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A parametrix for the fundamental solution of the Klein–Gordon equation on asymptotically de Sitter spaces

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Abstract

In this paper we construct a parametrix for the forward fundamental solution of the wave and Klein–Gordon equations on asymptotically de Sitter spaces without caustics. We use this parametrix to obtain asymptotic expansions for solutions of $(\Box - \lambda)u = f$ and to obtain a uniform L^p estimate for a family of bump functions traveling to infinity.

Keywords: Wave equation; Klein–Gordon equation; Parametrix; Microlocal

1. Introduction

De Sitter space is an exact solution of the vacuum Einstein equations with positive cosmological constant. In this paper, we study the forward fundamental solution of the wave and Klein–Gordon equations on asymptotically de Sitter spaces. This is the unique operator E_+ (which we identify with its Schwartz kernel) which satisfies $(\Box - \lambda)E_+ = I$ and is supported in the forward light cones, i.e., for a compactly supported smooth function f, the function $u = E_+ f$ satisfies

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$$\begin{aligned} (\Box - \lambda)u &= f, \\ u &\equiv 0 \quad \text{near past infinity.} \end{aligned} \tag{1}$$

Here λ is the Klein–Gordon parameter. If Eq. (1) is considered as a massive wave equation, the condition $\lambda \leqslant 0$ corresponds to positive mass. We construct a parametrix for this problem and establish asymptotic expansions for solutions. As an application of our parametrix, we prove a uniform L^p estimate for the operator applied to a family of bump functions tending toward infinity. We postpone to a future paper the consideration of Strichartz estimates and semilinear wave equations on asymptotically de Sitter spaces.

The study of the decay properties of the wave equation on various natural classes of spacetimes is an active area of research. For example, Dafermos and Rodnianski [3] and Melrose, Sá Barreto, and Vasy [19] have obtained decay results for solutions of the wave equation in the context of the de Sitter–Schwarzschild model of a black hole spacetime.

Our definition of asymptotically de Sitter spaces is given in [24] (called asymptotically de Sitter-like spaces in that manuscript) and follows the definition of asymptotically hyperbolic spaces given in [14,16]. An asymptotically de Sitter space is a compact manifold with boundary equipped with a Lorentzian metric having a prescribed asymptotic form near the boundary. This pushes the boundary off "to infinity".

The microlocal structure of the fundamental solution for general real principal type operators has been studied extensively. The solution operator for the Cauchy problem for general real principal type operators is a Fourier integral operator associated to a Lagrangian submanifold of phase space given by the flowout of the Hamilton vector field of the principal symbol of the operator. This was first described in this language by Duistermaat and Hörmander in [5]. In [20], Melrose and Uhlmann constructed the forward fundamental solution for a real principal type operator as a paired Lagrangian distribution. These are distributions associated to two cleanly intersecting Lagrangian submanifolds in phase space.

Guillemin and Uhlmann [8], Joshi [11], Melrose and Zworski [21], Hassell and Vasy [9], and others have all generalized the notion of paired Lagrangian distributions. Guillemin and Uhlmann defined a much more general class of paired Lagrangian distributions, which Joshi restricted slightly in order to construct a well-behaved calculus. Joshi then used this calculus to construct complex powers of the wave operator on Riemannian manifolds. Melrose and Zworski defined a class of distributions associated to an intersecting pair of Legendrians, while Hassell and Vasy later expanded this notion to describe the spectral projections on a scattering manifold.

Polarski [23] computed the propagator for the equation of the massless conformally coupled scalar field in the static de Sitter metric, which has been transformed to the Einstein open universe. Yagdjian and Galstian [25] computed the fundamental solutions for the Klein–Gordon equation in de Sitter spacetime transformed by the Lemaître–Robertson change of coordinates to the special case of the Friedmann–Robertson–Walker–Lemaître spacetime. They represented the fundamental solutions and solutions of the Cauchy problem using hypergeometric functions and proved $L^p L^q$ estimates.

Vasy [24] generalized and extended Polarski's result to asymptotically de Sitter spaces. He exhibited the well-posedness of the Cauchy problem and showed that on such spaces, the solution u of $(\Box - \lambda)u = 0$ with smooth Cauchy data has an asymptotic expansion at infinity. Indeed, if x is a boundary defining function for the conformal compactification of an asymptotically de Sitter space X and $\sqrt{\frac{(n-1)^2}{4} + \lambda}$ is not a half-integer, then u has an expansion

$$u = u_+ x^{\frac{n-1}{2} + \sqrt{\frac{(n-1)^2}{4} + \lambda}} + u_- x^{\frac{n-1}{2} - \sqrt{\frac{(n-1)^2}{4} + \lambda}},$$

where u_+ and u_- are smooth on X. (The difference between this expression and the corresponding one in [24] is due to differing sign conventions for Lorentzian metrics.) In the case of an integer coincidence, u_- is instead in $C^\infty(X) + x^{2\sqrt{\frac{(n-1)^2}{4}+\lambda}}(\log x)C^\infty(X)$. Vasy also showed that solutions of the wave equation exhibit scattering, i.e., that the data $u_\pm$ may be specified at one of $Y_\pm$ (future and past infinity, respectively), which fixes the data at $Y_\mp$.

Our result extends the work of Vasy to the study of the fundamental solution $(\Box - \lambda)E_+ = I$ on asymptotically de Sitter spaces. We require three global assumptions in our study of asymptotically de Sitter spaces:

(A1) $Y = Y_+ \cup Y_-$, with Y_+ and Y_- a union of connected components of Y,
(A2) each bicharacteristic γ of P converges to Y_+ as $t \to +\infty$ and to Y_- as $t \to -\infty$, or vice versa, and
(A3) the projection from T^*X to X of the flowout of the forward light cone from any point $p \in Y_-$ is an embedded submanifold of X (except at the point p, where it always has a conic singularity).

The first two of these assumptions are not particularly restrictive. They imply that the manifold is topologically a product $Y_+ \times \mathbb{R}$, but are reasonable from a physical viewpoint in that they imply a time orientation on the manifold. Colloquially, assumptions (A1) and (A2) prevent the breakdown of causality on X.

Assumption (A3) is needed only to obtain sharp global statements about the fundamental solution, but our construction works in some neighborhood of Y_+ (i.e., a neighborhood of future infinity) without this assumption.

Assumption (A3) prohibits the development of caustics, which significantly narrows the class of manifolds considered. De Sitter space just misses being covered by (A3), though a slight modification of our construction still applies here. Indeed, the projection of the flowout of the light cone from a point $p \in Y_-$ is a smooth embedded submanifold of the interior of de Sitter space, but intersects itself at Y_+. Section 16 discusses the minor modifications needed to handle this case.

If we slightly enlarge the spherical cross section of de Sitter space, then this new space satisfies the assumptions above. The assumptions above are stable under perturbation, so the construction applies to perturbations of this enlarged version of de Sitter space.

In order to remove the assumption (A3), we could combine the Poisson operator construction of [24] with the local description of the forward fundamental solution (given in [20]). Phrasing this in a geometric way is left to a future paper.

The main result of this paper is the following (we state it more precisely and define the relevant classes of distributions later):

Theorem 1. *Suppose (X, g) is an asymptotically de Sitter space. There is a compactification $\widetilde{X}_0^2$ of the interior of $X \times X$ to a compact manifold with corners such that the closures of the diagonal and the light cone both intersect all boundary hypersurfaces transversely. $\widetilde{X}_0^2$ is constructed by first blowing up the boundary of the diagonal in $X \times X$ and then blowing up the set where the projection of the flowout of the light cone hits the side face. On this compactification, the forward fundamental solution of $\Box - \lambda$ lifts to be the sum of a paired Lagrangian distribution,* *smooth down to the front face, and a conormal distribution associated to the light cone with polyhomogeneous expansions at the other faces of* $\widetilde{X}_0^2$.

We call a tempered distribution f on X *forward-directed* if it a smooth function on the interior of X that vanishes to all orders at Y_-. The work of Vasy [24] implies that if f is forward-directed, then we may apply E_+ to f.

A function f is *polyhomogeneous* with index set E on X if it has an asymptotic expansion of the form

$$\sum_{(r,l)\in E} x^r (\log x)^l a_{rl}(y)$$

near $Y_\pm$.

Let F_1 and F_2 be the following two index sets:

$$F_1 = \{(j,l)\colon\ j,l \in \mathbb{N}_0,\ l \leqslant j\},$$
$$F_2 = \{(s_\pm(\lambda) + m, 0)\colon\ m \in \mathbb{N}_0\}, \tag{2}$$

where $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \lambda}$.

A corollary of Theorem 1 describes the polyhomogeneity of the solutions of $P(\lambda)u = f$.

Theorem 2. *If f is forward-directed and polyhomogeneous on X with index set E at Y_+, then $E_+ f$ is forward-directed and polyhomogeneous with index set F, where*

$$F = F_1 \bar{\cup} F_2 \bar{\cup} E.$$

Here E_+ is regarded as an operator and $\bar{\cup}$ denotes the extended union of two index sets:

$$(z,p) \in E \bar{\cup} F \quad \Leftrightarrow \quad (z,p) \in E \cup F$$
$$\text{or} \quad p = p' + p'' + 1 \quad \text{with } (z,p') \in E \text{ and } (z,p'') \in F.$$

As an application of our parametrix, we obtain a uniform L^p estimate for a family of bump functions traveling to infinity. We leave more general L^p mapping properties and Strichartz estimates to a future manuscript.

Theorem 3. *Suppose $\phi \in C_c^\infty(\mathbb{R}_s^+ \times \mathbb{R}_z^{n-1})$ is supported near $(1,0)$. For $(\tilde{x},\tilde{y}) \in X$, let $f_{(\tilde{x},\tilde{y})}(x,y) = \phi(\frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}})$. Suppose that p, l, and r satisfy*

$$2 < p < \infty,$$
$$r > \max\left(\frac{1}{2}, \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right),$$

and

$$l \geqslant \max\left(0, -\frac{n-1}{2} + \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right).$$

Then

$$\| E_+ f_{(\tilde{x},\tilde{y})} \|_{x^{-l+2(l-r)/p} L^p(X;dg)} \leqslant C \tilde{x}^{-2r/p}. \tag{3}$$

Here the constant C *depends on the family* $f_{(\tilde{x},\tilde{y})}$ *but not on* $(\tilde{x}, \tilde{y})$.

In particular, for the wave equation, $\lambda = 0$*, and we may let* $l = 0$ *and* $r > \max(\frac{1}{2}, \frac{n-1}{2})$. *In this case,*

$$\| E_+ f_{(\tilde{x},\tilde{y})} \|_{x^{-2r/p} L^p(X;dg)} \leqslant C \tilde{x}^{-2r/p}.$$

Theorem 1 follows from a parametrix construction that comprises the bulk of this paper. The main ingredient in the proof of Theorem 2 is the pushforward theorem of Melrose. Theorem 3 follows from Theorem 1 and the L^2 estimates proved by Vasy [24].

The paper is broadly divided into three parts. The first part consists of Sections 2 through 9 and develops the tools necessary to construct the parametrix. The second part contains the construction, which begins with an outline in Section 10 and concludes in Section 15. The last part, consisting of Sections 17 and 18, proves Theorems 2 and 3.

2. Asymptotically de Sitter spaces

We start by describing the de Sitter space. Recall that hyperbolic space can be realized as one sheet of the two-sheeted hyperboloid in Minkowski space. It inherits a Riemannian metric from the Lorentzian metric in Minkowski space. De Sitter space, on the other hand, is the one-sheeted hyperboloid $\{-X_0^2 + \sum_{i=1}^n X_i^2 = 1\}$ in Minkowski space, but now the induced metric is Lorentzian. A good set of coordinates on this space, which is diffeomorphic to $S^n \times \mathbb{R}$, is

$$X_0 = \sinh\tau,$$
$$X_i = \omega_i \cosh\tau,$$

where ω_i are coordinates on the unit sphere. The de Sitter metric is then

$$-d\tau^2 + \cosh^2\tau \, d\omega^2.$$

If we let $T = e^{-\tau}$ near $\tau = +\infty$, then $\tau = +\infty$ corresponds to $T = 0$ and the metric now has the form

$$\frac{-dT^2 + \frac{1}{4}(T^2+1)^2 \, d\omega^2}{T^2}. \tag{4}$$

Our definition of an asymptotically de Sitter space is based on the form (4) of the de Sitter metric.

Definition 4. (X, g) is *asymptotically de Sitter* if X is an n-dimensional compact manifold with boundary Y, g is a Lorentzian metric on the interior of X, and, for a boundary defining function x, there is some collar neighborhood $[0, \epsilon)_x \times Y$ of the boundary on which g has the form

$$g = -\frac{dx^2}{x^2} + \frac{h(x, y, dy)}{x^2},$$

where h is a smooth symmetric $(0, 2)$-tensor on Y that is a metric on the boundary $\{0\} \times Y$.

A calculation similar to the ones in [14] or [16] shows that the sectional curvatures of asymptotically de Sitter spaces approach 1 as $x \to 0$.

Let $\Box$, the wave operator, be the wave operator (d'Alembertian) associated to g, and, for $\lambda \in \mathbb{R}$, let $P = P(\lambda) = \Box - \lambda$ be the Klein–Gordon operator. We are seeking the fundamental solution of the Klein–Gordon equation, i.e., a distribution $E_{+,\lambda}$ such that $P(\lambda)E_{+,\lambda} = I$. Note that the principal symbol $\sigma_2(P)$ is given by the dual metric function of g.

Definition 5. We say that the *bicharacteristics* of P over X°, the interior of X, are the integral curves of the Hamilton vector field of the principal symbol $\sigma_2(P)$ inside the characteristic set of P (the set where $\sigma_2(P)$ vanishes).

Throughout this paper, we assume that both (A1) and (A2) hold. These assumptions are introduced primarily to ensure that X exhibits a coherent causal structure.

Because g is conformal to the incomplete pseudo-Riemannian metric

$$\hat{g} = -dx^2 + h$$

near Y, the bicharacteristics of P (near Y) are reparametrizations of the bicharacteristics of $-dx^2 + h$. g is complete, so the global assumptions imply that each bicharacteristic $\gamma(t)$ has a limit in $S^*_Y X$ as $t \to \pm\infty$.

Physicists (e.g. [6]) have long known that these assumptions imply the existence of a global 'time' function $T \in C^\infty(\overline{X})$ such that $T|_{Y_\pm} = \pm 1$ and is monotone on the nullbicharacteristics of P. Note that $1 - x$ and $x - 1$ have the desired properties near Y_+ and Y_-, so the assumptions mean that such functions can be extended to all of X. Moreover, T gives a fibration $\overline{X} \to [-1, 1]$ and so X is diffeomorphic to $X \times S$ for a compact manifold S. This also shows that Y_+ and Y_- are both diffeomorphic to S.

We also assume that assumption (A3) holds. This assumption ensures that the boundary of the flowout of the light cone in the cotangent bundle is actually the cotangent bundle of a submanifold in the base. It allows us to work with an adapted class of conormal distributions rather than a class of Lagrangian distributions. Note that the projection of the flowout is automatically an embedded submanifold for small times. Moreover, because the nullbicharacteristics agree in the compact and non-compact settings (i.e., for $\hat{g}$ and g), there is a neighborhood of Y_+ such that the projection of the flowout of the forward light cone from any point in this neighborhood is an embedded submanifold. If we then restrict to data supported in this neighborhood, we may use our construction below even in the absence of assumption (A3).

In this paper we adopt the convention that $\int \delta(x - \tilde{x})\delta(y - \tilde{y}) f(\tilde{x}, \tilde{y})\, d\hat{g} = f(x, y)$. In particular, this breaks the formal self-adjointness of the operator $P(\lambda)$. When we seek a right parametrix, then, we must use the transpose of the operator $P(\lambda)$ with respect to this metric. We may write this out explicitly as

$$P^t(\lambda) = (x\partial_x)^2 + (n+1)x\partial_x + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}x\partial_x + x^2\Delta_y + n\left(1 + \frac{x\partial_x\sqrt{h}}{\sqrt{h}}\right) - \lambda. \tag{5}$$

We also adopt the convention that when we are applying a differential operator to the variables in the right factor, we use a subscript R.

In our consideration of the forward fundamental solution of $P(\lambda)$, it is useful to have the notion of *forward-directed* data, which are simply tempered distributions (or polynomially bounded smooth functions) on X that vanish to all orders at Y_-.

3. The Cauchy problem and scattering

In [24], Vasy studied both the Cauchy problem and the scattering problem on asymptotically de Sitter spaces. The Cauchy problem seeks a solution the equation

$$\begin{cases} P(\lambda)u = 0, \\ u|_{\Sigma_0} = \psi_0, \qquad Vu|_{\Sigma_0} = \psi_1, \end{cases} \tag{6}$$

where Σ_0 is a Cauchy hypersurface, V is a vector field transverse to Σ_0, and ψ_0, ψ_1 are smooth functions on Σ_0.

Via a positive commutator argument, Vasy showed global existence and uniqueness for solutions of the Cauchy problem (6) on asymptotically de Sitter spaces. One may use any "time slice" $\{T = \text{const}\}$ from the diffeomorphism $X \cong \mathbb{R} \times Y_+$ as the Cauchy hypersurface needed to pose the Cauchy problem.

Vasy further proved the following a global solvability result for the inhomogeneous equation on asymptotically de Sitter spaces under assumptions (A1) and (A2).

Theorem 6. *(See Theorem 5.4 of [24].) Suppose that $\lambda \in \mathbb{R}$, and that $l_\pm$ satisfy*

$$l_+ > \max\left(\frac{1}{2}, l(\lambda)\right), \qquad l_- < -\max\left(\frac{1}{2}, l(\lambda)\right).$$

Then for $f \in H_0^{0,l_+,l_-}(X)$, $Pu = f$ has a unique solution $u \in H_0^{1,l_+,l_-}(X)$.

Here $l(\lambda) = \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}$ while the $H_0^{m,q_+,q_-} = x_+^{q_+}x_-^{q_-}H_0^m$ measure both regularity and decay at Y_+, Y_- separately. The spaces H_0^m are weighted Sobolev spaces measuring regularity with respect to the 0-vector fields defined in the following section.

In particular, for any $f \in \dot{C}^\infty(X)$ (indeed, for any forward-directed tempered smooth function), there is a unique solution $u \in x_+^{-\infty}x_-^{\infty}H_0^1(X)$ to $Pu = f$. (Here $x_\pm$ is a defining function for $Y_\pm$, so the notation means that u is tempered at Y_+ and vanishes to infinite order at Y_-.) There is thus a distribution E_+ on $X \times X$ that can be called the forward fundamental solution of P. In other words, E_+ is such that for each $p \in X$, $PE_+(p) = \delta_p$ and $E_+(p)(q) \equiv 0$ for $q \in X$ not in the domain of influence of p. In terms of the identification $X \cong \mathbb{R}_t \times Y_+$, this means that $E_+(p)(q)$ vanishes when $t(q) < t(p)$.

Vasy (cf. Theorem 5.5 of [24]) proved also that solutions of the Cauchy problem (6) have unique asymptotic expansions near the boundary Y of X of the form

$$u = u_+ x^{\frac{n-1}{2}+\sqrt{\frac{(n-1)^2}{4}+\lambda}} + u_- x^{\frac{n-1}{2}-\sqrt{\frac{(n-1)^2}{4}+\lambda}},$$

where $u|_\pm$ are smooth up to the boundary of X. When the difference between the two exponents is an integer, then u_- is instead only in $C^\infty(X) + x^{2\sqrt{\frac{(n-1)^2}{4}+\lambda}}(\log x)C^\infty(X)$. Moreover, one may specify $u_\pm|_{Y_-}$, uniquely determining $u_\pm|_{Y_+}$. He showed (cf. Theorem 7.21 of [24]) that when there are no integer coincidences the map sending the data

$$(u_+|_{Y_-}, u_-|_{Y_-}) \mapsto (u_+|_{Y_+}, u_-|_{Y_+})$$

is a Fourier integral operator associated to the canonical relation given by bicharacteristic flow from Y_- to Y_+. This is via an explicit parametrix construction on an appropriate blow-up of $X \times Y$, and the static model of de Sitter space appears in the consideration of the rescaled normal operator at the front face of this blow-up.

Observe now that for $(\tilde{x}_0, \tilde{y}_0) \in X$ away from the boundary of X, the restriction of the kernel of the fundamental solution to the slice $\tilde{x} = \tilde{x}_0$ agrees with a multiple of the solution of the Cauchy problem

$$P(\lambda)u = 0,$$
$$u(\tilde{x}_0, \cdot) = 0,$$
$$Vu(\tilde{x}_0, \cdot) = \delta(y - \tilde{y}),$$

at least in the region to the future of $(\tilde{x}_0, \tilde{y}_0)$. (Here V is as in Eq. (6), i.e., a linear combination of $\tilde{x}\partial_{\tilde{x}}$ and $\tilde{x}\partial_{\tilde{y}}$ transverse to $\{\tilde{x} = \tilde{x}_0\}$.) In particular, we can understand the fundamental solution in this region by understanding the solution operator for the Cauchy problem, which Vasy studied.

The solution operator for the Cauchy problem is the composition of the Poisson operator and the Cauchy-to-scattering operator. Vasy showed that these are Fourier integral operators, and their canonical relations intersect transversely, so we can compose them. The result is an operator with canonical relation given by the restriction of the conormal bundle of the flowout of the light cone, restricted to $\tilde{x} = \tilde{x}_0$ but lifted to a blown-up space. This blown-up space agrees with a slice of the space we define later. Vasy's construction blows up $[X \times Y_+, \operatorname{diag} Y_+]$, which turns into what we call lcf_+ under this composition. Applying this operator to a delta distribution gives a conormal distribution on our space, and the log terms in Vasy's construction become the log terms in our construction, though this requires careful bookkeeping. Viewing this as a distribution parametrized by $\tilde{x}$ gives the fundamental solution in a neighborhood of the interior of what we call lf_+ away from the other boundary hypersurfaces.

4. 0-geometry

Recall from [16] the Lie algebra of 0-vector fields,

$$\mathcal{V}_0 = \{\text{vector fields vanishing at } \partial X\}.$$

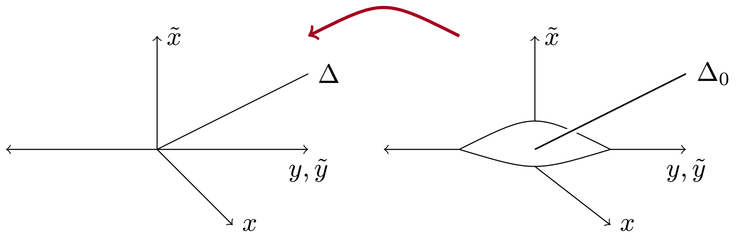


Fig. 1. Passing from $X \times X$ to the 0-double space $X_0^2 = [X^2, \partial\Delta]$.

The universal enveloping algebra of this Lie algebra is the algebra of 0-differential operators, $\mathrm{Diff}_0^*(X)$.

We may use the Lie algebra $\mathcal{V}_0(X)$ to define the natural tensor bundle ${}^0TX$, whose smooth sections are elements of $\mathcal{V}_0(X)$. Its dual, ${}^0T^*X$, has smooth sections spanned (over $C^\infty(X)$) by $\frac{dx}{x}$ and $\frac{dy_j}{x}$.

Recall that T^*X is endowed with a canonical 1-form given by

$$\alpha = \xi\, dx + \mu \cdot dy = x\xi \frac{dx}{x} + x\mu \cdot \frac{dy}{x}.$$

In particular, ${}^0T^*X$ is endowed with the canonical 1-form

$$^0\alpha = \tau \frac{dx}{x} + \eta \cdot \frac{dy}{x}. \tag{7}$$

Just as T^*X is endowed with a symplectic form given by $\omega = d\alpha$, ${}^0T^*X$ is endowed with the symplectic form ${}^0\omega = d^0\alpha$.

As in [14] or [16], we define the 0-double space $X \times_0 X = X_0^2$ as the blown-up space $[X \times X, \partial\Delta]$. $X \times_0 X$ is a manifold with corners agreeing with $X \times X$ on the interior. It has three boundary hypersurfaces – the left face $\mathrm{lf} = \{x = 0\}$, which is the lift of the boundary hypersurface $\{0\} \times X$ in $X \times X$; the right face $\mathrm{rf} = \{\tilde{x} = 0\}$, which is the lift of $X \times \{0\}$; and the front face ff, which is the boundary hypersurface introduced by the blow up construction. Recall also that X_0^2 is equipped with a blow-down map $\beta : X_0^2 \to X \times X$. A neighborhood of the front face ff is depicted in Fig. 1.

This construction is perhaps best thought of as an invariant way of introducing spherical coordinates near the boundary of the diagonal in $X \times X$. Indeed, a valid coordinate system near the front face ff is given by spherical coordinates:

$$\rho_{\mathrm{ff}} = \sqrt{x^2 + \tilde{x}^2 + |y - \tilde{y}|^2},$$
$$\theta_0 = \frac{x}{\rho_{\mathrm{ff}}}, \qquad \theta_n = \frac{\tilde{x}}{\rho_{\mathrm{ff}}}, \qquad \theta' = \frac{y - y'}{\rho_{\mathrm{ff}}},$$

where $\theta \in S^n_{++}$, a quarter sphere. It is often more convenient to work with projective coordinates near the front face:

$$s = \frac{x}{\tilde{x}}, \quad z = \frac{y - \tilde{y}}{\tilde{x}}, \quad \tilde{x}, \quad \tilde{y}.$$

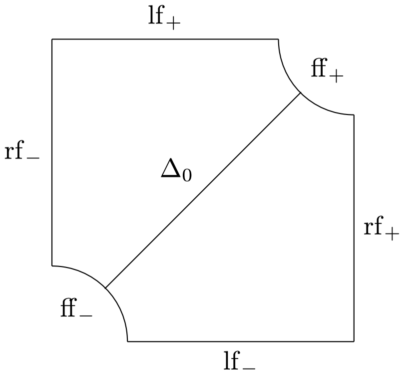


Fig. 2. A two-dimensional view of X_0^2.

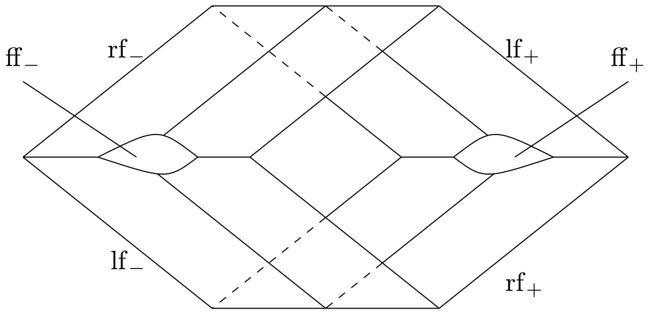


Fig. 3. Another view of X_0^2. The horizontal lines represent the $y, \tilde{y}$ axes.

In the projective coordinates $(s, z, \tilde{x}, \tilde{y})$, we may compute the lifts of the vector fields $x\partial_x, x\partial_y, \tilde{x}\partial_{\tilde{x}}$, and $\tilde{x}\partial_{\tilde{y}}$. Indeed, we have

$$x\partial_x \rightsquigarrow s\partial_s, \qquad x\partial_y \rightsquigarrow s\partial_z,$$
$$\tilde{x}\partial_{\tilde{x}} \rightsquigarrow \tilde{x}\partial_{\tilde{x}} - s\partial_s - z\cdot\partial_z, \qquad \tilde{x}\partial_{\tilde{y}} \rightsquigarrow \tilde{x}\partial_{\tilde{y}} - \partial_z.$$

Recall that the front face is the total space of a fibration. Indeed, it is the total space of a bundle over Y with quarter-sphere fibers. In local projective coordinates $(s, z, \tilde{x}, \tilde{y})$ near the front face, the fibers are given by $\{\tilde{x} = 0,\ \tilde{y} = \text{const}\}$. An asymptotically de Sitter metric on X induces a Lorentzian metric $-\frac{ds^2}{s^2} + \frac{h(0,\tilde{y},dz)}{s^2}$ on the interior of the fibers. By an affine change of coordinates in the fiber, this metric may be written as $-\frac{ds^2}{s^2} + \frac{dz^2}{s^2}$, which is the Wick rotation of the (negative definite) hyperbolic metric on an upper half-space.

Due to our assumptions (A1) and (A2), the space X has two boundary components Y_+ and Y_-, so it is useful at this point to give names to the components of the left face, right face, and front face. We call ff_+ the component of the front face that comes from blowing up the diagonal near $Y_+ \times Y_+$, and ff_- the other component. Similarly, we give the name lf_+ to the lift of the hypersurface $Y_+ \times X$ and lf_- to the lift of $Y_- \times X$ (and rf_+, rf_- denote the lifts of $X \times Y_+$ and $X \times Y_-$, respectively). This naming scheme is illustrated in Figs. 2 and 3.

We should also recall the trivial half-density bundles $\Omega^{\frac{1}{2}}(X)$ and $\Omega^{\frac{1}{2}}(X_0^2)$, which are trivial bundles. In local coordinates (x, y), $\Omega^{\frac{1}{2}}(X)$ is trivialized by the global section $\gamma = |d\hat{g}|^{1/2}$. The bundle $\Omega^{\frac{1}{2}}(X \times X)$ is trivialized by $\upsilon = |d\hat{g}_L d\hat{g}_R|^{1/2}$, where $d\hat{g}_L$ and $d\hat{g}_R$ are the lifts of the densities from the left and right factors of $X \times X$. Up to a nonvanishing factor, the Jacobian determinant of the blow-down map $X_0^2 \to X \times X$ is r^n, so υ lifts to $r^{n/2}\mu$, where μ is a nonvanishing section of the standard half-density bundle $\Omega^{\frac{1}{2}}(X_0^2)$. For example, we may take $\mu = |dr\, d\theta\, d\tilde{y}|^{1/2}$ in local polar coordinates. In projective coordinates, we may take $\mu = |ds\, dz\, d\tilde{x}\, d\tilde{y}|^{1/2}$, and then υ lifts to $\tilde{x}^{n/2}\mu$ (times a nonvanishing factor). In this paper, we adopt the convention that υ and γ are flat, i.e., for a differential operator L, $L(u\gamma) = (Lu)\gamma$ and $L(u\upsilon) = (Lu)\upsilon$. In particular,

$$L(u\mu) = L\big(\tilde{x}^{-n/2}u\upsilon\big) = \big(L\big(\tilde{x}^{-n/2}u\big)\big)\upsilon = \big(\tilde{x}^{n/2}L\big(\tilde{x}^{-n/2}u\big)\big)\mu.$$

Our operator $P(\lambda)$ commutes with $\tilde{x}$, so $P(\lambda)(u\mu) = (P(\lambda)u)\mu$.

On the fibers of the front face, elements of $\mathrm{Diff}_0^*(X)$ lift to differential operators on X_0^2 that restrict to differential operators on the fibers of the front face by setting $\tilde{x} = 0$. We call this restriction to the fiber over $p \in Y$ the *normal operator* and denote it as $N_p(A)$ for $A \in \mathrm{Diff}_0^*(X)$. As an example, in projective coordinates the normal operator of $P(\lambda)$ is given by

$$N_p\big(P(\lambda)\big) = (s\partial_s)^2 - (n-1)(s\partial_s) + s^2\Delta_z - \lambda. \tag{8}$$

The inclusions $\mathrm{ff} \subset X_0^2$ and $\mathrm{lf} \subset X_0^2$ induce inclusions on the tangent bundles $T\mathrm{ff} \subset T_{\mathrm{ff}}X_0^2$ and $T\mathrm{lf} \subset T_{\mathrm{lf}}X_0^2$. These inclusions induce natural projections on the cotangent bundle:

$$\pi_{\mathrm{ff}} : T_{\mathrm{ff}}^* X_0^2 \to T^*\mathrm{ff} \quad \text{and} \quad \pi_{\mathrm{lf}} T_{\mathrm{lf}}^* X_0^2 \to T^*\mathrm{lf}.$$

Let $2X_0^2$ denote the doubling of X_0^2 across ff, and let $T_{\mathrm{ff}}^* X_0^2$ denote the restriction of the cotangent space to the front face. As in [12], we say that a smooth closed conic Lagrangian submanifold $\Lambda \subset T^*X_0^2$ is *extendible* if it intersects $T_{\mathrm{ff}}^* X_0^2$ transversely. Recall that in this case there is a smooth closed conic Lagrangian submanifold $\Lambda^e \subset T^*2X_0^2$ such that

$$\Lambda = \Lambda^e \cap T^*X_0^2, \qquad \Lambda^0 = \pi_{\mathrm{ff}}\big(\Lambda \cap T_{\mathrm{ff}}^* X_0^2\big).$$

Note that there are many choices for the extension Λ^e.

We recall a result [12]:

Lemma 7. *(See Lemma 2.1 of [12].) Let $\Lambda \subset T^*X_0^2$ be an extendible Lagrangian. Then $\Lambda_0 = \pi_{\mathrm{ff}}(\Lambda \cap T_{\mathrm{ff}}^* X_0^2)$ is a Lagrangian submanifold of $T^*\mathrm{ff}$.*

For future use, we collect a bit more information about the symplectic structure of $T^*X_0^2$. The space ${}^0T^*X \times {}^0T^*X$ is endowed with the symplectic form

$$\omega = \pi_1^*\omega_X + \pi_2^*\omega_X, \tag{9}$$

where ω_X is the symplectic form on ${}^0T^*X$ (coming from the canonical 1-form (7)) and π_j the projection of the jth copy of ${}^0T^*X$. Moreover, the blow-down map β induces a smooth map

$$ {}^0T^*X \times {}^0T^*X \to T^*X_0^2 $$

which is an isomorphism on the interior.

The identification of T^*X and ${}^0T^*X$ over the interior of X then induces a smooth map ${}^0T^*X \times {}^0T^*X \to T^*X_0^2$ over the interior of $X \times X$. This map identifies the 1-forms

$$ \tau\frac{dx}{x} + \tilde{\tau}\frac{d\tilde{x}}{\tilde{x}} + \eta\cdot\frac{dy}{x} + \tilde{\eta}\cdot\frac{d\tilde{y}}{\tilde{x}} \quad \text{and} \quad \sigma\, ds + \xi\, d\tilde{x} + \zeta\cdot dz + \mu\cdot d\tilde{y}, $$

and so we must have

$$ \begin{aligned} \sigma &= \frac{\tau\tilde{x}}{x}, \qquad \xi = \frac{\tau}{\tilde{x}} + \frac{\tilde{\tau}}{\tilde{x}} + \frac{\eta}{x}\cdot\frac{y-\tilde{y}}{\tilde{x}}, \\ \zeta &= \frac{\tilde{x}\eta}{x}, \qquad \mu = \frac{\eta}{x} + \frac{\tilde{\eta}}{\tilde{x}}. \end{aligned} \tag{10} $$

Observe that in these coordinates, the symplectic form ω takes a familiar form:

$$ d\sigma\wedge ds + d\xi\wedge d\tilde{x} + d\zeta\wedge dz + d\mu\wedge d\tilde{y}. $$

5. Polyhomogeneity and conormal distributions

In order to consider the pushforward of a distribution, we need the notion of a b-fibration. We recall from [17] that an *interior b-map* $f : X \to Y$ of manifolds with corners is a function mapping $X \to Y$ and $X \setminus \partial X \to Y \setminus \partial Y$ such that each boundary defining function for Y pulls back to a sum of products of boundary defining functions for X. More precisely, suppose that $M_1(X)$ is the set of boundary hypersurfaces of X, and, for each $G \in M_1(X)$, let $I(G)$ be the ideal of functions vanishing at G. Suppose that H is a boundary hypersurface for Y and $I(H)$ is the ideal of functions vanishing on H. We say that $f : X \to Y$ is an interior b-map if

$$ f^*I(H) = \prod_{G\in M_1(X)} I(G)^{e_f(G,H)}. $$

Here $e_f(G,H)$ is a collection of nonnegative integers that we call the exponent matrix of f. We set $(e_f) = \{H \in M_1(X)\colon\ e_f(H,G) = 0,\ \forall G \in M_1(Y)\}$.

For an interior b-map f, the differential $f_* : T_pX \to T_{f(p)}Y$ extends by continuity to the b-differential

$$ f_* : {}^bT_pX \to {}^bT_{f(p)}Y. $$

Recall that a *b-fibration* between two manifolds with corners is one that is both b-normal and a *b-submersion*. A b-submersion is a b-map with surjective b-differential. A b-normal map is one such that the b-differential is surjective as a map ${}^bN_xH \to {}^bN_{f(x)}G$ on the interior of each boundary hypersurface H. Here G is the face that contains the image of H.

We say that a discrete subset $E \subset \mathbb{C} \times \mathbb{N}_0$ is an *index set* if $(z_j, k_j) \in E$ with $|(z_j, k_j)| \to \infty$ implies that $\Re z_j \to \infty$, $(z, k) \in E$ implies that $(z + p, k) \in E$ for all $p \in \mathbb{N}_0$, and if $(z, k) \in E$, then $(z, p) \in E$ for all $p \in \mathbb{N}_0$ with $0 \leqslant p < k$. We say that $\mathcal{E} = \{E_H\colon H$ is a boundary hypersurface of $M\}$ is an index family if each E_H is an index set.

Our construction below shows that certain distributions have asymptotic expansions. We make this notion more precise by recalling the definition of a polyhomogeneous distribution.

Definition 8. A *polyhomogeneous distribution* is a distribution that is smooth on the interior of M, and, near each boundary hypersurface H of M has an asymptotic expansion of the form

$$\sum_{(s,l)\in E_H} x_H^s (\log x_H)^l a_{sl}(x, y),$$

where x_H is a defining function for H, E_H is an index set for H, and a_{sl} are smooth functions independent of x_H. Near the corners of M, we require that polyhomogeneous distributions have an appropriate product-type expansion.

For an extended discussion of polyhomogeneous distributions, we refer the reader to [15,17], or [18]. We adopt the convention that a_{sl} should take values in the half-density bundles.

Recall that if f and g are polyhomogeneous with index families $\mathcal{F}$ and $\mathcal{G}$, then fg is polyhomogeneous with index set $\mathcal{F} + \mathcal{G}$.

We also require the following lemma, which can be found in [17] or [18], and allows us to understand the polyhomogeneity of the pushforward of a distribution:

Lemma 9. *If $f : X \to Y$ is a b-fibration between compact manifolds with corners, then for any index family $\mathcal{K}$ for X with*

$$\Re\mathcal{K}(H) > -1 \quad \text{if } H \in (e_f),$$

the pushforward gives

$$f_* : \mathcal{A}^{\mathcal{K}}_{\text{phg}}\big(X; \Omega \otimes f^*E\big) \to \mathcal{A}^{\mathcal{J}}_{\text{phg}}(Y; \Omega \otimes E),$$
$$\mathcal{J} = f_\#\mathcal{K}.$$

Here $\mathcal{J} = f_\#\mathcal{K}$ is the pushforward of the index family $\mathcal{K}$. In particular, for a boundary hypersurface H of Y,

$$\mathcal{J}(H) = \bar{\bigcup_{G\in M_1(X),\, e_f(G,H)\neq 0}} \big\{\big(z/e_f(G, H), p\big)\colon (z, p) \in \mathcal{K}(G)\big\},$$

where $\bar{\bigcup}$ is the extended union defined by $(z, p) \in K \mathbin{\bar{\cup}} J$ if and only if $(z, p) \in K \cup J$ or $p = p' + p'' + 1$ with $(z, p') \in K$ and $(z, p'') \in J$.

We recall the definition of the space of conormal distributions (see, for example, [10] or [17]). If M is a manifold with corners and Z is an interior p-submanifold, then $I^p(M; Z)$ is the space of conormal distributions of order p associated to Z. Here we only need the case when Z has codimension one and meets the boundary hypersurfaces transversely. In local coordinates (x, y)

where M is given by $x_i \geqslant 0$ for $i = 1, \dots, k$ and Z is given by $y_1 = 0$, then $u \in I^p(M; Z)$ has the form

$$\int_{\mathbb{R}} e^{iy_1\eta} a(x, y, \eta)\, d\eta,$$

with a a half-density valued classical symbol of order $p + n/4$ and $n = \dim M$ (note that we are using the order convention consistent with the orders of Lagrangian distributions in [20]). Because x behaves as a parameter, the boundaries cause no problems here.

We also need several refinements of this notion. The first is when we allow the symbol a to have an asymptotic expansion (in x) near the boundary of M. We need the space $\mathcal{A}^{\mathcal{E}}_{\mathrm{phg}} I^p(M; Z)$ of conormal distributions associated to Z with polyhomogeneous expansion at the boundary of M. These are merely conormal distributions in the sense above, but where we allow the symbol a to have a polyhomogeneous expansion in the x variables.

The second generalization we require is to allow the symbol to have a mild type of singularity at $\eta = 0$. In particular, we allow symbols of the form $\sum_j (\eta + i0)^{r-j} a_j$. The singularity at 0 only affects the growth of the distribution as $y \to \pm\infty$, and so we may avoid complications by multiplying our distribution by a cutoff function supported near $y = 0$. We need this generalization later in order to prove a lemma about the support of our distributions.

6. The Hamilton vector fields

By analogy with the definition of the Hamilton vector field, we recall from [12] the definition of the 0-*Hamilton vector field*.

Definition 10. Given $p \in C^\infty(T^*X)$, the 0-Hamilton vector field of p, denoted ${}^0H_p$, is defined by

$${}^0\omega\big(\cdot, {}^0H_p\big) = dp.$$

In local coordinates where ${}^0\alpha$ is given by (7), ${}^0H_p$ is given by

$${}^0H_p = x\frac{\partial p}{\partial \tau}\frac{\partial}{\partial x} + x\frac{\partial p}{\partial \eta}\cdot\frac{\partial}{\partial y} - \left(x\frac{\partial p}{\partial x} + \frac{\partial p}{\partial \eta}\cdot\eta\right)\frac{\partial}{\partial \tau} - \left(x\frac{\partial p}{\partial y} - \frac{\partial p}{\partial \tau}\eta\right)\cdot\frac{\partial}{\partial \eta}.$$

We are interested in the operator $P(\lambda)$, whose principal symbol is the length function

$$p = -x^2\xi^2 + x^2\big(h_0(y, \mu) + xh_1(x, y, \mu)\big) = -\tau^2 + h_0(y, \eta) + xh_1(x, y, \eta),$$

where we have written $h(x, y, \mu) = h_0(y, \mu) + xh_1(x, y, \mu)$. This is also the principal symbol of P when P is treated as an operator acting on the left factor of the product space $X \times X$.

In particular, in the coordinates given by (10), p pulls back to

$$\tilde{p} = -s^2\sigma^2 + s^2h_0(\tilde{y} + \tilde{x}z, \zeta) + s^3\tilde{x}h_1(\tilde{x}s, \tilde{y} + \tilde{x}z, \zeta).$$

Thus if $\tilde{\omega}$ is the lift of the symplectic form (9) (as in [12]) to $T^*X_0^2$, ${}^0H_p$ lifts to $H_{\tilde{p}}$, which is given by $\tilde{\omega}(\cdot, H_{\tilde{p}}) = d\tilde{p}$. In these coordinates, we may write

$$\begin{aligned} H_{\tilde{p}} = {} & \frac{\partial \tilde{p}}{\partial \sigma}\frac{\partial}{\partial s} - \frac{\partial \tilde{p}}{\partial s}\frac{\partial}{\partial \sigma} + \frac{\partial \tilde{p}}{\partial \zeta}\cdot\frac{\partial}{\partial z} - \frac{\partial \tilde{p}}{\partial z}\cdot\frac{\partial}{\partial \zeta} \\ & + \frac{\partial \tilde{p}}{\partial \xi}\frac{\partial}{\partial \tilde{x}} - \frac{\partial \tilde{p}}{\partial \tilde{x}}\frac{\partial}{\partial \xi} + \frac{\partial \tilde{p}}{\partial \mu}\cdot\frac{\partial}{\partial y} - \frac{\partial \tilde{p}}{\partial y}\cdot\frac{\partial}{\partial \mu}. \end{aligned}$$

An elementary computation then yields

$$\begin{aligned} H_{\tilde{p}} = {} & -2s^2\sigma\frac{\partial}{\partial s} - s\left(-2\sigma^2 + 2h_0 + 3s\tilde{x}h_1 + s^2\tilde{x}^2\frac{\partial h_1}{\partial x}\right)\frac{\partial}{\partial \sigma} \\ & + s^2\left(\frac{\partial h_0}{\partial \eta} + s\tilde{x}\frac{\partial h_1}{\partial \eta}\right)\cdot\frac{\partial}{\partial z} - s^2\left(\tilde{x}\frac{\partial h_0}{\partial y} + s\tilde{x}^2\frac{\partial h_1}{\partial y}\right)\cdot\frac{\partial}{\partial \zeta} \\ & - s^2\left(z\cdot\frac{\partial h_0}{\partial y} + sh_1 + s^2\tilde{x}\frac{\partial h_1}{\partial x} + s\tilde{x}z\cdot\frac{\partial h_1}{\partial y}\right)\frac{\partial}{\partial \xi} \\ & - s^2\left(\frac{\partial h_0}{\partial y} + s\tilde{x}\frac{pdh_1}{\partial y}\right)\cdot\frac{\partial}{\partial \mu}, \end{aligned}$$

where h_0 and h_1 are evaluated at $(\tilde{x}s, \tilde{y} + \tilde{x}z, \zeta)$.

Note that the characteristic set of P on X_0^2 is the set where $\tilde{p} = 0$, i.e., the set where $s^2\sigma^2 = s^2(h(\tilde{y} + \tilde{x}z, \zeta) + s\tilde{x}h_1(s\tilde{x}, \tilde{y} + \tilde{x}z, \zeta))$. In this set, $H_{\tilde{p}}$ has the form

$$\begin{aligned} H_{\tilde{p}} = s^2\Bigg(& -2\sigma\frac{\partial}{\partial s} - \left(\tilde{x}h_1 + s\tilde{x}^2\frac{\partial h_1}{\partial x}\right)\frac{\partial}{\partial \sigma} + \left(\frac{\partial h_0}{\partial \eta} + s\tilde{x}\frac{\partial h_1}{\partial \eta}\right)\cdot\frac{\partial}{\partial z} \\ & - \left(\tilde{x}\frac{\partial h_0}{\partial y} + s\tilde{x}^2\frac{\partial h_1}{\partial y}\right)\cdot\frac{\partial}{\partial \zeta} - \left(\frac{\partial h_0}{\partial y} + s\tilde{x}\frac{pdh_1}{\partial y}\right)\cdot\frac{\partial}{\partial \mu} \\ & - \left(z\cdot\frac{\partial h_0}{\partial y} + sh_1 + s^2\tilde{x}\frac{\partial h_1}{\partial x} + s\tilde{x}z\cdot\frac{\partial h_1}{\partial y}\right)\frac{\partial}{\partial \xi}\Bigg). \end{aligned} \tag{11}$$

In particular, we have proved the following proposition:

Proposition 11. *Inside the characteristic set of P on X_0^2, $H_L = H_{\tilde{p}} = s^2\widetilde{H}_L$, where $\widetilde{H}_L$ is a smooth vector field tangent to the front face that is nondegenerate at $s = 0$. Similarly, in coordinates $(x, y, \tilde{s} = s^{-1}, \tilde{z} = s^{-1}z)$, the Hamilton vector field for the lift from the right factor can be written $H_R = \tilde{s}^2\widetilde{H}_R$, where $\widetilde{H}_R$ is nondegenerate at $\tilde{s} = 0$.*

The fact that we may factor a power of s^2 from the Hamilton vector field is useful in the next section.

7. The Lagrangians

The two Lagrangians that interest us are the lift of the diagonal and the flowout from the characteristic set of P within this conormal bundle by the Hamilton vector field H_L.

The coordinates on phase space given by (10) imply that the conormal bundle of the diagonal,

$$\{x=\tilde{x},\ y=\tilde{y},\ \tau=-\tilde{\tau},\ \eta=-\tilde{\eta}\},$$

lifts to

$$\Lambda_0=\{s=1,\ z=0,\ \xi=0,\ \mu=0\}.$$

The Lagrangian submanifold Λ_0 intersects $T^*_{\mathrm{ff}}X^2_0$ transversely at

$$\Lambda^0_0=\{s=1,\ z=0,\ \xi=0,\ \mu=0,\ \tilde{x}=0\}=T^*_{\Delta_0\cap\mathrm{ff}}\mathrm{ff},$$

and so Λ_0 is extendible.

We now set Λ_1 to be the forward flowout of $H_{\tilde{p}}$ from $\Lambda_0\cap\Sigma(P)$. Because we may write $H_{\tilde{p}}=s^2\widetilde{H}_{\tilde{p}}$, with $\widetilde{H}_{\tilde{p}}$ a smooth nondegenerate vector field tangent to ff_+, Λ_1 is a smooth submanifold of $T^*X^2_0$, intersecting $T^*_{\mathrm{ff}_+}X^2_0$ transversely. Λ_1 is Lagrangian by general theory (see, for example, [4]). Thus Λ_1 is an extendible Lagrangian near ff_+.

To see that Λ_1 is extendible near ff_-, we require the following proposition:

Proposition 12. *The Lagrangian Λ_0 is invariant under the flow of H_L-H_R. Moreover, Λ_1 is also the flowout by H_R of $\Lambda_0\cap\Sigma(P)$.*

Proof. A straightforward calculation shows that H_L-H_R preserves Λ_0. Indeed, though this is a nonzero vector field, the coefficients of ∂_s, ∂_z, ∂_ξ, and ∂_μ all vanish at Λ_0. Note further that $\Sigma(P_L)\cap\Lambda_0=\Sigma(P_R)\cap\Lambda_0$, and so H_L-H_R preserves this set.

In order to see that Λ_1 is also the flowout of H_R, we observe that H_L and H_R must commute because they are lifts (from $X\times X$) of commuting vector fields. In particular, because H_R-H_L is tangent to $\Lambda_0\cap\Sigma(P_L)$, it is also tangent to Λ_1. In particular, H_R is also tangent to Λ_1. H_R is not tangent to Λ_0, so its flowout must also be Λ_1. □

An argument similar to the one above then shows that Λ_1 is extendible near ff_- (with H_L replaced by H_R). (For what we use below, the argument used above suffices, as we only need that Λ_1 is extendible in a neighborhood of the diagonal.)

The projection of H_L in $\Sigma(P)$ to $T^*\,\mathrm{ff}_+$ is given by

$$H_L=s^2\left(-2\sigma\frac{\partial}{\partial s}+2\frac{\partial h}{\partial\zeta}\cdot\frac{\partial}{\partial z}-\frac{\partial h}{\partial y}\cdot\frac{\partial}{\partial\mu}\right),$$

so that Λ^0_1 is given by the flowout of this vector field from Λ^0_0 in $T^*\mathrm{ff}_+$.

Note that the assumption (A3) implies that the projection of Λ_1 to the base X^2_0 is a smooth submanifold. We call this submanifold $\mathrm{LC}=\pi\Lambda_1$. This is a smooth codimension one submanifold. Note further that LC is the boundary of an open subset of X^2_0, which we call LC^{int}.

8. Paired Lagrangian distributions

We are interested in a calculus of paired Lagrangian distributions adapted to the 0-geometry. These are distributions with a model form

$$u = \int_0^\infty \int_{\mathbb{R}^n} e^{i(x-\tilde{x}-t\tilde{x})\frac{\xi}{\tilde{x}}+i(y-\tilde{y})\cdot\frac{\mu}{\tilde{x}}} a\left(t, \frac{x}{\tilde{x}}, \frac{y-\tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}, \xi, \mu\right) dt\, d\xi\, d\mu$$

$$= \int_0^\infty \int_{\mathbb{R}^n} e^{i(s-1-t)\xi+iY\cdot\mu} a(t, s, Y, \tilde{x}, \tilde{y}, \xi, \mu)\, dt\, d\xi\, d\mu, \tag{12}$$

where a is a symbol.

We briefly recall the definition of a paired Lagrangian distribution from [20]. Suppose M is a manifold and L_0, L_1 are conic Lagrangian submanifolds of T^*M intersecting cleanly in codimension 1. We say that $u \in I^m(M; L_0, L_1; \Omega^{\frac{1}{2}}(M))$ if it is a distributional $\frac{1}{2}$-density modeled microlocally on

$$\int_0^\infty \int_{\mathbb{R}^n} e^{i(x_1-s)\xi_1+ix'\cdot\xi'} a(s, x, \xi)\, d\xi\, ds,$$

where a is a symbol of order $m + \frac{1}{2} - \frac{n}{4}$. In the general case, we require the phase function to parametrize the Lagrangian pair (L_0, L_1) and a to be a symbol of order $m + \frac{1}{2} + \frac{n-2N}{4}$, where N is the number of variables required to parametrize L_1. In this model, the phase function parametrizes $L_0 = N^*\{0\}$, while $L_1 = \{x_1 \geqslant 0,\ x' = 0,\ \xi_1 = 0\}$.

We now recall what it means to parametrize a cleanly intersecting Lagrangian pair. Suppose that $L_0, L_1 \subset T^*M$ are two closed conic Lagrangian submanifolds intersecting cleanly in codimension 1 and that $\partial L_1 = L_1 \cap L_0$. Suppose $\phi : M_x \times \mathbb{R}^N_\xi \setminus \{0\} \to \mathbb{R}$ is homogeneous of degree 1 in ξ, and let

$$C = \big\{(x, \xi){:}\ x \in M,\ \xi \in \mathbb{R}^N \setminus \{0\},\ d_\xi \phi(x, \xi) = 0\big\}.$$

We say ϕ is nondegenerate if $d\frac{\partial\phi}{\partial\xi_j}$, $j = 1, \dots, N$, are linearly independent at any point in C. We say that ϕ parametrizes a single conic Lagrangian submanifold L if L is locally the image of the map $C \to T^*M$ given by $(x, \xi) \mapsto (x, d_x\phi(x, \xi))$. A simple example is the case when $L = N^*\{0\}$, in which case we may take $\phi(x, \xi) = x \cdot \xi$.

In the case of an intersecting Lagrangian pair, we say that ϕ is a nondegenerate parametrization of (L_0, L_1) near $q \in \partial L_1$ if

$$\phi(x, s, \xi) = \phi_0(x, \xi) + s\phi_1(x, s, \xi), \quad \xi \in \mathbb{R}^n,$$

where $q = (x_0, d_x\phi(x, 0, \xi))$, $d_{(s,\xi)}\phi(x, 0, \xi) = 0$, ϕ_0 is a nondegenerate parametrization of L_0 near q, and ϕ parametrizes L_1 near q for $s > 0$ with ds, $d\frac{\partial\phi}{\partial\xi_j}$, and $d\frac{\partial\phi}{\partial s}$ all linearly independent at q. In particular, the phase function $(1 - s - t)\xi + Y \cdot \mu$ parametrizes (Λ_0, Λ_1) near the front face of X_0^2.

Definition 13. We say that $u \in I^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$ is a *paired Lagrangian distribution* of order m associated to Λ_0 and Λ_1 on X_0^2 if it may be written as the restriction to X_0^2 of some distribution in $I^m(2X_0^2; \Lambda_0^e, \Lambda_1^e; \Omega^{\frac{1}{2}}(2X_0^2))$.

Definition 14. The class $I_0^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$ consists of the paired Lagrangian distributions that are supported away from the side faces of X_0^2.

We similarly define the classes $I_0^m(X_0^2; \Lambda_0; \Omega^{\frac{1}{2}}(X_0^2))$ and $I_0^m(X_0^2; \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$ as restrictions of distributions on $2X_0^2$ that are supported away from the side faces.

We typically shorten $I^m(X_0^2; \Lambda_0, \Lambda_1; \Omega^{\frac{1}{2}}(X_0^2))$ and $I_0^m(X_0^2; \Lambda; \Omega^{\frac{1}{2}}(X_0^2))$ by suppressing the half-density factor.

The following proposition follows immediately from Proposition 4.1 of [20].

Proposition 15. *If $u \in I_0^m(X_0^2; \Lambda_0, \Lambda_1)$, then $WF(u) \subset \Lambda_0 \cup \Lambda_1$. If B is a properly supported b-pseudodifferential operator of order 0 on X_0^2, then*

$$WF'(B) \cap \Lambda_0 = \emptyset \quad \Rightarrow \quad Bu \in I_0^m\big(X_0^2; \Lambda_1 \setminus \partial\Lambda_1\big),$$

and

$$WF'(B) \cap \Lambda_1 = \emptyset \quad \Rightarrow \quad Bu \in I_0^{m-1/2}\big(X_0^2; \Lambda_0 \setminus \partial\Lambda_1\big).$$

Note that the orders on Λ_0 and Λ_1 differ by $\frac{1}{2}$. This can be seen by integrating by parts once in s.

Just as in [20], this gives us two symbol maps. For $u \in I_0^m(X_0^2; \Lambda_0, \Lambda_1)$,

$$\sigma^{(1)}(u) \in C^\infty_{m+n/4}\big(\Lambda_1 \setminus \partial\Lambda_1; \Omega^{\frac{1}{2}} \otimes L_1\big),$$
$$\sigma^{(0)}(u) \in C^\infty_{m-1/2+n/4}\big(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{\frac{1}{2}} \otimes L_0\big),$$

where the subscript indicates the degree of homogeneity and L_i is the Maslov bundle over Λ_i. Admissible symbols are subject to a compatibility condition at $\partial\Lambda_1$ as in [20], but we do not need the explicit form of this condition here. We do, however, use the fact that if the principal symbol of u vanishes then u is one order better.

For these classes to be well defined, we must show that these classes are independent of the choice of extensions Λ_0^e and Λ_1^e. (It was already shown in [12] that the classes $I_0^m(X_0^2; \Lambda)$ are independent of the choice of extension.)

Lemma 16. *The class $I_0^m(X_0^2; \Lambda_0, \Lambda_1)$ is independent of the choice of extensions Λ_0^e and Λ_1^e.*

Proof. Suppose that Λ_0^e and $\widetilde{\Lambda}_0^e$ both extend Λ_0, while Λ_1^e and $\widetilde{\Lambda}_1^e$ extend Λ_1. Let $u \in I_0^m(2X_0^2; \widetilde{\Lambda}_0^e, \widetilde{\Lambda}_1^e)$. We may find $v \in I_0^m(2X_0^2; \Lambda_0^e, \Lambda_1^e)$ with the same symbol on $\Lambda_0 \cup \Lambda_1$.

Because u and v have the same symbol there, $u - v$ is order $m - 1$ on $\Lambda_0 \cup \Lambda_1$. We now iteratively solve away principal symbols to find w with $u - w$ of order $-\infty$ on $\Lambda_0 \cup \Lambda_1$, i.e., such that $u - w$ is a smooth function on X_0^2 up to the front face, but supported away from the side faces. $u - w$ is then clearly the restriction of an element of $I_0^{-\infty}(2X_0^2; \Lambda_0^e, \Lambda_1^e)$, proving the claim. □

The following proposition follows easily from Proposition 5.4 of [20]:

Proposition 17. *Suppose A is a properly supported b-differential operator of order m on X_0^2 and is such that $\sigma(A)$ vanishes on Λ_1. If $u \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0, \Lambda_1)$, then $Au = \tilde{x}^p h + \tilde{x}^p g$, with $h \in I_0^{m+k-1/2}(X_0^2; \Lambda_0)$ and $g \in I_0^{k+m-1}(X_0^2; \Lambda_0, \Lambda_1)$.*

Proof. We start by noting that the action of $\mathrm{Diff}_b^m(X_0^2)$ (extended to $2X_0^2$) commutes with restriction to X_0^2. This follows from the observation that $x\partial_x \chi_{\{x>0\}} = 0$, where $\chi_{\{x>0\}}$ is the characteristic function of a half-plane.

We start by writing $u = \tilde{x}^p \tilde{u}|_{X_0^2}$, where $\tilde{u} \in I_0^k(2X_0^2; \Lambda_0^e, \Lambda_1^e)$. We apply Proposition 5.4 of [20] to $\tilde{u}$, giving us $A\tilde{u} = \tilde{h} + \tilde{g}$.

In particular, we have that $Au = \tilde{x}^p h + \tilde{x}^p g + [A, \tilde{x}^p]u$. The operator A is a b-differential operator, so we know that $[A, \tilde{x}^p] = \tilde{x}^p B$, where B is a b-differential operator of order $m - 1$. We thus have that $Au = \tilde{x}^p h + \tilde{x}^p(g + Bu)$, where h and g are the restrictions of $\tilde{h}$ and $\tilde{g}$ to X_0^2. $g, Bu \in I_0^{k+m-1}(X_0^2; \Lambda_0, \Lambda_1)$, proving the claim. □

We also need another statement from Proposition 5.4 of [20], namely that the principal symbol of the distribution g above is given by the action of the Hamilton vector field of the principal symbol of A on the principal symbol of u. We only require this on the interior of the double space and so we do not prove it here.

Definition 18. We define the *normal operator* of a paired Lagrangian distribution (or a Lagrangian distribution, as in [12]) to be the restriction of its Schwartz kernel to the front face ff.

Note that this restriction is well defined by wavefront considerations and the transversality of Λ_0 and Λ_1 to ff. Moreover, by considering the model form (12), we find that the restriction is a paired Lagrangian distribution in the class $I^m(\mathrm{ff}; \Lambda_0^0, \Lambda_1^0)$.

Just as in [16], we have a short exact sequence.

Proposition 19. *The normal operator induces exact sequences*

$$0 \to \tilde{x} I_0^m\big(X_0^2; \Lambda_0\big) \to I_0^m\big(X_0^2; \Lambda_0\big) \to I_0^m\big(\mathrm{ff}; \Lambda_0^0\big) \to 0 \tag{13}$$

and

$$0 \to \tilde{x} I_0^m\big(X_0^2; \Lambda_0, \Lambda_1\big) \to I_0^m\big(X_0^2; \Lambda_0, \Lambda_1\big) \to I_0^m\big(\mathrm{ff}; \Lambda_0^0, \Lambda_1^0\big) \to 0 \tag{14}$$

such that for any differential operator $P \in \mathrm{Diff}_0^m(X)$ and any paired Lagrangian distribution u,

$$N_p(Pu) = N_p(P) N_p(u). \tag{15}$$

Proof. The exactness of the sequences (13) and (14) follows from Taylor expansion in $\tilde{x}$ near ff. The proof of (15) is identical to the one in [16]. □

Also note that because LC is an embedded submanifold, elements of $I_0^{k-1/2}(X_0^2; \Lambda_1)$ may be identified with distributions of order $k - 1/2$ conormal to LC.

9. Another blow-up

In order to understand the solutions of the transport equations, we must introduce another blow-up. Because our solutions eventually have differing asymptotic behaviors along the light cone and on the interior of the light cone, we blow up the boundary of the light cone (i.e., at the left and right faces). This blow-up always makes sense locally near the front face, but to make sense of it globally we use assumption (A3).

Because the boundary of the light cone meets the corner $\mathrm{lf}_+ \cap \mathrm{rf}_-$, the order in which we blow up the two submanifolds $\mathrm{LC} \cap \mathrm{lf}_+$ and $\mathrm{LC} \cap \mathrm{rf}_-$ matters. We deal with this situation by blowing up the submanifold $\mathrm{LC} \cap \mathrm{lf}_+ \cap \mathrm{rf}_-$ of the corner. We include this discussion primarily for completeness, as this piece of the construction is unnecessary as long as we restrict to forward-directed data. Indeed, the product of the pullback of a forward-directed function and any tempered distribution on $\widetilde{X}_0^2$ will vanish to all orders at this new face.

We first define what we refer to as the intermediate double space. This is the space on which we solve the transport equations for the conormal singularity. If we restrict to data supported away from Y_-, this space suffices for our entire construction.

Definition 20. We define the intermediate double space

$$X_{0,t}^2 = \left[X_0^2, \mathrm{LC} \cap \mathrm{rf}, \mathrm{LC} \cap \mathrm{lf}\right].$$

This is a new manifold with corners. We will call lcf_+ the lift of $\mathrm{LC} \cap \mathrm{lf}$ and lcf_- the lift of $\mathrm{LC} \cap \mathrm{rf}$.

Though we may think of this new manifold as an invariant way of introducing polar coordinates near $\mathrm{LC} \cap \mathrm{lf}$, projective coordinates are more convenient for our applications. Near lcf_+ and ff_+ but away from lf_+, we may use coordinates ρ/s, s, and $\tilde{x}$. Similarly, near lcf_- and ff_- and away from rf_-, we may use $\rho/\tilde{s}$, $\tilde{s}$, and x (and the remaining $\tilde{z}$ and y variables), where $\tilde{s} = \tilde{x}/x$.

Away from the front face and the corner $\mathrm{lf}_+ \cap \mathrm{rf}_-$, this is just the blow-up of an intersection of two hypersurfaces $\{\rho = 0\}$ and $\{x = 0\}$ (or $\{\tilde{x} = 0\}$). Near lcf_- but away from lcf_+ and ff_+, $\rho/\tilde{x}$ and $\tilde{x}$ are valid coordinates, while near lcf_+ away from lcf_- and ff_-, ρ/x and x are valid coordinates. Near their intersection but away from lf_+ and rf_-, $\rho/(\tilde{x}x)$, $\tilde{x}$, and x are valid.

Because lcf_+ and lcf_- intersect, the order in which we performed the blow-up matters. Near the interiors of the new faces the two spaces are locally diffeomorphic, but they are not globally diffeomorphic. This is relevant when understanding the behavior of our parametrix near intersections of these faces, so we instead work on what we call the *full double space*. In this space we first blow up $\mathrm{LC} \cap \mathrm{lf}_+ \cap \mathrm{rf}_-$ and then perform the other two blow-ups. Performing this first blow-up separates the lifts of $\mathrm{LC} \cap \mathrm{lf}_+$ and $\mathrm{LC} \cap \mathrm{rf}_-$, so the order in which we perform the other two blow-ups is now irrelevant.

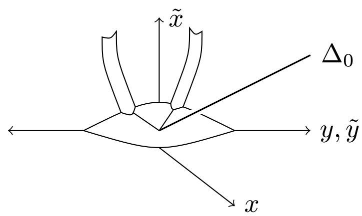


Fig. 4. The double space $\widetilde{X}_0^2$ near ff_+.

Definition 21. The full double space is given by

$$\widetilde{X}_0^2 = \left[X_0^2, \mathrm{LC} \cap \mathrm{lf}_+ \cap \mathrm{rf}_-, \mathrm{LC} \cap \mathrm{lf}_+, \mathrm{LC} \cap \mathrm{rf}_-\right].$$

This is another new manifold with corners and is the space on which the final parametrix lives. We call the additional corner scf, for scattering face. We give it this name because this face is related to Vasy's construction of the scattering operator in [24]. Near the interior of $\mathrm{scf} \cap \mathrm{lcf}_+$, we may use coordinates given by $\rho/(x\tilde{x})$, x/xt, and $\tilde{x}$, while near the interior of $\mathrm{scf} \cap \mathrm{lcf}_-$, we may use $\rho/(x\tilde{x})$, $\tilde{x}/x$, and x. We again emphasize that this blow-up is unnecessary if we restrict to data supported away from Y_-. Fig. 4 depicts a neighborhood of ff_+ in $\widetilde{X}_0^2$ (or, indeed, in $X_{0,t}^2$).

Up to a smooth nonvanishing factor, the Jacobian determinant of the blow-down map $X_{0,t}^2 \to X_0^2$ is given by $r_{\mathrm{lcf}_+} r_{\mathrm{lcf}_-}$, so sections of the bundle $\Omega^{\frac{1}{2}}(X_0^2)$ lift to sections of $(r_{\mathrm{lcf}_+} r_{\mathrm{lcf}_-})^{1/2} \Omega^{\frac{1}{2}}(X_{0,t}^2)$. Similarly, sections of $\Omega^{\frac{1}{2}}(X_0^2)$ lift to sections of $r_{\mathrm{scf}} r_{\mathrm{lcf}_+}^{1/2} r_{\mathrm{lcf}_-}^{1/2} \Omega^{\frac{1}{2}}(\widetilde{X}_0^2)$.

One can see that the fundamental solution does not lift nicely to X_0^2 by considering the forward fundamental solution for the Klein–Gordon equation on the half-space $\mathbb{R}_+^n = (0, \infty)_s \times \mathbb{R}_z^{n-1}$ equipped with the Lorentzian metric $\frac{-ds^2 + dz^2}{s^2}$. Finding the forward fundamental solution requires solving the equation

$$P(\lambda)u = \delta(s-1)\delta(z),$$
$$u \equiv 0 \quad \text{for } s > 1.$$

This solution can be constructed explicitly by taking the Fourier transform in the z variables, which transforms the equation into a Bessel ordinary differential equation for each value of the dual variable to z. By applying a stationary phase argument to the inverse Fourier transform of this family of Bessel functions, one can see that the asymptotic behavior of the forward fundamental solution is qualitatively different from its behavior on the interior of the light cone, justifying the blow-up.

10. Outline of the construction

We present now an outline of the parametrix construction, which consists of five steps. At each step we construct an approximation of the fundamental solution that captures the "worst" remaining singularity and yields an error term that is less singular.

We simultaneously construct a distribution that is both a left and a right parametrix for our operator. In other words, we construct a distribution K such that $P(\lambda)K = I + R_1$ and

$P(\lambda)^t_R K = I + R_2$. Here the subscript R denotes the operator acting on the right variables and $P(\lambda)^t$ is the transpose of $P(\lambda)$ as in Eq. (5). The distributions R_1 and R_2 are smoothing operators that vanish to infinite order at relevant boundary hypersurfaces and should be considered "negligible".

First we show that the construction in [20] is valid in a neighborhood of the diagonal, smooth down to the front face of X_0^2. The Schwartz kernel of the identity is a distribution conormal to the lift of the diagonal to X_0^2, and so the fundamental solution of $P(\lambda)$ is a paired Lagrangian distribution associated to the lift of the diagonal and its flowout via the Hamilton vector field, uniformly down to the front face. This is the most singular piece of the distribution but is the most standard portion of the construction. After cutting off this distribution outside a neighborhood of the lifted diagonal, this step yields a remainder term that is Lagrangian on the interior, uniformly down to the front face, and supported away from the side faces. Our assumption (A3) (which is unnecessary if we are only considering the construction in a neighborhood of Y_+) guarantees that the Lagrangian remainder term is in fact a conormal distribution associated to the projection of the flowout of the light cone.

In the second step of the construction, we solve away the conormal error term to obtain a remainder that is smooth on the interior of the double space. Because the light cone is characteristic for $P(\lambda)$, this reduces to solving a sequence of transport ordinary differential equations. These may be solved on the interior, but solving each successive transport equation on X_0^2 causes a loss of an order of decay at the boundary. In other words, with each successive improvement of regularity at the interior, the behavior of the solution at the boundary becomes worse.

We instead solve the transport equations on the intermediate double space $X_{0,t}^2$. We solve these equations on $X_{0,t}^2$ rather than the full double space $\widetilde{X}_0^2$ because transport equations have a slightly simpler form here. Indeed, the main modification needed to handle de Sitter space is to solve the equations on $\widetilde{X}_0^2$ rather than $X_{0,t}^2$. This step then yields a remainder term that is smooth on the interior and polyhomogeneous at the light cone faces lcf_+ and lcf_- and the scattering face scf.

The third step solves away part of the polyhomogeneous remainder at the light cone faces. We write down a formal series expansion for the solution and solve term by term, which yields a sequence of ordinary differential equations. The numbers $s_\pm(\lambda)$ are the indicial roots of these equations near the side faces of $\widetilde{X}_0^2$, so $x^{s_\pm(\lambda)}$ solve the equations to first order. In other words, the fundamental solution has an expansion at the side faces starting with $x^{s_\pm(\lambda)}$. The remainder term from this step vanishes to all orders at the light cone face and has an expansion at the side faces.

In the fourth step we solve away the remaining error term at the front face and side faces. This reduces to another formal power series calculation, and was carried out in [24]. The distribution we obtain by adding the distributions found in the four steps then solves the left (right) equation up to an error term that is smooth and vanishing to all orders at the "plus" ("minus") boundary faces.

In the fifth and final step, we remove this last error term via an argument showing that the fundamental solution itself must be in our class of distributions.

11. The diagonal singularity

The aim of this section is to solve away the diagonal singularity of the fundamental solution, leaving us with a Lagrangian error. The method here is similar to the one found in [12]. We use the normal operator to solve away the paired Lagrangian singularity at the front face up to $O(\tilde{x}^\infty)$, which allows us to invoke the construction in [20].

Proposition 22. *Given $f \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0)$, there is a paired Lagrangian distribution $u \in \tilde{x}^p I_0^{-3/2+k}(X_0^2; \Lambda_0, \Lambda_1)$ such that*

$$P(\lambda)u - f \in \tilde{x}^p I_0^{-1/2+k}\big(X_0^2; \Lambda_1\big).$$

In particular, if f is the Schwartz kernel of the identity operator, then

$$f = \delta(s-1)\delta(z)\tilde{x}^{-n/2}\mu \in \tilde{x}^{-n/2} I_0^0\big(X_0^2; \Lambda_0\big),$$

and there is a paired Lagrangian distribution $u \in \tilde{x}^{-n/2} I_0^{-3/2}(X_0^2; \Lambda_0, \Lambda_1)$ such that $P(\lambda)u - f \in \tilde{x}^{-n/2} I_0^{-1/2}(X_0^2; \Lambda_1)$.

Proof. On the interior of the manifold we may use the construction in [20], so we localize near the front face. Because we are only considering distributions supported away from the side faces for now, we do not need separate arguments for dealing with ff_+ and ff_-.

We start by fixing a cutoff function $\chi \in C^\infty(X_0^2)$ such that $\chi \equiv 1$ in a neighborhood of the diagonal, but supported away from the side faces of X_0^2. We also note now that $P(\lambda)$ commutes with multiplication by $\tilde{x}$.

Consider now the normal operator $N_p(P(\lambda))$ acting on the fiber ff_p over the front face. In the projective coordinates $(s, z, \tilde{x}, \tilde{y})$, this is given by Eq. (8).

It is easy to verify that $N_p(P(\lambda))$ satisfies the assumptions required to apply Proposition 6.6 of [20]. We may thus find $\kappa \in I^{k-3/2}(\mathrm{ff}; \Lambda_0^0, \Lambda_1^0)$ such that $N_p(P(\lambda))\kappa(p) - N_p(\tilde{x}^{-p} f) \in C^\infty(\mathrm{ff})$. Multiplying κ by the cutoff function χ gives

$$N_p\big(P(\lambda)\big)\big(\chi\kappa(p)\big) - N_p\big(x^{-r} f\big) \in I_0^{k-1/2}\big(\mathrm{ff}; \Lambda_1^0\big).$$

By appealing to the short exact sequence (14), we may find some paired Lagrangian distribution $u_0 \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$ such that $N(\tilde{x}^{-p} u_0) = \tilde{x}^{-p}\chi\kappa$. $P(\lambda)$ is characteristic on Λ_1, so we may appeal to Propositions 17 and 19 to write

$$P(\lambda)u_0 - f = v_0 + w_0,$$

where $v_0 \in \tilde{x}^{p+1} I_0^k(X_0^2; \Lambda_0)$ and $w_0 \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$.

The distribution v_0 is supported away from the side face, so $\tilde{x}$ is bounded away from 0 and thus $\tilde{x}^{-1} v_0 \in \tilde{x}^p I_0^k(X_0^2; \Lambda_0)$. We may now again use Eq. (13), Proposition 6.6 of [20], and Eq. (14) to find $u_1 \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$ such that

$$N\big(P(\lambda)\big)N\big(\tilde{x}^{-p} u_1\big) = N\big(\chi\tilde{x}^{-1} v_0\big).$$

In particular, we may then again appeal to Propositions 17 and 19 to write

$$P(\lambda)\big(\chi(u_0 + \tilde{x}u_1)\big) = v_1 + w_0 + \tilde{x}w_1,$$

with $v_1 \in \tilde{x}^{p+2} I_0^k(X_0^2; \Lambda_0)$ and $w_i \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$.

Iterating this process gives us distributions u_j such that

$$P(\lambda)\left(\chi \sum_{j=0}^{N} \tilde{x}^j u_j\right) = v_N + \sum_{j=0}^{N} \tilde{x}^j w_j$$

with $v_N \in \tilde{x}^{p+N+1} I_0^k(X_0^2; \Lambda_0)$ and $w_j \in \tilde{x}^p I_0^{k-1/2}(X_0^2; \Lambda_1)$.

We may now asymptotically sum the $\tilde{x}^j u_j$ to find $u \in \tilde{x}^p I_0^{k-3/2}(X_0^2; \Lambda_0, \Lambda_1)$ such that

$$P(\lambda)(\chi u) - f = v + w, \quad v \in \tilde{x}^\infty I_0^k\big(X_0^2; \Lambda_0\big),\ w \in \tilde{x}^p I_0^{k-1/2}\big(X_0^2; \Lambda_1\big).$$

We now extend v by zero to $2X_0^2$, and use Proposition 6.6 of [20] to find

$$\tilde{u} \in I_0^{k-3/2}\big(X_0^2; \Lambda_0, \Lambda_1\big)$$

such that $P(\lambda)\tilde{u} + v \in C^\infty(2X_0^2)$. In particular,

$$P(\lambda)(\chi \tilde{u}) - v = \tilde{w} \in \tilde{x}^p I_0^{k-1/2}\big(X_0^2; \Lambda_1\big).$$

The distribution $\chi u + \chi \tilde{u}$ then satisfies

$$P(\lambda)(\chi u + \chi \tilde{u}) - f \in \tilde{x}^p I_0^{k-1/2}\big(X_0^2; \Lambda_1\big). \qquad \square \tag{16}$$

We should note here that solving the transport equations for the symbols on Λ_1 (i.e., when invoking Proposition 6.6 of [20]) actually fixes the behavior of the solution in both components of $\Lambda_1 \setminus 0$. In particular, we may arrange it so that the symbol on Λ_1 has an expansion in decreasing powers of $(\eta + i0)$ (here η is the fiber variable in $\Lambda_1 = N^*\mathrm{LC}$). This allows us to guarantee that our parametrix u and the error term in (16) are supported on the interior of the light cone. This is not surprising because the exact solution must also be supported on the interior of the light cone due to the finite speed of propagation for the wave equation.

The following lemma is also useful:

Lemma 23. *When $f = \kappa_I$ is the Schwartz kernel of the identity operator, then the distribution u constructed in this section is also a left parametrix for P, i.e., $P_R^t u - f = \tilde{x}^{-n/2} r$, where r is smooth in a neighborhood of the diagonal. Here P_R^t denotes the transpose operator for P acting on the right factor.*

Proof. This is an application of the symbol calculus for paired Lagrangian distributions.

We start by observing that $(P_L - P_R^t)\kappa_I = 0$ because the identity operator commutes with any operator. In other words, we must have that

$$\int (P_L \kappa_I) v = P v = \int \kappa_I (P v) = \int \big(P_R^t \kappa_I\big) v.$$

Now let $v = (P_L - P_R^t) u \in \tilde{x}^{-n/2} I_0^k(X_0^2; \Lambda_0, \Lambda_1)$. We know that $P_L v$ is smooth in a neighborhood of the diagonal, down to the front face, because P_L and P_R^t commute. We now invoke the symbol calculus:

$$0=\sigma(P_L v)|_{\Lambda_0\setminus\partial\Lambda_1}=\sigma(P_L)\sigma(v)|_{\Lambda_0\setminus\partial\Lambda_1},$$

and $\sigma(P_L)\neq 0$ on $\Lambda_0\setminus\partial\Lambda_1$, so $\sigma(v)=0$ on this set. This fixes an initial condition for $\sigma(v)$, i.e., $\sigma(v)|_{\partial\Lambda_1}=0$. We now use the form of the transport equation

$$0=\sigma(P_L v)|_{\Lambda_1}=(i\mathcal{L}_{H_L}+c)\sigma(v)|_{\Lambda_1},$$

with initial condition 0, to conclude that $\sigma(v)|_{\Lambda_1}=0$. We may use this argument for any k, so we must have that $v\in\tilde{x}^{-n/2}I_0^{-\infty}(X_0^2;\Lambda_0,\Lambda_1)$, proving the claim. □

12. The transport equation

We now wish to solve away the error from Proposition 22. We call this error r and note that $r\in r_{\mathrm{ff}}^{p}I_0^{k-1/2}(X_0^2;\Lambda_1)$. By solving a transport equation for some finite time and multiplying by a cutoff function, we may assume that this error is supported in a neighborhood of the side faces. Viewed as a conormal distribution near ff_+, r may be written as

$$\int e^{i\frac{\rho}{s}\eta}a\left(s,\frac{\rho}{s},\theta,\tilde{x},\tilde{y},\eta\right)d\eta, \tag{17}$$

where ρ is a defining function for the light cone and a is a classical symbol of order $(k-\frac{1}{2})+\frac{n-1}{2}$. In fact, as noted in Section 11, we may assume that the symbol has an expansion in decreasing powers of $(\eta+i0)$. Near $\mathrm{lcf}_+\cap\mathrm{lcf}_-$, we must replace $\frac{\rho}{s}$ with $\frac{\rho}{x\tilde{x}}$.

We first fix a defining function ρ for the submanifold LC. Assumption (A3) guarantees that LC is an embedded submanifold of X_0^2. Because $N^*\mathrm{LC}$ is characteristic for $P(\lambda)$ and $d\rho$ spans $N^*\mathrm{LC}$, we must have that $\hat{g}(d\rho,d\rho)=0$ at LC, i.e.

$$\hat{g}(d\rho,d\rho)=\rho b, \tag{18}$$

where b is a smooth function.

In coordinates near $\partial X=Y$, we may write

$$P(\lambda)=(s\partial_s)^2-(n-1)s\partial_s+\frac{\tilde{x}s\partial_s\sqrt{h}}{\sqrt{h}}s\partial_s+s^2\Delta_h-\lambda.$$

Our ansatz is that u is polyhomogeneous at lcf_+ and lcf_- and conormal to LC, and so we seek an expression of the form (17) with a a classical symbol of order $k-\frac{3}{2}+\frac{n-1}{2}=k+\frac{n}{2}-2$ with an expansion in powers of $(\eta+i0)$, i.e., of the form

$$u\sim\sum_{j\geqslant 0}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{k+\frac{n}{2}-2-j}a_j\,d\eta. \tag{19}$$

This is because our error r from Section 11 is a Lagrangian distribution of order $k-\frac{1}{2}$ associated to the conormal bundle of LC. LC is characteristic for $P(\lambda)$, so we expect the solution of $P(\lambda)u=r$ to be conormal of one order better, i.e., of order $k-\frac{3}{2}$. Moreover, at each step, we multiply the symbols a_j by a compactly supported smooth function in $\frac{\rho}{s}$ that is identically

1 near 0, which makes the singularity of $(\eta+i0)^{k+\frac{n}{2}-2-j}$ at 0 superfluous. Note that we could equivalently insist on expressing our ansatz in powers of $(\frac{\rho}{s})_+$. If $k=0$, then the top power seen here would be $-1+2-\frac{n}{2}=1-\frac{n}{2}$, the same powers seen in the construction of the fundamental solution of the wave equation on Minkowski space (in our convention, n is the total dimension of the spacetime).

Lemma 24. *Suppose u is of the form* (19), *and $a_j=\tilde{a}_j\nu$, where ν is a fixed nonvanishing section of $\Omega^{\frac{1}{2}}(X^2_{0,t})$. If we write $\gamma_j=k+\frac{n}{2}-2-j$, then near* lcf_+ *away from* $\mathrm{lcf}_+\cap\mathrm{lcf}_-$ *we have*

$$P(\lambda)u\sim\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{\gamma_j+1}\bigl(-2i\hat{g}(d\rho,sd\tilde{a}_j) \tag{20}$$

$$-i(\partial_s\rho)\bigl(n-3-2\gamma_j+O(s)\bigr)\tilde{a}_j\bigr)\,d\eta\,\nu$$

$$+\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{\gamma_j}\biggl(P(\lambda)-(\gamma_j+1)(n-3-2\gamma_j)$$

$$+\bigl(\gamma_j+O(s)\bigr)s\partial_s+\frac{1}{2}\biggl(n-\frac{1}{2}\biggr)+O(s)s\partial_z+O(s)\biggr)\tilde{a}_j\,d\eta\,\nu \tag{21}$$

where $O(s)$ is taken to mean an element of sC^∞ and $(s=\frac{x}{\tilde{x}},\ z=\frac{y-\tilde{y}}{\tilde{x}})$ are projective coordinates near the front face.

Proof. We first show the result near $\mathrm{ff}_+\cap\mathrm{lf}_+$.

Write $u_j=\tilde{u}_j\nu$, where ν is a fixed trivialization of $\Omega^{\frac{1}{2}}(X^2_{0,t})$. Say, for concreteness, that $\nu=r_{\mathrm{lcf}_+}^{-1/2}\tilde{x}^{-n/2}\upsilon$. We then have that

$$P(\lambda)(\tilde{u}_j\nu)=\bigl(P(\lambda)\tilde{u}_j\bigr)\upsilon+r_{\mathrm{lcf}_+}^{1/2}\tilde{x}^{n/2}\bigl(\bigl[P(\lambda),r_{\mathrm{lcf}_+}^{-1/2}\tilde{x}^{-n/2}\bigr]\bigr)\tilde{u}_j.$$

We note also that near lcf_+ but away from lf_+, $r_{\mathrm{lcf}_+}=s(1+\alpha s)$, where α is smooth, and so we may easily calculate this commutator:

$$\bigl(r_{\mathrm{lcf}_+}^{1/2}\tilde{x}^{n/2}\bigr)\bigl[P(\lambda),r_{\mathrm{lcf}_+}^{-1/2}\tilde{x}^{-n/2}\bigr]=\bigl(-1+O(s)\bigr)s\partial_s+O(s)s\partial_z+\frac{1}{2}\biggl(n-\frac{1}{2}+O(s)\biggr).$$

We now use this calculation to drop the density factor. We apply $P(\lambda)$ to our ansatz and use Eq. (18). Integration by parts allows us to exchange powers of $\frac{\rho}{s}$ for decreasing powers of $(\eta+i0)$, as $\frac{\rho}{s}e^{i\frac{\rho}{s}\eta}=\frac{1}{i}\partial_\eta e^{i\frac{\rho}{s}\eta}$. If we write $\gamma_j=k+\frac{n}{2}-2-j$, this yields

$$P(\lambda)\tilde{u}=\biggl(P(\lambda)+\bigl(-1+O(s)\bigr)s\partial_s+\frac{1}{2}\biggl(n-\frac{1}{2}+O(s)\biggr)\biggr)\tilde{u}$$

$$\sim\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{\gamma_j+2}\hat{g}(d\rho,d\rho)\tilde{a}_j\,d\eta$$

$$
\begin{aligned}
&+\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{\gamma_j+1}\big(-2i\hat{g}(d\rho, sd\tilde{a}_j)\\
&-i(\partial_s\rho)(n-1-2\gamma_j-3)\tilde{a}_j - i(\partial_s\rho)\tilde{a}_j + O(s)\tilde{a}_j\big)\,d\eta\\
&+\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{s}\eta}(\eta+i0)^{\gamma_j}\bigg(P(\lambda)+\big(2\gamma_j+1+O(s)\big)s\partial_s\\
&-(\gamma_j+1)(n-3-2\gamma_j)+\frac{1}{2}\bigg(n-\frac{1}{2}\bigg)+O(s)s\partial_z+O(s)\bigg)\tilde{a}_j\,d\eta,
\end{aligned}
$$

where $O(s)$ is taken to mean an element of sC^∞. We now use that $\hat{g}(d\rho, d\rho) = \rho b$ and integrate by parts to prove the first part of the claim.

We finish the proof with a similar calculation, where we change our operator to P_R^t and our ansatz to be one of the following two forms:

$$
u \sim \sum_{j=0}^{\infty}\int e^{i\frac{\rho}{\tilde{s}}\eta} a_j(\eta+i0)^{\gamma_j}\,d\eta,
$$

$$
u \sim \sum_{j=0}^{\infty}\int e^{i\frac{\rho}{x\tilde{x}}\eta} a_j(\eta+i0)^{\gamma_j}\,d\eta,
$$

where $\tilde{s} = \frac{\tilde{x}}{x}$, and P_R^t is given by Eq. (5). We use here that $r_{\mathrm{lcf}_-} = \tilde{s}(1+\alpha\tilde{s})$ in the first case and that $r_{\mathrm{lcf}_-} = \tilde{x}(1+\tilde{\alpha}\tilde{x})$ and $r_{\mathrm{lcf}_+} = x(1+\alpha x)$ in the second case. □

Lemma 25. *Similarly, we may compute the right operator near* lcf$_-$ *away from the corner*:

$$
\begin{aligned}
P(\lambda)_R^t u = &\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{\tilde{s}}\eta}(\eta+i0)^{\gamma_j+1}\big(-2i\hat{g}(d_R\rho, \tilde{s}d_R\tilde{a}_j)\\
&+i(\partial_{\tilde{s}}\rho)\big(n+3+2\gamma_j+O(\tilde{s})\big)\tilde{a}_j\big)\,d\eta\,\nu\\
&+\sum_{j=0}^{\infty} e^{i\frac{\rho}{\tilde{s}}\eta}(\eta+i0)^{\gamma_j}\bigg(P(\lambda)_R^t+(\gamma_j+1)(n+3+2\gamma_j)\\
&+\big(\gamma_j+O(\tilde{s})\big)\tilde{s}\partial_{\tilde{s}}+\frac{1}{2}\bigg(n-\frac{1}{2}\bigg)+O(\tilde{s})\tilde{s}\partial_{\tilde{z}}+O(\tilde{s})\bigg)\tilde{a}_j\,d\eta\,\nu.
\end{aligned}
$$

We may also compute the behavior of the left and right operators near lcf$_+ \cap$ lcf$_-$:

$$
\begin{aligned}
P(\lambda)u \sim &\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{x\tilde{x}}\eta}(\eta+i0)^{\gamma_j+1}\frac{1}{\tilde{x}}\big(-2i\hat{g}(d\rho, xd\tilde{a}_j)\\
&-i(\partial_x\rho)\big(n-3-2\gamma_j+O(x)\big)\tilde{a}_j\big)\,d\eta\,\nu
\end{aligned}
$$

$$+\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{x\tilde{x}}\eta}(\eta+i0)^{\gamma_j}\biggl(P(\lambda)-(\gamma_j+1)(n-3-2\gamma_j)$$
$$+\bigl(\gamma_j+O(x)\bigr)x\partial_x+\frac{1}{2}\left(n-\frac{1}{2}\right)+O(x)x\partial_y+O(x)\biggr)\tilde{a}_j\,d\eta\,\nu,$$
$$P(\lambda)^t_R u=\sum_{j=0}^{\infty}\int e^{i\frac{\rho}{x\tilde{x}}\eta}(\eta+i0)^{\gamma_j+1}\frac{1}{x}\bigl(-2i\hat{g}(d_R\rho,\tilde{x}d_R\tilde{a}_j)$$
$$+i(\partial_{\tilde{x}}\rho)\bigl(n+3+2\gamma_j+O(\tilde{x})\bigr)\tilde{a}_j\bigr)\,d\eta\,\nu$$
$$+\sum_{j=0}^{\infty}e^{i\frac{\rho}{x\tilde{x}}\eta}(\eta+i0)^{\gamma_j}\biggl(P(\lambda)^t_R+(\gamma_j+1)(n+3+2\gamma_j)$$
$$+\bigl(\gamma_j+O(\tilde{x})\bigr)\tilde{x}\partial_{\tilde{x}}+\frac{1}{2}\left(n-\frac{1}{2}\right)+O(\tilde{x})\tilde{x}\partial_{\tilde{y}}+O(\tilde{x})\biggr)\tilde{a}_j\,d\eta\,\nu.$$

Because we wish to solve $P(\lambda)u=r$ up to smooth terms, we wish to iteratively solve away the terms in the above expansions. The first transport equation we must solve is then

$$-2\hat{g}(d\rho,sd\tilde{a}_0)-(\partial_s\rho)\bigl(n-3-2\gamma_0+O(s)\bigr)\tilde{a}_0=r_0,$$

where r_0 is compactly supported and comes from the inhomogeneous term.

Because $\partial_s\rho$ is nonzero, we may divide by it to obtain the transport equation

$$-\frac{h^{kl}(\partial_{z_k}\rho)s\partial_{z_l}\tilde{a}_0}{\partial_s\rho}+s\partial_s\tilde{a}_0-\left(\frac{n}{2}-\frac{3}{2}-\gamma_0+O(s)\right)\tilde{a}_0=\frac{r_0}{2\partial_s\rho}.$$

Note that $\gamma_0=k+\frac{n}{2}-2$, so the coefficient of $\tilde{a}_0$ is just $k-\frac{1}{2}+O(s)$.

Near the face lcf_+, given by $s=0$, we may use a parameter t along the light cone LC. The parameter t is then equivalent to s, so we may change coordinates to

$$t\partial_t\tilde{a}_0+\left(k-\frac{1}{2}\right)\tilde{a}_0=O(t)\tilde{a}_0+\tilde{r}.$$

We note that the solution $\tilde{a}_0$ of this equation must have a polyhomogeneous expansion in t. In fact, we prove a more precise version of this statement.

Lemma 26. *Suppose that v solves the differential equation*

$$t\partial_t v-\left(j-k+\frac{1}{2}\right)v=t\cdot c(t)v+b,$$

where c is smooth in t, and b is polyhomogeneous in t with index set

$$\mathcal{E}_{j-1}=\left\{\left(-k+\frac{1}{2}+l,i\right):\ l\in\mathbb{N}_0,\ i\leqslant l\text{ if }l\leqslant j-1,\ i=j-1\text{ if }l\geqslant j\right\}$$

when $j \neq 0$ and b is supported away from $t = 0$ for $j = 0$. Then v has a polyhomogeneous expansion in t with index set

$$\mathcal{E}_j = \left\{ \left(-k + \frac{1}{2} + l, i \right) : l \in \mathbb{N}_0,\ i \leqslant l \text{ if } l \leqslant j,\ i = j \text{ if } l > j \right\}.$$

Remark 27. We may prove a similar lemma for solutions of the transport equations on the right, with appropriate modifications for the index sets.

Proof of Lemma 26. This lemma follows as a simple exercise in the b-calculus of Melrose (cf. [7] or [18]) or as an exercise in the theory of hyperbolic Fuchsian operators (cf. [2,13], or [22]). Because the proof of this lemma is elementary, we include it here.

We show this by constructing a formal power series solution.

We start with the case $j = 0$ so that near $t = 0$, $b \equiv 0$. We seek a formal power series solution of the form

$$t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} v_l t^l.$$

Indeed, with this ansatz, the equation becomes

$$t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} l v_l = t^{-k+\frac{1}{2}} \sum_{l=0}^{\infty} q_{l-1},$$

where q_{l-1} depends only on $c(t)$ and the first $l - 1$ coefficients v_i (so $q_{-1} = 0$). The coefficient v_0 is fixed by the initial condition of the differential equation, and then the remaining coefficients may be found iteratively.

We then sum this series with Borel summation to find a function with this power series at $t = 0$ and the difference between this function and the solution of the differential equation vanishes to all orders at $t = 0$.

For general j, we write

$$b = \sum_{l=0}^{j-1} \sum_{i=0}^{\min(l,j-1)} b_{li} t^l (\log t)^i$$

with a similar expression for v. A similar calculation then reduces the equation to

$$\sum_{l=0}^{\infty} \sum_{i=0}^{\min(l,j)} t^{-k+\frac{1}{2}+l} (\log t)^i \big((l - j) v_{li} + (i + 1) v_{l,i+1} \big)$$
$$= \sum_{l=1}^{\infty} \sum_{i=0}^{l} t^{-k+\frac{1}{2}+l} (\log t)^i (q_{l-1,i} + b_{li}),$$

where $q_{l-1,i}$ depends on the function $c(t)$ and the coefficients $v_{l'i'}$ where $l' \leqslant l-1$ or $l' = l$ and $i' > i$. In particular, we may again iteratively solve for each coefficient and then Borel sum the result. □

We may now apply Lemma 26 to the first transport equation

$$t\partial_t \tilde{a}_0 - \left(j - k + \frac{1}{2}\right)\tilde{a}_0 + tc(t)\tilde{a}_0 = b(t),$$

where $c(t)$ is smooth in t. We find that a_0 is polyhomogeneous in t with index set $\mathcal{E}_0$. Changing coordinates back to s tells us that a_0 is polyhomogeneous in s with index set $\mathcal{E}_0$.

Letting Q_j be the operator acting on $\tilde{a}_j$ in the coefficient of $(\eta + i0)^{\gamma_0 - j}$ in Eq. (20), the jth transport equation is then

$$-\frac{h^{kl}(\partial_{z_k}\rho)s\partial_{z_l}\tilde{a}_j}{\partial_s \rho} + s\partial_s \tilde{a}_j - \left(j - k + \frac{1}{2}\right)\tilde{a}_j + O(s)\tilde{a}_j = -Q_j \tilde{a}_{j-1}.$$

By applying Lemma 26 again, we may conclude that a_j is polyhomogeneous in s with index set $\mathcal{E}_j$.

Now let $\mathcal{E} = \bigcup_j \mathcal{E}_j$. By repeating the process above, we obtain conormal distributions u_j in $I_{\mathcal{E}}^{k-3/2-j}(X_{0,t}^2; \mathrm{LC})$ with symbols a_j such that

$$P(\lambda)\left(\sum_{j=0}^{N} u_j\right) - r \in I_{\mathcal{E}}^{k-1/2-N-1}\left(X_{0,t}^2; \mathrm{LC}\right).$$

We now wish to asymptotically sum this expression to find u such that $P(\lambda)u - f$ is smooth on the interior of $X_{0,t}^2$.

Lemma 28. *There is a distribution $u \in I_{\mathcal{E}}^{k-3/2}(X_{0,t}^2; \mathrm{LC})$, supported on the interior of the light cone, such that*

$$u \sim \sum_{j=0}^{\infty} u_j.$$

Proof. Each symbol $(\eta + i0)^{\gamma_0 - j} a_j$ is analytic (in η) and exponentially decreasing in the upper half plane $\Im \eta > 0$, so the Paley–Wiener theorem tells us that each u_j is supported in the region $\{\frac{\rho}{s} \geqslant 0\}$. A standard Borel summation argument completes the proof. □

Putting our factors of r_{ff} from Section 11 back in, we have thus proved:

Proposition 29. *Given $r \in r_{\mathrm{ff}}^{p} I_0^{k-1/2}(X_0^2; \mathrm{LC})$, there is a conormal distribution,*

$$u \in r_{\mathrm{ff}}^{p} \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}_1} I^{k-3/2}\left(X_{0,t}^2; \mathrm{LC}\right),$$

such that

$$P(\lambda)u - r \in r_{\mathrm{ff}}^{p} \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}_1}\left(X_{0,t}^2; \mathrm{lcf}_+\right),$$

where u is supported away from the corner $\mathrm{ff}_+ \cap \mathrm{lf}_+$ and $\mathcal{F}_1$ is given by

$$\mathcal{F}_1 = \left\{\left(j - k + \frac{1}{2}, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\}.$$

We may further arrange that the distribution is supported in the interior of the light cone.

We observe here that an argument similar to the one given in the previous section shows that $P_R^t u - r$ is again smooth on the interior when we are constructing a parametrix for the fundamental solution. In particular, the symbols of u satisfy the transport equations for the actions on the right factor, and so the following proposition holds:

Proposition 30. *There is a conormal distribution $u \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}} I^{-3/2}(X_{0,t}^2; \mathrm{LC})$ such that*

$$P(\lambda)u - r \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}_L}\left(X_{0,t}^2\right),$$
$$P(\lambda)_R^t u - r \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}_R}\left(X_{0,t}^2\right).$$

Here the index families $\mathcal{F}$ and $\mathcal{E}$ are given by

$$F_{\mathrm{lcf}_+} = \left\{\left(j + \frac{1}{2}, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\},$$
$$F_{\mathrm{lcf}_-} = \left\{\left(j + \frac{1}{2} - n, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\},$$
$$E_{\mathrm{lcf}_+, L} = \left\{\left(j + \frac{1}{2}, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\},$$
$$E_{\mathrm{lcf}_-, L} = \left\{\left(j + \frac{1}{2} - n - 1, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\},$$
$$E_{\mathrm{lcf}_+, R} = \left\{\left(j + \frac{1}{2} - 1, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\},$$
$$E_{\mathrm{lcf}_-, R} = \left\{\left(j + \frac{1}{2} - n, l\right):\ j, l \in \mathbb{N}_0,\ l \leqslant j\right\}.$$

Note that the decrease of -1 for two of the error terms in the above proposition come from the factors of $\frac{1}{x}$ in the expression given in Lemma 25. The $-n$ on the "minus" faces comes from taking the transpose of our operator. This is because we are using sections of the standard half-density bundle rather than the 0-half-density bundle.

13. The light cone face

We now wish to solve away the error from Proposition 30. We aim to solve away the left error at lcf_+ and the right error at lcf_-. We show only the left calculation here and observe that the right calculation is nearly identical.

We again call the error r, which is in $r_{\mathrm{ff}}^{p}\mathcal{A}_{\mathrm{phg}}^{\mathcal{E}_L}(X_{0,t}^2)$, i.e., near ff_+, r has an expansion of the form

$$r \sim \tilde{x}^p \sum_{j}^{\infty} \sum_{l=0}^{j} s^{\gamma_j} (\log s)^l r_{jl} \nu,$$

where $\gamma_j = j - k + \frac{1}{2}$. We drop the power of $\tilde{x}$ for now because $P(\lambda)$ commutes with $\tilde{x}$.

We first claim that this error lifts to be polyhomogeneous on the full double space $\widetilde{X}_0^2$.

Lemma 31. *Suppose that r is a polyhomogeneous distribution on $X_{0,t}^2$ supported in a small neighborhood of* LC *with index family $\mathcal{F}$. Then r lifts to a polyhomogeneous distribution on $\widetilde{X}_0^2$ with index family $\mathcal{G}$, where $\mathcal{G}$ is given by*

$$G_{\mathrm{lcf}_+} = F_{\mathrm{lcf}_+},$$
$$G_{\mathrm{lcf}_-} = F_{\mathrm{lcf}_-},$$
$$G_{\mathrm{scf}} = F_{\mathrm{lcf}_+} + F_{\mathrm{lcf}_-} + 1.$$

Remark 32. Here the notation $F + 1$ is shorthand for

$$(\alpha, l) \in F + 1 \quad \text{if and only if} \quad (\alpha - 1, l) \in F.$$

Proof of Lemma 31. The result follows because the two possible orders of the blow-up are locally diffeomorphic near $\mathrm{LC} \cap \mathrm{lcf}_+ \cap \mathrm{lcf}_-$. The extra 1 in G_{scf} is because sections of $\Omega^{\frac{1}{2}}(X_{0,t}^2)$ lift to sections of $r_{\mathrm{scf}}\Omega^{\frac{1}{2}}(\widetilde{X}_0^2)$. □

We may thus consider r as a polyhomogeneous function on $\widetilde{X}_0^2$.

We now proceed in two steps. The first is to solve the away near lcf_+ and the second is to show that it has the desired form at lf_+. This only away from the scattering face scf, though the computation near scf is nearly identical. In this section there are many terms that come from differentiating our ansatz. We attempt to indicate the origin of the important terms.

Note that the statement about the support in Proposition 30 means that r_{jl} is supported on the interior of the light cone and vanishes to infinite order at LC.

Because we are working near lcf_+, we first use projective coordinates $(s, w = \frac{\rho}{s}, \theta)$, where ρ is a defining function for LC (as above) and θ are the remaining variables. In these coordinates, derivatives of the function ρ appear as coefficients of ∂_w in our operator $P(\lambda)$. In particular, $s\partial_s$ lifts to $s\partial_s + (\partial_s \rho)\partial_w - w\partial_w$.

We again expect a polyhomogeneous expansion in s. In other words, we expect an expansion of the form

$$u \sim \sum_{j=0}^{\infty} \sum_{l=0}^{j} s^{\gamma_j} (\log s)^l u_{jl}, \tag{22}$$

where u_{jl} is regarded as a function of w, θ, $\tilde{x}$, and $\tilde{y}$.

Write $u_{jl} = \tilde{u}_{jl}\nu$. We again note that we may take r_{lcf_+} equivalent to s in this region, giving us an extra $-\frac{1}{2}s\partial_s - \frac{1}{2}(n-\frac{1}{2})$ in our operator. Applying $P(\lambda)$ to our ansatz yields

$$\begin{aligned}
P(\lambda)u \sim \sum_j \sum_{l=0}^{j} s^{\gamma_j}(\log s)^l \Bigg[&\big(-\hat{g}(d\rho, d\rho) - 2w(\partial_s\rho) + w^2\big)\big(\partial_w^2 \tilde{u}_{jl}\big) \\
&- \left(\left(n + \frac{1}{2} - 2\gamma_j\right)(\partial_s\rho) - \left(n + \frac{1}{2}\right)w\right)\partial_w \tilde{u}_{jl} \\
&- \left((n-1-\gamma_j)\gamma_j + \frac{1}{2}\left(n - \frac{1}{2}\right) - \lambda\right)\tilde{u}_{jl} \\
&+ A_{j-2}\tilde{u}_{j-2,l} + B_{j-1}\tilde{u}_{j-1,l} + B'_{l+1}\tilde{u}_{j-1,l+1} + X_{jl}\tilde{u}_{j,l+1} + Y_l \tilde{u}_{j,l+2}\Bigg]\nu,
\end{aligned}$$

where

$$A_{j+2} = -\sum_q (\Delta_h \theta_q)\partial_{\theta_q} - \sum_{q,r,i,k}\left(\frac{\partial\theta_r}{\partial z_i}\right)\left(\frac{\partial\theta_q}{\partial z_k}\right)\partial_{\theta_r}\partial_{\theta_q}, \qquad B'_{l+1} = \frac{\tilde{x}\partial_s\sqrt{h}}{\sqrt{h}}(l+1),$$

$$B_{j+1} = -\sum_{i,k,r} 2h^{ik}\left(\frac{\partial\theta_r}{\partial z_i}\right)\left(\frac{\partial\rho}{\partial z_k}\right)\partial_{\theta_r}\partial_w + (\Delta_h\rho)\partial_w + \big(\partial_s^2\rho\big)\partial_w,$$

$$X_{jl} = 2(l+1)(\partial_s\rho - w)\partial_w - (l+1)\left(n - \frac{1}{2} - 2\gamma_j\right), \qquad Y_l = (l+1)(l+2).$$

The constants above come from the operator and from $s\partial_s$ landing on the powers of s. The leading ∂_w terms come from the $(\partial_s\rho)\partial_w$ terms when we lift $s\partial_s$ and the $h^{kl}(\partial_{z_k}\rho)(\partial_{z_l}\rho)\partial_w^2$ term in the lift of the Laplacian in z.

Note that because LC is characteristic for $\Box$, we must have that $\hat{g}(d\rho, d\rho) = \rho b$. $\rho = sw$, so by replacing B_{j+1} with $\widetilde{B}_{j+1} = B_{j+1} - wb\partial_w^2$, we may write

$$\begin{aligned}
P(\lambda)u \sim \sum_j \sum_{l=0}^{j} s^{\gamma_j}(\log s)^l \big[&Q_j \tilde{u}_{jl} + A_{j-2}\tilde{u}_{j-2,l} + \widetilde{B}_{j-1}\tilde{u}_{j-1,l} \\
&+ B'_{l+1}\tilde{u}_{j-1,l+1} + X_{jl}\tilde{u}_{j,l+1} + Y_l\tilde{u}_{j,l+2}\big]\nu,
\end{aligned}$$

where

$$\begin{aligned}
Q_j = \big(-(\partial_s\rho) + w\big)w\partial_w^2 &- \left((\partial_s\rho)\left(n + \frac{1}{2} - 2\gamma_j\right) - \left(n + \frac{1}{2}\right)w\right)\partial_w \\
&- (n-1-\gamma_j)\gamma_j - \frac{1}{2}\left(n - \frac{1}{2}\right) - \lambda.
\end{aligned}$$

We thus wish to solve a sequence of inhomogeneous ordinary differential equations $Q_j \tilde{u}_{jl} = \tilde{r}_{jl}$, where

$$\tilde{r}_{jl} = r_{jl} - X_{jl}\tilde{u}_{j,l+1} - Y_l \tilde{u}_{j,l+2} - A_{j-2}\tilde{u}_{j-2,l} - \widetilde{B}_{j-1}\tilde{u}_{j-1,l} - B'_{l+1}\tilde{u}_{j-1,l+1},$$

where all terms are supported in $\{w \geqslant 0\}$, vanishing to infinite order at $w = 0$. (We know already that r_{jl} has this property, and we show at each step that $\tilde{u}_{jl}$ does as well.)

It is clear that 0 is a regular singular point of the differential operator Q_j, so the solutions of $Q_j v = 0$ have formal power series expansions at $w = 0$ with first term given by w^{μ_i}. Here μ_i are the roots of the indicial equation $-\mu(\mu - 1) - (n + \frac{1}{2} - 2\gamma_j)\mu = 0$ (see, for example, [1]), i.e.,

$$\mu_1 = 0, \qquad \mu_2 = 2j - 2k - n.$$

Here we have used that $\gamma_j = j - k + \frac{1}{2}$. Standard ODE techniques (i.e., variation of parameters) then give us solutions $\tilde{u}_{jl}$ to $Q_j \tilde{u}_{jl} = \tilde{r}_{jl}$ in terms of a basis of solutions for $Q_j v = 0$. Moreover, because $\tilde{r}_{jl}$ vanishes to all orders at $w = 0$, we may also guarantee that $\tilde{u}_{jl}$ is supported in $\{w \geqslant 0\}$, vanishing to all orders at $w = 0$. Indeed, if v_1 and v_2 are a basis for the solutions of $Q_j v = 0$, then

$$\tilde{u}_{jl}(w) = -v_1(w) \int_0^w \frac{v_2(w')\tilde{r}_{jl}(w')}{W(v_1, v_2)(w')}\, dw' + v_2(w) \int_0^w \frac{v_1(w')\tilde{r}_{jl}(w')}{W(v_1, v_2)(w')}\, dw'.$$

Because $\tilde{r}_{jl}$ vanishes to all orders at $w = 0$ (and v_i are bounded by w^{-N} for some N), the integrals make sense and vanish to all orders at $w = 0$. Multiplication by v_i then preserves this property.

We may thus solve these equations and now wish to consider their asymptotics near the corner $\mathrm{lcf}_+ \cap \mathrm{lf}_+$. Near this corner, the coordinates (s, w, θ) are invalid and so we must use the other set of projective coordinates $(\rho = sw, W = w^{-1}, \theta)$. In these coordinates, $s\partial_s$ lifts to $(\partial_s \rho)\partial_\rho + W\partial_W - (\partial_s \rho)W^2 \partial_W$. The top order terms below then come from the $W\partial_W$ and the constants come only from the operator.

Because polyhomogeneous distributions are independent of our choice of coordinate systems, we may also express our ansatz (22) in terms of these coordinates (ρ, W, θ). In this case, expansions in s are equivalent to expansions in ρ, so our ansatz here has the form

$$u \sim \sum_j \sum_{l=0}^{j} \rho^{\gamma_j} (\log \rho)^{\ell} v_{jl}.$$

In the computation that follows, the important point is that $s_\pm(\lambda)$ are the indicial roots of our operator $P(\lambda)$, and that this behavior is dominant away from LC because the fundamental solution is a smooth solution of the homogeneous equation here.

Apply $P(\lambda)$ to such an ansatz to see

$$P(\lambda)u \sim \sum_{(j,l)\in\mathcal{F}_1} \rho^j (\log\rho)^l \big[(1-q_2W)W^2\partial_W^2\tilde{v}_{jl} - (n-2+q_1W)W\partial_W\tilde{v}_{jl} - (\lambda+q_0W)\tilde{v}_{jl} + WA_{j-1,l}\tilde{v}_{j-1,l} + WA'_{j-1,l+1}\tilde{v}_{j-1,l+1} + WB_{j-2,l}\tilde{v}_{j-2,l} + WB'_{j-2,l+1}\tilde{v}_{j-2,l+1} + WB''_{j-2,l+2}\tilde{v}_{j-2,l+2}\big],$$

where we have (via a similar, but more tedious, calculation)

$$q_2 = 2(\partial_s\rho) - (\partial_s\rho)^2W + |d_z\rho|_h^2W,$$
$$q_1 = -(n+2\gamma_j-4)(\partial_s\rho) + (2\gamma_j-2)\hat{g}(d\rho,d\rho)W,$$
$$q_0 = (n-2)\gamma_j(\partial_s\rho) - \gamma_j(\gamma_j-1)\hat{g}(d\rho,d\rho)W,$$

while

$$A_{j-1,l} = 2h^{ik}(\partial_{z_i}\theta_p)(\partial_{z_k}\rho)W^2\partial_{\theta_p}\partial_W - 2\gamma_j h^{ik}(\partial_{z_i}\rho)(\partial_{z_k}\theta_p)W\partial_{\theta_p} - (\Box_{\hat{g}}\rho)W^2\partial_W - \gamma_j(\Box_{\hat{g}}\rho)W,$$
$$A'_{j-1,l+1} = (l+1)\big[\big(-2(\partial_s\rho) + 2\hat{g}(d\rho,d\rho)\big)W\partial_W - (\partial_s\rho)(n-2) - (2\gamma_j-2)\hat{g}(d\rho,d\rho)W\big],$$

and

$$B_{j-2,l} = (\Delta_h\theta_p)W\partial_{\theta_p} - h^{ik}(\partial_{z_i}\theta_p)(\partial_{z_k}\theta_q)W\partial_{\theta_p}\partial_{\theta_q},$$
$$B'_{j-2,l+1} = -2(l+1)h^{ik}(\partial_{z_k}\rho)(\partial_{z_i}\theta_p)W\partial_{\theta_p} + (l+1)(\Box_{\hat{g}}\rho)W,$$
$$B''_{j-2,l+2} = -(l+1)(l+2)\hat{g}(d\rho,d\rho)W.$$

We must still take into account the density factor here. In this region, r_{lcf_+} is equivalent to ρ. A similar computation to the one above shows that this leads to two types of terms. One of these is $O(W)W\partial_W$, while the other is $O(W)$. This means that they may be absorbed into q_1 and q_0. Let us now write

$$Q_j = (1-q_2W)W^2\partial_W^2 - (n-2+q_1W)W\partial_W - (\lambda+q_0W).$$

We wish to solve a sequence of transport equations (which are really the same equations as above written in these coordinates) given by

$$Q_j\tilde{v}_{jl} = \tilde{r}_{jl} - WA_{j-1,l}\tilde{v}_{j-1,l} - WA'_{j-1,l+1}\tilde{v}_{j-1,l+1} - WB_{j-2,l}\tilde{v}_{j-2,l} - WB'_{j-2,l+1}\tilde{v}_{j-2,l+1} - WB''_{j-2,l+2}\tilde{v}_{j-2,l+2} = \tilde{r}'_{jl}.$$

The solutions of these transport equations are polyhomogeneous conormal functions. Here the indicial roots of Q_j are $s_\pm(\lambda) = \frac{n-1}{2} \pm \sqrt{\frac{(n-1)^2}{4} + \lambda}$. More precisely, we have

Lemma 33. *Suppose that u_{jl} solves $Q_j u_{jl} = \tilde{r}'_{jl}$ as above. Suppose first that $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$. Suppose that $\tilde{r}'_{jl}$ is polyhomogeneous conormal at* lf_+ *with index set $F + 1 = (F^+ + 1) \cup (F^- + 1)$, where*

$$F^{\pm} = \{(s_{\pm}(\lambda) + m, 0)\colon m \in \mathbb{N}_0\},$$

then u_{jl} is polyhomogeneous with index set $F = F^+ \cup F^-$.

If $s_+(\lambda) - s_-(\lambda) = N \in \mathbb{Z}$, then if $\tilde{r}'_{jl}$ is polyhomogeneous with index set $\widetilde{F} + 1$, where

$$\widetilde{F} = \{(s_-(\lambda) + m, l)\colon m \in \mathbb{N}_0,\ l = 0 \text{ for } l < N,\ l = 1 \text{ for } l \geqslant N\},$$

then u_{jl} must be polyhomogeneous with index set $\widetilde{\mathcal{F}}$.

Remark 34. Note that the index set above means that if $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$, then $u_{jl} \in W^{s_+(\lambda)} C^\infty + W^{s_-(\lambda)} C^\infty$. If $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$, then

$$u_{jl} - \sum_{m=0}^{s_+(\lambda)-s_-(\lambda)-1} W^{s_-(\lambda)+m} u_{jl}^{(m)} \in (1 + \log W) C^\infty.$$

Proof of Lemma 33. We may again construct a formal power series solution (or apply Lemma 5.44 of Melrose [18]). We omit this here because it has been described in detail already. The key point is that the operators $A_{j-1,l}$, $A'_{j-1,l+1}$, $B_{j-2,l}$, $B'_{j-2,l+1}$, and $B''_{j-2,l+2}$ are all elements of $W \operatorname{Diff}_b^*$ (products of W and differential operators tangent to the boundary). This ensures that the terms we are solving away vanish to one order better at $W = 0$. □

Asymptotically summing $\sum \sum \rho^{\nu_j} (\log \rho)^l u_{jl}$ then solves away the error r at lcf_+. Note that because $s_\pm(\lambda)$ are the indicial roots of $N(P(\lambda))$, we in fact have that the error term vanishes to one order better.

When we are constructing a parametrix for the fundamental solution, we may perform the same construction on the right and the left. We may also add a distribution solving away the right error at lcf_-.

Remembering our factors of r_{ff} now, we have now proved the following proposition.

Proposition 35. *Given $r_1 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}_{\mathrm{phg}}^{\mathcal{E}_L}(\widetilde{X}_0^2)$ and $r_2 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}_{\mathrm{phg}}^{E_R}(\widetilde{X}_0^2)$ as above, we may find a smooth function $u \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}}(\widetilde{X}_0^2)$ vanishing outside the light cone such that $P(\lambda)u - r_1$ vanishes to all orders at* lcf_+ *and is polyhomogeneous with index family $\mathcal{G}_1$, while $P(\lambda)^t_R u - r_2$ vanishes to all orders at* lcf_- *and is polyhomogeneous with index set $\mathcal{G}_2$. Here we have that*

$$F_{\mathrm{lf}_+} = G_{\mathrm{lf}_+,2} = \{(s_\pm(\lambda) + m, 0)\colon m \in \mathbb{N}_0\},$$

$$F_{\mathrm{lcf}_+} = E_{\mathrm{lcf}_+,L},$$

$$F_{\mathrm{rf}_+} = G_{\mathrm{rf}_+,1} = \{(-n + s_\pm(\lambda) + m, 0)\colon m \in \mathbb{N}_0\},$$

$$F_{\mathrm{lcf}_-} = E_{\mathrm{lcf}_-,R},$$

$$G_{\mathrm{lcf}_-,1} = F_{\mathrm{lcf}_-} - 1,$$

$$G_{\mathrm{lf}_+,1} = F_{\mathrm{lf}_+} + 1,$$
$$G_{\mathrm{lcf}_+,2} = F_{\mathrm{lcf}_+} - 1,$$
$$G_{\mathrm{rf}_+,2} = F_{\mathrm{rf}_+} + 1,$$
$$G_{\mathrm{lcf}_+,1} = G_{\mathrm{lcf}_-,2} = E_{\mathrm{lf}_+,L} = E_{\mathrm{rf}_-,L} = E_{\mathrm{lf}_+,R} = E_{\mathrm{rf}_+,R} = \emptyset,$$
$$F_{\mathrm{scf}} = E_{\mathrm{scf},L} = E_{\mathrm{scf},R} = G_{\mathrm{scf},1} = G_{\mathrm{scf},2} = \big\{(-n+j,l)\colon\ l \leqslant j\big\}, \tag{23}$$

if $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$. If $s_+ - s_-(\lambda) = N \in \mathbb{Z}$, then the index sets become

$$\widetilde{F}_{\mathrm{lf}_+} = \big\{\big(s_-(\lambda)+m, l\big)\colon\ m \in \mathbb{N}_0,\ l = 0 \text{ for } m < N,\ l = 1 \text{ for } m \geqslant N\big\},$$
$$\widetilde{F}_{\mathrm{rf}_+} = \big\{\big(-n + s_-(\lambda)+m, l\big)\colon\ m \in \mathbb{N}_0,\ l = 0 \text{ for } m < N,\ l = 1 \text{ for } m \geqslant N\big\},$$

with corresponding changes for G_{lf_+} and G_{rf_+}.

14. The front face

We now wish to solve away the errors on the front face, which we again call r, from the previous step. We show how to solve away the error term at ff_+ for the operator acting on the left, and the corresponding calculation at ff_- for the operator acting on the right is nearly identical. We now suppose that r is the error term from Proposition 35 for the operator acting on the left.

Because r vanishes to all orders at lcf_+, we may blow down lcf_+ to solve away r. In this view, $\tilde{x}^{-p} r$ is smooth on ff_+, supported inside the light cone, and has an expansion at lf_+ of the form

$$r \sim \tilde{x}^p \sum_{(\alpha_j,l)\in\mathcal{G}} s^{\alpha_j} (\log s)^l r_{jl}\mu.$$

The generic case here is that $l = 0$ when $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$. We again drop the powers of $\tilde{x}$ from our notation.

We wish to solve this error away with a function of the same form:

$$u \sim \sum_{(\alpha_j,l)\in\mathcal{G}} s^{\alpha_j} (\log s)^l u_{jl}\mu.$$

Applying $N(P(\lambda))$ to such an ansatz yields

$$\begin{aligned} N\big(P(\lambda)\big)u &\sim \sum_j \sum_l s^{\alpha_j} (\log s)^l \big[\alpha_j(\alpha_j + 1 - n)u_{jl} + 2\alpha_j(l+1)u_{j,l+1} \\ &\quad + (l+1)(l+2)u_{j,l+2} - (n-1)(l+1)u_{j,l+1} + \Delta_z u_{j-2,l} - \lambda u_{jl}\big]\nu \\ &= \sum_j \sum_l s^{\alpha_j} (\log s)^l \big[-\big((n-1-\alpha_j)\alpha_j + \lambda\big)u_{jl} \\ &\quad - (n-1-2\alpha_j)(l+1)u_{j,l+1} + (l+1)(l+2)u_{j,l+2} + \Delta_z u_{j-2,l}\big]\mu. \end{aligned} \tag{24}$$

The coefficient of u_{jl} in this expression vanishes precisely when $\alpha_j = s_\pm(\lambda)$, so we may solve away the error to all orders at lf_+ because the expansions of r begin at $s_\pm(\lambda)+1$.

Proposition 36. *For $s_+(\lambda) - s_-(\lambda) \notin \mathbb{Z}$, there is a smooth function u on ff_+, polyhomogeneous at lf_+ with index set F_{lf_+}, where F_{lf_+} is defined in Eq.* (23)*, such that $N(P(\lambda))u - \tilde{x}^{-p} r$ vanishes to all orders at $\mathrm{lf}_+ \cap \mathrm{ff}_+$. An identical statement holds when $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$, with F_{lf_+} replaced by $\widetilde{F}_{\mathrm{lf}_+}$.*

Proof. This is clear from the expression (24) for $N(P(\lambda))u$. □

On the front face, we are then left with an error term r_0 vanishing to all orders at $\mathrm{lf}_+ \cap \mathrm{ff}_+$ and supported inside the light cone. Because ff_+ is an asymptotically de Sitter space, we may now use Corollary 3.6 of Vasy [24] to find a smooth function on ff_+, vanishing to all orders at lf_+, such that $N(P(\lambda))u = \tilde{x}^{-p} r_0$.

By iterating this construction (and extending it to the interior of lf_+ as well), we may find $u \in \tilde{x}^p \mathcal{A}_{\mathrm{phg}}^{\mathcal{F}_2}(X_0^2; \mathrm{lf}_+)$ such that $P(\lambda)u - r$ vanishes to all orders at ff_+ and lf_+. By gluing two of these functions together, we may simultaneously solve the left error near the "plus" faces and the right error near the "minus" faces. This proves the following proposition.

Proposition 37. *We may solve away the left error at ff_+ and lf_+, and the right error at rf_- and ff_- with a function $u \in \mathcal{A}_{\mathrm{phg}}^{\mathcal{H}}(\widetilde{X}_0^2)$, where $\mathcal{H}$ is given by*

$$H_{\mathrm{lcf}_+} = H_{\mathrm{lcf}_-} = \emptyset, \qquad H_* = F_* \quad \text{for the other index sets.}$$

The remaining error terms are in $\mathcal{A}_{\mathrm{phg}}^{\mathcal{K}_{L,R}}(\widetilde{X}_0^2)$, where $\mathcal{K}_L$ and $\mathcal{K}_R$ are given by

$$K_{\mathrm{lf}_+,L} = K_{\mathrm{rf}_-,R} = K_{\mathrm{lcf}_+} = K_{\mathrm{lcf}_-} = \emptyset, \qquad K_{\mathrm{lf}_+,R} = F_{\mathrm{lf}_+},$$
$$K_{\mathrm{rf}_-,L} = F_{\mathrm{rf}_-}, \qquad K_{\mathrm{scf},L} = K_{\mathrm{scf},R} = H_{\mathrm{scf}}.$$

15. The full parametrix

We now take the various pieces of the parametrix constructed in Sections 11, 12, 13, and 14 to construct a parametrix for the fundamental solution of P.

Putting together the results of Propositions 22, 30, 35, and 37, we have proved the following theorem:

Theorem 38. *Suppose that X is an asymptotically de Sitter space, satisfying assumptions* (A1)*,* (A2)*, and* (A3)*. We may find a left parametrix K such that $P(\lambda)K = I + R_1$ and $KP(\lambda) = I + R_2$, where the Schwartz kernels of R_1 and R_2 are smooth on the interior of $\widetilde{X}_0^2$ and are polyhomogeneous with index families $\mathcal{E}_L$ and $\mathcal{E}_R$ on $\widetilde{X}_0^2$. We may write $K = K_1 + K_2 + K_3$, where K_1 is supported in a neighborhood of the diagonal, K_2 is supported in a small neighborhood of the light cone* LC *away from the diagonal, and all three pieces are supported on the interior of the light cone. Moreover,*

$$K_1 \in r_{\mathrm{ff}}^{-n/2} I_0^{-3/2}\big(X_0^2; \Lambda_0, \Lambda_1\big),$$

$$K_2 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^{-3/2}\big(X^2_{0,t}; \mathrm{LC}\big),$$
$$K_3 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}\big(\widetilde{X}^2_0\big). \tag{25}$$

The index families $\mathcal{E}_L$, $\mathcal{E}_R$, and $\mathcal{F}$ are given by

$$F_{\mathrm{lcf}_+} = \left\{\left(j + \frac{1}{2}, l\right)\colon\ l \leqslant j,\ j \in \mathbb{N}_0\right\},$$
$$F_{\mathrm{lcf}_-} = F_{\mathrm{lcf}_+} - n,$$
$$F_{\mathrm{lf}_+} = E_{\mathrm{lf}_+,R} = \big\{\big(s_\pm(\lambda) + m, 0\big)\colon\ m \in \mathbb{N}_0\big\},$$
$$E_{\mathrm{rf}_-,L} = F_{\mathrm{rf}_-} = F_{\mathrm{lf}_+} - n,$$
$$E_{\mathrm{lcf}_-,L} = F_{\mathrm{lcf}_-} - 1,$$
$$E_{\mathrm{lcf}_+,R} = F_{\mathrm{lcf}_+} - 1,$$
$$F_{\mathrm{scf}} = E_{\mathrm{scf},L} = E_{\mathrm{scf},R} = \big\{(-n + j, l)\colon\ l \leqslant j\big\},$$
$$E_{\mathrm{ff}_-,L} = E_{\mathrm{ff}_+,R} = \left\{\left(-\frac{n}{2} + m, 0\right)\colon\ m \in \mathbb{N}_0\right\},$$
$$E_{\mathrm{lcf}_+,L} = E_{\mathrm{lcf}_-,R} = E_{\mathrm{lf}_+,L} = E_{\mathrm{rf}_-,R} = E_{\mathrm{ff}_+,L} = E_{\mathrm{ff}_-,R} = \emptyset.$$

If $s_+(\lambda) - s_-(\lambda) \in \mathbb{Z}$, then we must modify F_{lf_+} and F_{rf_+} (*and the index sets depending on them*) *as described earlier.*

Remark 39. As observed earlier, this theorem holds without the assumption (A3) as long as we are willing to multiply our distribution by a cutoff function supported in a neighborhood of the front face. In this case, the extra blow-up to obtain the scattering face scf is unnecessary.

Because our remainder terms lose one order of decay at the light cone faces, we lose an order of decay there when we pass to the exact fundamental solution. The following is a precise statement of the main result (Theorem 1).

Theorem 40. *The exact forward fundamental solution E_+ is in this class of distributions, but with index sets $F'_{\mathrm{lcf}_+} = F_{\mathrm{lcf}_+} - 1$ and $F'_{\mathrm{lcf}_-} = F_{\mathrm{lcf}_-} - 1$. In other words, we may write $E_+ = K_1 + K_2 + K_3$ with*

$$K_1 \in r_{\mathrm{ff}}^{-n/2} I_0^{-3/2}\big(X^2_0; \Lambda_0, \Lambda_1\big),$$
$$K_2 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}} I^{-3/2}\big(X^2_{0,t}; \mathrm{LC}\big),$$
$$K_3 \in r_{\mathrm{ff}}^{-n/2} \mathcal{A}^{\mathcal{F}}_{\mathrm{phg}}\big(\widetilde{X}^2_0\big). \tag{26}$$

Here the index family $\mathcal{F}$ is given by

$$F_{\mathrm{lcf}_+} = \left\{\left(j - \frac{1}{2}, l\right)\colon\ l \leqslant j,\ j \in \mathbb{N}_0\right\}, \qquad F_{\mathrm{lcf}_-} = F_{\mathrm{lcf}_+} - n,$$
$$F_{\mathrm{lf}_+} = \big\{\big(s_\pm(\lambda) + m, 0\big)\colon\ m \in \mathbb{N}_0\big\}, \qquad F_{\mathrm{rf}_-} = F_{\mathrm{lf}_+} - n,$$

$$F_{\mathrm{scf}} = \{(-n-1+j,l)\colon\ l \leqslant j\},$$

with modifications to F_{lf_+} *and* F_{rf_-} *when* $s_+(\lambda) - s_-(\lambda)$ *is an integer.*

Remark 41. If we instead adopt the convention that the 0-densities are flat, that

$$\int_X f(\tilde{x},\tilde{y})\delta(x-\tilde{x})\delta(y-\tilde{y})\,d\hat{g}(x,y) = f(x,y),$$

and we write K as a section of the pullback bundle of ${}^0\Omega^{\frac{1}{2}}(X\times X)$, then the index sets change somewhat. Indeed, if $K = \hat{K}\tilde{\nu}$, where $\tilde{\nu}$ is a nonvanishing section of the pullback of ${}^0\Omega^{\frac{1}{2}}(X\times X)$, then the same theorem holds, but with index sets

$$F_{\mathrm{lcf}_+} = F_{\mathrm{lcf}_-} = \{(j-1,l)\colon\ l\leqslant j,\ j\in\mathbb{N}_0\},$$
$$F_{\mathrm{lf}_+} = F_{\mathrm{rf}_-} = \{(s_\pm(\lambda)+m,0)\colon\ m\in\mathbb{N}_0\},$$
$$F_{\mathrm{scf}} = \{(j-1,l)\colon\ l\leqslant j,\ j\in\mathbb{N}_0\},$$
$$F_{\mathrm{ff}_+} = F_{\mathrm{ff}_-} = \{(j,0)\colon\ j\in\mathbb{N}_0\}.$$

Proof of Theorem 40. E_+ is the forward fundamental solution, so if f is a compactly supported smooth function on X, then $P(\lambda)E_+f = f$ and $E_+P(\lambda)f = f$. Moreover, continuity allows us to extend this to any forward-oriented distribution. In particular, if f is any smooth function on the interior of X vanishing to all orders at Y_- that is also a tempered distribution on X, then

$$P(\lambda)E_+f = f,$$
$$E_+P(\lambda)f = f.$$

Let K be the parametrix for E_+ constructed in Theorem 38. If f is a smooth function on X, vanishing to all orders at Y, then Kf vanishes to all orders at Y_- because K is identically zero in a neighborhood of lf_- and the lift of f vanishes to all orders at lcf_-, ff_-, scf, and rf_-. A similar argument applies to R_1f and R_2f. We may then write

$$(Kf) = E_+P(\lambda)Kf = E_+f + E_+R_1f,$$
$$(Kf) = KP(\lambda)E_+f = E_+f + R_2E_+f.$$

In particular,

$$E_+ = K - KR_1 + R_2E_+R_1.$$

We then observe that the error terms KR_1 and $R_2E_+R_1$ have the desired properties, finishing the proof. □

16. Modifications for de Sitter space

De Sitter space does not satisfy assumption (A3) because the projection of the forward flowout of the light cone from a point at past infinity intersects itself at future infinity (though not on the interior of the spacetime). We briefly discuss the modifications to our construction needed for de Sitter space. The most important modification is to solve the transport equations on $\widetilde{X}_0^2$ rather than $X_{0,t}^2$. (Indeed, we could have done this from the outset, but chose not to because solutions of the transport equations are easier to understand on $X_{0,t}^2$.)

Observe that the construction detailed above works without difficulty away from the corner $\mathrm{lf}_+ \cap \mathrm{rf}_-$. As mentioned earlier, LC intersects itself in this corner. This intersection is given by $\{(0, y, 0, -y)\colon\ y \in Y\}$. Near this intersection, we may write $\rho = x + \tilde{x} - |y + \tilde{y}|$, plus terms vanishing to higher order at the boundary. Note that the function ρ is no longer smooth at this intersection because $|y + \tilde{y}| = 0$ there.

We resolve this singularity by blowing up the submanifold $\mathrm{LC} \cap \mathrm{lf}_+ \cap \mathrm{rf}_-$ as in Definition 21. Although this submanifold had codimension 3 when X satisfied (A3), it has codimension $n + 1$ here. Indeed, this blow-up is almost the same as the one in Section 4 that defined X_0^2. The function ρ now lifts to be smooth on this new space $[X_0^2, \mathrm{LC} \cap \mathrm{lf}_+ \cap \mathrm{rf}_-]$. Indeed, after the blow-up we may write

$$\rho = (1 + s) - |z| + O(\tilde{x}) \tag{27}$$

near $\mathrm{scf} \cap \mathrm{lf}_+$. This is now smooth near $\rho = 0$, and so we may blow up $\mathrm{LC} \cap \mathrm{lf}_+$ and $\mathrm{LC} \cap \mathrm{rf}_-$ to obtain $\widetilde{X}_0^2$.

We must also modify the manner in which we solve the transport equations. The behavior at lcf_+ and lcf_- may be obtained in the same way as in Section 12, but the behavior at scf requires a slightly different approach. In Section 9, we showed that the symbol of the conormal distribution was polyhomogeneous at scf by constructing it on $X_{0,t}^2$ and lifting it to $\widetilde{X}_0^2$. Because ρ is not smooth on X_0^2 when X is the de Sitter space, we cannot solve the transport equations on the intermediate double space $X_{0,t}^2$ up to the corner and instead we must solve the transport equation along lcf_- and lcf_+.

Solving the transport equation on these faces requires using the semi-explicit form of ρ. The terms where the operator lands entirely on ρ in Eq. (20) can no longer be ignored. Using the form (27) for ρ, we observe that these now contribute a constant term to the equation. Because $\Box\rho = s - (n - 1)s + s^2(n - 1) + O(\tilde{x})$ and $\partial_s \rho = 1 + O(\tilde{x})$.

A computation in the same spirit as those in Section 12 shows that the symbol of K_2 is also polyhomogeneous at scf with index set

$$F_{\mathrm{scf}} = \{(-n + m, 0)\colon\ m \in \mathbb{N}_0\}.$$

Note that this is the same index set we found before.

The rest of the construction proceeds without change.

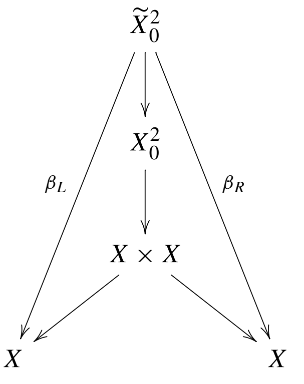
The modifications to the construction for de Sitter space correspond to allowing the location of the pole p in $P(\lambda)u = \delta_p$ tend to past infinity. If we require that the point p is uniformly bounded away from past infinity, no modification is necessary.

In [23,25], the authors do not consider sending this pole to past infinity. Our unmodified construction recovers a slightly weaker version of the results of these authors when the pole is in the interior but the modified construction also describes the behavior of the fundamental solution when the pole is at past infinity.

17. Polyhomogeneity

The aim of this section is to prove Theorem 2, which was stated in Section 1.

We begin by considering the maps β_L and β_R, where $\beta_{L,R}$ are given by composing the blow-down maps with projections onto each factor, as in the diagram here:



We require the following four lemmata.

Lemma 42. *The maps β_L and $\beta_L|_{\mathrm{LC}^{int}}$ are b-fibrations.*

Lemma 43. *The fibers of β_L and $\beta_L : \mathrm{LC}^{int} \to X$ are transverse to* LC.

Lemma 44. *For V a b-vector field on X, there is a b-vector field $\widetilde{V}$ on $\widetilde{X}_0^2$ such that $(\beta_L)_* \widetilde{V} = V$ and $\widetilde{V}$ is tangent to* LC.

Lemma 45. *Suppose that M and N are manifolds with corners, $F : M \to N$ is a b-fibration, H is a boundary hypersurface of M, and $F(H) = N$. Suppose also that $K \in I^0(M; H)$ is polyhomogeneous at the other boundary hypersurfaces of M (satisfying the hypotheses of Lemma* 9). *Then $F_* K$ is a polyhomogeneous distribution on N.*

Proof of Lemma 42. Here LC^{int} is taken as an open manifold away from the cone edge near the diagonal. K_2 is supported away from the cone point, so we may restrict our attention to this region.

That β_L and $\beta_L|_{\mathrm{LC}^{int}}$ are b-fibrations (defined in Section 5) follows from a more general statement: if $F : M \to N$ is a b-fibration, and $Z \subset \partial M$ is a p-submanifold, then $F \circ \beta : [M; Z] \to N$ is a b-fibration. In particular, the blow-down map is a composition of b-fibrations. □

Proof of Lemma 43. Because Λ_1 is the flowout Lagrangian in the right factor, its intersection with the fibers $\tilde{x}s = x_0$, $\tilde{y} + \tilde{x}z = y_0$ is the flowout of the light cone with cone point at (x_0, y_0). In particular, this is an embedded submanifold of X, and so the intersection is transverse. □

Proof of Lemma 44. This is really a consequence of Lemma 43. Indeed, we may choose a local basis of vector fields given by ∂_ν and V_j, where ∂_ν is tangent to the fibers of β_L and V_j are tangent to LC. Then $(\beta_L)_*\partial_\nu = 0$ because ∂_ν is tangent to the fibers of β_L, and so we may choose a lift of the vector fields so that the ∂_ν component vanishes at LC. □

Proof of Lemma 45. We choose K^ϵ polyhomogeneous on M, supported away from H such that K^ϵ are uniformly bounded in $I^0(M;H)$ and converge to K in $I^\delta(M;H)$ for any $\delta > 0$. The pushforward theorem of Melrose (Lemma 9 of this paper) tells us that $F_* K^\epsilon$ are polyhomogeneous with fixed index set.

We claim now that the $F_* K^\epsilon$'s are Cauchy and so converge to $F_* K$. This guarantees that $F_* K$ is polyhomogeneous. The key observation here is that if $\widetilde{V}$ is a lift of V, then

$$V\big(F_*\big(K^{\epsilon_1} - K^{\epsilon_2}\big)\big) = F_*\big(\widetilde{V}\big(K^{\epsilon_1} - K^{\epsilon_2}\big)\big).$$

The expression $\widetilde{V}(K^{\epsilon_1} - K_{\epsilon_2})$ tends to 0 because $K^\epsilon \to K$ as distributions conormal to H. □

Proof of Theorem 2. A simple wavefront set argument gives us that the $K_2 f$ piece is smooth.

Thinking functorially, we write

$$K(v\gamma) = (\beta_L)_*\big(K \cdot \beta_R^*(v\gamma)\big) = u\gamma.$$

In particular, if we write $K = \widetilde{K}\nu$, then

$$u\gamma^2 = u|dx\,dy| = (\beta_L)_*\big(\widetilde{K}\big(\beta_R^* v\big) r_{\mathrm{ff}}^{n/2} (r_{\mathrm{lcf}_+} r_{\mathrm{lcf}_-})^{1/2} r_{\mathrm{scf}} \nu^2\big).$$

We now use the decomposition $K = K_1 + K_2 + K_3$, where K_i are as in Theorem 40. The lemma above shows that β_L is a b-fibration, and so we may use the pushforward theorem of Melrose in Lemma 9 to treat the contribution from K_3. In particular, we note that if v has index set E at Y_+ and vanishes to all orders at Y_-, then $\beta_R^* v$ is polyhomogeneous on $\widetilde{X}_0^2$, smooth at lcf_+ and lf, with index set E at ff_+, and vanishing to all orders at ff_-, lcf_-, and scf because it is forward-directed. Index sets add when functions are multiplied, and so we know that $r_{\mathrm{ff}}^{n/2}(r_{\mathrm{lcf}_+} r_{\mathrm{lcf}_-})^{1/2} r_{\mathrm{scf}} \widetilde{K}_3 \cdot \beta_R^* f$ has index family $\mathcal{G}$ given by

$$G_{\mathrm{ff}_+} = E, \qquad G_{\mathrm{ff}_-} = \emptyset,$$
$$G_{\mathrm{lf}_+} = \big\{\big(s_\pm(\lambda) + m, 0\big): m \in \mathbb{N}_0\big\}, \qquad G_{\mathrm{lf}_-} = \emptyset,$$
$$G_{\mathrm{rf}_-} = \emptyset, \qquad G_{\mathrm{rf}_+} = \emptyset,$$
$$G_{\mathrm{lcf}_+} = \big\{(j,l): l \leqslant j,\ j \in \mathbb{N}_0\big\}, \qquad G_{\mathrm{lcf}_-} = \emptyset,$$
$$G_{\mathrm{scf}} = \emptyset.$$

G_{lf_+} must be modified when $s_+(\lambda) - s_-(\lambda)$ is an integer. We may now use Lemma 9 to conclude that $K_3 f$ is polyhomogeneous on X with index set

$$E \mathbin{\bar{\cup}} G_{\mathrm{lf}_+} \mathbin{\bar{\cup}} G_{\mathrm{lcf}_+}.$$

To handle the contribution from K_2, we use the lemmas above. Indeed, Lemmas 42, 44, and 45 show that $K_2 f$ is polyhomogeneous with index set $G_{\mathrm{lcf}_+} \bar{\cup} E$.

This leaves only the contribution from K_1. This is just a consequence of the local theory of paired Lagrangian distributions. Indeed, the work of Joshi in [11] implies that the pushforward of $K_1 \beta_R^* f$ exists as a smooth function. The uniformity of K_1 down to the front face ff then tells us that $K_1 f$ is polyhomogeneous with index set E. $\square$

18. An L^p estimate

As another application of Theorem 38, we consider the behavior of the L^p norms of a family of smooth compactly supported functions with support tending towards Y_+.

Suppose first that $R \in \Psi_0^{-\infty}(X)$ is a smoothing 0-pseudodifferential operator in the small calculus of [14] or [16], supported near ff_+. In other words, the Schwartz kernel of R is a smooth function on X_0^2, supported away from lf and rf and near ff_+. Concretely, let ϕ is a smooth, compactly supported function on $\mathbb{R}_+^n$, supported near $(1, 0)$, and with $\int_0^\infty \int_{\mathbb{R}^{n-1}} \phi \frac{ds\,dz}{s^n} = 1$. Let χ be a smooth function on X that is identically zero near Y_-. If $R(s, z, \tilde{x}, \tilde{y}) = \phi(s, z)\chi(\tilde{x}, \tilde{y})$, then $R \in \Psi_0^{-\infty}(X)$ is such an operator.

Define now a family of compactly supported functions f given by

$$f_{(\tilde{x},\tilde{y})}(x, y) = R\left(\frac{x}{\tilde{x}}, \frac{y - \tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}\right) = Rv,$$

where $v = \delta(x' - \tilde{x})\delta(y' - \tilde{y})$. Each $f_{(\tilde{x},\tilde{y})}$ is a smooth function on X supported in a compact neighborhood of $(\tilde{x}, \tilde{y})$ with unit $L^1(X; dg)$-norm.

Because $f_{(\tilde{x},\tilde{y})}$ is given by applying R to a δ function, pointwise bounds for $E_+ f_{(\tilde{x},\tilde{y})}$ are equivalent to pointwise bounds on the Schwartz kernel of $E_+ R$.

The following lemma is useful for obtaining pointwise bounds.

Lemma 46. *Suppose that $K = \hat{K}v$, where K is a polyhomogeneous function on $\widetilde{X}_0^2$ with index family $\mathcal{F}$ and supported near* ff_+. *Suppose that $\mathcal{F}$ satisfies*

$$\begin{aligned}
F_{\mathrm{ff}_+} &= \{(s_{\mathrm{ff}} + j, p)\colon\ j \in \mathbb{N}_0,\ p \leqslant p_j\},\\
F_{\mathrm{lcf}_+} &= \{(s_{\mathrm{lcf}} + j, p)\colon\ j \in \mathbb{N}_0,\ p \leqslant p_j\},\\
F_{\mathrm{lf}_+} &= \{(s_{\mathrm{lf}} + j, p)\colon\ j \in \mathbb{N}_0,\ p \leqslant p_j\},\\
F_{\mathrm{rf}_+} &= \{(s_{\mathrm{rf}} + j, p)\colon\ j \in \mathbb{N}_0,\ p \leqslant p_j\},
\end{aligned}$$

and $p_0 = 0$ for each index set (i.e., no log terms appear in the top order part of the expansion). Suppose further that

$$s_{\mathrm{ff}} \geqslant \frac{n}{2}, \qquad s_{\mathrm{lcf}} \geqslant \frac{1}{2}, \qquad s_{\mathrm{lf}} \geqslant 0, \qquad s_{\mathrm{rf}} \geqslant 0.$$

Then K (considered as an operator) satisfies

$$\|Ku\|_{L^\infty(X)} \leqslant C\|u\|_{L^1(X;d\hat{g})}.$$

If $s_{\mathrm{ff}} \geqslant -\frac{n}{2}$ instead, then K is bounded $L^1(X; dg) \to L^\infty(X)$.

Proof. The half-density $|d\hat{g}_L d\hat{g}_R|^{1/2}$ lifts to a nonvanishing smooth multiple of $r_{\mathrm{lcf}_+}^{1/2} r_{\mathrm{ff}_+}^{n/2}\nu$ near ff_+, so we may write

$$\hat{K}\nu = \hat{K} r_{\mathrm{lcf}_+}^{-1/2} r_{\mathrm{ff}_+}^{-n/2} \beta^*\big(|d\hat{g}_L d\hat{g}_R|^{1/2}\big),$$

where β^* is the pullback by the blow-down map $\widetilde{X}_0^2 \to X \times X$. The assumptions above mean that $\hat{K} r_{\mathrm{lcf}_+}^{-1/2} r_{\mathrm{ff}_+}^{-n/2}$ is a bounded function on $\widetilde{X}_0^2$ and so is the pullback of a bounded function on $X \times X$.

The second statement of the lemma follows from the observation that $d\hat{g} = x^n dg$. □

Let K_1, K_2, and K_3 be the decomposition of E_+ given in Theorem 38. The Schwartz kernel of $K_j R$ is given by

$$\int K_j\left(\frac{x}{s'\tilde{x}}, \frac{y - (\tilde{y} + \tilde{x}z')}{s'\tilde{x}}, s'\tilde{x}, \tilde{y} + \tilde{x}z'\right) R\big(s', z', \tilde{x}, \tilde{y}\big)\, ds'\, dz' \tag{28}$$

where $s' = \frac{x'}{\tilde{x}}$ and $z' = \frac{y' - \tilde{y}}{\tilde{x}}$. This corresponds to writing the composition of operators A and B on $X \times X$ as

$$\kappa_{AB}(x, y, \tilde{x}, \tilde{y}) = \int \kappa_A\big(x, y, x', y'\big)\kappa_B\big(x', y', \tilde{x}, \tilde{y}\big)\, dx'\, dy',$$

where κ_A here denotes the Schwartz kernel of A. If κ_A and κ_B are instead functions of $s, z, \tilde{x}, \tilde{y}$, where $s = \frac{x}{\tilde{x}}$ and $z = \frac{y - \tilde{y}}{\tilde{x}}$, this becomes

$$\begin{aligned}
&\int \kappa_A\left(\frac{x}{x'}, \frac{y - y'}{x'}, x', y'\right)\kappa_B\left(\frac{x'}{\tilde{x}}, \frac{y' - \tilde{y}}{\tilde{x}}, \tilde{x}, \tilde{y}\right) dx'\, dy' \\
&\quad = \int \kappa_A\left(\frac{x}{s'\tilde{x}}, \frac{y - (\tilde{y} + \tilde{x}z')}{s'\tilde{x}}, s'\tilde{x}, \tilde{y} + \tilde{x}z'\right)\kappa_B\big(s', z', \tilde{x}, \tilde{y}\big)\, ds'\, dz',
\end{aligned}$$

which yields Eq. (28).

Note that K_1, K_2, and K_3 all vanish identically near rf_+. Because the family $f_{(\tilde{x},\tilde{y})}$ is supported away from Y_-, we may assume that K_i are supported away from the "minus" faces ff_-, lcf_-, rf_-, and scf.

Consider first K_1, the piece corresponding to the paired Lagrangian singularity of E_+. By Theorem 38, the Schwartz kernel of K_1 is given by $(x')^{-n/2}\widetilde{K}_1$, where $\widetilde{K}_1$ is a paired Lagrangian distribution on X_0^2 supported away from lf and rf.

Because the fibers of integration in Eq. (28) are transverse to the diagonal and to the light cone, it follows that $K_1 R = \tilde{x}^{-n/2} u_1$, where u_1 is a smooth function on X_0^2 supported away from lf and rf. In particular, u_1 is bounded, and Lemma 46 implies that

$$\|K_1 f_{(\tilde{x},\tilde{y})}\|_{L^\infty} \leqslant C.$$

Because R is defined only near ff_+, we are free to use the improved parametrix of Theorem 38 rather than distribution of Theorem 40. In particular, the symbol of the conormal distribution K_2

is bounded by $r_{\mathrm{lcf}_+}^{1/2}$ and so $K_2 R$ satisfies the conditions of the second part of Lemma 46 and so

$$\|K_2 f_{(\tilde{x},\tilde{y})}\|_{L^\infty(X)} \leqslant C.$$

Consider finally the polyhomogeneous term K_3. The Schwartz kernel of $r_{\mathrm{lf}_+}^{-s_-(\lambda)} K_3 R$ satisfies the conditions of the lemma, and so

$$\|K_3 f_{(\tilde{x},\tilde{y})}\|_{x^{s_-(\lambda)} L^\infty(X)} \leqslant C.$$

Now let $l = \max(0, -\Re s_-(\lambda))$. Putting the estimates for K_1, K_2, and K_3 together yields

$$\|E_+ f_{(\tilde{x},\tilde{y})}\|_{x^{-l} L^\infty(X)} \leqslant C.$$

An L^2 estimate for this family follows from Theorem 6 (from [24]). By reversing the roles of Y_- and Y_+ in this theorem, we conclude that for any forward-directed $f \in x^{-r} L^2(X)$ and $r > \max(\frac{1}{2}, l(\lambda))$, there is a unique $u \in x^{-r} H_0^1(X)$ such that $P(\lambda)u = f$, and

$$\|u\|_{x^{-r} L^2(X;dg)} \leqslant \|u\|_{x^{-r} H_0^1(X)} \leqslant C \|f\|_{x^{-r} L^2(X;dg)}.$$

Here H_0^1 is the 0-Sobolev space of order one, i.e., it measures regularity with respect to the $x\partial_x$ and $x\partial_y$ vector fields.

In order to apply this estimate to the family $f_{(\tilde{x},\tilde{y})}$, it is important to understand how the L^2-norms of the functions vary. Indeed, a simple calculation shows that

$$\|f_{(\tilde{x},\tilde{y})}\|_{x^{-r} L^2(X)}^2 = \int_X \left|f_{(\tilde{x},\tilde{y})}(x,y)\right|^2 \frac{dx\,dy}{x^{n+r}} = \int \left|\phi(s,z)\right|^2 \frac{ds\,dz}{s^{n+r}\tilde{x}^r} = \tilde{x}^{-r} C_r^2,$$

where C_r depends on r, but not on $\tilde{x}$ or $\tilde{y}$. In particular, $\|f\|_{x^{-r} L^2(X;dg)} = \tilde{x}^{-r/2} C_r$.

We may now prove Theorem 3.

Proof. Interpolating between the L^∞ and L^2 estimates provides an L^p estimate for $p \in (2, \infty)$. Indeed, if

$$r > \max\left(\frac{1}{2}, \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right),$$

$$l = \max\left(0, -\frac{n-1}{2} + \Re\sqrt{\frac{(n-1)^2}{4} + \lambda}\right),$$

and $\frac{1}{p} = \frac{\theta}{2}$, $\theta \in [0, 1]$, then

$$\|E_+ f_{(\tilde{x},\tilde{y})}\|_{x^{-r\theta - l(1-\theta)} L^{2/\theta}(X;dg)} \leqslant C\tilde{x}^{-r\theta}, \tag{29}$$

which finishes the proof. □

Remark 47. The proof of Theorem 3 uses the inclusion $H_0^1 \subset L^2$. In particular, we ignore one derivative of $E_+ f$ and so we could modify Eq. (29) to include a fractional derivative.

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