_\lambda)$ with $\lambda<k$ satisfies either one of*

(a) $j_\lambda = i_k$ *and* $[x_k, b_\lambda] \in E$,

(b) $j_\lambda \ne i_k$ *and* $x_k = b_\lambda$;

(6) *the sequence σ is maximal with respect to the rules* (1) *and* (3), *i.e. no $b_{p-1}(j_{p+1})$ can be added without violating these rules.*

Proof. (1) *If there exists such a sequence* σ put $B=\{b_0, b_1, \dots, b_p\}$, $A=\{x_0, x_1, \dots, x_{p-1}\}$. First, we remove the colors of the vertices in B, and then we assign to $x_k \in A$ the color i_k for $k=0,1,2,\dots,p+1$.

The result is a coloring of G, because the new color assigned to x_k does not conflict with the new color assigned to x_λ if $\lambda<k$ (rule (4)); it does not conflict with the color of an unchanged vertex, because otherwise such a vertex b_{p+1} could be added to the sequence, and this contradicts the maximality of σ. On the other hand, the number of colored vertices in σ was $|B|$ in the old coloring (by rule (3) and the maximality of σ), and is $|A|=|B|+1$ in the new coloring. Thus we have obtained a complete q-coloring of G.

(2) *If G is q-colorable*, put $Q=\{1,2,\dots,q\}$, and consider the complete graph K_q on Q. The Cartesian sum $G+K_q$ is a graph on $X\times Q$ where (x,i) and (y,j) are joined if and only if $[x,y]\in E$ and $i=j$, or $x=y$ and $i\neq j$. Every stable set S_0 of $G+K_q$ defines a partial q-coloring of G as follows: color the vertex x of G with color i if and only if $(x,i)\in S_0$. Conversely, every partial q-coloring of G defines a stable set of $G+K_q$.

If G is q-colorable, then a q-coloring $\mathscr{S}_q=(S_1, S_2, \dots, S_q)$ of $G-\{x_0\}$ corresponds in $G+K_q$ to a stable set B_0 which is not maximum, and an optimal q-coloring $\mathscr{S}_q'$ of G corresponds to a maximum stable set A_0. Then, a known result [1, Chapter 13, Theorem 2] shows that there exists in $G+K_q$ an alternating sequence

$$\bar{\sigma}=(a_1, b_1, a_2, b_2, \dots, a_p)$$

with $a_i\in A_0$, $b_j\in B_0$, such that

(a) $a_1=x_0\in A_0$;
(b) $b_k\in B_0-\{b_1, b_2, \dots, b_{k-1}\}$ and $\Gamma_G(b_k)\cap\{a_1, b_1, a_2, b_2, \dots, a_k\}\neq\emptyset$;
(c) $a_k\in A_0-\{a_1, a_2, \dots, a_{k-1}\}$ and $\Gamma(a_k)\cap\{b_1, b_2, \dots, b_{k-1}\}\neq\emptyset$;
(d) the sequence $\bar{\sigma}$ *is maximal.*

It is easy to check that such a sequence corresponds in G to a sequence σ defined by the rules (1)–(6).

The procedure is much simpler for a claw-free graph (i.e. a graph without an induced $K_{1,3}$).

Let G be a claw-free graph with only one vertex x_0 which is left uncolored in a partial q-coloring $\mathscr{S}_q$ and consider an odd alternating sequence $\sigma=[x_0(i_0), b_0(j_0), x_1(i_1), \dots]$ as before. We shall consider an arborescence H with vertices $x_0(i_0)$, $b_0(j_0)$, $x_1(i_1), \dots$, whose arcs are the "heavy arcs" and the "light arcs" defined as follows:

(i) if $b_k(j_k)$ has been obtained by rule (3a) from an element $x_\lambda(i_\lambda)$, $\lambda<k$, draw a directed heavy line from $x_\lambda(i_\lambda)$ to $b_k(j_k)$;

(ii) if $b_k(j_k)$ has been obtained by rule (3b), draw a directed light line from $x_\lambda(i_\lambda)$ to $b_k(j_k)$;

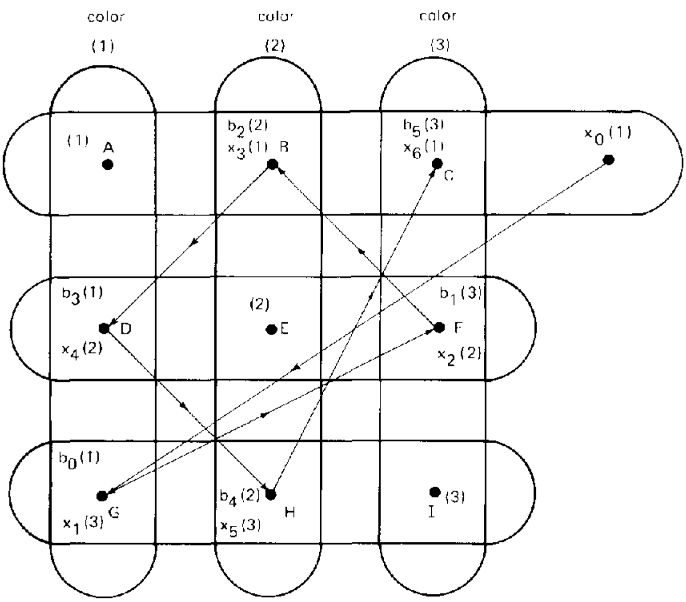

Fig. 1.

(iii) if $x_k(i_k)$ has been obtained by rule (5a) from an element $b_\lambda(j_\lambda)$, draw a directed heavy line from $b_\lambda(j_\lambda)$ to $x_k(i_k)$;

(iv) if $x_k(i_k)$ has been obtained by rule (5b), draw similary a directed light line from $b_\lambda(j_\lambda)$ to $x_k(b_k)$.

Consider with a partial 3-coloring a graph G consisting of 10 vertices $A, B, C, D, E, F, G, H, I, x_0$ as in Fig. 1; all the pairs of vertices which are not on the same row or on the same column are joined by an edge, so that the sets ADG, BEH, CFI are stable and are respectively of color 1, 2, 3.

No color is available for the vertex x_0. A sequence of interchanges does not help, because interchanging the colors i and j in a bicolor component C_{ij} gives a new coloring which is equivalent to the previous one. Thus the Kempe method is hopeless.

But an iterative procedure to construct an arborescence H which is a path gives immediately an optimal 3-coloring of G.

For instance try: $i_0 = 1$. No choice is left for the alternating sequence $x_0(1), b_0(1), x_1(3), b_1(3), x_2(2), \ldots, x_6(1)$ corresponding to a path: $x_0GFBDHC$. This odd sequence gives immediately an optimal 3-coloring (see Fig. 1).

If we try $i_0 - 2$, we obtain also an odd alternating sequence corresponding to a path $x_0HFGCEA$, and an optimal 3-coloring.

Open questions. Does Theorem 1 allow to obtain a good estimate for the complexity of the coloring problem (in a claw-free graph)?

Does Theorem 1 allow to simplify the proof of Vizing's theorem?

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Reference

[1] C. Berge, Graphs, North-Holland Mathematical Library 6(2) (North-Holland, Amsterdam, 1985) Chapter 13.