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Basic Properties of Segal Algebras

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I.

Recently, considerable interest has been shown in Segal algebras and their generalizations.

DEFINITION [5, p. 127]. Let G denote a locally compact abelian group. A Segal algebra $S = S(G)$ is a proper subalgebra of $L^1 = L^1(G)$ satisfying the following conditions:

S1. S is dense in L^1.

S2. S is a Banach algebra with respect to a norm $\| \ \|_S$.

It is also assumed that S is translation invariant and, in addition,

(a) $\|L_a f\|_S = \|f\|_S$ $(f \in S)$.

(b) The map $a \to f_a$ is continuous from G to $(S, \| \ \|_S)$. (Here $f_a(x) = f(x - a)$ and $L_a f = f_a$.)

On the basis of these assumptions the following properties of S may be deduced [5, p. 128]:

S3. There exists $M > 0$ such that

$$\|f\|_{L^1} \leqslant M \|f\|_S \qquad (f \in S).$$

S4. The algebra S is an ideal in L^1 and

$$\|f * g\|_S \leqslant \|f\|_{L^1} \|g\|_S \qquad (f \in L^1, g \in S).$$

A reason for the interest in Segal algebras is the fact that there are so many interesting special cases ([5, p. 12], [7]).

For example, if T denotes the circle group, then for $1 < p < \infty$ all of

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the spaces $L^p(T)$ are Segal algebras. (The space $L^\infty(T)$ is not a Segal algebra since $a \to f_a$ is not continuous.)

On the real line R, the space $L^1 \cap L^p$ with norm $\| \|_{L^1} + \| \|_{L^p}$ is a Segal algebra. So is the space A_p $(1 \leqslant p < \infty)$ consisting of all $f \in L^1(R)$ such that $f^\wedge$ (the Fourier transform of f) belongs to $L^p(R)$. The norm for A_p is $\| f \|_{L^1} + \| f^\wedge \|_{L^p}$. Both of these examples apply to more general groups.

An example of a Segal algebra on R which does not have an analogue for more general groups is the algebra L_A consisting of all $f \in L^1$ which are locally absolutely continuous and have Df, the derivative of f, belonging to L^1. The norm for $f \in L_A$ is $\| f \|_{L^1} + \| Df \|_{L^1}$.

II.

A fascinating feature of Segal algebras is that all of them inherit some important properties from $L^1(G)$ and yet all of them fail to inherit others. For example, the (closed) ideal structure of any Segal algebra $S \subset L^1$ is precisely that of L^1 itself. Every closed ideal I in S is the intersection with S of a unique closed ideal J in L^1. (Indeed, J is the L^1 closure of I.) Conversely, if J is a closed ideal in L^1 then $I = J \cap S$ is a closed ideal in S. The maximal ideal spaces of S and L^1 are (algebraically and topologically) the same, and the Wiener–Ditkin condition (the crucial property necessary for the Wiener Tauberian theorem and more general Spectral Synthesis theorems to hold for L^1) is also true for S.

On the other hand, L^1 has a (norm) bounded approximate identity. Consequently, according to the theorem of Cohen [3], L^1 has the factorization property: Every $f \in L^1$ can be expressed as $f = g * h$ for some $g, h \in L^1$. However, as shown by H. C. Wang [7], all of the Segal algebras mentioned in I and a great many more fail to have this factorization property. (Although, as this implies, these Segal algebras have no norm bounded approximate identity, they do possess (unbounded) approximate identities [7].)

Thus, all Segal algebras have the same ideal theory as L^1, but, as far as is known, all are different from L^1 in the matter of factorization. As to the question of *multipliers* it is known that the multipliers of the Segal algebras $L^p(T)$ are *different* from those of $L^1(T)$. As opposed to this it seems that on the real line most of the standard Segal algebras $S(R)$ have the *same* multipliers as $L^1(R)$. This has been established for a good many cases including $L^1 \cap L^p$ and A_p.

III.

One generalization of Segal algebra is the concept of *normed ideal* due to Cigler:

Definition [2]. An ideal $N \subset L^1(G)$ is called a normed ideal if

N1. N is dense in $L^1(G)$.

N2. N is a Banach space with respect to a norm $\| \|_N$, and

$$\|f\|_{L^1} \leqslant M \|f\|_N \qquad (f \in N). \tag{*}$$

N3. $\|f * g\|_N \leqslant \|f\|_{L^1} \|g\|_N$ $(f \in L^1,\ g \in N)$.

Cigler makes the important observation that these axioms make sense for a nonabelian group. In fact the definition of *Segal algebra* makes sense for a nonabelian group. We will require that (a) and (b) hold for both right and left translations and assume S3 in addition. He also raises the interesting QUESTION: Can the inequality (*) be proved, instead of assumed, if instead one assumes in N2 that N is a Banach *algebra*?

IV.

In [1] Burnham introduced the notion of A(bstract) Segal algebra.

Definition. Let $(A, \| \|_A)$ be a Banach algebra. The proper subalgebra B of A is called an A-Segal algebra if

A1. B is a dense left ideal of A.

A2. B is a Banach algebra with respect to a norm $\| \|_B$.

A3. There exists $M > 0$ such that

$$\|f\|_A \leqslant M \|f\|_B \qquad (f \in B).$$

A4. There exists $C > 0$ such that

$$\|fg\|_B \leqslant C \|f\|_A \|g\|_B \qquad (f \in A, g \in B).$$

Here there is no group involved in the definition and hence, of course, no translation invariance. Moreover A, B are not assumed to be commutative. This definition generalizes that of normed ideal and (classical) Segal algebra. (Just take $A = L^1(G)$ for a (not necessarily abelian) group G.) In [1] Burnham showed that the ideal structure of B is precisely that of A, thus generalizing the similar result for Segal algebras mentioned in II.

An interesting example of an A-Segal algebra which is not a (classical) Segal algebra is given as follows:

Let A be the algebra of all completely continuous operators on $L^2(R)$ with the usual operator norm and let B be the algebra of all operators of Hilbert–Schmidt type. That is, $T \in B$ if there exists $K = K(x, y) \in L^2(R \times R)$ such that $K(x, y) = \overline{K(y, x)}$ and

$$Tf(x) = \int_R K(x, y) f(y) \, dy \qquad \text{a.e.}$$

For $T \in B$ let $\| T \|_B^2 = \int_R \int_R | K(x, y)|^2 \, dx \, dy$. It may be varified that B is an A-Segal algebra, [6, p. 34].

The purpose of this paper is to give, in the context of the general definition of A-Segal algebra,

(a) An affirmative answer to the question of Cigler mentioned in III, and

(b) Conditions under which one of A3, A4, implies the other.

We now begin the exposition.

V.

LEMMA A. *Let $(X, \| \|_X)$ be a normed algebra with a right approximate identity $\langle e_\alpha \rangle$. If $x \in X$ is such that $x \cdot X = \{0\}$, then $x = 0$.*

Proof. By assumption $xe_\alpha = 0$ for all α. But $x = \lim_\alpha xe_\alpha$ since $\langle e_\alpha \rangle$ is an approximate identity. Hence $x = 0$.

The lemma is, of course, trivial. However, as a side issue, it is interesting to compare it to the following theorem.

THEOREM B. *There exists a sequence $\{g_n\}$ in $L^1(R)$ such that*

$$\lim_{n \to \infty} \| g_n * f \|_{L^1} = 0, \qquad \text{for all } f \in L^1, \tag{1}$$

but

$$\| g_n \|_{L^1} = 1, \qquad \text{for all } n. \tag{2}$$

(Thus, a sequence not converging to 0 can annihilate all of L^1 even though, by the lemma, a single nonzero element of L^1 cannot annihilate L^1.)

Proof. Simply let $\delta(t) = \Pi^{-1} t^{-2}(1 - \cos t)$. That is, δ is the familiar Fejér kernel. Define $g_n(x) = e^{inx}\delta(x)$. If $f \in L^1$ then

$$\begin{aligned} g_n * f(x) &= \int_{-\infty}^{\infty} e^{in(x-t)} \delta(x - t) f(t) \, dt \\ &= e^{inx} \int e^{-int} \delta(x - t) f(t) \, dt. \end{aligned} \tag{3}$$

Since δ is bounded, for each fixed x the function $\delta(x-t)f(t)$ is in L^1. Moreover, by the Riemann–Lebesgue theorem, for fixed x

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} e^{-int}\delta(x-t)f(t)\,dt = 0.$$

This implies, by (3),

$$\lim_{n\to\infty} |g_n * f(x)| = 0 \qquad (-\infty < x < \infty). \tag{4}$$

But

$$|g_n * f(x)| \leqslant \int_{-\infty}^{\infty} \delta(x-t)|f(t)|\,dt, \qquad \text{a.e.} \quad (-\infty < x < \infty)$$

and the function on the right is integrable over $(-\infty, \infty)$ and is independent of n. This, (4), and application of the Lebesgue dominated convergence theorem yields

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} |g_n * f(x)|\,dx = 0,$$

which proves (1). But

$$\| g_n \|_{L^1} = \int_{-\infty}^{\infty} | e^{inx}\delta(x)|\,dx = \| \delta \|_{L^1} = 1,$$

which proves (2). This completes the proof.

We now return to our main project.

LEMMA C. *Let B be a dense left ideal in the normed algebra $(A, \|\ \|_A)$. If A contains a right approximate identity, and if $b \cdot B = \{0\}$ for some $b \in A$, then $b = 0$.*

Proof. Obvious from Lemma A.

The following theorem will yield our answer to the question of Cigler mentioned in III.

THEOREM D. *Let $(B, \|\ \|_B)$ be a Banach algebra which is a dense left ideal in the Banach algebra $(A, \|\ \|_A)$. Suppose A contains a right approximate identity. Then if* A4 *(of* IV*) holds, so does* A3.

Proof. We wish to prove that the identity map from $(B, \|\ \|_B)$ to $(A, \|\ \|_A)$ is continuous. By the closed graph theorem it is sufficient to show that if b_n, $b \in B$, $B' \in A$

$$\| b_n - b \|_B \to 0 \tag{1}$$

and

$$\| b_n - b' \|_A \to 0, \tag{2}$$

then

$$b' = b. \tag{3}$$

Fix $f \in B$. Then

$$\begin{aligned}\|(b - b')f\|_B &\leqslant \|(b - b_n)f\|_B + \|(b_n - b')f\|_B \\ &\leqslant \| b - b_n \|_B \|f\|_B + C \| b_n - b' \|_A \|f\|_B ,\end{aligned}$$

where A4 was used in the last step. Letting $n \to \infty$ and using (1), (2), we obtain

$$\|(b - b')f\|_B = 0,$$

for all $f \in B$. Lemma C thus implies (3) which is what we wished to show.

Remark E. Since $L^1(G)$ always contains a right approximate identity, Theorem D has the affirmative answer to Cigler's question as a corollary.

We conclude with a result which gives conditions when A3 implies A4.

THEOREM F. *Let* $(B, \| \|_B)$ *be a Banach algebra which is a left ideal in the Banach algebra* $(A, \| \|_A)$. *Then if* A3 *holds, so does* A4.

Proof. For fixed $g \in B$ define the operator $T_g: A \to B$ by $T_g f = fg$. We shall use the closed graph theorem to show that T_g is continuous. Suppose $\langle f_n \rangle \in A$ and $\langle T_g f_n \rangle$, $h \in B$ satisfy

$$\|f_n\|_A \to 0, \tag{1}$$

$$\| T_g f_n - h \|_B \to 0. \tag{2}$$

We must show $h = 0$. We have $\| T_g f_n - h \|_B = \| f_n g - h \|_B \to 0$ by (2). Hence, by A3, $\| f_n g - h \|_A \to 0$. But (1) implies $\| f_n g \|_A \to 0$. Thus $h = 0$, so T_g is continuous. Hence there exists $C_g > 0$ such that

$$\| gf \|_B = \| T_g f \|_B \leqslant C_g \|f\|_A \qquad (f \in A, g \in B). \tag{3}$$

For each $f \in A$ with $\|f\|_A = 1$ let $S_f g = gf$ $(g \in B)$. Let $\mathscr{S}$ denote the family of all such S_f. By (3), if $S_f \in \mathscr{S}$ then

$$\| S_f g \|_B \leqslant C_g \qquad (g \in B).$$

The uniform boundedness principle thus implies the existence of $M > 0$ such that

$$\| S_f g \|_B \leqslant M \| g \|_B \qquad (S_f \in \mathscr{S}, g \in B).$$

That is, if $\|f\|_A = 1$, then

$$\|fg\|_B \leqslant M\|g\|_B \qquad (g \in B). \tag{4}$$

For arbitrary $f \in A$, $f \neq 0$ substitute $f/\|f\|_A$ for f in (4). We obtain

$$\|fg\|_B \leqslant M\|f\|_A\|g\|_B \qquad (f \in A, g \in B).$$

But this is A4, so the proof is complete.

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