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A Class of Vector Fields on Path Spaces

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In this paper we show that the vector field $X^{\nabla, h}$ on a based path space $W_o(M)$ over a Riemannian manifold M defined by parallel translating a curve h in the initial tangent space $T_o M$ via an affine connection ∇ induces a solution flow which preserves the Wiener measure on the based path space $W_o(M)$ provided the affine

1. INTRODUCTION

Let (M, g) be a compact, d-dimensional Riemannian manifold, let Δ be the Laplace–Beltrami operator, and let $W(M)$ be the path space of all continuous paths in M with time length $[0, 1]$. For any $o \in M$, there is a unique probability measure $\mathbb{P}^o$ on the paths x with $x(0) = o$ such that $(W(M), x(t), \mathbb{P}^o)$ is a $\frac{1}{2}\Delta$-diffusion process. We fix a base point $o \in M$, and for simplicity use ν to denote the diffusion measure $\mathbb{P}^o$. The measure ν is called canonical Wiener measure, and is supported on the based path space $W_o(M)$.

Suppose we are given a metric connection ∇. By parallel translating a curve h in the initial tangent space $T_o M$, we obtain a vector field X^h on the path space $W_o(M)$: by definition, for any $\sigma \in W_o(M)$, define $X^h(\sigma)$ to be the vector field along the curve σ given by $X^h(\sigma)_s = \gamma_s h_s$, where γ is the unique horizontal lift of σ, based on a fixed point u_o in the orthonormal frame bundle $O(M)$, $\pi(u_o) = o$. Note that X^h is only well defined ν-almost surely.

It is natural to ask if there is a *solution flow* on the based path space $W_o(M)$ induced by the vector field X^h. Because X^h is only well defined ν-almost surely, one should ask that does the solution flow leave the Wiener measure ν quasi-invariant? These problems are considered and solved by B. Driver [7] for a metric connection ∇ on the Riemannian manifold (M, g).

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B. Driver [7] has proved that the vector field X^h induces a quasi-invariant flow on the based path space $W_o(M)$ provided the metric connection ∇ is torsion skew-symmetric (TSS) with respect to the metric g.

B. Driver [8, 9, 10] further considered a class of so-called geometric vector fields on the based path space—those vector fields are obtained by parallel translating adapted vector fields on the Wiener space via a metric and torsion skew-symmetric connection ∇. More precisely, if K is an adapted vector field on T_oM, i.e. K is a d-dimensional continuous semimartingale which has form

$$K_s = \int_0^s C_u \, db_u + \int_0^s R_u \, du,$$

where C_u is a Lie algebra $o(d)$-valued adapted process, R_u is an $\mathbb{R}^d$-valued adapted process, and b is a standard d-dimensional Brownian motion, then we formally define a geometric vector field X^K on the based path space $W_o(M)$ by setting $X^K(\sigma)_t = \gamma_t K_t$, where γ is the horizontal lift of σ. Essentially an adapted vector fields on the Wiener space is the tangent vector field of a progressive measurable transformation flow which leaves the Wiener measure quasi-invariant.

It is showed in B. Driver [7, 8, 9, 10] that the pull-back vector field M^K on the Wiener space of a geometric vector field X^K by the Itô's development map (defined by the connection ∇) is still an adapted vector field, so that such a geometric vector field X^K induces a solution flow on the based path space leaving the Wiener measure quasi-invariant.

In this paper we consider vector fields $X^{\nabla, h}$ on the based path space $W_o(M)$ defined via any affine connection ∇, and we shall prove that the vector field $X^{\nabla, h}$ induces a quasi-invariant flow on the path space $W_o(M)$ provided ∇ is adjoint skew-symmetric. That is we will show that $X^{\nabla, h}$ is a geometric vector field in the sense of B. Driver [8, 9, 10], provided ∇ is adjoint skew-symmetric. If the affine connection ∇ preserves the Riemannian metric g, then our adjoint skew-symmetric condition is equivalent to that ∇ is torsion skew-symmetric. However an adjoint skew-symmetric affine connection does not need to be metric.

The study of vector fields on the piece-wisely smooth path space $W^\infty(M)$ of a smooth manifold M is a classical topic in differential geometry, for example, in the study of geodesics on a Riemannian manifold, cf. [28, 31]. The study of adapted vector fields on the continuous path space $W(M)$ (a reason to study continuous path space, rather than the smooth path space, is that there is not any natural measure on smooth path space) has been motivated by attempts to establish a geometric analysis on the path space $W(M)$ endowed with for example a Brownian motion measure, cf. [7, 13, 15, 17, 22, 24, 26] etc., where it is needed to extend classical Cameron–

Martin theorem for Wiener measure [3, 4, 5] to the case of curved spaces. Equivalently we need a nonlinear extension of Cameron–Martin quasi-invariance theorem of Wiener measure under translation in directions in the Cameron–Martin space, cf. [25, 30] and the references in [7].

Let $(W_o(\mathbb{R}^d), \mu)$ be the Wiener space and let T_ω be the measurable transformation from $W_0(\mathbb{R}^d)$ to itself; $T_\omega z = z + \omega$. Then the measure $T_\omega^{-1} \circ \mu$ is absolutely continuous with respect to the Wiener measure μ if and only if $\omega \in \mathbb{H}^1(\mathbb{R}^d)$: the Cameron–Martin space of all $\mathbb{R}^d$-valued absolutely continuous functions on $[0, 1]$ with L^2-integrable first order derivatives. Instead of considering the transformation T_ω we may consider a transformation flow $(T_\omega^t)_{t \in \mathbb{R}}$ on $W_0(\mathbb{R}^d)$ defined by

$$T_\omega^t z_. = z_. + t\omega_., \qquad \forall z \in W_0(\mathbb{R}^d),$$

so that $T_\omega^1 = T_\omega$, and

$$\frac{\partial T_\omega^t z}{\partial t} = \omega, \qquad \forall t \in \mathbb{R},\ z \in W_0(\mathbb{R}^d).$$

That is, $(T_\omega^t)_{t \in \mathbb{R}}$ is the solution flow of the vector field $M^\omega(z) = \omega$.

Thus it is natural to study nonlinear transformations on the Wiener space generated by a vector field. A. B. Cruzeiro [6] first solved a class of differential equations on the Wiener space induced by a special class of vector fields, and proved quasi-invariance of the solution flows. P. Malliavin [22] proposed two classes of natural vector fields on the path space over a Riemannian manifold, and studied their induced solution flows [24].

A significant breakthrough has been made by B. Driver [7] in which he has proved the pull-back on the Wiener space of the vector field X^h obtained by parallel translating a curve in the initial tangent space via a metric connection is an adapted vector field, so that he was able to solve the geometric flow equation, and extend the Cameron–Martin quasi-invariance theorem to the case of the based path space.

The paper is organised as follows. In Section 2, we recall several basic facts in differential geometry and introduce the notion of adjoint skew-symmetric connection. Section 3 is the main part of this paper, in which we deduce a formula for the pull-back vector field. We define a vector field on the based path space over a Riemannian manifold via an affine connection. The main difference from that of B. Driver [7] is that we always pull back a vector field via the Levi–Civita's connection rather than the affine connection ∇, because the law of the diffusion generated by the projection of the Bochner–Laplacian defined by an affine connection may not be absolutely continuous with respect to the Wiener measure. As soon as a pull-back formula of a vector field is obtained, we can employ the same strategy in B. Driver [7] to construct a solution flow. Thus in Section 4, we recall a result about solution flow on the Wiener space induced by a so-called Itô's vector field. Finally in Section 5 we use results in Sections 4 to obtain a solution flow on the based path space, and followed by an integration by parts formula.

2. AFFINE CONNECTIONS

In this section we recall several basic notions and facts about affine connections, and establish notations as well. Our standard references are [11, 29].

Let M be a d-dimensional smooth manifold, and let $\pi: L(M) \to M$ be its frame bundle. $L(M)$ is a principal fibre bundle with its structure group $GL(d)$. Each $u \in L(M)$ is a frame of the tangent space $T_x M$ at $x = \pi(u)$, so that u is an isomorphism from $\mathbb{R}^d$ to $T_x M$ determined by

$$u\xi = \xi^i E_i, \qquad \text{if} \quad u = (x, (E_1, ..., E_d)),$$

where (E_i) is a base of $T_x M$, and we use the convention of summation over repeated upper-lower indices.

The canonical 1-form θ on $L(M)$ is the $\mathbb{R}^d$-valued 1-form defined by

$$\theta(V) = u^{-1}\pi_*(V), \qquad \forall V \in T_u L(M),$$

where π_* is the differential of the bundle map π.

The vertical tangent bundle $VTL(M)$ is the kernel of the canonical 1-form θ.

An affine connection thus is an assignment of a complementary horizontal tangent bundle $HTL(M)$ such that

$$T_u L(M) = VT_u L(M) \oplus HT_u L(M), \qquad \forall u \in L(M),$$

each $HT_u L(M)$ is invariant under the natural right action of $GL(d)$ on the frame bundle $L(M)$, and the field $u \to HT_u L(M)$ is smooth, for detail, see [29].

An affine connection can be described via a connection form ω; by definition which is a $gl(d)$-valued 1-form on $L(M)$ satisfying the following conditions:

(1) $\omega \circ (R_a)_* = \mathrm{ad}(a^{-1}) \circ \omega$, for any $a \in GL(d)$, R_a is the right action, and $\mathrm{ad}(a^{-1})$ is the adjoint action of $GL(d)$ on its Lie algebra $gl(d)$.

(2) $\omega(A^*(u)) = A$, for any $A \in gl(d), u \in L(M)$, where $A^*(u)$ the fundamental tangent vector of A, i.e. $A^*(u)$ is the differential of $t \to R_{\exp(tA)} u$ at $t = 0$.

It is clear that $A^*(u) \in VT_uL(M)$. Given a connection form ω on $L(M)$, the horizontal tangent bundle is defined to be the kernel of the connection 1-form ω.

We will use $h\colon TL(M) \to HTL(M)$ to denote the projection along the vertical tangent bundle $VTL(M)$.

Given a connection, the torsion form Θ and the curvature form Ω are 2-forms on $L(M)$ defined by

$$\Theta(V_1, V_2) = d\theta(hV_1, hV_2), \qquad \forall V_1, V_2 \in TL(M)$$

and

$$\Omega(V_1, V_2) = d\omega(hV_1, hV_2), \qquad \forall V_1, V_2 \in TL(M)$$

respectively, where d denotes the exterior differential, and for any 1-form α we use the convention that

$$d\alpha(V_1, V_2) = V_1\alpha(V_2) - V_2\alpha(V_1) - \alpha([V_1, V_2]),$$

if V_1, V_2 are two vector fields.

The first structure equation:

$$d\theta(V_1, V_2) = -\omega(V_1)\theta(V_2) + \omega(V_2)\theta(V_1) + \Theta(V_1, V_2), \qquad \forall V_1, V_2 \in TL(M). \tag{1}$$

The second structure equation:

$$d\omega(V_1, V_2) = -[\omega(V_1), \omega(V_2)] + \Omega(V_1, V_2), \qquad \forall V_1, V_2 \in TL(M). \tag{2}$$

We say an affine connection ω is torsion free if $\Theta = 0$.

Given a C^1-curve σ in M, $\sigma(0) = x$ and a base point $u \in L(M)$ such that $\pi(u) = x$, there is a unique horizontal lift γ of σ such that $\gamma(0) = u$, where we say γ is the horizontal lift of σ if $\pi(\gamma(t)) = \sigma(t)$ and $\dot\gamma(t) \in HTL(M)$. Hence any tangent vector $X \in T_xM$ there is a unique horizontal lift $\tilde X \in HT_uL(M)$ such that $\pi_*(\tilde X) = X$, for any $u \in L(M)$, $\pi(u) = x$.

For any $u \in L(M)$, and $\xi \in \mathbb{R}^d$, we use $B(u)\xi$ (or $B_u\xi$) to denote the unique horizontal lift in $T_uL(M)$ of $u\xi \in T_{\pi(u)}M$. Then $B\colon L(M) \to \mathbb{L}(\mathbb{R}^d, TL(M))$ is a smooth section of the fibre bundle $\pi\colon \mathbb{L}(\mathbb{R}^d, TL(M)) \to L(M)$. B is called the standard horizontal vector field, although it defines d vector fields on $L(M)$.

By definition $\omega(B_u\xi) = 0$, and

$$\theta(B_u\xi) = \xi, \qquad (R_a)_*(B_u\xi) = B_{R_au}(a^{-1}\xi),$$

for any $u \in L(M)$, $\xi \in \mathbb{R}^d$ and $a \in GL(d)$.

Given an affine connection ω, the covariant derivative ∇ is defined by

$$\nabla_X Y(x) = u \frac{d}{dt} \gamma_t^{-1} Y(\gamma(t)) \bigg|_{t=0}, \qquad \forall u \in L(M), \pi(u) = x,$$

for $X \in T_x M$, a vector field Y on M, where γ is the unique horizontal lift (with based point u) of σ, and any curve σ in M such that $\sigma(0) = x$, $\dot{\sigma}(0) = X$. Also we call a covariant derivative ∇ an affine connection.

We next assume that M is endowed with a Riemannian metric g. An affine connection ω is called a metric connection if its corresponding covariant derivative ∇ preserves the metric g, i.e.

$$dg(X, Y)(Z) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$$

for any vector fields X, Y, Z on M.

If ω is a metric connection, then ω restricted on the orthonormal frame bundle $O(M)$ of (M, g) is $o(d)$-valued, so is its curvature form Ω. $O(M)$ is a principal bundle with its structure group $O(d)$; $o(d)$ is the Lie algebra of $O(d)$.

The unique torsion free, metric connection is called the Levi–Civita connection, denoted by D.

If ∇, $\tilde{\nabla}$ are two affine connections, then

$$S(X, Y) = \nabla_X Y - \tilde{\nabla}_X Y,$$

defines a (1,2)-type tensor on the manifold M. In this case we write $\nabla = \tilde{\nabla} + S$.

Definition 1. Let S be a (1, 2)-type tensor on M, and g be a Riemannian metric. Then we say S is skew-symmetric with respect to the metric g if

$$g(S(Z, X), Y) + g(S(Z, Y), X) = 0, \qquad \forall X, Y, Z \in TM,$$

and we say S is adjoint skew-symmetric with respect to the metric g if

$$g(S(X, Z), Y) + g(S(Y, Z), X) = 0, \qquad \forall X, Y, Z \in TM,$$

respectively.

Proposition 1. (1) *Let $\nabla = D + S$ be an affine connection on the Riemannian manifold (M, g). Then ∇ is a metric connection if and only if S is skew-symmetric.*

(2) *Let $\nabla = D + S$ be a metric connection on (M, g). Then S is adjoint skew-symmetric if and only if ∇ is torsion skew-symmetric: torsion tensor of the connection ∇ is skew-symmetric with respect to the metric g.*

Remark. 1. If a metric connection ∇ is torsion skew-symmetric, then the projection on M of the Bochner–Laplacian $L_\nabla = \sum_i B_i B_i$ is the Laplace–Beltrami operator, $B_i = Be_i$, where (e_i) is the standard base of $\mathbb{R}^d$ and B is the standard horizontal vector field defined by ∇.

2. If ∇ is a metric connection and $\nabla_X Y = D_X Y + S(X, Y)$, then $\tilde{\nabla}_X Y = D_X Y + S(Y, X)$ is an adjoint skew-symmetric affine connection.

3. We say an affine connection is adjoint skew-symmetric if $S = \nabla - D$ is adjoint skew-symmetric.

3. NATURAL VECTOR FIELDS

In the sequel we are working with a Riemannian manifold (M, g) with its Levi–Civita connection D. Points $o \in M$ and $u_o \in O(M)$, $\pi(u_o) = o$, are fixed. The curvature form, development map, horizontal lift etc. are those defined by the Levi–Civita connection D, and the based point u_o if appropriate, except otherwise specified.

For a manifold N and a point $u \in N$, we will use $W_u^\infty(N)$ (resp. $W_u(N)$) to denote the family of all continuous, piece-wisely smooth paths (resp. the space of all continuous paths) with time length $[0, 1]$ starting at u.

In order to introduce a class of vector fields on the path space $W_o(M)$, we first recall the development map associated to the Levi–Civita connection D.

Let $\sigma \in W_o^\infty(M)$. Then its unique horizontal lift γ is the unique solution of the differential equation:

$$d\gamma_t = B(\gamma_t)\,\gamma_t^{-1} \circ d\sigma_t, \qquad \gamma_0 = u_o.$$

Let $z_t = \int_0^t \theta(\circ\, d\gamma_s)$. Then it is easily seen that γ is the unique solution to the differential equation:

$$d\gamma_t = B(\gamma_t) \circ dz_t, \qquad \gamma_0 = u_o, \tag{3}$$

and $\sigma_t = \pi(\gamma_t)$. We will denote by $\gamma = I(z)$, $z = J^{-1}(\sigma)$, and by $\sigma = J(z)$. I (or J) is called the development map (or Itô map if involved paths are semimartingales).

Given a $W \in T_\sigma W_o^\infty(M)$, i.e. W is a smooth vector field along the path $\sigma \in W_o(M)$ starting at zero, so that $W(t) \in T_{\sigma(t)} M$. Let $z = J^{-1}(\sigma)$. Then we define

$$M = J_*^{-1} W,$$

the pull-back of the tangent vector W by the development map J^{-1}. By definition, if $\sigma^{\varepsilon} \in W_o(M)$ such that $\sigma^0 = \sigma$ and

$$\left.\frac{d\sigma^{\varepsilon}(t)}{d\varepsilon}\right|_{\varepsilon=0} = W(t), \qquad \forall t \in [0, 1],$$

then

$$M_t = \left.\frac{dz^{\varepsilon}(t)}{d\varepsilon}\right|_{\varepsilon=0}, \qquad z^{\varepsilon} = J^{-1}(\sigma^{\varepsilon}).$$

Using the first and the second structure equations, we can obtain the following formula

$$M_t = K_t - \int_0^t \int_0^s \Omega(B_{\gamma_u} \circ dz_u, B_{\gamma_u} K_u) \circ dz_s, \tag{4}$$

where $K_t = \gamma_t^{-1} W_t$, and γ is the unique solution of Eq. 3. Note that K is $\mathbb{R}^d$-valued. Formula 4 can be deduced by the use of the same method as in [7, 18]. If $W(t) = \gamma_t h_t$, then $K_t = h_t$.

We are now given another affine connection $\tilde{\omega}$ with its corresponding covariant derivative ∇. We use $\tilde{B}$ to denote the standard horizontal vector field defined by $\tilde{\omega}$. Let $\sigma \in W_o(M)$, and $z = J^{-1}(\sigma)$, and γ be the unique horizontal lift of σ, i.e. γ is the unique solution of Eq. 3. Using $\tilde{\gamma}$ to denote the unique horizontal lift of σ under the connection $\tilde{\omega}$, so that $\tilde{\gamma}$ is the unique solution of the differential equation:

$$d\tilde{\gamma}_t = \tilde{B}(\tilde{\gamma}_t)\, \tilde{\gamma}_t^{-1} \circ d\sigma_t, \qquad \gamma_0 = u_o. \tag{5}$$

Note that $\gamma_t \in O(M)$, $\tilde{\gamma}_t \in L(M)$, and

$$\pi(\gamma_t) = \pi(\tilde{\gamma}_t) = \sigma_t.$$

Define $X^{\nabla, h}$ to be a vector field on $W_o(M)$ for $h \in \mathbb{H}(\mathbb{R}^d)$ by

$$X^{\nabla, h}(\sigma)_t = \tilde{\gamma}_t h_t, \qquad \forall \sigma \in W_o(M). \tag{6}$$

Let $z = J^{-1}(\sigma)$ and define

$$M^{\nabla, h}(z) = J_*^{-1} X^{\nabla, h}(\sigma). \tag{7}$$

By formula 4, we need to calculate $K(z)_t = \gamma_t^{-1} \tilde{\gamma}_t h_t$.

To this end, we set $H_t = \gamma_t^{-1} \tilde{\gamma}_t$. Then $H_t \in \mathbb{L}(\mathbb{R}^d, \mathbb{R}^d)$, and $H_0 = \mathrm{id}$.

Let S be the (1,2)-type tensor defined by

$$\nabla_X Y = D_X Y + S(X, Y).$$

In a local chart $(x^1, ..., x^d)$, let $ds^2 = g_{ij}\,dx^i \otimes dx^j$, $D = (\Gamma^k_{ij})$, and $\nabla = (\tilde{\Gamma}^k_{ij})$, so that $S^k_{ij} = \tilde{\Gamma}^k_{ij} - \Gamma^k_{ij}$.

We can write

$$\gamma(t) = (\sigma_t, (E_1(t), ..., E_d(t))), \qquad E_i(t) \in T_{\sigma_t} M,$$

and

$$\tilde{\gamma}(t) = (\sigma_t, (\tilde{E}_1(t), ..., \tilde{E}_d(t))), \qquad \tilde{E}_i(t) \in T_{\sigma_t} M.$$

Note that $g(E_i, E_j) = \delta_{ij}$. For any $\xi \in \mathbb{R}^d$, we have

$$H_t \xi = H^j_i(t)\, \xi^i e_j,$$

where $(e_1, ..., e_d)$ is the standard base of $\mathbb{R}^d$, and

$$\begin{aligned} H^j_i(t) &= g(\tilde{E}_i(t), E_j(t)) \\ &= g_{kl}(\sigma(t))\, E^l_j(t) \tilde{E}^k_i(t). \end{aligned}$$

Using these notations, we can write Eq. 3 and Eq. 5 in local chart (cf. [27]):

$$\begin{aligned} d\sigma^i(t) &= E^i_\alpha(t) \circ dz^\alpha(t), \\ dE^i_j(t) &= -\Gamma^i_{mk}(\sigma(t))\, E^k_j(t) \circ d\sigma^m(t), \end{aligned}$$

and

$$\begin{aligned} d\sigma^i(t) &= \tilde{E}^i_\alpha(t) \circ d\tilde{z}^\alpha(t), \\ d\tilde{E}^i_j(t) &= -\tilde{\Gamma}^i_{mk}(\sigma(t))\, \tilde{E}^k_j(t) \circ d\sigma^m(t), \end{aligned}$$

respectively, where $\tilde{z} = \tilde{J}^{-1}(\sigma)$, $\tilde{J}$ is the development map defined via the connection $\tilde{\omega}$. Hence

$$d\tilde{E}^i_j(t) = -\tilde{\Gamma}^i_{mk}(\sigma(t))\, \tilde{E}^k_j(t)\, E^m_\alpha(t) \circ dz^\alpha(t). \tag{8}$$

Since E_i is an orthonormal base, we have

$$\tilde{E}_j(t) = g(\tilde{E}_j(t), E_i(t))\, E_i(t) = H^i_j(t)\, E_i(t),$$

so that

$$\tilde{E}^k_j(t) = H^i_j(t)\, E^k_i(t). \tag{9}$$

Inserting Eq. 9 into Eq. 8, we get

$$d\tilde{E}_j^i = -\tilde{\Gamma}^i_{mk}(\sigma(t))\, H_j^p(t)\, E_p^k(t)\, E_\alpha^m(t) \circ dz^\alpha(t).$$

Thus we have

$$\begin{aligned} dH_i^j &= g_{kl}\tilde{E}_i^k \circ dE_j^l + g_{kl}E_j^l \circ d\tilde{E}_i^k + \frac{\partial g_{kl}}{\partial x^q}\tilde{E}_i^k E_j^l \circ d\sigma^q \\ &= -g_{kl}\Gamma^l_{mp} H_i^\beta E_\beta^k E_j^p E_\alpha^m \circ dz^\alpha - g_{kl}\Gamma^l_{mp} H_i^\beta E_\beta^p E_j^k E_\alpha^m \circ dz^\alpha \\ &\quad - g_{kl}S^l_{mp} H_i^\beta E_\beta^p E_j^k E_\alpha^m \circ dz^\alpha + \frac{\partial g_{mn}}{\partial x^q} H_i^\beta E_\beta^m E_j^n E_\alpha^q \circ dz^\alpha. \end{aligned}$$

Since D is the Levi–Civita connection,

$$\frac{\partial g_{mn}}{\partial x^q} = g_{ln}\Gamma^l_{qm} + g_{ml}\Gamma^l_{qn},$$

so that

$$\begin{aligned} dH_i^j &= -g_{kl}S^l_{mp} H_i^\beta E_\beta^p E_j^k E_\alpha^m \circ dz^\alpha, \\ H_i^j(0) &= \delta_i^j. \end{aligned}$$

That is,

$$\begin{aligned} dH_t &= -g(S(\gamma_t \circ dz_t, \gamma_t(H_t -)), \gamma_t -), \\ H_0 &= \mathrm{id}. \end{aligned}$$

In other words,

$$\begin{aligned} d\langle H_t\xi, \eta\rangle_{\mathbb{R}^d} &= -g(S(\gamma_t \circ dz_t, \gamma_t(H_t\xi)), \gamma_t\eta), \qquad \forall \xi, \eta \in \mathbb{R}^d, \\ H_0 &= \mathrm{id}. \end{aligned}$$

Hence we have proved the following

THEOREM 1. *Let (M, g) be a Riemannian manifold with its Levi–Civita connection, and let $\nabla = D + S$ be an affine connection. For any $\sigma \in W_o^\infty(M)$, and $h \in \mathbb{H}^1(\mathbb{R}^d)$, define $X^{\nabla, h}(\sigma)_s = \tilde{\gamma}_s h_s$, where $\tilde{\gamma}$ is the unique horizontal lift* *of σ under the connection ∇. Let $z = J^{-1}(\sigma)$, and $M^{\nabla, h}(z) = J_{-1}^{*} X^{\nabla, h}(\sigma)$. Then*

$$M^{\nabla, h}(z)_t = \int_0^t H_s \, dh_s - \int_0^t g(S(\gamma_s \circ dz_s, \gamma_s(H_s h_s)), \gamma_s -) - \int_0^t \int_0^s \Omega(\gamma_u \circ dz_u, \gamma_u(H_u h_u)) \circ dz_s, \tag{10}$$

where γ and H are the unique solutions of the equations:

$$d\gamma_s = B(\gamma_s) \circ dz_s, \qquad \gamma_0 = u_o,$$

and

$$dH_s = -g(S(\gamma_s \circ dz_s, \gamma_s(H_s -)), \gamma_s -), \qquad H_0 = \mathrm{id},$$

respectively.

4. ITÔ'S VECTOR FIELDS ON WIENER SPACE

Adapted vector fields on the classical Wiener space $(W_0(\mathbb{R}^d), \mu)$ are proposed and discussed in detail by B. Driver [8, 9, 10], although various authors [3, 4, 5, 6, 24, 22] have noticed several progressive measurable transformation flows on the Wiener space $(W_0(\mathbb{R}^d), \mu)$ which leave the Wiener measure μ quasi-invariant.

In this section we introduce a sub-class of adapted vector fields on the Wiener space $(W_0(\mathbb{R}^d), \mu)$; which consist of those vector fields defined by solving differential equations. The vector fields in this class are called Itô's vector fields.

For making sense of our next discussion we regard a path in $W_0(\mathbb{R}^d)$ as a continuous semimartingale on $(W_0(\mathbb{R}^d), \mathscr{F}, \mathscr{F}_t, \omega, \mu)$, where ω is a standard $\mathbb{R}^d$-valued Brownian motion under the measure μ, and $\mathscr{F}$, $\mathscr{F}_t$ are the natural filtration of ω. A vector field M on $W_0(\mathbb{R}^d)$ by definition is a map from $W_0(\mathbb{R}^d)$ to the tangent bundle $TW_0(\mathbb{R}^d)$, i.e. for any $\sigma \in W_0(\mathbb{R}^d)$, $M(\sigma)$ is a vector field along σ, so that $M(\sigma)_t \in T_{\sigma_t}\mathbb{R}^d$. However $\mathbb{R}^d$ is flat, we identify $T_p(\mathbb{R}^d)$ with $\mathbb{R}^d$, so that $M(\sigma)$ is a continuous curve in $\mathbb{R}^d$. Given such a vector field M, its solution flow $(\xi^t)_{t \in \mathbb{R}}$ by definition is a family of progressive measurable maps ξ^t from $W_0(\mathbb{R}^d)$ to $W_0(\mathbb{R}^d)$, such that

$$\frac{\partial \xi^t_{\cdot}}{\partial t} = M(\xi^t_{\cdot}), \qquad \xi^o z_{\cdot} = z, \qquad \forall z \in W_0(\mathbb{R}^d).$$

Now we assume that solution flow $(\xi^t)_{t\in\mathbb{R}}$ exists and ξ^t leaves the Wiener measure quasi-invariant for each t, i.e. the law of the semimartingale ξ^t is absoultly continuous with respect to the measure μ. Then the local martingale part of ξ^t must be a d-dimensional standard Brownian motion, so that z^t has the following Doob–Meyer's decomposition:

$$\xi^t_s = W^t_s + \text{variation process}, \qquad \forall t\in\mathbb{R}.$$

By martingale representation theorem (cf. [32], p. 187), we can write

$$W^t_s = \int_0^s O^t_u \, d\omega_u.$$

Since W^t is a Brownian motion, so that O^t is $O(d)$-valued. Hence

$$M(z)_s = \left.\frac{dz^t_s}{dt}\right|_{t=0} = \int_0^s \left.\frac{dO^t_u}{dt}\right|_{t=0} d\omega_u + \text{variation process};$$

and $C = dO^t/dt|_{t=0}$ is $o(d)$-valued. Thus $M(z)$ has the following decomposition:

$$M(\omega)_s = \int_0^s C_u \, d\omega_u + \text{variation process},$$

where C is $o(d)$-valued.

Proposition 2. *If M is a vector field on $(W_0(\mathbb{R}^d), \mu)$ which induces a quasi-invariant, progressive measurable flow, and*

$$M(\omega)_s = \int_0^s C_u \, d\omega_u + \textit{variation process}.$$

Then C_u is $o(d)$-valued.

Now we consider an Itô's vector field on $(W_0(\mathbb{R}^d), \mu)$. Let $M(z)$ be the unique strong solution of the following stochastic differential equation:

$$dE_t = f_1(E_t, z_t, h_t, t)\circ dz_t + f_2(E_t, z_t, h_t, t)\, dh_t, \qquad E_0 = e_0,$$

$$dM_t = f_3(M_t, E_t, z_t, h_t, t)\circ dE_t, \qquad M_0 = 0,$$

where h is an $\mathbb{R}^N$-valued continuous function with finite variation, f_1 is $\mathbb{L}(\mathbb{R}^d, \mathbb{R}^m)$-valued, f_2 is $\mathbb{L}(\mathbb{R}^N, \mathbb{R}^m)$-valued, f_3 is $\mathbb{L}(\mathbb{R}^m, \mathbb{R}^d)$ valued, and they are all smooth. $M(z)$ is an $\mathbb{R}^d$-valued continuous semimartingale for any

d-dimensional semimartingale z, so that $z \to M(z)$ defines an adapted vector field on the Wiener space $(W_0(\mathbb{R}^d), \mu)$. Since

$$dM_t = f_3 f_1(M_t, E_t, z_t, h_t, t) \circ dz_t$$
$$+ f_3 f_2(M_t, E_t, z_t, h_t, t)\, dh_t, \qquad M_0 = 0,$$

we have (cf. [7, 8) 9, 10])

THEOREM 2. *If in addition $f_3 f_1$; which is $\mathbb{L}(\mathbb{R}^d, \mathbb{R}^d)$-valued; is valued in the Lie algebra $o(d)$, $f_3 f_2$ has bounded derivatives and $h \in \mathbb{H}^1(\mathbb{R}^d)$, then there is a unique quasi-invariant solution flow of the Itô's vector field M. That is, there is a unique measurable map ξ^t: $W_o(\mathbb{R}^d) \to W_0(\mathbb{R}^d)$ for each $t \in \mathbb{R}$, such that*

(1) *ξ^t is a d-dimensional continuous semimartingale for each $t \in \mathbb{R}$,*

(2) *The law $(\xi^t)^{-1} \circ \mu$ of the semimartingale ξ^t is equivalent to the Wiener measure μ for each $t \in \mathbb{R}$,*

(3) *$\xi^t \circ \xi^s = \xi^{t+s}$ for any $t, s \in \mathbb{R}$,*

(4) *$(\xi^t)_{t \in \mathbb{R}}$ is the solution flow, i.e.*

$$\frac{d\xi_s^t}{dt} = M(\xi^t)_s, \qquad \xi^0 = id, \qquad \mu - a.e.\ \forall t \in \mathbb{R},\ s \in [0, 1].$$

$(\xi^t)_{t \in \mathbb{R}}$ is called the Driver's flow of the adapted vector field M.

5. DRIVER'S FLOW ON PATH SPACE

In this section we are going to look for a solution flow induced by a vector field $X^{\nabla, h}$ on the based path space $W_o(M)$, i.e. to solve the geometric flow equation:

$$\frac{d\zeta^t}{dt} = X^{\nabla, h}(\zeta^t), \qquad \zeta^0 = \mathrm{id}.$$

We assume that (M, g) is a d-dimensional, compact Riemannian manifold with Levi–Civita connection D. Let ν be the unique probability measure on the based path space $W_o(M)$ such that the coordinate process $(x_t)_{t \in [0,1]}$ is a Brownian motion on the Riemannian manifold (M, g) starting from o, and let $\mathscr{F}$, $\mathscr{F}_t$ be the natural filtration of (x_t).

Let ∇ be an affine connection, and $\nabla = D + S$, and let $h \in \mathbb{H}^1(\mathbb{R}^d)$. Define $X^{\nabla,h}(\sigma)_s = \tilde{\gamma}_s h_s$, where $\tilde{\gamma}_s$ is the solution of the differential equation:

$$d\tilde{\gamma}_s = \tilde{B}(\tilde{\gamma}_s)\, \tilde{\gamma}_s^{-1} \circ d\sigma_s, \qquad \tilde{\gamma}_0 = u_o,$$

for any continuous semimartingale σ on M.

By Th. 1, we define an Itô's vector field $M^{\nabla,h}$ by

$$M^{\nabla,h}(z)_t = \int_0^t H_s\, dh_s - \int_0^t g(S(\gamma_s \circ dz_s, \gamma_s(H_s h_s)), \gamma_s -)$$
$$- \int_0^t \int_0^s \Omega(\gamma_u \circ dz_u, \gamma_u(H_u h_u)) \circ dz_s,$$

for any continuous $\mathbb{R}^d$-valued semimartingale z, where γ and H are the solutions of the stochastic differential equations:

$$d\gamma_s = B(\gamma_s) \circ dz_s, \qquad \gamma_0 = u_o,$$

and

$$dH_s = -g(S(\gamma_s \circ dz_s, \gamma_s(H_s -)), \gamma_s), \qquad H_0 = \mathrm{id},$$

respectively, and $\circ d$ denotes the Stratonovich's differential.

If furthermore z is a Brownian semimartingale, i.e. the local martingale part of z is a d-dimensional Brownian motion, then we can write $M^{\nabla,h}$ in Itô's form:

$$M^{\nabla,h}(z)_t = \int_0^t C(z)_s\, dz_s + \int_0^t R(z)_s\, ds, \tag{11}$$

where

$$C(z)_s = \int_0^s \Omega(\gamma_u \circ dz_u, \gamma_u(H_u h_u)) + g(S(\gamma_s \cdot, \gamma_s(H_s h_s)), \gamma_s -), \tag{12}$$

and

$$R(z)_s = H_s \dot{h}_s + \tfrac{1}{2}\Theta_S(\gamma_s)(\gamma_s(H_s h_s), -) - \tfrac{1}{2}\nabla\Theta_S(\gamma_s)(\gamma_s(H_s h_s), -)$$
$$+ \tfrac{1}{2}\mathrm{Ric}(\gamma_s(H_s h_s)). \tag{13}$$

We have used the following definitions in the above formulas. Given a (1,2)-type tensor S on the Riemannian manifold (M, g), we define two maps

$$f_S: O(M) \to \mathbb{L}((\mathbb{R}^d)^{\otimes 4}, \mathbb{R}),$$
$$\tilde{f}_S: O(M) \to \mathbb{L}((\mathbb{R}^d)^{\otimes 6}, \mathbb{R})$$

by

$$f_S(u)(\xi_1, ..., \xi_4) = g((D_{u\xi_2} S)(u\xi_1, u\xi_3), u\xi_4), \tag{14}$$

and

$$\tilde{f}_S(u)(\xi_1, ..., \xi_6) = g(S(u\xi_1, u\xi_2), u\xi_3)\ g(S(u\xi_4, u\xi_5), u\xi_6), \tag{15}$$

respectively. Then $\Theta_S; \nabla\Theta_S: O(M) \to \mathbb{L}((\mathbb{R}^d)^{\otimes 2}, \mathbb{R})$ defined by

$$\Theta_S(u)(\xi, \eta) = \sum_{i,\, j=1}^{d} \tilde{f}_S(u)(e_i, e_j, \eta, e_i, \xi, e_j), \tag{16}$$

and

$$\nabla\Theta_S(u)(\xi, \eta) = \sum_{i=1}^{d} f_S(u)(e_i, e_i, \xi, \eta), \tag{17}$$

respectively, for any $\xi, \eta \in \mathbb{R}^d$, where (e_i) is any orthonormal base of $\mathbb{R}^d$.

We recall also that Ric denotes the Ricci tensor defined by

$$\mathrm{Ric}(X) = \sum_{i=1}^{d} \Omega(B_u(X), ue_i)\ e_i, \forall X \in T_x M, u \in O(M), \pi(u) = x,$$

where (e_i) is any orthonormal base of $\mathbb{R}^d$.

Since Ω is $o(d)$-valued, so that we have

PROPOSITION 3. *If the (1,2)-type tensor $S = \nabla - D$ is adjoint skew-symmetric, then $C(z)$ is $o(d)$-valued.*

In the remainder of this paper we always assume $S = \nabla - D$ is adjoint skew-symmetric. By Th. 2, there is a unique solution flow of $M^{\nabla, h}$ on the Wiener space $W_0(\mathbb{R}^d)$, i.e. there exists a family $(\xi^t)_{t \in \mathbb{R}}$ of measurable maps from $W_0(\mathbb{R}^d)$ into $W_0(\mathbb{R}^d)$, such that each ξ^t is an $\mathbb{R}^d$-valued continuous, Brownian semimartingale with law equivalent to the Wiener measure μ; $\xi^t \circ \xi^s = \xi^{t+s}$ almost surely with respect to μ; and

$$\frac{d\xi^t}{dt} = M^{\nabla, h}(\xi^t), \qquad \xi^0 = \mathrm{id}, \ \mu\text{-a.e..}$$

Define ζ^t: $W_o(M) \to W_o(M)$ by

$$\zeta^t = J \circ \xi^t \circ J^{-1}.$$

Then ζ^t is ν-a.e. well defined, and we have (cf. [8]):

THEOREM 3. *The family* $(\zeta^t)_{t \in \mathbb{R}}$ *is a solution flow of the vector field* $X^{\nabla, h}$. *That is,*

1. ζ^t *is an M-valued continuous semimartingale with law equivalent to the Brownian measure* ν,
2. $\zeta^t \circ \zeta^s = \zeta^{s+t}$, *$\nu$-a.e.,*
3. $(\zeta^t)_{t \in \mathbb{R}}$ *is a solution flow of* $X^{\nabla, h}$:

$$\frac{d\zeta^t_{\cdot}}{d} t = X^{\nabla, h}(\zeta^t)., \qquad \zeta^0 = id, \ \nu\text{-}a.e.$$

In the final we write down an integration by parts formula in our context, using Bismut's method (cf. [2]).

Let $b = J^{-1}(x)$ be the standard Brownian motion. For any $\varepsilon \in \mathbb{R}$, let

$$b^\varepsilon_s = \int_0^s e^{\varepsilon C(b)_u} \, db_u + \varepsilon \int_0^s R(b)_u \, du.$$

Then b^ε is a Brownian semimartingale, and

$$\left. \frac{db^\varepsilon_s}{d\varepsilon} \right|_{\varepsilon=0} = M^{\nabla, h}(b)_s. \tag{18}$$

Let $\sigma^\varepsilon = J(b^\varepsilon)$. Then by Th. 1, we have

$$\left. \frac{d\sigma^\varepsilon}{d\varepsilon} \right|_{\varepsilon=0} = X^{\nabla, h}(\sigma). \tag{19}$$

We say a real valued function F on the based path space $W_o(M)$ is a smooth function, if

$$F(\sigma) = \tilde{F}(\sigma_{t_1}, ..., \sigma_{t_k}), \qquad 0 \leqslant t_1 < \cdots < t_k \leqslant 1, \tag{20}$$

for some smooth function $\tilde{F}$ on M^k. For such a function F, we define

$$D_{[\nabla, h]} F(\sigma) = \sum_{i=1}^{k} d\tilde{F}_i(\sigma_{t_1}, ..., \sigma_{t_k})(\tilde{\gamma}_{t_i} h_{t_i}). \tag{21}$$

Note that the right hand side of Eq. 21 is well defined ν-almost surely. It is easy to see that

$$D_{[\nabla, h]} F(\sigma) = \frac{d}{d\varepsilon} F(\sigma^{\varepsilon}), \qquad \nu - \text{a.e.}, \tag{22}$$

so that $D_{[\nabla, h]} F$ is well defined, i.e. it does not depend on the special representation of F. Note we can rewrite Eq. 22:

$$\begin{aligned} D_{[\nabla, h]} F(\sigma) &= \sum_{i=1}^{k} d\tilde{F}_i(\sigma_{t_1}, ..., \sigma_{t_k})(\gamma_{t_i}(H_{t_i} h_{t_i})) \\ &= \sum_{i=1}^{k} \langle \nabla \tilde{F}_i(\sigma_{t_1}, ..., \sigma_{t_k}), \gamma_{t_i}(H_{t_i} h_{t_i}) \rangle \\ &= \sum_{i=1}^{k} \langle \gamma_{t_i}^{-1} \nabla \tilde{F}_i(\sigma_{t_1}, ..., \sigma_{t_k}), H_{t_i} h_{t_i} \rangle_{\mathbb{R}^d}, \end{aligned}$$

where γ and H are the solutions of the differential equations:

$$d\gamma_s = B(\gamma_s) \circ db_s, \qquad \gamma_0 = u_o,$$

and

$$dH_s = -g(S(\gamma_s \circ db_s, \gamma_s(H_s -), \gamma_s -), \qquad H_0 = \text{id},$$

respectively.

Let μ^{ε} be the probability measure on $W_0(\mathbb{R}^d)$ such that

$$\frac{d\mu^{\varepsilon}}{d\mu} = \exp\left[\varepsilon \int_0^1 \exp(-\varepsilon C(b)_u)\, R(b)_u\, db_u - \frac{1}{2}\varepsilon^2 \int_0^s |R(b)|_u^2\, du\right],$$

and let K^{ε} denote the right side of the above equation.

Let $Y^{\varepsilon} = K^{\varepsilon} \circ J$. Then

$$\left.\frac{dY^{\varepsilon}}{d\varepsilon}\right|_{\varepsilon=0} = \int_0^1 R(b)_u\, db_u.$$

Using P. Lévy characterisation theorem (cf. [32], p. 141) and Girsanov's theorem (cf. [32], p. 303), one can show that b^ε is a d-dimensional standard Brownian motion under μ^ε, so that

$$\frac{d}{d\varepsilon}\mathbb{E}^{\mu^\varepsilon}[F\circ J(b^\varepsilon)]=0.$$

Hence we have

$$\frac{d}{d\varepsilon}\mathbb{E}^{\nu}(F(\sigma^\varepsilon)\ Y^\varepsilon)=0,$$

which yields that

$$\mathbb{E}^{\nu}(D_{[\nabla,h]}F)=-\mathbb{E}^{\nu}\left(F\int_0^1 R(b)_u\,db_u\right), \tag{23}$$

where $b=J^{-1}(x)$ is a standard Brownian motion.

There are several papers by various authors about integration by parts formulas, e.g see [2, 6–8, 12, 13, 15, 16, 18–20, 23] etc.

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Note added in proof. In [12], and using a different method, Elworthy and Li also proved an integraton by parts formula for the vector fields considered in this paper.

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