Discrete Applied Mathematics 13 (1986) 33-39
North-Holland

ON PRE-PERIODS OF DISCRETE INFLUENCE SYSTEMS

Svatopluk POLJAK
Technical University, Dept. of System Engineering, Thákurova 7, 166 29 Praha 6, Czechoslovakia

Daniel TURZÍK
Institute of Chemistry and Technology, Dept. of Mathematics, Suchbátarova 5, 166 28 Praha 6, Czechoslovakia

Received 5 July 1984
Revised 20 May 1985

We investigate mappings of the form $g=fA$ where f is a cyclically monotonous mapping of finite range and A is a linear mapping given by a symmetric matrix. We give some upper bounds on the pre-period of g, i.e. the maximum q for which all $g(x), g^2(x), \ldots, g^q(x)$ are distinct.

Let S be a subset of the m-dimensional Euclidean space R^m and $g:S\to S$ be a mapping of finite range. The *period* and the *pre-period* of g are minimal $p\ge 1$ and $q\ge 0$ such that $g^{t+p}(y)=g^t(y)$ for all $t\ge q$ and $y\in S$. A mapping $f:S\to S$ is *cyclically monotonous* (abbreviated as c.m.) if

$$\sum_{i=1}^{n} (f(x_i)-f(x_{i-1}))x_i\ge 0 \tag{1}$$

for every $n\ge 2$ and every $x_1, x_2, \ldots, x_n=x_0\in S$. This notion was introduced by R.T. Rockafellar (see [10]) who proved that c.m. mappings are just subgradients of convex functions. Let us mention that a function $f:R\to R$ is c.m. if and only if it is nondecreasing.

Let us say that a mapping f is *strongly c.m.* if $f(x_1)=\cdots=f(x_n)$ whenever the equality occurs in (1) for some $x_1, \ldots, x_n$. We proved in [7] and [8] that the period of fA is at most 2 when f is a strongly c.m. mapping and A is a symmetric matrix.

In the present paper we give some upper bounds on the pre-period of $g=fA$ in three special cases. (Z denotes the set of integers.)

(a) $f=f_1\times\cdots\times f_m$ where $f_i:Z\to\{-1,1\}$ are threshold functions,

(b) $f=f_1\times\cdots\times f_m$ where $f_i:Z\to Z$ are multi-threshold functions,

(c) $f=f_1\times\cdots\times f_r$ where $f_i:Z^{m_i}\to Z^{m_i}$ are strongly c.m. and $m_1+\cdots+m_r=m$.

Clearly, (c) is more genaral than (b), and (b) than (a), but the more special form of f enables us to give better estimates. The theorems of the paper are formulated rather to illustrate some different methods of estimation, than to cover all possible variation of premises. The integrality is not important for computing bounds but it enables us to simplify the statements and the proofs.

The decomposition of f into Cartesian product $f_1 \times \cdots \times f_r$ has also a reasonable interpretation. Consider a society of r members where each member has some initial opinion represented by a vector y_0^i, $i = 1, \ldots, r$. The members change simultaniously their opinions in discrete steps by the rule

$$y_{t+1}^i = f_i\left(\sum_{j=1}^{r} A_{ji} y_t^j\right)$$

where y_t^j is the opinion of the i-th member at time t, A_{ji} is a matrix of size (m_i, m_j) which represents the influence of the j-th member on the i-th one, and f_i is the evaluation mapping used by the i-th member to compute his new opinion from influences of other members.

The discrete influence systems as described above were studied first by Harary [14] and French [2] (see also [9]), and similar models also appeared in study of neural networks [1]. First results on 'period 2' were special evaluation mappings: in [3] and [4] for multi-threshold functions, and in [6] for the 'choice of the most spread opinion'. The unifying approach with the states of an influence system encoded by vectors of an Euclidean space was started in [7], where the Corollary of Theorem 2 was proved. The important role of convex functions in discrete influence systems was established first in [8]. The limit behaviour of discrete influence systems with an infinite number of states was studied in [5]. Some applications to social models with ranking alternatives were considered in [15].

We were informed by one of the referees that some results on pre-periods were proved in [11], [12] and [13].

Throughout the paper we use the following notation. For a vector $x \in R^m$, x^i denotes the i-th component, and $\|x\| = \sum_{i=1}^m |x^i|$. The scalar product of vectors x and y is written as xy. If A is a matrix, then $\|A\| = \sum_{i,j} |a_{ij}|$. We also use A to denote the linear mapping $x \mapsto Ax$. If $f_i : X_i \to Y_i$, $i = 1, 2$, are mappings, then $f_1 \times f_2 : X_1 \times X_2 \to Y_1 \times Y_2$ denotes their Cartesian product.

Theorem 1. *Let $f : R^m \to \{-1, 1\}^m$ be defined by $f(x^1, \ldots, x^m) = (y^1, \ldots, y^m)$ where $y^i = 1$ if $x^i \geq 0$ and $y^i = -1$ if $x^i < 0$. Let $A = (a_{ij})$ be a symmetric matrix of size m with integral entries. Then the pre-period of $fA : \{-1, 1\}^m \to \{-1, 1\}^m$ is at most $\frac{1}{2}(\|A\| + 3s - m)$ where $s = |\{i : \sum_{j=1}^m a_{ij}$ is even$\}|$.*

Proof. Define the matrix $B = (b_{ij})$ of size $m+1$ by

$$b_{ij} = a_{ij} \quad \text{for } i, j \leq m,$$

$$b_{m+1,i} = b_{i,m+1} = \begin{cases} 1 & \text{for } \sum_j a_{ij} \text{ even, } i \leq m \\ 0 & \text{for } \sum_j a_{ij} \text{ odd, } i \leq m, \end{cases}$$

$$b_{m+1,m+1} = \|A\| + 1.$$

Define $h: R^{m+1} \to \{-1, 1\}^{m+1}$ by $h(x^1, \dots, x^m, x^{m+1}) = (f(x^1, \dots, x^m), 1)$. It is easy to see that the period and the pre-period of hB and fA are the same. Let q be the pre-period of hB. Let an initial $y_0 \in \{-1, 1\}^{m+1}$ be given. Put $y_{t+1} = hB(y_t)$ for $t \geq 0$. If $t < q$, then $y_{t+2}By_{t+1} = \|By_{t+1}\|$, and because of $y_{t+2} \neq y_t$ it follows that $y_tBy_{t+1} \leq \|By_{t+1}\| - 2$. Thus,

$$y_{t+2}By_{t+1} - y_tBy_{t+1} \geq 2 \quad \text{for } t < q. \tag{2}$$

Set

$$V = \sum_{t=0}^{q-1} (y_{t+2}By_{t+1} - y_tBy_{t+1}) = y_{q+1}By_q - y_0By_1.$$

Using (2) we get

$$V \geq 2q. \tag{3}$$

On the other hand, $y_{q+1}By_q = \|By_q\| \leq 2\|A\| + 2s + 1$ and $y_0By_1 = y_1By_0 = \|By_0\| \geq \|A\| - s + m + 1$. Hence

$$V \leq \|A\| + 3s - m. \tag{4}$$

Finally, combining (3) and (4) we get $q \leq \frac{1}{2}(\|A\| + 3s - m)$. □

Example. Let $A = (a_{ij})$ be the matrix defined by $a_{i,i+1} = a_{i+1,1} = 1$ for $i = 1, \dots, m-1$, $a_{m,m} = 1$, and $a_{ij} = 0$ otherwise. Let f be as in Theorem 1, and $y_0 = (1, -1, -1, \dots, -1)$. One can easily check that $(fA)^{t+1}(y_0) \neq (fA)^t(y_0)$ for $t \leq 2m - 2$, and $(fA)^{2m-1}(y_0) = (fA)^{2m-2}(y_0)$. Hence the pre-period of fA is at least $2m - 2$ which agrees with the upper bound by Theorem 1. (This example was suggested by J. Demel.)

Theorem 2. *Let $m_1, \dots, m_r$ be positive integers, and A be a symmetric matrix of size $m = \sum m_i$ with integral entries. Let $f_i: Z^{m_i} \to Z^{m_i}$, $i = 1, \dots, r$, be strongly c.m. mappings, each attaining at most p distinct values. Then the pre-period of $(f_1 \times \cdots \times f_r)A$ is at most $2M^2\|A\|p(4p+1) + 2pr$ where $M = \max\{\|f_i(x)\|: i = 1, \dots, r, x \in Z^{m_i}\}$.*

Proof. Set $f = f_1 \times \cdots \times f_r$. For $x \in Z^{m_1} \times \cdots \times Z^{m_r}$, let x^i denote the vector consisting of those components of x belonging to Z^{m_i}. Thus $x = (x^1, \dots, x^r) \in Z^{m_1} \times \cdots \times Z^{m_r}$. Let $y_0 \in Z^m$ be a given initial vector. Set $y_{t+1} = f(Ay_t)$ for $t \geq 0$. Let q be the pre-period of fA. Set

$$V = \sum_{t=1}^{q} (y_{t+1} - y_{t-1})Ay_t = y_{q+1}Ay_q - y_0Ay_1.$$

Clearly,

$$V \leq 2M^2\|A\|. \tag{5}$$

Let $d=\lceil q/2 \rceil$ (the upper integer approximation). Set

$$V_{\text{odd}}=\sum_{t=1}^{d}(y_{2t+1}-y_{2t-1})Ay_{2t},$$

and

$$V_{\text{even}}=\sum_{t=1}^{d}(y_{2t}-y_{2t-2})Ay_{2t-1}.$$

Put $x_t=Ay_t$, $t\geq 0$, and set

$$V_{\text{odd}}^i=\sum_{t=1}^{d}(f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_{2t}^i.$$

Clearly $V_{\text{odd}}=\sum_{i=1}^{r}V_{\text{odd}}^i$. To every t_1,t_2, $0\leq t_1<t_2\leq d$ we assign the interval $I=I(t_1,t_2)=\{t:t_1<t\leq t_2\}$. Define

$$S_i(I(t_1,t_2))=\{t\in I(t_1,t_2):f_i(x_{2t}^i)\neq f_i(x_{2t-2}^i),$$

and denote by $s_i(I(t_1,t_2))$ the cardinality of the set S_i.

Fix some $i=1,\dots,r$. We give a lower bound on V_{odd}^i. We say that an interval $I(t_1,t_2)$ is *admissible* if $f_i(x_{2t_1}^i)=f(x_{2t_2}^i)$. We claim that there is a system $\mathscr{I}$ of pairwise disjoint admissible intervals with the properties

$$|\mathscr{I}|\geq\left\lceil\frac{1}{p}\,s_i(I(0,d))\right\rceil-1, \tag{6}$$

and

$$\left|I(0,d)\setminus\bigcup_{I\in\mathscr{I}}I\right|\leq 2p. \tag{7}$$

As f_i attains at most p distinct values, there is some y such that the set $U_y=\{t\in S_i(I(1,d)):f_i(x_{2t}^i)=y^i\}$ has at least $u=\lceil(1/p)s_i(I(0,d))\rceil$ elements. Let $t_1<t_2<\dots<t_u$ be elements of U_y. Set $\mathscr{I}_1=\{I(t_i,t_{i+1}):i=1,\dots,u-1\}$. Let $\mathscr{I}_2$ be a system (possibly empty) of pairwise disjoint admissible intervals such that

$$\bigcup_{I\in\mathscr{I}_2}I\subset I(0,t_1)\quad\text{and}\quad\left|I(0,t_1)\setminus\bigcup_{I\in\mathscr{I}_2}I\right|\leq p.$$

Similarly, let $\mathscr{I}_3$ be a system of pairwise disjoint admissible intervals such that

$$\bigcup_{I\in\mathscr{I}_3}I\subset I(t_u,d)\quad\text{and}\quad\left|I(t_u,d)\setminus\bigcup_{I\in\mathscr{I}_3}I\right|\leq p.$$

(The existence of both $\mathscr{I}_2$ and $\mathscr{I}_3$ follows from the pigeon-hole principle.) Clearly, $\mathscr{I}=\mathscr{I}_1\cup\mathscr{I}_2\cup\mathscr{I}_3$ satisfies (6) and (7). As f_i is strongly c.m. we get

$$\sum_{t=t'+1}^{t''}(f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_{2t-1}^i\geq 1 \tag{8}$$

for every admissible interval $I(t',t'')\in\mathscr{I}$. On the other hand,

$$(f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_{2t-1}^i \le 2M^2\Big(\sum_{l=m_{i-1}+1}^{m_i} \sum_{j=1}^{m} |a_{lj}|\Big) \tag{9}$$

$(m_0=0)$. Clearly (6), (7), (8) and (9) give together

$$\begin{aligned} V_{\text{odd}}^i &= \sum_{t\in\mathscr{I}_1} (f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_t + \sum_{t\in\mathscr{I}_2\cup\mathscr{I}_3} (f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_t \\ &\quad + \sum_{t\notin\bigcup\mathscr{I}} (f_i(x_{2t}^i)-f_i(x_{2t-2}^i))x_t \\ &\ge \left\lceil \frac{1}{p} s_i(I(0,d)) \right\rceil - 1 + 0 - 2p\cdot 2M^2 \sum_{l=m_{i-1}+1}^{m_i} \sum_{j=1}^{m} |a_{lj}|. \end{aligned}$$

Thus

$$V_{\text{odd}} \ge \frac{1}{p} \sum_{i=1}^{r} s_i(I(0,d)) - r - 4pM^2\|A\|. \tag{10}$$

As for every $t=1,\ldots,d$ there is some i such that $f_i(x_{2t}^i) \ne f_i(x_{2t-2}^i)$, we have

$$\sum_{i=1}^{r} s_i(I(0,d)) \ge d. \tag{11}$$

Hence (10) and (11) give

$$V_{\text{odd}} \ge \frac{1}{p} d - r - 4pM^2\|A\|.$$

It is possible to estimate V_{even} in the same way. Thus

$$V = V_{\text{even}} + V_{\text{odd}} \ge 2\left(\frac{1}{p} d - r - 4pM^2\|A\|\right),$$

which together with (5) give $d \le M^2\|A\|p(4p+1)+pr$. □

Corollary [8]. *If f is strongly c.m. and A a symmetric matrix, then the period of fA is at most 2.* □

Lemma. *Let $f: Z\to Z$ be a nondecreasing mapping. Then*

$$\sum_{i=1}^{n} (f(x_i)-f(x_{i-1}))x_i \ge \tfrac{1}{2}|\{i: f(x_i)\ne f(x_{i-1}), i=1,\ldots,n\}|$$

for every $n\ge 2$ and $x_1,\ldots,x_n=x_0\in Z$.

Proof. By induction on n. It is trivial for $n=2$. Let $n>2$ and $x_1,\ldots,x_n=x_0$ be given. We can assume that $x_i\ne x_{i-1}$ for $i=1,\ldots,n$, and also $x_n\ge x_i$ for $i=1,\ldots,n$. Put $y_i=x_i$ for $i=1,\ldots,n-1$, $y_0=x_{n-1}$. Then

$$\sum_{i=1}^{n} (f(x_i)-f(x_{i-1}))x_i = \sum_{i=1}^{n-1} (f(y_i)-f(y_{i-1}))y_i + (f(x_n)-f(x_{n-1}))(x_n-x_1).$$

If $f(x_n)=f(x_{n-1})$, then the statement follows immediately, and if $f(x_n)\neq f(x_{n-1})$, then $(f(x_n)-f(x_{n-1}))(x_n-x_1)\geq 1$ and

$$|\{i=1,\dots,n : f(x_i)\neq f(x_{i-1})\}|\leq |\{i=1,\dots,n-1 : f(y_i)\neq f(y_{i-1})\}|+2.$$

$\square$

Theorem 3. *Let m be a positive integer, A be a symmetric matrix of size m, $f_1,\dots,f_m$ be nondecreasing real functions, each attaining at most p distinct values. Set $f=f_1\times\cdots\times f_m$. Then the pre-period of $g=fA$ is at most $2mp+4M^2\|A\|(2p+1)$ where $M=\max\{|f_i(x)|: i=1,\dots,m,\ x\in R\}$.*

Proof. We follow the proof of Theorem 2 with $m_1=\cdots=m_r=1$ and $m=r$. We give here a better bound on V^i_{odd} (and V^i_{even}). Fix arbitrary $i=1,\dots,m$. It follows from the pigeon-hole principle that there is some system $\mathscr{I}$ of pairwise disjoint admissible intervals such that $|\{1,\dots,d\}\setminus\bigcup\mathscr{I}|\leq p$. Using the Lemma we get

$$\sum_{t=t_1+1}^{t_2}(f_i(x^i_{2t})-f_i(x^i_{2t-2}))x_{2t}\geq \tfrac{1}{2}s_i(I(t_1,t_2))$$

for every admissible interval $I(t_1,t_2)\in\mathscr{I}$. Thus

$$\begin{aligned}V^i_{\text{odd}} &\geq \tfrac{1}{2}\sum_{I\in\mathscr{I}} s_i(I)-2pM^2\sum_{j=1}^{m}|a_{ij}|\\ &\geq \tfrac{1}{2}(s_i(I(1,d))-p)-2pM^2\sum_{j=1}^{m}|a_{ij}|.\end{aligned}$$

Then using $s_1(I(1,d))+\cdots+s_m(I(1,d))\geq d$, we get

$$V_{\text{odd}}\geq \tfrac{1}{2}(d-mp)-2pM^2\|A\|,$$

and finally $d\leq mp+2M^2\|A\|(2p+1)$. $\square$

Remark. Theorem 2 when applied to mappings in Theorem 3 gives the estimation $2M^2\|A\|p(4p+1)+2mp$, which is weaker than the bound given by Theorem 3.

References

[1] E.R. Caianello, De Luca and L. Ricciardi, Reverbations and control of neural networks, Kybernetik 4 (1967).

[2] J.R.P. French, A formal theory of social power, Psych. Review 63 (1956) 181–194.

[3] E. Goles and J. Olivos, Comportement itératif des fonctions à multiseuil, Information and Control 45 (1980) 300–313.

[4] E. Goles and J. Olivos, Comportement périodique des fonctions à seuil binaries et applications, Discrete Appl. Math. 3 (1981).

[5] J. Pelant, S. Poljak and D. Turzík, Cyclically monotonous evaluation in social influence models, submitted to Math. Oper. Res.

[6] S. Poljak and M. Šůra, On periodical behaviour in societies with symmetric influences, Combinatorica 3 (1983) 119–121.
[7] S. Poljak and D. Turzík, On systems, periods, and semipositive mappings, Comm. Math. Univ. Carolinae 25 (4) (1984) 597–614.
[8] S. Poljak and D. Turzík, On an application of convexity to discrete systems, Discrete Appl. Math., 13 (1986) 27–32.
[9] F.S. Roberts, Discrete Mathematical Models, with Application to Social, Biological, and Environmental Problems (Prentice-Hall, Englewood Cliffs, NJ, 1976).
[10] R.T. Rockafellar, Convex Analysis (Princeton Univ. Press, Princeton, NJ, 1970).
[11] F. Fogelman, E. Goles and G. Weisbuch, Transient length in sequential iterations of threshold functions, Discrete Appl. Math. 6 (1983) 95–98.
[12] E. Goles, F. Fogelman and D. Pellegrin, Decreasing energy functions as a tool for studying threshold networks, Discrete Appl. Math. 12 (1985) 261–277.
[13] E. Goles, Dynamics on positive automata networks, Theor. Comput. Sci., to appear.
[14] F. Harary, A criterion for unanimity in French's theory of social power, in: D. Cartwright, ed., Studies in Social Power, (Inst. Soc. Res., Ann Arbor, MI, 1959) 168–182.
[15] S. Poljak and D. Turzík, Social influence models with ranking alternatives and local election rules, Math. Soc. Sci. 10 (1985) 189–198.