

Perpendicular Categories with Applications to Representations and Sheaves

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Communicated by Walter Feit

Received March 20, 1989

DEDICATED TO PROFESSOR HIROYUKI TACHIKAWA
ON THE OCCASION OF HIS 60TH BIRTHDAY

This paper is concerned with the omnipresence of the formation of the subcategories right (left) perpendicular to a subcategory of objects in an abelian category. We encounter these subcategories in various contexts:

- the formation of quotient categories with respect to localizing subcategories (cf. Section 2);
- the deletion of vertices and shrinking of arrows (see [37]) in the representation theory of finite dimensional algebras (cf. Section 5);
- the comparison of the representation theories of different extended Dynkin quivers (cf. Section 10);
- the theory of tilting (cf. Sections 4 and 6);
- the study of homological epimorphisms of rings (cf. Section 4);
- the passage from graded modules to coherent sheaves on a possibly weighted projective variety or scheme (cf. Section 7 and [21]);
- the study of (maximal) Cohen–Macaulay modules over surface singularities (cf. Section 8);
- the comparison of weighted projective lines for different weight sequences (cf. Section 9);
- the formation of affine and local algebras attached to path algebras of extended Dynkin quivers, canonical algebras, and weighted projective lines (cf. Section 11 and [21] and the concept of universal localization in [40]).

Formation of the perpendicular category has many aspects in common with localization and allows one to dispose of localization techniques in situations not accessible to any of the classical concepts of localization. This applies in particular to applications in the domain of finite dimensional algebras and their representations. Several applications of the methods presented in this paper are already in existence, partly published, or appearing in print in the near future (see, for instance, [40, 39, 4, 45, 26, 49, 46]) and have show the versatility of the notion of a perpendicular category.

It seems that (right) perpendicular categories first appeared—as the subcategories of so-called closed objects—in the process of the formation of the quotient category of an abelian category with respect to a localizing Serre subcategory (see [18, 47, 34]). Another natural occurrence is encountered in Commutative Algebra, forming the possibly infinitely generated modules of depth ≥ 2 (cf. Section 7). The concept and some of the central applications were first presented in a talk given by the first author at the Honnef meeting in January 1985.

We also note that the perfectly matching nomination “perpendicular category” was coined by A. Schofield, who discovered independently the usefulness of this concept in dealing with hereditary algebras (see [39], cf. also Section 7). The authors further acknowledge the support of the Deutsche Forschungsgemeinschaft (SPP “Darstellungstheorie von endlichen Gruppen und endlichdimensionalen Algebren”).

Throughout this paper rings are associative with unit and modules are unitary right modules. $\text{Mod}(R)$ (respectively $\text{mod}(R)$) denotes the category of all (respectively all finitely presented) right R -modules. © 1991 Academic Press, Inc.

1. DEFINITIONS AND BASIC PROPERTIES

If \mathcal{S} denotes a system of objects in an abelian category \mathcal{A} —usually viewed as full subcategory of \mathcal{A} —the categories \mathcal{S}^\perp and ${}^\perp\mathcal{S}$ right (resp. left) perpendicular to \mathcal{S} are defined as the full subcategories of \mathcal{A} consisting of all objects $A \in \mathcal{A}$ satisfying the following two conditions:

1. $\text{Hom}(S, A) = 0$ (resp. $\text{Hom}(A, S) = 0$) for all $S \in \mathcal{S}$,
2. $\text{Ext}^1(S, A) = 0$ (resp. $\text{Ext}^1(A, S) = 0$) for all $S \in \mathcal{S}$.

Here, for objects A and B in \mathcal{A} , $\text{Ext}^n(A, B)$ denotes the group formed by the equivalence classes of all n -extensions from B by A taken in the sense of Yoneda (cf., for instance Mitchell [31]). We say that an object A in \mathcal{A} has projective dimension $\leq n$ ($\text{proj dim } A \leq n$) if $\text{Ext}^k(A, -) = 0$ for all $k \geq n + 1$. If $\text{proj dim } S \leq n$ for all $S \in \mathcal{S}$ we write $\text{proj dim } \mathcal{S} \leq n$.

In the following we concentrate mainly on right perpendicular categories. The case of left perpendicular categories is dual.

PROPOSITION 1.1. *Let \mathcal{S} be a system of objects in an abelian category \mathcal{A} . Then the category \mathcal{S}^\perp right perpendicular to \mathcal{S} is closed under the formation of kernels and extensions.*

If additionally $\text{proj dim } \mathcal{S} \leq 1$, \mathcal{S}^\perp is an exact subcategory of \mathcal{A} ; i.e., \mathcal{S}^\perp is abelian and the inclusions $\mathcal{S}^\perp \rightarrow \mathcal{A}$ is exact.

Proof. Let $f: A \rightarrow B$ be a morphism in \mathcal{S}^\perp and denote by K , I , and C the kernel, image, and cokernel of f , respectively. The corresponding sequences $0 \rightarrow K \rightarrow A \rightarrow I \rightarrow 0$ and $0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0$ yield long exact sequences with S in \mathcal{S} :

- (1) $0 \rightarrow \text{Hom}(S, K) \rightarrow \text{Hom}(S, A) \rightarrow \text{Hom}(S, I)$
 $\rightarrow \text{Ext}^1(S, K) \rightarrow \text{Ext}^1(S, A) \rightarrow \text{Ext}^1(S, I),$
- (2) $0 \rightarrow \text{Hom}(S, I) \rightarrow \text{Hom}(S, B) \rightarrow \text{Hom}(S, C)$
 $\rightarrow \text{Ext}^1(S, I) \rightarrow \text{Ext}^1(S, B) \rightarrow \text{Ext}^1(S, C).$

Since A and B are in \mathcal{S}^\perp , $\text{Hom}(S, K) = 0$ by the exactness of (1) and $\text{Hom}(S, I) = 0$ by the exactness of (2); thus $\text{Ext}^1(S, K) = 0$ since (1) is exact and $K \in \mathcal{S}^\perp$ follows. The fact that \mathcal{S}^\perp is closed under extension also follows from the associated Ext-sequence.

Now suppose $\text{proj dim } \mathcal{S} \leq 1$. Then additionally $\text{Ext}^1(S, I) = 0$ due to the exactness of (1) and $\text{Hom}(S, C) = 0 = \text{Ext}^1(S, C)$ follows from the exactness of (2). Hence I and C are in \mathcal{S}^\perp . ■

In general \mathcal{S}^\perp is neither closed under cokernels nor an abelian category (for explicit examples we refer to Section 8).

LEMMA 1.2. *Let \mathcal{S} and \mathcal{T} be systems of objects of an abelian category \mathcal{A} . Then:*

- (i) $\mathcal{S} \subset \mathcal{T} \Rightarrow \mathcal{T}^\perp \subset \mathcal{S}^\perp;$
- (ii) $\mathcal{S} \subset {}^\perp(\mathcal{S}^\perp);$
- (iii) $\mathcal{S}^\perp = ({}^\perp(\mathcal{S}^\perp))^\perp.$

Proof. Properties (i) and (ii) are obvious. By applying (i) to inclusion (ii), we obtain $({}^\perp(\mathcal{S}^\perp))^\perp \subset \mathcal{S}^\perp$. By using the left perpendicular version of (ii) for \mathcal{S}^\perp we get the converse inclusion. ■

Let \mathcal{A} be an abelian category and \mathcal{S} be a system of objects of \mathcal{A} . An object A in \mathcal{A} is called (*finitely*) \mathcal{S} -generated if there is a (finite) index set I and an epimorphism $\bigoplus_{i \in I} S_i \rightarrow A$ with $S_i \in \mathcal{S}$ for all $i \in I$. A is called (*finitely*) \mathcal{S} -presented if there exist (finite) index sets I and J and an exact sequence $\bigoplus_{j \in J} S_j \rightarrow \bigoplus_{i \in I} S_i \rightarrow A \rightarrow 0$ with $S_i, S_j \in \mathcal{S}$.

By means of the preceding notions Proposition 1.1 can be slightly sharpened. In view of the applications, we express the left perpendicular version.

PROPOSITION 1.3. *Let $f: A \rightarrow B$ be a morphism, where $B \in {}^\perp\mathcal{S}$ and A admits a finite filtration with ${}^\perp\mathcal{S}$ -generated factors. Then the cokernel of f belongs to ${}^\perp\mathcal{S}$.*

If \mathcal{A} is a Grothendieck category, then ${}^\perp\mathcal{S}$ is closed under arbitrary direct sums and cokernels of morphisms $f: A \rightarrow B$, where $B \in {}^\perp\mathcal{S}$ and A is the union of a smooth well-ordered chain (A_α) , whose factors $A_{\alpha+1}/A_\alpha$ are ${}^\perp\mathcal{S}$ -generated.

Proof. Denote by I (resp. C) the image (resp. cokernel) of f . Then exactness of $0 \rightarrow I \rightarrow B \rightarrow C \rightarrow 0$ yields $\text{Hom}(C, S) = 0$ and $\text{Hom}(I, S) \cong \text{Ext}^1(C, S)$ for any S in \mathcal{S} . The assumption on A implies that $\text{Hom}(A, S) = 0$, hence $\text{Hom}(I, S) = 0$ for any $S \in \mathcal{S}$. This proves $C \in {}^\perp\mathcal{S}$. The proof of the second claim is similar. ■

We denote by $\text{cl}(\mathcal{S})$ (closure of \mathcal{S}) the smallest subcategory \mathcal{S}' of \mathcal{A} which contains \mathcal{S} and is closed under extensions and under cokernels of morphisms $f: A \rightarrow B$, where B is in \mathcal{S}' and A admits a finite filtration with \mathcal{S}' -generated factors.

If \mathcal{A} is a Grothendieck category, $\text{Cl}(\mathcal{S})$ denotes the smallest subcategory \mathcal{S}' of \mathcal{A} which contains \mathcal{S} and is closed under extensions, (arbitrary) direct sums, and cokernels of morphisms $f: A \rightarrow B$, where B is in \mathcal{S}' and A is the union of a smooth well-ordered chain with \mathcal{S}' -generated factors. The definition implies that $\text{Cl}(\mathcal{S})$ is also closed under direct limits.

In view of Lemma 1.2 any subcategory \mathcal{S}' with $\mathcal{S} \subset \mathcal{S}' \subset {}^\perp(\mathcal{S}^\perp)$ has the same right perpendicular category as \mathcal{S} .

PROPOSITION 1.4. *Let \mathcal{A} be an abelian category and \mathcal{S} a system of objects in \mathcal{A} . Then ${}^\perp(\mathcal{S}^\perp)$ is closed under extensions and cokernels of morphisms $f: A \rightarrow B$, where B is in ${}^\perp(\mathcal{S}^\perp)$ and A has a finite filtration with ${}^\perp(\mathcal{S}^\perp)$ -generated factors. In particular, $\text{cl}(\mathcal{S})$ is contained in ${}^\perp(\mathcal{S}^\perp)$ and so is any object from \mathcal{A} which admits a finite filtration with finitely \mathcal{S} -presented factors. Moreover $\mathcal{S}^\perp = \text{cl}(\mathcal{S})^\perp$.*

If moreover \mathcal{A} is a Grothendieck category, then ${}^\perp(\mathcal{S}^\perp)$ is also closed under arbitrary direct sums and cokernels of morphisms $f: A \rightarrow B$, where B is in ${}^\perp(\mathcal{S}^\perp)$ and A is the union of a smooth well-ordered chain (A_α) with \mathcal{S} -generated factors. In particular, $\text{Cl}(\mathcal{S})$ is contained in ${}^\perp(\mathcal{S}^\perp)$ and so is any object from \mathcal{A} which is the union $A = \bigcup A_\alpha$ of a smooth well-ordered chain with \mathcal{S} -presented factors $A_{\alpha+1}/A_\alpha$. Moreover $\mathcal{S}^\perp = \text{Cl}(\mathcal{S})^\perp$.

In general the inclusion $\text{cl}(\mathcal{S}) \subset {}^\perp(\mathcal{S}^\perp)$ is strict. Let, for instance, \mathcal{A} (resp. \mathcal{S}) denote the category of all finitely generated (resp. all finite) abelian groups. Then $\mathcal{S}^\perp = 0 = {}^\perp\mathcal{S}$, hence ${}^\perp(\mathcal{S}^\perp) = \mathcal{A}$ but $\text{cl}(\mathcal{S}) = \mathcal{S}$.

Let \mathcal{S} be a Serre subcategory; i.e., a full subcategory closed under subobjects, quotient objects, and extensions. In order to determine the category right perpendicular to \mathcal{S} it is convenient to dispose of a “small” subsystem \mathcal{S}' of \mathcal{S} with the property $(\mathcal{S}')^\perp = \mathcal{S}^\perp$. For that purpose we state several easy applications of Proposition 1.4.

COROLLARY 1.5. *Let \mathcal{S} be a Serre subcategory of an abelian category \mathcal{A} . Further suppose that \mathcal{S} is a length category; i.e., every object in \mathcal{S} has finite length. Then:*

(a) if \mathcal{S}' is the system of all objects which are simple in \mathcal{S} , then $(\mathcal{S}')^\perp = \mathcal{S}^\perp$;

(b) if \mathcal{S}' is a system of objects in \mathcal{S} such that for each simple objects $S \in \mathcal{S}$ there is an epimorphism $L \rightarrow S$ with $L \in \mathcal{S}'$, then $(\mathcal{S}')^\perp = \mathcal{S}^\perp$.

COROLLARY 1.6. *Let R be a right noetherian ring, $\mathcal{A} = \text{mod}(R)$, and $I \subset R$ be a two-sided ideal in R . If \mathcal{S} is the Serre subcategory of all objects S in \mathcal{A} annihilated by some finite power I^n (with n depending on S), then $(R/I)^\perp = \mathcal{S}^\perp$.*

Proof. $I^n S = 0$ for some n implies that S has a composition series with finitely presented R/I -modules, and the assertion follows. ■

We note that Corollary 1.6 has a group-graded version. In this case we have to replace R/I by the system of all H -shifts $R/I(h)$. See Section 7 for further details.

COROLLARY 1.7. *Let \mathcal{G} be a Grothendieck category and \mathcal{S} the Serre subcategory of all artinian objects in \mathcal{G} . If \mathcal{S}' is the system of all simple objects in \mathcal{S} , then $(\mathcal{S}')^\perp = \mathcal{S}^\perp$.*

2. LOCALIZING SUBCATEGORIES

In this section we investigate the interrelations between perpendicular categories and *localizing* categories.

Let \mathcal{A} be an abelian category. Recall that a Serre subcategory \mathcal{S} of \mathcal{A} is a subcategory closed under forming subobjects, quotients, and extensions. Moreover, we may form the *quotient category* \mathcal{A}/\mathcal{S} of \mathcal{A} with respect to \mathcal{S} , and $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ denotes the *quotient functor*. For the definition and properties of quotient categories we refer to [18, 47, 34].

LEMMA 2.1. *Let \mathcal{S} be a Serre subcategory of an abelian category \mathcal{A} . For $A \in \mathcal{A}$ and $B \in \mathcal{S}^\perp$, the natural homomorphism $T_{A,B}: \text{Hom}_{\mathcal{A}}(A, B) \rightarrow \text{Hom}_{\mathcal{A}/\mathcal{S}}(TA, TB)$ is an isomorphism. In particular the functor $T: \mathcal{S}^\perp \rightarrow \mathcal{A}/\mathcal{S}$ is a full embedding.*

Proof. Since B has no subobject belonging to \mathcal{S} , $T_{A,B}$ is injective. Let $\varphi: TA \rightarrow TB$ be a morphism in \mathcal{A}/\mathcal{S} . Then $\varphi = (Tv)^{-1}Tf(Tu)^{-1}$, where $u: U \rightarrow A$ is a monomorphism with cokernel in \mathcal{S} , $v: B \rightarrow V$ is an epimorphism with kernel in \mathcal{S} , and $f: U \rightarrow V$ is a morphism. Since B is in \mathcal{S}^\perp , v is an isomorphism; thus we assume $B = V$. Moreover $f: U \rightarrow B$ can be extended to A . Hence $T_{A,B}$ is surjective. ■

Recall that a Serre subcategory \mathcal{S} of \mathcal{A} is called *localizing* if the quotient functor $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ has a right adjoint $\Sigma: \mathcal{A}/\mathcal{S} \rightarrow \mathcal{A}$, called the *section functor*. Note that $T\Sigma$ is isomorphic to the identity functor on \mathcal{A}/\mathcal{S} .

PROPOSITION 2.2. *For a Serre subcategory \mathcal{S} of an abelian category \mathcal{A} the following conditions are equivalent:*

- (a) \mathcal{S} is a localizing subcategory.
- (b) For each object $A \in \mathcal{A}$ there is an exact sequence

$$0 \rightarrow S \rightarrow A \rightarrow \bar{A}$$

with $\bar{A} \in \mathcal{S}^\perp$ and $S \in \mathcal{S}$.

- (c) For each object $A \in \mathcal{A}$ there is an exact sequence

$$0 \rightarrow S_1 \rightarrow A \rightarrow \bar{A} \rightarrow S_2 \rightarrow 0$$

with $\bar{A} \in \mathcal{S}^\perp$ and $S_1, S_2 \in \mathcal{S}$.

- (d) $T: \mathcal{S}^\perp \rightarrow \mathcal{A}/\mathcal{S}$ is an equivalence of categories.

Moreover, in the presence of these conditions, an object $A \in \mathcal{A}$ belongs to \mathcal{S}^\perp if and only if $A \cong \Sigma B$ for some object $B \in \mathcal{A}/\mathcal{S}$. Further, inclusion $j: \mathcal{S}^\perp \rightarrow \mathcal{A}$ admits the functor $\Sigma T: \mathcal{A} \rightarrow \mathcal{S}^\perp$ as a left adjoint.

Proof. (a) \Rightarrow (b): For $A \in \mathcal{A}$ the object $\Sigma T(A)$ is contained in \mathcal{S}^\perp . Adjointness implies the existence of an exact sequence $0 \rightarrow S \rightarrow A \rightarrow \bar{A}$ with $S \in \mathcal{S}$.

(b) \Rightarrow (c): For $A \in \mathcal{A}$ there exists an exact sequence $0 \rightarrow S_1 \rightarrow A \rightarrow A' \rightarrow C \rightarrow 0$ with $S_1 \in \mathcal{S}$ and $A' \in \mathcal{S}^\perp$. Further there is an exact sequence $0 \rightarrow S_2 \rightarrow C \rightarrow \bar{C}$ with $S_2 \in \mathcal{S}$ and $\bar{C} \in \mathcal{S}^\perp$. Let \bar{A} be the inverse image of S_2 in A' . Then there is an exact sequence $0 \rightarrow S_1 \rightarrow A \rightarrow \bar{A} \rightarrow S_2 \rightarrow 0$.

It remains to show that \bar{A} is contained in \mathcal{S}^\perp . Since $\text{Hom}(S, A') = 0$ for all $S \in \mathcal{S}$ the same holds true for \bar{A} . From the exact sequence $0 \rightarrow \bar{A} \rightarrow A' \rightarrow C/S_2 \rightarrow 0$ we obtain the following long exact sequence for $S \in \mathcal{S}$:

$$\text{Hom}(S, C/S_2) \rightarrow \text{Ext}^1(S, \bar{A}) \rightarrow \text{Ext}^1(S, A') \rightarrow \text{Ext}^1(S, C/S_2).$$

Since C/S_2 is contained in \bar{C} , $\text{Hom}(S, C/S_2) = 0$ for all $S \in \mathcal{S}$; thus $\text{Ext}^1(S, \bar{A}) = 0$ for all $S \in \mathcal{S}$ and $\bar{A} \in \mathcal{S}^\perp$ follows.

(c) \Rightarrow (d): By Lemma 2.1, $T|_{\mathcal{S}^\perp}$ is a full embedding and it follows from (c) that this functor is representative.

(d) \Rightarrow (a): Let $\Sigma: \mathcal{A}/\mathcal{S} \rightarrow \mathcal{S}^\perp \subset \mathcal{A}$ be an inverse equivalence of $T|_{\mathcal{S}^\perp}$. Then for $A \in \mathcal{A}$ and $B \in \mathcal{A}/\mathcal{S}$ we have functorial isomorphisms

$\text{Hom}(A, \Sigma(B)) \rightarrow \text{Hom}(T(A), T\Sigma(B)) \rightarrow \text{Hom}(T(A), B)$; thus Σ is right adjoint to T . ■

COROLLARY 2.3. *If \mathcal{S} is a localizing subcategory of an abelian category \mathcal{A} , then $\mathcal{S} = {}^\perp(\mathcal{S}^\perp)$.*

Proof. Let X be an object in ${}^\perp(\mathcal{S}^\perp)$. Then $\text{Hom}_{\mathcal{A}/\mathcal{S}}(T(X), B) = \text{Hom}_{\mathcal{A}}(X, \Sigma(B)) = 0$ for all $B \in \mathcal{A}/\mathcal{S}$. Thus $T(X) = 0$ and $X \in \mathcal{S}$ follows. ■

COROLLARY 2.4. *Let \mathcal{S} be a localizing subcategory of \mathcal{A} . We assume that there exists a subsystem \mathcal{S}' of \mathcal{S} with $\text{proj dim } \mathcal{S}' \leq 1$ and $\mathcal{S}'^\perp = \mathcal{S}^\perp$; then the following assertions hold:*

(i) *For any two objects A, B in \mathcal{S}^\perp the natural homomorphism*

$$\text{Ext}_{\mathcal{A}}^1(A, B) \rightarrow \text{Ext}_{\mathcal{A}/\mathcal{S}}^1(TA, TB), \quad \eta \mapsto T\eta,$$

is an isomorphism.

(ii) *For each integer $i \geq 0$ the homomorphism*

$$K_i(\mathcal{S}) \oplus K_i(\mathcal{S}^\perp) \rightarrow K_i(\mathcal{A})$$

for the Quillen K -groups, induced by the inclusion functors, is an isomorphism.

By means of Theorem 7.5 it is not difficult to construct examples showing that assertion (i) does not hold for localizing subcategories in general.

Proof. In view of Proposition 1.1, \mathcal{S}^\perp is an exact subcategory of \mathcal{A} which is closed under extensions; therefore the inclusion functor from \mathcal{S}^\perp into \mathcal{A} induces an isomorphism $\text{Ext}_{\mathcal{S}^\perp}^1(A, B) \rightarrow \text{Ext}_{\mathcal{A}}^1(A, B)$.

With regard to assertion (ii) we note that the inclusion functor $j: \mathcal{S}^\perp \rightarrow \mathcal{A}$ and the quotient functor $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ are exact, and $T \circ j$ is an equivalence. Hence composition

$$K_i(\mathcal{S}^\perp) \xrightarrow{K_i(j)} K_i(\mathcal{A}) \xrightarrow{K_i(T)} K_i(\mathcal{A}/\mathcal{S})$$

is an isomorphism. The assertion now follows from the exactness of Quillen's localization sequence

$$\dots \longrightarrow K_i(\mathcal{S}) \xrightarrow{K_i(j)} K_i(\mathcal{A}) \xrightarrow{K_i(T)} K(\mathcal{A}/\mathcal{S}) \rightarrow K_{i-1}(\mathcal{S}) \longrightarrow \dots$$

(see [35, p. 113]). ■

In the case of a Grothendieck category Corollary 2.3 can be extended to (see also [18, 34]):

PROPOSITION 2.5. *For a Serre subcategory \mathcal{S} of a Grothendieck category \mathcal{G} the following conditions are equivalent:*

- (a) \mathcal{S} is a localizing subcategory.
- (b) $\mathcal{S} = {}^\perp(\mathcal{S}^\perp)$.
- (c) \mathcal{S} is closed under arbitrary direct sums.

Proof. (a) \Rightarrow (b) follows from Corollary 2.3, while (c) holds for any left perpendicular category.

(c) \Rightarrow (a): We show condition (b) of Proposition 2.2. Let $G \in \mathcal{G}$. Then there exists $S_1 \subset G$ with $S_1 \in \mathcal{S}$ and $\text{Hom}(S, G/S_1) = 0$ for all $S \in \mathcal{S}$. Let I be an injective envelope of G/S_1 . Then $\text{Hom}(S, I) = 0$ for all $S \in \mathcal{S}$, and hence $I \in \mathcal{S}^\perp$ and $0 \rightarrow S_1 \rightarrow G \rightarrow I$ is the wanted sequence. ■

For later reference we include

LEMMA 2.6. *Let \mathcal{G} be a locally noetherian Grothendieck category and $\mathcal{L} \subset \mathcal{G}$ be a localizing subcategory. Suppose there is a subsystem $\mathcal{S} \subset \mathcal{L}$ with $\mathcal{S}^\perp = \mathcal{L}^\perp$ and the property that $\text{Hom}(S, -)$, $S \in \mathcal{S}$, commutes with arbitrary direct sums. Then \mathcal{L}^\perp is closed under arbitrary direct sums in \mathcal{G} . In particular, the section functor $\Sigma: \mathcal{G}/\mathcal{L} \rightarrow \mathcal{G}$ commutes with arbitrary direct sums.*

Proof. Since \mathcal{G} is locally noetherian it is easy to see also that $\text{Ext}^1(S, -)$, $S \in \mathcal{S}$, commutes with arbitrary direct sums. Thus, for any system of objects G_i in \mathcal{S}^\perp ($i \in I$), we also have $\bigoplus_{i \in I} G_i \in \mathcal{S}^\perp$. ■

3. EXISTENCE OF LEFT ADJOINTS

LEMMA 3.1. *Let S be an object of an abelian category \mathcal{A} with $\text{Ext}^1(S, S) = 0$. For each object X in \mathcal{A} such that $\text{Ext}^1(S, X)$ has finite length over $\text{End}(S)$ there exists an exact sequence*

$$0 \rightarrow X \rightarrow \bar{X} \rightarrow S^n \rightarrow 0$$

with $\text{Ext}^1(S, \bar{X}) = 0$. If additionally $\text{End}(S)$ is a skew field, $\text{Hom}(S, X) = \text{Hom}(S, \bar{X})$.

Proof. Let l be the length of $\text{Ext}^1(S, X)$ over $\text{End}(S)$. If $l = 0$, there is nothing to show. If $l > 0$, there is a non-split exact sequence $\eta: 0 \rightarrow X \rightarrow X' \rightarrow S \rightarrow 0$, and thus an exact sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}(S, X) &\rightarrow \text{Hom}(S, X') \rightarrow \text{Hom}(S, S) \\ &\rightarrow \text{Ext}^1(S, X) \rightarrow \text{Ext}^1(S, X') \rightarrow 0 \end{aligned}$$

of $\text{End}(S)$ -modules. Since η does not split, the morphism $\text{Hom}(S, S) \rightarrow \text{Ext}^1(S, X)$ is non-zero, and thus $\text{lg Ext}^1(S, X') < l$, where lg refers to the length. If additionally $\text{End}(S)$ is a skew field, $\text{Hom}(S, X) \rightarrow \text{Hom}(S, X')$ is an isomorphism. Now the assertion follows by induction. ■

PROPOSITION 3.2. *Assume that S is an object of an abelian category \mathcal{A} that satisfies the following assumptions:*

- (a) $\text{Ext}^1(S, S) = 0$.
- (b) For each A in \mathcal{A} , $\text{Hom}(S, A)$ and $\text{Ext}^1(S, A)$ have finite length over the endomorphism ring $\text{End}(S)$.
- (c) For each A in \mathcal{A} we have $\text{Ext}^2(S, A) = 0$.

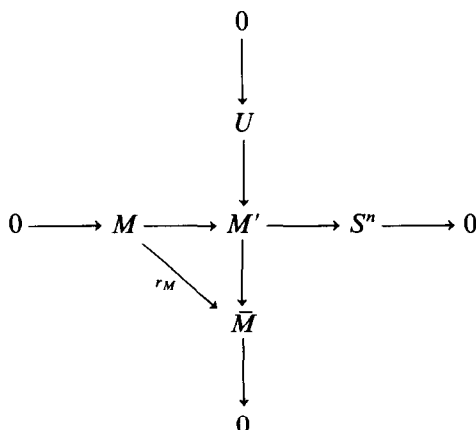
Then S^\perp is an exact subcategory in \mathcal{A} and there exists a functor $l: \mathcal{A} \rightarrow S^\perp$ which is left adjoint to the conclusion functor $i: S^\perp \rightarrow \mathcal{A}$.

Proof. Let M be an arbitrary object of \mathcal{A} . Then by Lemma 3.1 there exists an exact sequence

$$0 \rightarrow M \rightarrow M' \rightarrow S^n \rightarrow 0$$

with $\text{Ext}^1(S, M') = 0$. Next, we choose a generating system f_1, \dots, f_m for $\text{Hom}(S, M')$ over $\text{End}(S)$ and define U as the image of the map $(f_1, \dots, f_m): S^m \rightarrow M'$. As is easily checked, the quotient $\bar{M} = M'/U$ belongs to S^\perp . And, clearly, $r_M: M \rightarrow \bar{M}$, being defined as the composition $M \rightarrow M' \rightarrow \bar{M}$, is the universal homomorphism from M into an object of S^\perp . ■

The situation is depicted by the diagram



Under the assumptions of Proposition 3.2 we now give a different interpretation of the category S^\perp right perpendicular to S .

Let $\lambda: K_0(\mathcal{A}) \rightarrow \mathbf{Z}$ be a linear function on the Grothendieck group $K_0(\mathcal{A})$ of \mathcal{A} . Then the category $\mathcal{A}(\lambda)$ determined by λ is by definition the full subcategory of \mathcal{A} consisting of all objects X with the properties

- (1) $\lambda(X) = 0$;
- (2) $\lambda(X') \leq 0$ for every subobject X' of X .

$\mathcal{A}(\lambda)$ is an exact subcategory of \mathcal{A} closed under extensions.

Let S be an object in \mathcal{A} of finite projective dimension such that for every $A \in \mathcal{A}$ the $\text{End}(S)$ -module $\text{Ext}^i(S, A)$ is of finite length for all $i \geq 0$. Then the linear form $\lambda_S: K_0(\mathcal{A}) \rightarrow \mathbf{Z}$ is defined by

$$\lambda_S(A) = \sum_{i \geq 0} (-1)^i \text{lg Ext}^i(S, A),$$

where lg denotes the length over $\text{End}(S)$.

PROPOSITION 3.3. *Under the assumptions of Proposition 3.2 we have $S^\perp = \mathcal{A}(\lambda_S)$.*

Proof. Let A be an object in S^\perp . Then clearly $\lambda_S(A) = 0$. If A' is a subobject of A , we have $\text{Hom}(S, A') = 0$; hence $\lambda_S(A') = -\text{lg Ext}^1(S, A') \leq 0$ and $A \in \mathcal{A}(\lambda_S)$ follows.

Conversely, let A be an object in $\mathcal{A}(\lambda_S)$. We suppose that $\text{Hom}(S, A)$ is non-zero. Let A' be the image of a non-zero morphism from S to A . Since S has projective dimension ≤ 1 and no self-extensions we get $\text{Ext}^1(S, A') = 0$. On the other hand $\lambda_S(A') \leq 0$ implies that $\text{Ext}^1(S, A') \neq 0$, a contradiction. Hence $\text{Hom}(S, A) = 0$ and $\text{Ext}^1(S, A) = 0$ follows; thus $A \in S^\perp$. ■

LEMMA 3.4. *Let \mathcal{A} be an abelian category and \mathcal{S} be a system of objects in \mathcal{A} . Suppose the embedding $i: \mathcal{S}^\perp \rightarrow \mathcal{A}$ admits a left adjoint $l: \mathcal{A} \rightarrow \mathcal{S}^\perp$. Then $l(M) = 0$ for all $M \in {}^\perp(\mathcal{S}^\perp)$.*

Proof. If $M \in {}^\perp(\mathcal{S}^\perp)$ we have

$$0 = \text{Hom}(M, il(M)) = \text{Hom}(l(M), l(M))$$

and $l(M) = 0$ follows. ■

PROPOSITION 3.5. *Let \mathcal{A} be an abelian category, where we assume all objects to be noetherian. Assume that S is an object of \mathcal{A} satisfying the following assumptions:*

(a) $\text{Ext}^1(S, S) = 0$.

(b) For each A in \mathcal{A} with $\text{Hom}(S, A) = 0$, $\text{Ext}^1(S, A)$ has finite length over the endomorphism ring $\text{End}(S)$.

(c) The endomorphism ring of S is a skew field.

Then there exists a functor $l: \mathcal{A} \rightarrow S^\perp$ which is left adjoint to the inclusion $i: S^\perp \rightarrow \mathcal{A}$.

If additionally $\text{proj dim } S \leq 1$ holds, each object in ${}^\perp(S^\perp)$ has the form S^n ; in particular $\text{cl}(S) = {}^\perp(S^\perp)$.

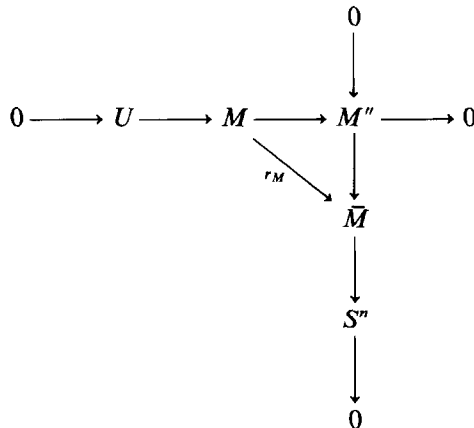
Proof. Here, our strategy of proof will differ from that of Proposition 3.2. By the noetherianness of $M \in \mathcal{A}$ we first choose a subobject U of M which admits a finite filtration with finitely S -generated quotients and such that $M'' = M/U$ satisfies $\text{Hom}(S, M'') = 0$. Then Lemma 3.1 yields an exact sequence

$$0 \rightarrow M'' \rightarrow \bar{M} \rightarrow S^n \rightarrow 0$$

with $\text{Ext}^1(S, \bar{M}) = 0$ and $\text{Hom}(S, \bar{M}) = \text{Hom}(S, M'') = 0$; hence \bar{M} belongs to S^\perp .

Finally let M belong to ${}^\perp(S^\perp)$. In particular $l(M) = 0$, so the above construction of l shows that M admits a finite filtration, whose factors are finitely S -generated. Because $\text{proj dim } S \leq 1$ and $\text{Ext}^1(S, S) = 0$ this implies the existence of an exact sequence $0 \rightarrow K \rightarrow S^a \rightarrow M \rightarrow 0$. Invoking the Hom-Ext-sequence induced by $\text{Hom}(-, Y)$ with Y in S^\perp we see that $lK = 0$. By the preceding argument we thus arrive at an exact sequence $S^b \rightarrow S^a \rightarrow M \rightarrow 0$. In view of (c) this shows that $M \cong S^c$ and proves the last assertion. ■

The following picture reminds the reader of the scheme of thought in this case:



Remark. In the situation of Proposition 3.5 the subcategory S^\perp is not necessarily abelian. If, however, we assume additionally that every subobject of S is finitely S -generated, the category $\text{cl}(S)$, which in this case consists of all finitely S -presented objects, will form a Serre subcategory which will be localizing by the proposition just proved. So we may invoke Proposition 2.2 to conclude that S^\perp is isomorphic to $\mathcal{A}/\text{cl}(\mathcal{S})$, hence an abelian category.

Next, we prove a variant of Proposition 3.5 which is important for most of the applications we have in view. It is possible to prove a similar variant of Proposition 3.2, where we leave the details to the reader.

THEOREM 3.6. *Let \mathcal{A} be an abelian category, where we assume all objects to be noetherian. Further let $\mathcal{S} = \{S_i\}_{i \in I}$ be a subsystem of \mathcal{A} satisfying the following conditions:*

- (a) $\text{End}(S_i)$ is a skew field for all $i \in I$.
- (b) $\text{Ext}^1(S_i, S_i) = 0$ for all $i \in I$.
- (c) $\text{Hom}(S_i, S_j) = 0$ for all $i, j \in I, i \neq j$.
- (d) I admits an ordering such that for each $i \in I$ the set of predecessors of i is finite and $\text{Ext}^1(S_i, S_j) \neq 0$ implies $i < j$.
- (e) For each $A \in \mathcal{A}$ with $\text{Hom}(S_i, A) = 0$ for all $i \in I$, the right $\text{End}(S_i)$ -module $\text{Ext}^1(S_i, A)$ always has finite length and is non-zero for only finitely many $i \in I$.

Then there exists a functor $l: \mathcal{A} \rightarrow \mathcal{S}^\perp$ which is left adjoint to the inclusion $\mathcal{S}^\perp \rightarrow \mathcal{A}$. Further, for any object $M \in \mathcal{A}$ the adjunction homomorphism $r_M: M \rightarrow l(M)$ has a cokernel (resp. kernel) which admits a finite filtration whose factors belong to \mathcal{S} (resp. are finitely \mathcal{S} -generated).

If we assume additionally the two conditions

- (f) $\text{projdim } S_i \leq 1$ for all $i \in I$ and
- (g) $\text{Ext}^1(S_i, S_j)$ is of finite length as a left module over $\text{End}(S_j)$ for each $j \in J$

then each object in ${}^\perp(\mathcal{S}^\perp)$ is the cokernel of a morphism $X_1 \rightarrow X_0$, where X_0 and X_1 admit a finite filtration with factors from \mathcal{S} . In particular ${}^\perp(\mathcal{S}^\perp) = \text{cl}(\mathcal{S})$.

Proof. The proof of the first assertion is analogous to the proof of Proposition 3.5. So given an object $M \in \mathcal{A}$ we first choose by the noetherianness of M a maximal subobject U admitting a finite filtration by \mathcal{S} -generated objects. The quotient $M'' = M/U$ now satisfies $\text{Hom}(S_i, M'') = 0$ for ever $i \in I$.

Next, we prove that any object $M \in \mathcal{A}$ with $\text{Hom}(S_i, M) = 0$ for all $i \in I$ embeds into an object \bar{M} from \mathcal{S}^\perp by forming successive extensions with objects from \mathcal{S} , if such extensions exist. (Here, conditions (b) and (c) are needed.) That the extension process stops is seen by induction on the cardinal number of the finite set of all predecessors of elements $i \in I$ with $\text{Ext}^1(S_i, M) \neq 0$. This proves the first assertion.

With regard to the claim on ${}^\perp(\mathcal{S}^\perp)$ we need some preparation. Let J be any finite subset of I which is closed under predecessors. We first show that it is possible to replace $\mathcal{S}_J = \{S_j\}_{j \in J}$ by a system $\mathcal{T}_J = \{T_j\}_{j \in J}$ with the same closure $\text{cl}(\mathcal{S}_J) = \text{cl}(\mathcal{T}_J)$ and satisfying moreover the conditions

- (1) $\text{Ext}^1(T_i, T_j) = 0$ for all $i, j \in J$;
- (2) $\text{Ext}^1(S_i, T_j) = 0$ for all $j \in J$ and $i \in I \setminus J$;
- (3) $\text{Ext}^1(T_j, S_i)$ has finite length over $\text{End}(S_i)$ for all $j \in J$ and $i \in I \setminus J$;
- (4) $\text{Hom}(T_j, S_i) = 0$ for all $j \in J$ and $i \in I \setminus J$;
- (5) $\text{proj dim } T_j \leq 1$ for all $j \in J$;
- (6) each $S_j, j \in J$, is finitely \mathcal{T}_J -generated.

We prove the assertion by induction on the cardinality of J . If $J = \{j\}$ then j is a minimal element of I ; further $\mathcal{T}_J := \mathcal{S}_J = \{S_j\}$ has the same closure as \mathcal{S}_J and satisfies properties (1)–(6).

Now, let $\text{card}(J) > 1$, k be a maximal element of J , and $J' = J \setminus \{k\}$. Further we assume that there is a system $\mathcal{T}_{J'} = \{T'_j\}_{j \in J'}$ satisfying conditions (1)–(6) and having the same closure as $\mathcal{S}_{J'}$. From the dual version of Lemma 3.1 we obtain for each $j \in J'$ an exact sequence

$$\eta_j: 0 \rightarrow S_k^l \rightarrow T_j \rightarrow T'_j \rightarrow 0$$

with $T_j \in {}^\perp S_k$. We set $T_k = S_k$ and $\mathcal{T}_J = \{T_j\}_{j \in J}$. It is straightforward to check that \mathcal{T}_J satisfies conditions (1)–(6) and further \mathcal{S}_J and \mathcal{T}_J have the same closure.

We are now in a position to determine the structure of an object $A \in {}^\perp(\mathcal{S}^\perp)$. Since $l(A) = 0$, it follows from the preceding construction of l that there exists a finite subset J_0 of I , closed under predecessors, such that A has a finite filtration with finitely \mathcal{S}_{J_0} -generated, hence also finitely \mathcal{T}_{J_0} -generated, factors. Since $\text{proj dim } T_j \leq 1$ and $\text{Ext}^1(T_i, T_j) = 0$ for all $i, j \in J_0$ we conclude that A is already finitely \mathcal{T}_{J_0} -generated. Let

$$0 \rightarrow K \rightarrow X_0 \rightarrow A \rightarrow 0$$

be an exact sequence, where X_0 is a direct sum of objects from \mathcal{T}_{J_0} . For each B in \mathcal{S}^\perp we obtain $\text{Hom}(K, B) = 0$, hence $l(K) = 0$. From the

preceding argument we conclude that K is finitely \mathcal{T}_{J_1} -generated, for a suitable finite subset J_1 of J , which proves our assertion on ${}^\perp(\mathcal{S}^\perp)$. ■

In view of Proposition 2.2 the proof of Theorem 3.6 yields:

LEMMA 3.7. *If in addition to the assumptions of Theorem 3.6 the objects S_i are simple, then the Serre subcategory \mathcal{C} generated by the S_i , $i \in I$, is localizing.*

As was shown in Proposition 1.1 for a subsystem \mathcal{S} of an abelian category \mathcal{A} with $\text{proj dim } \mathcal{S} \leq 1$, \mathcal{S}^\perp is an exact subcategory of \mathcal{A} . This leads to an investigation on exact subcategories where the embedding has a left adjoint. We concentrate on the case of a module category over a ring.

PROPOSITION 3.8. *Let R be a ring, $\mathcal{A} = \text{Mod}(R)$ be the category of right R -modules, and \mathcal{A}' be an exact subcategory of \mathcal{A} which is closed under arbitrary direct sums that the embedding functor $j: \mathcal{A}' \rightarrow \mathcal{A}$ has a left adjoint functor $l: \mathcal{A} \rightarrow \mathcal{A}'$. Further let $R' = \text{End}(lR)$.*

Then $\text{Hom}(lR, -): \mathcal{A}' \rightarrow \text{Mod}(R')$ is an equivalence of categories with inverse equivalence $- \otimes_{R'} lR: \text{Mod}(R') \rightarrow \mathcal{A}'$.

Moreover, there exists an epimorphism of rings $\varphi: R \rightarrow R'$ such that $j \circ - \otimes_{R'} lR \cong \varphi_$ and $\text{Hom}(lR, -) \circ l \cong \varphi^*$, where φ_* denotes the natural functor $\text{Mod}(R') \rightarrow \text{Mod}(R)$ and $\varphi^* = - \otimes_R R'$ the left adjoint of φ_* .*

Proof. The functor l , as a left adjoint, is right exact and commutes with arbitrary direct sums. Further, since j is a full embedding, $l(M) \cong M$ for all $M \in \mathcal{A}'$. If $R^{(I)} \rightarrow M$ is an epimorphism with $M \in \mathcal{A}'$, application of l yields an epimorphism $l(R)^{(I)} \rightarrow M$. Thus $l(R)$ is a generator in \mathcal{A}' . Moreover, by adjunction, $\text{Hom}_R(lR, M) \cong \text{Hom}_{R'}(R, M)$ for all $M \in \mathcal{A}'$. It follows that $l(R)$ is small and projective in \mathcal{A}' . This proves that $\text{Hom}(lR, -): \mathcal{A}' \rightarrow \text{Mod}(R')$ is an equivalence.

Let $F = \text{Hom}(lR, -)$ and $G = - \otimes_{R'} lR$. We have isomorphisms $GF(lR) \cong lR$ and $FG(R') \cong R'$. Since, as right exact functors, GF and FG are determined by their values on lR and R' , respectively, we have $GF \cong 1_{\mathcal{A}'}$ and $FG \cong 1_{\text{Mod}(R')}$.

Let $\varphi: R \rightarrow R'$ be the homomorphism

$$R = \text{End}(R_R) \rightarrow \text{End}(R'_R) = R', \quad \alpha \mapsto l(\alpha),$$

induced by l . By adjunction we have isomorphisms of right R -modules

$$lR \cong \text{Hom}_R(R, lR) \cong \text{Hom}_R(lR, lR) = R',$$

and thus $\varphi_*(R') \cong j(G(R'))$. Further we get $F(l(R)) \cong R' \cong \varphi^*(R)$. Again

φ_* and $j \circ G$ are determined by their values on R' and $F \circ l$ and φ^* by their values on lR . Hence, $\varphi_* \cong j \circ G$ and $\varphi^* \cong F \circ l$.

Since j is a full embedding, the same holds true for φ_* . This implies that φ is an epimorphism of rings [43]. ■

COROLLARY 3.9. *Let R be a right noetherian ring, $\mathcal{A} = \text{mod}(R)$, the category of finitely presented right R -modules, and \mathcal{A}' be an exact subcategory of \mathcal{A} such that the embedding functor $j: \mathcal{A}' \rightarrow \mathcal{A}$ has a left adjoint functor $l: \mathcal{A} \rightarrow \mathcal{A}'$. Further let $R' = \text{End}(lR)$.*

Then $\text{Hom}(lR, -): \mathcal{A}' \rightarrow \text{mod}(R')$ is an equivalence of categories with inverse equivalence $- \otimes_{R'} lR: \text{mod}(R') \rightarrow \mathcal{A}'$.

Moreover, there exists an epimorphism of rings $\varphi: R \rightarrow R'$ such that $j \circ - \otimes_{R'} lR \cong \varphi_$ and $\text{Hom}(lR, -) \circ l \cong \varphi^*$, where φ_* denotes the natural functor $\text{mod}(R') \rightarrow \text{mod}(R)$ and $\varphi^* = - \otimes_{R'} R'$ the left adjoint of φ_* . In particular, R' is right noetherian.*

For a small category \mathcal{A} we denote by $\text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab})$ the category of all left-exact functors from \mathcal{A}^{op} to the category of abelian groups. If \mathcal{A} is noetherian, $\text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab})$ is a locally noetherian Grothendieck category with \mathcal{A} as the full subcategory of all noetherian objects, where \mathcal{A} is considered as a full exact subcategory of $\text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab})$ by the Yoneda embedding $A \mapsto (-, A)$. If, moreover, $\mathcal{A} = \text{mod}(R)$ for a right noetherian ring R , we have $\text{Mod}(R) \cong \text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab})$, given by the functor $M \mapsto (-, M)|_{\text{mod}(R)}$.

Let $T: \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor. Then there exists a functor $\bar{T}: \text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab}) \rightarrow \text{Lex}(\mathcal{B}^{\text{op}}, \text{Ab})$, unique up to isomorphism, making the diagram

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{T} & \mathcal{B} \\
 \downarrow & & \downarrow \\
 \text{Lex}(\mathcal{A}^{\text{op}}, \text{Ab}) & \xrightarrow{\bar{T}} & \text{Lex}(\mathcal{B}^{\text{op}}, \text{Ab})
 \end{array}$$

commutative and commuting with direct limits. If T is a full embedding, the same holds true for \bar{T} . If \mathcal{A} is noetherian, T exact implies \bar{T} exact. Further, if $T: \mathcal{A} \rightarrow \mathcal{B}$ and $S: \mathcal{B} \rightarrow \mathcal{A}$ is an adjoint pair of functors, (\bar{T}, \bar{S}) also form an adjoint pair of functors.

Proof of Corollary 3.9. The full exact embedding $j: \mathcal{A}' \rightarrow \text{mod}(R)$ induces a full exact embedding $\bar{j}: \text{Lex}((\mathcal{A}')^{\text{op}}, \text{Ab}) \rightarrow \text{Mod}(R)$ with left adjoint $\bar{l}: \text{Mod}(R) \rightarrow \text{Lex}((\mathcal{A}')^{\text{op}}, \text{Ab})$. Let $R' = \text{End}(lR)$. By Proposition 3.8 there is an equivalence $\text{Hom}(lR, -): \text{Lex}((\mathcal{A}')^{\text{op}}, \text{Ab}) \rightarrow \text{Mod}(R')$ with inverse equivalence $- \otimes_{R'} lR: \text{Mod}(R') \rightarrow \text{Lex}((\mathcal{A}')^{\text{op}}, \text{Ab})$ and an epimorphism $\varphi: R \rightarrow R'$ such that $j \circ - \otimes_{R'} lR \cong \varphi_*$ and $\text{Hom}(lR, -) \circ l \cong \varphi^*$.

All these functors map finitely presented objects to finitely presented objects. Since $\text{mod}(R')$ is a noetherian category, R' is a right noetherian ring. ■

4. HOMOLOGICAL EPIMORPHISMS OF RINGS

In this section we study properties of ring homomorphisms $\varphi: R \rightarrow U$. In order to simplify proofs, we start with the following abstract setting.

Let \mathcal{A} be an abelian category. A subcategory \mathcal{C} is called *thick* if for each short-exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ the fact that two terms belong to \mathcal{C} implies that the third term also belongs to \mathcal{C} .

We say that \mathcal{C} *covers* (resp. *finitely covers*) \mathcal{A} if the smallest thick subcategory \mathcal{C}' of \mathcal{A} containing \mathcal{C} which is closed under the formation of arbitrary (resp. finite) direct sums is equal to \mathcal{A} .

Further we say that $A \in \mathcal{A}$ *admits a resolution by objects from \mathcal{C}* in case there is an exact sequence

$$\cdots \rightarrow C_n \rightarrow \cdots \rightarrow C_2 \rightarrow C_1 \rightarrow C_0 \rightarrow A \rightarrow 0.$$

We say that \mathcal{C} *weakly covers* (resp. *finitely weakly covers*) \mathcal{A} if the smallest thick subcategory \mathcal{C}' of \mathcal{A} containing all objects admitting a resolution by arbitrary (resp. finite) direct sums of objects from \mathcal{C} equals \mathcal{A} .

For the notion of an exact connected sequence of covariant functors $G_n: \mathcal{A} \rightarrow \mathcal{B}$, $n \in \mathbb{Z}$, we refer to [12].

LEMMA 4.1. *Let \mathcal{A} and \mathcal{B} be abelian categories and $\eta: (G_n) \rightarrow (H_n)$ be a morphism of exact connected sequences of additive functors (G_n) , (H_n) from \mathcal{A} to \mathcal{B} .*

(a) *Suppose that \mathcal{A}' finitely covers \mathcal{A} and $\eta_n(A): G_n(A) \rightarrow H_n(A)$ is an isomorphism for all $A \in \mathcal{A}'$ and all n ; then η is an isomorphism.*

(b) *Suppose \mathcal{A}' finitely weakly covers \mathcal{A} , $G_n = 0 = H_n$ for all $n < 0$, and further for any $A \in \mathcal{A}'$ the morphism $\eta_0(A): G_0(A) \rightarrow H_0(A)$ is an isomorphism and $G_n(A) = 0 = H_n(A)$ for all $n \neq 0$. Then η is an isomorphism.*

Proof. (a) Let \mathcal{C} be the subcategory consisting of all objects $A \in \mathcal{A}$ such that $\eta_n(A)$ is an isomorphism for all n . By assumption, \mathcal{A}' is contained in \mathcal{C} and by the Five-Lemma, \mathcal{C} is a thick subcategory. Since \mathcal{A}' finitely covers \mathcal{A} we obtain $\mathcal{C} = \mathcal{A}$.

(b) We prove that η_n is an isomorphism by induction on n . We first deal with the case $n = 0$. Let A be in \mathcal{A} and $X_1 \rightarrow X_0 \rightarrow A \rightarrow 0$ be an exact sequence with X_0, X_1 in \mathcal{A}' . From the right exactness of G_0 and H_0 it follows that $\eta_0(A)$ is an isomorphism.

Now suppose $n > 0$ and η_{n-1} is an isomorphism. Let A be an object in \mathcal{A} and $0 \rightarrow K \rightarrow X \rightarrow A \rightarrow 0$ be exact, where X is a finite direct sum of objects in \mathcal{A}' . Since $G_i(X) = 0 = H_i(X)$ for all $i > 0$ we get a commutative diagram

$$\begin{array}{ccccccc}
 0 & \longrightarrow & G_n(A) & \longrightarrow & G_{n-1}(K) & \longrightarrow & G_{n-1}(X) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & H_n(A) & \longrightarrow & H_{n-1}(K) & \longrightarrow & H_{n-1}(X)
 \end{array}$$

and by the Five-Lemma, $G_n(A) \rightarrow H_n(A)$ is an isomorphism. ■

If additionally we suppose that G_n and H_n commute with arbitrary direct sums then Lemma 4.4 remains valid if we replace “finitely covers” (resp. “finitely weakly covers”) by “covers” (resp. “weakly covers”).

We now discuss an analogue of Lemma 4.1 for derived categories. For an abelian category \mathcal{A} , $D^b(\mathcal{A})$ denotes the *derived category of bounded complexes* in \mathcal{A} . We refer to [50, 28] for the definition and properties of *triangulated* and *derived categories*. We consider \mathcal{A} as a full subcategory of $D^b(\mathcal{A})$, viewing $A \in \mathcal{A}$ as a complex concentrated at 0. We note that $D^b(\mathcal{A})$ is equipped with a translation functor T given by $(T(X))^n = X^{n+1}$ and $(Td_X)^n = -d_X^{n+1}$ for $X \in D^b(\mathcal{A})$.

A functor $G: \mathcal{C} \rightarrow \mathcal{B}$ from a triangulated category \mathcal{C} to an abelian category \mathcal{A} is called a (covariant) *exact* functor if for each triangle $X \rightarrow Y \rightarrow Z \rightarrow TX$ in \mathcal{C} the induced sequence

$$\dots \rightarrow G(T^i X) \rightarrow G(T^i Y) \rightarrow G(T^i Z) \rightarrow G(T^{i+1} X) \rightarrow \dots$$

in \mathcal{B} is exact.

Let \mathcal{C} be a triangulated category. A subcategory \mathcal{D} is called *thick* if for each triangle $X \rightarrow Y \rightarrow Z \rightarrow TX$ in \mathcal{C} the fact that two terms belong to \mathcal{D} implies that the third term also belongs to \mathcal{D} . (This implies in particular that \mathcal{D} is stable under the translation functor T .)

We say that \mathcal{D} *covers* (resp. *finitely covers*) \mathcal{C} if the smallest thick subcategory \mathcal{D}' of \mathcal{C} containing \mathcal{D} which is closed under the formation of arbitrary (resp. finite) direct sums is equal to \mathcal{C} .

Further we say that $C \in \mathcal{C}$ *admits a resolution by objects from \mathcal{D}* in case there is a sequence of triangles $K_{i+1} \rightarrow D_i \rightarrow K_i$, $i = -1, 0, 1, \dots$, with $K_{-1} = C$ and $D_i \in \mathcal{D}$. Finally we say that \mathcal{D} *weakly covers* (resp. *finitely weakly covers*) \mathcal{C} if the smallest thick subcategory \mathcal{D}' of \mathcal{C} containing all objects admitting a resolution by arbitrary (resp. finite) direct sums of objects from \mathcal{D} equals \mathcal{C} .

The same argument as in Lemma 4.1 yields a variant of Beilinson’s lemma [7]:

LEMMA 4.2. *Let \mathcal{C} be a triangulated category, \mathcal{B} be an abelian category, and $\eta: G \rightarrow H$ be a morphism of exact functors G, H from \mathcal{C} to \mathcal{B} . Let $G_n = G \circ T^{-n}$, $H_n = H \circ T^{-n}$.*

(a) *Suppose that \mathcal{C}' finitely covers \mathcal{C} and $\eta_n(C): G_n(C) \rightarrow H_n(C)$ is an isomorphism for all $C \in \mathcal{C}'$ and all n ; then η is an isomorphism.*

(b) *Suppose that \mathcal{C}' finitely weakly covers \mathcal{C} , $G_n = 0 = H_n$ for all $n < 0$, and further for any $C \in \mathcal{C}'$ the morphism $\eta_0(C): G_0(C) \rightarrow H_0(C)$ is an isomorphism and $G_n(C) = 0 = H_n(C)$ for all $n \neq 0$. Then η is an isomorphism.*

PROPOSITION 4.3. *Let $j: \mathcal{A}' \rightarrow \mathcal{A}$ be an exact embedding of abelian categories. Then the following conditions are equivalent:*

(1) *The natural morphism*

$$\text{Ext}_{\mathcal{A}'}^n(A, B) \rightarrow \text{Ext}_{\mathcal{A}}^n(jA, jB)$$

is an isomorphism for all $A, B \in \mathcal{A}'$ and $n \geq 0$.

(2) *The induced functor of derived categories*

$$D^b(j): D^b(\mathcal{A}') \rightarrow D^b(\mathcal{A})$$

is a full embedding.

Proof. The implication (2) \Rightarrow (1) follows from the formula $\text{Ext}_{\mathcal{A}}^n(A, B) = \text{Hom}_{D^b(\mathcal{A})}(A, T^n B)$.

(1) \Rightarrow (2): For each $A \in \mathcal{A}'$ we have a morphism of exact functors $\eta_A: \text{Hom}_{D^b(\mathcal{A}')} (A, -) \rightarrow \text{Hom}_{D^b(\mathcal{A})} (jA, -) \circ D^b(j)$, which by assumption is an isomorphism on all objects of \mathcal{A}' . Since \mathcal{A}' finitely covers $D^b(\mathcal{A}')$, we deduce that η_A is an isomorphism by Lemma 4.2.

Now, for an object $X \in D^b(\mathcal{A}')$ we have a morphism of exact functors $\eta_X: \text{Hom}_{D^b(\mathcal{A}')} (-, X) \rightarrow \text{Hom}_{D^b(\mathcal{A})} (-, D^b(j)A) \circ D^b(j)$, which by the argument above is an isomorphism on all objects of \mathcal{A}' , and hence an isomorphism. ■

Let $\varphi: R \rightarrow U$ be a homomorphism of rings and $\varphi_*: \text{Mod}(U) \rightarrow \text{Mod}(R)$, $M \mapsto M$, be the functor induced by φ . If M is a U -module, we often write M for $\varphi_*(M)$ and it becomes clear from the context whether M is viewed as a module over U or R .

If M and M' are right (resp. left) U -modules we have a natural homomorphism $\text{Hom}_U(M, M') \rightarrow \text{Hom}_R(M, M')$ of abelian groups. This homomorphism induces natural morphisms $\text{Ext}_U^i(M, M') \rightarrow \text{Ext}_R^i(M, M')$ for all i .

If M is a right and N is a left U -module the morphism

$M \otimes_R N \rightarrow M \otimes_U N$ induces natural morphism $\text{Tor}_i^R(M, N) \rightarrow \text{Tor}_i^U(M, N)$ for all i .

THEOREM 4.4. *For a homomorphism of rings $\varphi: R \rightarrow U$ the following conditions are equivalent:*

(1) *The multiplication map $U \otimes_R U \rightarrow U$ is an isomorphism and $\text{Tor}_i^R(U, U) = 0$ for all $i \geq 1$.*

(2) *For all right U -modules M the multiplication map $M \otimes_R U \rightarrow M$ is an isomorphism and $\text{Tor}_i^R(M, U) = 0$ for all $i \geq 1$.*

(2') *For all left U -modules N the multiplication map $U \otimes_R N \rightarrow N$ is an isomorphism and $\text{Tor}_i^R(U, N) = 0$ for all $i \geq 1$.*

(3) *For all right U -modules M and all left U -modules N the natural map $\text{Tor}_i^R(M, N) \rightarrow \text{Tor}_i^U(M, N)$ is an isomorphism for all $i \geq 0$.*

(4) *For all right U -modules M the natural map $\text{Hom}_R(U_R, M_R) \rightarrow M_R$ is an isomorphism and $\text{Ext}_R^i(U_R, M_R) = 0$ for all $i \geq 1$.*

(4') *For all left U -modules N the natural map $\text{Hom}_R({}_R U, {}_R N) \rightarrow {}_R N$ is an isomorphism and $\text{Ext}_R^i({}_R U, {}_R N) = 0$ for all $i \geq 1$.*

(5) *For all right U -modules M and M' the natural map $\text{Ext}_U^i(M_U, M'_U) \rightarrow \text{Ext}_R^i(M_R, M'_R)$ is an isomorphism for all $i \geq 0$.*

(5') *For all left U -modules N and N' the natural map $\text{Ext}_U^i({}_U N, {}_U N') \rightarrow \text{Ext}_R^i({}_R N, {}_R N')$ is an isomorphism for all $i \geq 0$.*

(6) *The induced functor of derived categories*

$$D^b(\varphi_*): D^b(\text{Mod}(U)) \rightarrow D^b(\text{Mod}(R))$$

is a full embedding.

(6') *The induced functor of derived categories*

$$D^b(\varphi_*^{\text{op}}): D^b(\text{Mod}(U^{\text{op}})) \rightarrow D^b(\text{Mod}(R^{\text{op}}))$$

is a full embedding.

Proof. (1) \Rightarrow (2): For each right U -module M we have a sequence of natural isomorphisms

$$M \otimes_R U \simeq M \otimes_U U \otimes_R U \simeq M \otimes_U U \simeq M$$

whose composition is just the multiplication map. The assertion now follows from Lemma 4.1(b).

(2) \Rightarrow (3): Let M be a right U -module and N be a left U -module. Consider $M \otimes_R -$ as a functor from $\text{Mod}(U^{\text{op}})$ to the category of abelian groups. The natural transformation $M \otimes_R - \rightarrow M \otimes_U -$ is an isomorphism

on U , and hence an isomorphism since both functors are right exact. Again the assertion now follows from Lemma 4.1(b).

The implication $(3) \Rightarrow (1)$ is obvious. In the same way we prove $(1) \Rightarrow (2') \Rightarrow (3)$. In a manner similar to that of the proof of $(2) \Rightarrow (3)$ we obtain $(4) \Rightarrow (5)$ and $(4') \Rightarrow (5')$. The implications $(5) \Rightarrow (4)$ and $(5') \Rightarrow (4')$ are obvious.

$(2') \Rightarrow (4)$: For a right U -module M we have a sequence of isomorphisms

$$\begin{aligned} \text{Hom}_R(U, M) &\simeq \text{Hom}_R(U, \text{Hom}_U(U, M)) \simeq \text{Hom}_U(U \otimes_R U, M) \\ &\simeq \text{Hom}_U(U, M) \simeq M \end{aligned}$$

whose composition is just the natural map. In particular, $\text{Hom}_R(U, -)$ is exact on sequences of right U -modules.

For a right U -module M we denote by DM the left U -module $\text{Hom}_Z(M, \mathbf{Q}/\mathbf{Z})$. By the duality isomorphism [12] we get

$$\text{Ext}_R^i(U, DDM) \cong D \text{Tor}_i^U(T, DM) = 0$$

for all $i \geq 1$. Since M is a submodule of DDM and $\text{Hom}_R(U, -)$ is exact on sequences of right U -modules, we conclude $\text{Ext}_R^1(U, M) = 0$ for all right U -modules M . The assertion now follows by induction.

$(4) \Rightarrow (2')$: Let N be a left U -module. Then by duality $D \text{Tor}_i^R(U, N) \cong \text{Ext}_R^i(U, DN)$ for all $i \geq 0$. Hence, $\text{Tor}_i^R(U, N) = 0$ for all $i \geq 1$ and we get

$$\text{Hom}_Z(N, \mathbf{Q}/\mathbf{Z}) \cong \text{Hom}_R(U, \text{Hom}_Z(N, \mathbf{Q}/\mathbf{Z})) \cong \text{Hom}_Z(U \otimes_R N, \mathbf{Q}/\mathbf{Z}).$$

The latter isomorphism is induced by the multiplication, hence the multiplication is itself an isomorphism.

Analogously, we prove the equivalence of (2) and (4'). The equivalence of (5) and (6) (resp. (5') and (6')) follows from Proposition 4.3. This finishes the proof of the theorem. ■

We note that the multiplication $U \otimes_R U \rightarrow U$ is an isomorphism if and only if φ is an epimorphism of rings [43]. Epimorphisms of rings for finite dimensional algebras have been considered recently in [20].

We further note that the presence of condition (1) implies that it is only necessary to require conditions (2), (2'), and (3) for finitely presented U -modules.

DEFINITION 4.5. A homomorphism $\varphi: R \rightarrow U$ satisfying the equivalent conditions of Theorem 4.4 is called a *homological epimorphism* of rings.

COROLLARY 4.6. *Let $\varphi: R \rightarrow U$ be a homological epimorphism of rings. Then*

- (1) $\varphi^{\text{op}}: R^{\text{op}} \rightarrow U^{\text{op}}$ is a homological epimorphism of rings;
- (2) $\text{gl dim } U \leq \text{gl dim } R$.

COROLLARY 4.7. (1) *Let $\varphi: R \rightarrow U$ be an epimorphism and suppose that U is flat as a right R -module. Then φ is a homological epimorphism.*

(2) *If R is a commutative ring and S a multiplicative subset of R , the natural ring homomorphism $\varphi: R \rightarrow S^{-1}R$ is a homological epimorphism.*

Proof. (1) Since φ is an epimorphism the multiplication $U \otimes_R U \rightarrow U$ is an isomorphism. Since U is flat over R , $\text{Tor}_i^R(U, U) = 0$ for all $i \geq 1$.

Now (2) follows from (1). ■

COROLLARY 4.8. *Let R be a ring and \mathcal{A} be a full exact subcategory of $\text{Mod}(R)$ closed under arbitrary direct sums and extensions such that the embedding $j: \mathcal{A} \rightarrow \text{Mod}(R)$ admits a left adjoint functor $l: \text{Mod}(R) \rightarrow \mathcal{A}$.*

If $\text{proj dim } lR \leq 1$ then the ring homomorphism $\varphi: R \rightarrow R' = \text{End}(lR)$ induced by l is a homological epimorphism.

Proof. By Proposition 3.8 we have $\varphi_* \cong j \circ - \otimes_R lR$. Hence for all right R' -modules M the natural homomorphism $\text{Hom}_{R'}(R', M) \rightarrow \text{Hom}_R(R', M)$ is an isomorphism since φ_* is full and $\text{Ext}_R^i(R', M) = 0$ for all $i \geq 1$ since $\text{proj dim } R'_R \leq 1$ and $\text{Mod}(R')$ can be considered as a full subcategory of $\text{Mod}(R)$ which is closed under extensions. ■

We now consider the case where φ induces an embedding $\varphi_*: \text{mod}(U) \rightarrow \text{mod}(R)$.

PROPOSITION 4.9. *Let R be a right coherent (for instance, right noetherian) ring and $\varphi: R \rightarrow U$ be a ring homomorphism such that U is finitely presented and of finite projective dimension as a right R -module. Then the following conditions are equivalent:*

- (1) φ is a homological epimorphism of rings.
- (2) *The natural map $\text{Hom}_R(U_R, U_R) \rightarrow U$ is an isomorphism and $\text{Ext}_R^i(U_R, U_R) = 0$ for all $i \geq 1$.*
- (3) *For all finitely presented right U -modules M the natural map $\text{Hom}_R(U_R, U_R) \rightarrow M$ is an isomorphism and $\text{Ext}_R^i(U_R, M_R) = 0$ for all $i \geq 1$.*
- (4) *For all finitely presented right U -modules M and M' the natural map $\text{Ext}_U^i(M_U, M'_U) \rightarrow \text{Ext}_R^i(M_R, M'_R)$ is an isomorphism for all $i \geq 0$.*

(5) *The induced functor*

$$D^b(\varphi_*) : D^b(\text{mod}(U)) \rightarrow D^b(\text{mod}(R))$$

is a full embedding.

Proof. The implications (1) \Rightarrow (4) \Rightarrow (3) \Rightarrow (2) are obvious.

Since U is finitely presented as a right R -module the functor $\text{Hom}_R(U_R, -)$ commutes with arbitrary direct sums and since R is right noetherian, the same holds true for the functors $\text{Ext}_R^i(U_R, -)$ for all $i \geq 1$.

Now, suppose $\text{proj dim } U_R = n$. Then the functor $\text{Ext}_R^n(U, -)$ is right exact and $\text{Ext}_R^n(U_R, U_R) = 0$ implies $\text{Ext}_R^n(U_R, M_R) = 0$ for all U -modules M . In particular, the functor $\text{Ext}_R^{n-1}(U_R, -)$ is right exact on sequences of right U -modules. By induction we conclude that $\text{Ext}_R^i(U_R, M) = 0$ for all right U -modules M and all $i \geq 1$. Thus $\text{Hom}_R(U_R, -)$ is exact on sequences of right U -modules M and $\text{Hom}_R(U_R, M) \cong M$ for all right U -modules follows. This proves (2) \Rightarrow (1).

The equivalence of (4) and (5) follows by Proposition 4.3. \blacksquare

As in Corollary 4.8 we get:

COROLLARY 4.10. *Let R be a right noetherian ring and \mathcal{A} be a full exact subcategory of $\text{mod}(R)$ closed under extensions such that the embedding $j : \mathcal{A} \rightarrow \text{mod}(R)$ admits a left adjoint functor $l : \text{mod}(R) \rightarrow \mathcal{A}$.*

If $\text{proj dim } lR \leq 1$ then the ring homomorphism $\varphi : R \rightarrow R' = \text{End}(lR)$ induced by l is a homological epimorphism.

Homological epimorphisms of rings which are also injective frequently occur in applications (see below and Sections 10 and 11) and are now studied further.

PROPOSITION 4.11. *Let R be a right noetherian ring and S a finitely presented right R -module satisfying the following conditions:*

- (1) $\text{proj dim } S \leq 1$.
- (2) $\text{Ext}_R^1(S, S) = 0$.
- (3) $\text{Hom}_R(S, M)$ and $\text{Ext}_R^1(S, M)$ are $\text{End}(S)$ -modules of finite length for all finitely presented right R -modules M .
- (4) $\text{End}(S)$ is a skew field.
- (5) $\text{Hom}_R(S, R) = 0$.

Then the embedding $j : S^\perp \rightarrow \text{mod}(R)$ has a left adjoint functor $l : \text{mod}(R) \rightarrow S^\perp$ and the ring homomorphism $\varphi : R \rightarrow R' = \text{End}(lR)$ induced by l is injective and a homological epimorphism.

Proof. The existence of a left adjoint functor l follows from Proposition 3.5. Since $R' = lR$ as right R -modules, the same proposition shows the existence of an exact sequence $0 \rightarrow R \xrightarrow{\varphi} R' \rightarrow S^n \rightarrow 0$. In particular, φ is injective. Moreover, $\text{proj dim } R'_R \leq 1$, and hence φ is a homological epimorphism by Corollary 4.10. ■

A variant of Proposition 4.11 may be proved along the same lines with Theorem 3.6 invoked instead of Proposition 3.5. We leave the details to the reader.

For the rest of this section we assume that k is a commutative noetherian ring, $\varphi: R \rightarrow U$ is an injective homological epimorphism of k -algebras, which are finitely generated k -modules, and $\text{proj dim } U_R \leq 1$.

Note that we always view $\text{Mod}(U)$, accordingly $\text{mod}(U)$, as a full subcategory of $\text{Mod}(R)$. Since U/R is an (R, R) -bimodule, left multiplication of R on U/R defines a ring homomorphism

$$\psi: R \rightarrow V := \text{End}((U/R)_R), \quad r \mapsto [x \mapsto r.x],$$

which we call *the ring homomorphism associated to φ* . The exact sequence of (R, R) -bimodules

$$0 \longrightarrow R \xrightarrow{\varphi} U \longrightarrow U/R \longrightarrow 0 \tag{*}$$

induces for each right R -module X an exact sequence

$$\begin{aligned} 0 \longrightarrow \text{Hom}_R(U/R, X) \longrightarrow \text{Hom}_R(U, X) \xrightarrow{\sigma_X} X \\ \xrightarrow{\tau_X} \text{Ext}_R^1(U/R, X) \longrightarrow \text{Ext}_R^1(U, X) \longrightarrow 0, \end{aligned} \tag{**}$$

again of right R -modules.

PROPOSITION 4.12. *For any right R -module X the following conditions are equivalent:*

- (1) $X \in (U/R)^\perp$.
- (2) $\sigma_X: \text{Hom}_R(U, X) \rightarrow X$ is an isomorphism and $\text{Ext}_R^1(U, X) = 0$.
- (3) $\sigma_X: \text{Hom}_R(U, X) \rightarrow X$ is an isomorphism.
- (4) X is a right U -module.

In particular, $(U/R)^\perp$ formed in $\text{mod}(R)$ coincides with $\text{mod}(U)$. Also $\text{Hom}_R(U, R/R) = 0$.

Proof. (1) \Rightarrow (2) follows from the exactness of (**), while the implications (2) \Rightarrow (3) \Rightarrow (4) are obvious. Finally (4) \Rightarrow (2) follows from Theorem 4.4. ■

PROPOSITION 4.13. *The ring homomorphism $\psi: R \rightarrow V$ associated to φ is a homological epimorphism and $\text{proj dim}_R V \leq 1$. Further for any right R -module X the following conditions are equivalent:*

- (1) $X \in U^\perp$.
- (2) $\tau_X: X \rightarrow \text{Ext}_R^1(U/R, X)$ is an isomorphism and $\text{Hom}_R(U/R, X) = 0$.
- (3) $\tau_X: X \rightarrow \text{Ext}_R^1(U/R, X)$ is an isomorphism.
- (4) X is a right V -module.

In particular, the category U^\perp formed in $\text{mod}(R)$ coincides with $\text{mod}(V)$.

Proof. The equivalence (1) \Leftrightarrow (2) follows from the exactness of (**), while the implications (2) \Rightarrow (3) \Rightarrow (4) are obvious.

Next, we show that $\text{Ext}_R^p(U/R, X) \in U^\perp$ for each $X \in \text{Mod}(R)$. Because φ is a homological epimorphism, tensoring of (*) with U_R leads to the exactness of

$$0 = \text{Tor}_1^R(U, U) \rightarrow \text{Tor}_1^R(U, U/R) \rightarrow U \xrightarrow{\cong} U \otimes_R U \rightarrow U \otimes_R U/R \rightarrow 0.$$

We therefore have $\text{Tor}_p^R(U, U/R) = 0$ for all p and thus

$$\text{Ext}_R^p(U, \text{Hom}_R(U/R, Q)) = \text{Hom}_R(\text{Tor}_p^R(U, U/R), Q) = 0$$

for any injective right R -module Q . Note that for $p = 0$ the above formula holds without any restriction for Q . Embedding X into an injective R -module Q yields an exact sequence

$$0 \rightarrow \text{Hom}_R(U/R, X) \rightarrow \text{Hom}_R(U/R, Q) \rightarrow C \rightarrow 0$$

with $C \subset \text{Hom}_R(U/R, Q/X)$ and in turn the exactness of

$$0 \rightarrow \text{Hom}_R(U, C) \rightarrow \text{Ext}_R^1(U, \text{Hom}_R(U/R, X)) \rightarrow 0.$$

Since $\text{Hom}_R(U, C) \subset \text{Hom}_R(U, \text{Hom}_R(U/R, Q/X)) = 0$ we thus obtain $\text{Hom}_R(U/R, X) \in U^\perp$. Therefore also $\text{Ext}_R^p(U/R, X) \in U^\perp$ because U^\perp is an exact subcategory of $\text{Mod}(R)$. (Alternatively, the property $\text{Ext}_R^p(U/R, X) \in U^\perp$, $p \geq 0$, may be derived from Cartan and Eilenberg's associativity spectral sequence [12, p. 345].)

Since U is contained in $(U/R)^\perp$, application of $\text{Hom}_R(U/R, -)$ to sequence (*) shows that

$$V = \text{Hom}_R(U/R, U/R) \cong \text{Ext}_R^1(U/R, R)$$

as (R, V) -bimodules. In particular V , hence also $\text{mod}(V)$, is contained in U^\perp , which proves that (4) \Rightarrow (1).

Next we show that ψ is an epimorphism. By invoking $V = \text{Ext}_R^1(U/R, R)$ and the right exactness of $\text{Ext}_R^1(U/R, -)$, we obtain an equivalence of functors $\text{Ext}_R^1(U/R, -) \cong - \otimes_R V$, and so in view of the established equivalence (3) \Leftrightarrow (4), an isomorphism $V \cong \text{Ext}_R^1(U/R, V) \cong V \otimes_R V$. By noetherianness the isomorphism $V \cong V \otimes_R V$ implies that the surjective multiplication map $V \otimes_R V \rightarrow V$ is already an isomorphism, and hence $\psi: R \rightarrow V$ is an epimorphism of rings. In particular, the category $\text{mod}(V)$ may be regarded as a full subcategory of $\text{mod}(R)$, which in view of the equivalence (1) \Leftrightarrow (4) equals U^\perp .

It remains to show that ψ is also homological: By passing to the left derived functors of $\text{Ext}_R^1(U/R, -) = - \otimes_R V$ and invoking $\text{proj dim}((U/R)_R) \leq 1$ as well as $\text{Hom}_R(U/R, R) = 0$, we deduce that $\text{Hom}_R(U/R, -) \cong \text{Tor}_1^R(-, V)$ and $\text{Tor}_i^R(-, V) = 0$ for every $i \geq 2$. Due to noetherianness this proves $\text{proj dim}_R V \leq 1$. Since $V \in U^\perp$ we further have $\text{Tor}_1^R(V, V) \cong \text{Hom}_R(U/R, V) \subset \text{Hom}_R(U, V) = 0$. In view of Theorem 4.4 this proves that the epimorphism $\psi: R \rightarrow V$ is homological. ■

THEOREM 4.14. *For a commutative noetherian ring k let $\varphi: R \rightarrow U$ be an injective homological epimorphism of k -algebras, which are finitely generated k -modules. We also suppose $\text{proj dim } U_R \leq 1$. Then*

- (1) $T = U \otimes U/R$ is a tilting module;
- (2) $(U/R)^\perp = \text{mod}(U)$ and $U^\perp = \text{mod}(V)$;
- (3) the embeddings $\varphi_*: \text{mod}(U) \rightarrow \text{mod}(R)$ and $\psi_*: \text{mod}(V) \rightarrow \text{mod}(R)$ induce an isomorphism

$$K_0(\text{mod}(U)) \oplus K_0(\text{mod}(V)) \simeq K_0(\text{mod}(R)), \quad ([X], [Y]) \mapsto [X \oplus Y].$$

For the definition and the properties of tilting modules we refer to the papers of Happel and Ringel [27], Bongartz [9], and Miyashita [32].

Proof. (1) The sequence $0 \rightarrow R \rightarrow U \rightarrow U/R \rightarrow 0$ defines a T -core-resolution of R ; further $\text{proj dim } U_R \leq 1$ implies $\text{proj dim } T_R \leq 1$. It thus remains to show that T has no self-extensions.

First, $\text{Ext}_R^1(U/R, U) = 0$ because $U \in (U/R)^\perp$. Since φ is a homological epimorphism we also have $\text{Ext}_R^1(U, U) = 0$, which in turn implies that $\text{Ext}_R^1(U, U/R) = 0$ because $\text{proj dim } U_R \leq 1$. The remaining assertion $\text{Ext}_R^1(U/R, U/R) = 0$ follows by application of $\text{Ext}_R^1(U/R, -)$ to the sequence (*), observing that $\text{proj dim}(U/R)_R \leq 1$. Assertion (2) is covered by Propositions 4.12 and 4.13.

(3) Inclusion $i := \varphi_*$ is exact and hence induces a homomorphism $i_*: K_0(\text{mod}(U)) \rightarrow K_0(\text{mod}(R))$, $[X] \mapsto [X]$. Since $\text{proj dim } U_R \leq 1$ we may define a homomorphism $m_*: K_0(\text{mod}(R)) \rightarrow K_0(\text{mod}(U))$ on classes of

modules by $m_* = [\text{Hom}_R(U, -)] - [\text{Ext}_R^1(U, -)]$. Since φ is a homological epimorphism we obtain $m_* \circ i_* = 1_{K_0(\text{mod}(U))}$.

Also $j = \psi_*: \text{mod}(V) \rightarrow \text{mod}(R)$ is exact and hence induces a homomorphism $j_*: K_0(\text{mod}(V)) \rightarrow K_0(\text{mod}(R))$, $[Y] \mapsto [Y]$. Let $l = -\otimes_R V = \text{Ext}_R^1(U/R, -)$ be the left adjoint of j . In view of $\text{proj dim}(U/R)_R \leq 1$, the functor l induces a homomorphism $l_*: K_0(\text{mod}(R)) \rightarrow K_0(\text{mod}(U))$ given on classes of modules by

$$l_* = [-\otimes_R V] - [\text{Tor}_1^R(-, V)] = -[\text{Hom}_R(U/R, -)] + [\text{Ext}_R^1(U/R, -)].$$

Since also ψ is a homological epimorphism we obtain $l_* \circ j_* = 1_{K_0(\text{mod}(V))}$.

In order to prove the assertion on the K -groups it remains to show:

(a) $l_* \circ i_* = 0$, (b) $m_* \circ j_* = 0$, (c) $i_* \circ m_* + j_* \circ l_* = 1_{K_0(\text{mod}(R))}$.

(a) and (b) follow from $(U/R)^\perp = \text{mod}(U)$ and $U^\perp = \text{mod}(V)$, respectively.

From the exact sequence (**) we finally obtain that

$$\begin{aligned} [X] &= [\text{Hom}_R(U, X)] - [\text{Ext}_R^1(U, X)] - [\text{Hom}_R(U/R, X)] \\ &\quad + [\text{Ext}_R^1(U/R, X)], \end{aligned}$$

and hence $[X] = (i_* \circ m_* + j_* \circ l_*)([X])$. ■

COROLLARY 4.15. *Assume that S is a right R -module satisfying the conditions of Proposition 4.11. If $\lambda_S: K_0(R) \rightarrow \mathbf{Z}$ denotes the linear form defined by*

$$\lambda_S([M]) = \dim_{\text{End}(S)} \text{Hom}_R(S, M) - \dim_{\text{End}(S)} \text{Ext}_R^1(S, M),$$

the sequence of abelian groups

$$0 \longrightarrow K_0(S^\perp) \xrightarrow{i_*} K_0(\text{mod}(R)) \xrightarrow{\lambda_S} \mathbf{Z} \longrightarrow 0$$

is exact.

Proof. We apply Theorem 4.14 to the homological epimorphism $\varphi: R \rightarrow U$, $U = \text{End}_R(l(R))$, induced by the functor l left adjoint to the inclusion $S^\perp \subset \text{mod}(R)$. From $\text{proj dim } S_R \leq 1$ and $\text{Hom}_R(S, R) = 0$ we deduce that $U/R = S^n$ for some integer $n \geq 1$ hence $S^\perp = (U/R)^\perp$. With the above notations this yields an exact sequence

$$0 \longrightarrow K_0(S^\perp) \xrightarrow{i_*} K_0(\text{mod}(R)) \xrightarrow{l_*} K_0(\text{mod}(V)) \longrightarrow 0.$$

Since $V = \text{End}(S^n)$ is the ring of all $(n \times n)$ -matrices over the skew field $\text{End}(S)$, we may identify $K_0(\text{mod}(V))$ with \mathbf{Z} , accordingly l_* with λ_S , which proves the assertion. ■

The situation described in the foregoing frequently occurs in the study of representations of finite dimensional algebras:

THEOREM 4.16. *Let A be a finite dimensional algebra over a field k and $S \in \text{mod}(A)$ such that*

- (a) $S = S_1 \oplus \dots \oplus S_p$ with $\text{End}(S_i)$ a skew field and $S \in \text{mod}(A)$ such that $\text{Hom}(S_i, S_j) = 0$ for $i \neq j$ and $\text{Ext}^1(S_i, S_j) = 0$ for all i and j ;
- (b) $\text{proj dim } S \leq 1$;
- (c) $\text{Hom}(S, A) = 0$.

Then there exists a finite dimensional algebra A' and a homological epimorphism $\varphi: A \rightarrow A'$ which is also injective such that

- (1) $\text{mod}(A') = S^\perp$;
- (2) $T = S \oplus \varphi_*(A')$ is a tilting module in $\text{mod}(A)$;
- (3) $\text{gl dim } A' \leq \text{gl dim } A$;
- (4) *the number of (isomorphism classes of) simple A' -modules is equal to the number of (isomorphism classes of) simple A -modules $- p$. More precisely, φ_* allows us to identify $K_0(A')$ with a direct summand of $K_0(A)$ such that*

$$K_0(A) = K_0(A') \oplus (\mathbf{Z}[S_1] \oplus \dots \oplus \mathbf{Z}[S_k]).$$

5. CATEGORIES PERPENDICULAR TO PROJECTIVE OR SIMPLE MODULES

This section deals with the category right perpendicular to a system of projective or simple R -modules in $\text{Mod}(R)$ or $\text{mod}(R)$, where in the latter case we assume R to be right noetherian. The questions are linked to the study of those homological epimorphisms of rings which are surjective. In the same context we deal with Serre subcategories of $\text{mod}(A)$ for an Artin algebra A . We start with the case of projective modules.

Let \mathcal{A} be a small additive category. Then the category of all (resp. all finitely presented) right \mathcal{A} -modules is by definition the category $(\mathcal{A}^{\text{op}}, \mathbf{Ab})$ (resp. $\mathbf{fp}(\mathcal{A}^{\text{op}}, \mathbf{Ab})$) of all (resp. all finitely presented) contravariant additive functors from \mathcal{A} to the category of abelian groups and is denoted by $\text{Mod}(\mathcal{A})$ (resp. $\text{mod}(\mathcal{A})$).

For a full subcategory \mathcal{B} of \mathcal{A} , $\mathcal{A}/[\mathcal{B}]$ denotes the *factor category* of \mathcal{A} by \mathcal{B} . The category $\mathcal{A}/[\mathcal{B}]$ has the same objects as \mathcal{A} , while the morphisms of two objects A_1 and A_2 in $\mathcal{A}/[\mathcal{B}]$ are given by

$$\text{Hom}_{\mathcal{A}/[\mathcal{B}]}(A_1, A_2) = \text{Hom}_{\mathcal{A}}(A_1, A_2)/\mathcal{B}(A_1, A_2),$$

where $\mathcal{B}(A, A_2)$ is the subgroup of all morphisms from A_1 to A_2 factoring through a finite direct sum of objects of \mathcal{B} . Finally, composition is induced by the composition in \mathcal{A} . Clearly, $[\mathcal{B}]$ is an idempotent two-sided ideal of \mathcal{A} .

For a ring R and a right R -module M the *trace ideal* $\text{Tr}(M)$ of M in R is the two-sided ideal consisting of the sum of all $\text{im}(f)$ with $f \in \text{Hom}(M, R)$. For any projective module P the trace ideal is an idempotent ideal with the property $P \cdot \text{Tr}(P) = P$. If e is an idempotent in R then $\text{Tr}(eR) = Re$. If for some two-sided ideal \mathfrak{a} the natural mapping $\varphi: R \rightarrow R/\mathfrak{a}$ is a homological epimorphism, then \mathfrak{a} is idempotent. Note moreover that each idempotent two-sided ideal \mathfrak{a} which is projective as a right R -module coincides with its trace ideal.

PROPOSITION 5.1. *Let R be a ring (resp. a right noetherian ring) and P be a projective right R -module. Then P^\perp formed in $\text{Mod}(R)$ (resp. in $\text{mod}(R)$) consists of all modules X in $\text{Mod}(R)$ (resp. in $\text{mod}(R)$) which are annihilated by $\text{Tr}(P)$ and is a localizing subcategory of $\text{Mod}(R)$ (resp. $\text{mod}(R)$).*

If $U = \text{End}_R(P)$, the functor $\text{Hom}_R(P, -)$ induces equivalences

$$(P^\perp)^\perp \simeq \text{Mod}(U) \quad \text{and} \quad \text{Mod}(R)/\text{Mod}(R/\text{Tr}(P)) \simeq \text{Mod}(U)$$

as well as

$$(P^\perp)^\perp \simeq \text{mod}(U) \quad \text{and} \quad \text{mod}(R)/\text{mod}(R/\text{Tr}(P)) \simeq \text{mod}(U).$$

If moreover $\text{Tr } P$ is projective as a right R -module—for instance if P is simple projective—the natural mapping $\varphi: R \rightarrow R/\text{Tr}(P)$ is a homological epimorphism of rings.

Proof. Since $\text{Hom}_R(P, M) = 0$ holds if and only if $M \cdot \text{Tr } P = 0$, we may identify P^\perp with the category of all (resp. all finitely generated) $R/\text{Tr}(P)$ -modules. By means of this identification, the functor $M \mapsto M/M \cdot \text{Tr}(P)$ serves as a left adjoint to the inclusion $P^\perp \subset \text{mod}(R)$.

Since for any idempotent ideal \mathfrak{a} of R we have $\text{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{a}) = 0$ the last assertion is a consequence of Theorem 4.4. For the remaining assertions we refer to the next proposition. ■

In the context of functor categories a slightly different formulation for Proposition 5.1 is preferable. In terms of representation theory both propositions deal with the deletion of vertices [37]:

PROPOSITION 5.1*. *Let \mathcal{A} be a small additive category (resp. a small additive category which is right noetherian) and let \mathcal{B} be a full subcategory of \mathcal{A} which we also view as the system of all $\text{Hom}_{\mathcal{A}}(-, B)$ with B in \mathcal{B} .*

Then the category right perpendicular to \mathcal{B} is the localizing subcategory of $\text{Mod}(\mathcal{A})$ (resp. of $\text{mod}(\mathcal{A})$) consisting of all functors M which are zero on \mathcal{B} and hence may be identified with $\text{Mod}(\mathcal{A}/[\mathcal{B}])$ (resp. with $\text{mod}(\mathcal{A}/[\mathcal{B}])$). Moreover, restriction to \mathcal{B} induces equivalences

$$\text{Mod}(\mathcal{A}/[\mathcal{B}])^\perp \simeq \text{Mod}(\mathcal{B}) \quad \text{and} \quad \text{Mod}(\mathcal{A})/\text{Mod}(\mathcal{B}) \simeq \text{Mod}(\mathcal{B})$$

as well as

$$\text{mod}(\mathcal{A}/[\mathcal{B}])^\perp \simeq \text{mod}(\mathcal{B}) \quad \text{and} \quad \text{mod}(\mathcal{A})/\text{mod}(\mathcal{B}) \simeq \text{mod}(\mathcal{B}).$$

Proof. In either case the restriction functor is exact and, using Kan extension, also representative. The full embedding $\text{Mod}(\mathcal{A}/[\mathcal{B}]) \subset \text{Mod}(\mathcal{A})$ induced by the natural functor $\mathcal{A} \rightarrow \mathcal{A}/[\mathcal{B}]$ allows one to identify $\text{Mod}(\mathcal{A}/[\mathcal{B}])$ with the full subcategory of all functors vanishing on all objects of \mathcal{B} . Finally, the existence of a right adjoint to r , namely the right Kan extension, shows that r induces the claimed equivalence. ■

A Serre subcategory \mathcal{S} of an abelian category \mathcal{A} is called *colocalizing* if the quotient functor $T: \mathcal{A} \rightarrow \mathcal{A}/\mathcal{S}$ has a left adjoint. Note that \mathcal{S} is colocalizing in \mathcal{A} if and only if \mathcal{S}^{op} is localizing in \mathcal{A}^{op} .

COROLLARY 5.2. *There are equivalences*

$$(\mathcal{B}^\perp)^\perp = \text{Mod}(\mathcal{A}/[\mathcal{B}])^\perp \cong \text{Mod}(\mathcal{B}) \cong {}^\perp(\mathcal{B}^\perp) = {}^\perp\text{Mod}(\mathcal{A}/[\mathcal{B}]),$$

where the perpendicular categories are formed in $\text{Mod}(\mathcal{A})$.

If additionally \mathcal{A} is right noetherian and for each $A \in \mathcal{A}$ the left resp. right \mathcal{B} -modules $\text{Hom}_{\mathcal{A}}(A, -)|_{\mathcal{B}}$ and $\text{Hom}_{\mathcal{A}}(-, A)|_{\mathcal{B}}$ are finitely generated, then

$$(\mathcal{B}^\perp)^\perp = \text{mod}(\mathcal{A}/[\mathcal{B}])^\perp \cong \text{mod}(\mathcal{B}) \cong {}^\perp(\mathcal{B}^\perp) = {}^\perp\text{mod}(\mathcal{A}/[\mathcal{B}]),$$

with the perpendicular categories formed in $\text{mod}(\mathcal{A})$.

Note that in either case $(\mathcal{B}^\perp)^\perp$ will differ from ${}^\perp(\mathcal{B}^\perp)$.

Proof. Left and right Kan extension provide left and right adjoints to the kernel of the restriction functor $r: \text{Mod}(\mathcal{A}) \rightarrow \text{Mod}(\mathcal{B})$ (resp. $r: \text{mod}(\mathcal{A}) \rightarrow \text{mod}(\mathcal{B})$). Therefore $\text{Mod}(\mathcal{A}/[\mathcal{B}])$ (resp. $\text{mod}(\mathcal{A}/[\mathcal{B}])$) is a localizing and colocalizing subcategory of $\text{Mod}(\mathcal{A})$ (resp. $\text{mod}(\mathcal{A})$). By virtue of Proposition 2.2 both the left and the right perpendicular category of $(\text{Mod}(\mathcal{A}/[\mathcal{B}])$ (resp. $\text{mod}(\mathcal{A}/[\mathcal{B}])$) are equivalent to $\text{Mod}(\mathcal{B})$ (resp. $\text{mod}(\mathcal{B})$). ■

We finish this section with a digression on Serre subcategories of $\text{mod}(A)$ for an Artin algebra A and those homological epimorphism $\varphi: A \rightarrow A'$ which are surjective mappings.

PROPOSITION 5.3. *Let A be an Artin algebra and \mathcal{S} be a full subcategory of $\text{mod}(A)$. Then the following assertions are equivalent:*

- (i) \mathcal{S} is a Serre subcategory.
- (ii) \mathcal{S} is a Serre subcategory generated by simple modules S_1, \dots, S_k .
- (iii) \mathcal{S} is localizing and colocalizing in $\text{mod}(A)$.
- (iv) $\mathcal{S} = \text{mod}(A/\mathfrak{a})$ for some two-sided ideal \mathfrak{a} which is idempotent (resp. the trace ideal of some projective right A -module).
- (v) $\mathcal{S} = P^\perp$, where P is a (finitely generated) projective right A -module.

In this situation moreover the following properties hold true:

(a) *If P denotes the direct sum of a representative system of indecomposable projective right A -modules P^i satisfying $\text{Hom}_A(P^i, S_i) = 0$ (equivalently $P^i \mathfrak{a} = P^i$) for $i = 1, \dots, k$, then \mathcal{S}^\perp consists of all $M \in \text{mod}(A)$ such that $\text{Hom}_A(P, M) \rightarrow \text{Hom}_A(\text{rad } P, M)$ is an isomorphism.*

(b) *Let $A' = \text{End}_A(P)$; then the functor $\text{Hom}_A(P, -)$ induces equivalences*

$$\text{mod}(A)/\mathcal{S} \cong \text{mod}(A') \cong {}^\perp \mathcal{S} \cong \mathcal{S}^\perp \cong \text{mod}(\Sigma).$$

Note that in general ${}^\perp \mathcal{S}$ differs from \mathcal{S}^\perp . In terms of representation theory the passage from $\text{mod}(A)$ to $\text{mod}(A')$ is linked to the shrinking of arrows (cf. [37]).

Proof. (iii) \Rightarrow (i) is obvious.

(i) \Rightarrow (ii): Let S_1, \dots, S_k be a representative system of those simple modules which occur as a composition factor of a module in \mathcal{S} ; then, clearly, S_1, \dots, S_k generate \mathcal{S} .

(ii) \Rightarrow (v): Let S_{k+1}, \dots, S_n be the remaining simple right A -modules and denote by P_{k+1}, \dots, P_n their projective hulls. The module $P = P_{k+1} \oplus \dots \oplus P_n$ satisfies $\text{Hom}_A(P, S_i) = 0$ for $i = 1, \dots, k$ and $\text{Hom}_A(P, S_j) \neq 0$ for each $j = k+1, \dots, n$. Hence \mathcal{S} consists of all X in $\text{mod}(A)$ with the property $\text{Hom}_A(P, X) = 0$, and $\mathcal{S} = P^\perp$ follows.

(v) \Rightarrow (iv): follows from Proposition 5.1.

(iv) \Rightarrow (iii): The functor $\text{Hom}_A(P, -): \text{mod}(A) \rightarrow \text{mod}(\Sigma)$, where $\Sigma = \text{End}_A(P)$, may be viewed as the quotient functor $T: \text{mod}(A) \rightarrow \text{Mod}(A)/\mathcal{S}$. Since $- \otimes_\Sigma P: \text{mod}(\Sigma) \rightarrow \text{mod}(A)$ (resp. $\text{Hom}_\Sigma(P^*, -)$) serves as a left (resp. right) adjoint to $\text{Hom}_A(P, -) = - \otimes_\Sigma P^*$, where $P^* = \text{Hom}_A(P, A)$, we see that \mathcal{S} is colocalizing and localizing in $\text{mod}(A)$.

The remaining assertions follow from Proposition 5.1. ■

By the preceding proposition $\text{mod}(A)$ has only finitely many Serre subcategories; accordingly:

COROLLARY 5.4. *For an Artin algebra A with n non-isomorphic simple modules, there exist—up to isomorphism—at most 2^n homological surjective epimorphisms with domain A .*

For further and non-trivial examples of homological epimorphisms whose domain is an Artin algebra we refer to Sections 10 and 11.

6. PERPENDICULAR CATEGORIES UNDER TILTING

Tilting theory has been a central theme of the representation theory of finite dimensional algebras for quite a number of years (see, for instance, [11, 27, 48, 32]). The interpretation of the tilting process as providing an equivalence of the attached derived categories is due to Happel [25]. Tilting from sheaves to representations first occurred in the paper by Beilinson [7].

LEMMA 6.1. *Let \mathcal{A} be an abelian category and $(\mathcal{T}, \mathcal{F})$ be a torsion theory for \mathcal{A} . Further let $S \in \mathcal{A}$ be an object of projective dimension ≤ 1 . If, moreover, $S \in \mathcal{T}$, then $(\mathcal{T} \cap S^\perp, \mathcal{F} \cap S^\perp)$ is a torsion theory for S^\perp .*

Proof. Obviously, $(\mathcal{T} \cap S^\perp) \cap (\mathcal{F} \cap S^\perp) = 0$, $\mathcal{T} \cap S^\perp$ is closed under quotients, and $\mathcal{F} \cap S^\perp$ is closed under subobjects in S^\perp . Now, let $A \in S^\perp$. By assumption there exists an exact sequence

$$0 \rightarrow A_0 \rightarrow A \rightarrow A_1 \rightarrow 0$$

with $A_0 \in \mathcal{T}$ and $A_1 \in \mathcal{F}$. We have $\text{Ext}^1(S, A_1) = 0$ since $\text{proj dim } S \leq 1$ and $\text{Hom}(S, A_1) = 0$ since $S \in \mathcal{T}$. Thus $A_1 \in \mathcal{F} \cap S^\perp$ and $A_0 \in \mathcal{T} \cap S^\perp$ follows. ■

Let \mathcal{A} be a small noetherian category; i.e., we assume that the isomorphism classes of objects from \mathcal{A} form a set and, moreover, all objects are noetherian. An object $T \in \mathcal{A}$ is called a *tilting object* in \mathcal{A} if

- (1) $\text{proj dim } T < \infty$,
- (2) $\text{Ext}^i(T, T) = 0$ for all $i > 0$,
- (3) T generates $D^b(\mathcal{A})$ as a triangulated category, i.e., $D^b(\mathcal{A})$ is the smallest triangulated subcategory of $D^b(\mathcal{A})$ containing all direct factors of finite direct sums of T , and
- (4) $A = \text{End}(T)$ is a right noetherian ring of finite global dimension.

Let T be a tilting object and let F and G denote the functors $(\text{Hom}(T, -): \mathcal{A} \rightarrow \text{mod}(A))$ and $(-\otimes_A T: \text{mod}(A) \rightarrow \mathcal{A})$, respectively. The functors F and G induce derived functors (see [28, 50])

$$RF: D^b(\mathcal{A}) \rightarrow D^b(\text{mod}(A))$$

and

$$LG: D^b(\text{mod}(A)) \rightarrow D^b(\mathcal{A})$$

which are equivalences mutually inverse to each other (see, for instance, [5]).

If, generally, $H^j: D^b(\mathcal{C}) \rightarrow \mathcal{C}$ denotes the j th homology functor, then

$$H^j RF = R^j F = \text{Ext}^j(T, -),$$

and

$$H^{-j} LG = L_j G = \text{Tor}_j(-, T).$$

If $\mathcal{X}_i = \{A \in \mathcal{A} \mid R^j F(A) = 0 \text{ for } i \neq j\}$ and $\mathcal{Y}_i = \{M \in \text{mod}(A) \mid L_j G(M) = 0 \text{ for } i \neq j\}$, the functors RF and LG induce equivalences

$$R^i F = \text{Ext}^i(T, -): \mathcal{X}_i \rightarrow \mathcal{Y}_i$$

and

$$L_i G = \text{Tor}_i(-, T): \mathcal{Y}_i \rightarrow \mathcal{X}_i$$

mutually inverse to each other for all $i \geq 0$.

Moreover, we have the formula

$$\text{Ext}'_A(\text{Ext}^i_{\mathcal{A}}(T, A_i), \text{Ext}^j_{\mathcal{A}}(T, A_j)) = \text{Ext}^{l-i+j}_{\mathcal{A}}(A_i, A_j)$$

for all i, j, l and all $A_i \in \mathcal{X}_i$ and $A_j \in \mathcal{X}_j$ (compare [21]).

If T has projective dimension ≤ 1 the categories \mathcal{X}_i and \mathcal{Y}_i are zero for $i \geq 2$. Further $R^i F \circ L_j G$ and $L_j G \circ R^i F$ are zero for $i \neq j$, and $(\mathcal{X}_0, \mathcal{X}_1)$ and $(\mathcal{Y}_1, \mathcal{Y}_0)$ are torsion theories for \mathcal{A} and $\text{mod}(A)$, respectively. If additionally $\text{gl dim } \mathcal{A} \leq 1$, the torsion theory $(\mathcal{Y}_1, \mathcal{Y}_0)$ is splitting, i.e., each Y in $\text{mod}(A)$ has the form $Y = Y_0 \oplus Y_1$ with $Y_0 \in \mathcal{Y}_0$ and $Y_1 \in \mathcal{Y}_1$.

THEOREM 6.2. *Let T be a tilting object of projective dimension ≤ 1 in a small noetherian category \mathcal{A} , let $\Lambda = \text{End}(T)$, and let $S \in \mathcal{A}$ be an object of projective dimension ≤ 1 . If S is contained in \mathcal{X}_0 , the functors RF and LG induce equivalences*

$$R^*F: D_{S^\perp}^b(\mathcal{A}) \rightarrow D_{F(S)^\perp}^b(\text{mod}(A))$$

and

$$L_* G: D_{F(S)^\perp}^b(\text{mod}(A)) \rightarrow D_{S^\perp}^b(\mathcal{A})$$

mutually inverse to each other.

Here, $D_{S^\perp}^b(\mathcal{A})$ (resp. $D_{F(S)^\perp}^b(\text{mod}(A))$) denotes the full triangulated subcategory of $D^b(\mathcal{A})$ (resp. $D^b(\text{mod}(A))$) of all complexes with cohomology in S^\perp (resp. $F(S)^\perp$).

Proof. (1) Let M_j be a A -module contained in \mathcal{Y}_i , hence of the form $\text{Ext}_{\mathcal{A}}^j(T, A_j)$ with $A_j \in \mathcal{X}_j$. Then

$$\text{Ext}_{\mathcal{A}}^l(\text{Hom}(T, S), \text{Ext}_{\mathcal{A}}^j(T, A_j)) = \text{Ext}_{\mathcal{A}}^{l+j}(S, A_j) = 0$$

for all $l \geq 2$. Since $(\mathcal{Y}_1, \mathcal{Y}_0)$ is a torsion theory for $\text{mod}(A)$, every A -module is an extension of a module in \mathcal{Y}_1 by a module in \mathcal{Y}_0 . This proves that $\text{Hom}(T, S)$ has projective dimension at most 1. In particular, $\text{Hom}(T, S)^\perp$ is a full exact subcategory of $\text{mod}(A)$ which is closed under extensions.

(2) Let A_j be an object contained in $\mathcal{X}_j \cap S^\perp$. Then the formula in (1) shows that $\text{Ext}_{\mathcal{A}}^j(T, A_j)$ is contained in $\text{Hom}(T, S)^\perp$. Now, let $A \in S^\perp$ be an arbitrary object. By Lemma 6.1 there is an exact sequence $0 \rightarrow A_0 \rightarrow A \rightarrow A_1 \rightarrow 0$ with $A_j \in \mathcal{X}_j \cap S^\perp$ for $j=0, 1$. Thus $\text{Hom}(T, A) \cong \text{Hom}(T, A_0)$ and $\text{Ext}_{\mathcal{A}}^1(T, A) \cong \text{Ext}_{\mathcal{A}}^1(T, A_1)$ are contained in $\text{Hom}(T, S)^\perp$.

(3) Consider the sequence

$$T_0 \xrightarrow{\alpha_0} T_1 \xrightarrow{\alpha_1} T_2$$

with $\alpha_1 \circ \alpha_0$ and where all T_i are direct factors of finite direct sums of T . We denote by K_i and I_i the kernel and the image of α_i , respectively. We further assume that the cohomology $H = K_1/I_0$ is contained in S^\perp . We have the exact sequences

$$0 \rightarrow K_0 \rightarrow T_0 \rightarrow I_0 \rightarrow 0$$

and

$$0 \rightarrow I_0 \rightarrow K_1 \rightarrow H \rightarrow 0.$$

Application of the functor $\text{Hom}(T, -)$ yields the exactness of

$$0 \rightarrow \text{Hom}(T, K_0) \rightarrow \text{Hom}(T, T_0) \rightarrow \text{Hom}(T, I_0) \rightarrow \text{Ext}^1(T, K_0) \rightarrow 0$$

and

$$\begin{aligned} 0 \rightarrow \text{Hom}(T, I_0) \rightarrow \text{Hom}(T, K_1) \rightarrow \text{Hom}(T, H) \\ \rightarrow 0 \rightarrow \text{Ext}^1(T, K_1) \rightarrow \text{Ext}^1(T, H) \rightarrow 0. \end{aligned}$$

Note that the condition $\text{Ext}^1(T, T) = 0$ implies $\text{Ext}^1(T, I_i) = 0$.

If I denotes the image of the map $\text{Hom}(T, T_0) \rightarrow \text{Hom}(T, T_1)$, we obtain the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 0 & \longrightarrow & I & \longrightarrow & \text{Hom}(T, K_1) & \longrightarrow & \bar{H} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & \text{Hom}(T, I_0) & \longrightarrow & \text{Hom}(T, K_1) & \longrightarrow & \text{Hom}(T, H) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \text{Ext}^1(T, K_0) & & 0 & & 0 & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
 \end{array}$$

We conclude that $\text{Ext}^1(T, K_1) \cong \text{Ext}^1(T, H)$ is contained in $\text{Hom}(T, S)^\perp$ and that for the cohomology $\bar{H} = \text{Hom}(T, K_1)/I$ we have an exact sequence $0 \rightarrow \text{Ext}^1(T, K_0) \rightarrow \bar{H} \rightarrow \text{Hom}(T, H) \rightarrow 0$

(4) Let X^\bullet be a complex in $D_{S^\perp}^b(\mathcal{A})$. Since T generates $D^b(\mathcal{A})$, X^\bullet is isomorphic to a complex T^\bullet of objects which are direct factors of finite direct sums of T . Then by (3), $R \text{Hom}(T, -)(T^\bullet) = \text{Hom}(T, T^\bullet)$ is contained in $D_{\text{Hom}(T, S)^\perp}^b(\text{mod}(A))$.

(5) Let M_j be a A -module contained in $\mathcal{Y}_j \cap \text{Hom}(T, S)^\perp$. Then we have that

$$\begin{aligned}
 \text{Ext}^l(S, \text{Tor}_j(T, M_j)) &\cong \text{Ext}^{l-j}(\text{Hom}(T, S), \text{Ext}^j(T, \text{Tor}_j(T, M_j))) \\
 &\cong \text{Ext}^{l-j}(\text{Hom}(T, S), M_j) = 0;
 \end{aligned}$$

hence $\text{Tor}_j(T, M_j)$ is contained in S^\perp . Now, let M be an arbitrary module in S^\perp . Then there is an exact sequence $0 \rightarrow M_1 \rightarrow M \rightarrow M_0 \rightarrow 0$ with $M_i \in \mathcal{Y}_i$. Obviously, $\text{Hom}(\text{Hom}(T, S), M_1) = 0$. Further,

$$\text{Ext}^1(\text{Hom}(T, S), M_1) \cong \text{Ext}^2(S, \text{Tor}_1(T, M_1)) = 0$$

since $\text{proj dim } S \leq 1$. Hence M_1 and M_0 are contained in $\text{Hom}(T, S)^\perp$. This shows that $M \otimes T \cong M_0 \otimes T$ and $\text{Tor}_1(M, T) \cong \text{Tor}_1(M_1, T)$ are contained in S^\perp .

(6) Any complex in $D^b(\text{mod}(A))$ is isomorphic to a complex consisting of finitely generated projective A -modules. By a proof dual to (3), the functor $L_* G$ maps such complexes with cohomology in $\text{Hom}(T, S)^\perp$ to complexes with cohomology in S^\perp .

We now proved that the functors R^*F and L_*G are properly defined. Since RF and LG are equivalences mutually inverse to each other the same holds true for the functors R^*F and L_*G . ■

COROLLARY 6.3. *Under the assumptions of Theorem 6.2 the following assertions hold true:*

- (1) $\text{proj dim } F(S) \leq 1$. In particular, $F(S)^\perp$ is an exact subcategory of $\text{mod}(A)$ closed under extensions.
- (2) $(\mathcal{Y}_1 \cap F(S)^\perp, \mathcal{Y}_0 \cap F(S)^\perp)$ is a torsion theory for $F(S)^\perp$.
- (3) The functors

$$\text{Ext}^i(T, -): \mathcal{X}_i \cap S^\perp \rightarrow \mathcal{Y}_i \cap F(S)^\perp$$

and

$$\text{Tor}_i(-, T): \mathcal{Y}_i \cap F(S)^\perp \rightarrow \mathcal{X}_i \cap S^\perp$$

are equivalences of categories mutually inverse to each other.

- (4) If S^\perp is contained in \mathcal{X}_i ($i=0$ or $i=1$), then

$$\text{Ext}^i(T, -): S^\perp \rightarrow F(S)^\perp$$

is an equivalence of categories.

COROLLARY 6.4. *Let T be a tilting object of projective dimension ≤ 1 in a small noetherian category \mathcal{A} . Suppose that $T \cong S \oplus T'$ with $\text{Hom}(S, T') = 0$ and the property that $\text{Ext}^1(S, A) = 0$ implies $\text{Ext}^1(T, A) = 0$ for all $A \in \mathcal{A}$. Then $S^\perp \cong \text{mod}(\text{End}(T'))$.*

Proof. If A is an object of S^\perp we have $\text{Ext}^1(T, A) = 0$, and hence $A \in \mathcal{X}_0$. By Corollary 6.3(4) the functor $\text{Hom}(T, -): S^\perp \rightarrow \text{Hom}(T, S)^\perp$ is an equivalence of categories. Since $\text{Hom}(T, S)$ is a projective $\text{End}(T)$ -module, $\text{Hom}(T, S)^\perp \cong \text{mod}(A')$, where $A' = \text{End}(T)/\text{Tr}(\text{Hom}(T, S))$. Since $\text{Hom}(\text{Hom}(T, S), \text{Hom}(T, T')) = \text{Hom}(S, T') = 0$ we get $A' = \text{End}(T')$. ■

PROPOSITION 6.5. *Let \mathcal{A} be a small noetherian category, T be a tilting object in \mathcal{A} of projective dimension ≤ 1 , and $A = \text{End}(T)$. Further let $S \in \mathcal{A}$ be an object satisfying the following properties:*

- (a) $S \in \mathcal{X}_0$.
- (b) $\text{proj dim}_{\mathcal{A}} S \leq 1$.
- (c) $\text{Ext}^1(S, S) = 0$
- (d) For all $A \in \mathcal{A}$ the $\text{End}(S)$ -module $\text{Hom}(S, A)$ and $\text{Ext}^1(S, A)$ are of finite length.

- (e) $\text{End}(S)$ is a skew field.
- (f) $\text{Hom}(S, T) = 0$.

Moreover, let $l: \mathcal{A} \rightarrow S^\perp$ be the left adjoint to the embedding $j: S^\perp \rightarrow \mathcal{A}$. If $\text{proj dim}_{S^\perp} lT \leq 1$ then lT is a tilting object in S^\perp .

Proof. Since $\text{Hom}(S, T) = 0$, there is an exact sequence

$$0 \rightarrow T \rightarrow lT \rightarrow S^n \rightarrow 0$$

(Proposition 3.5), and hence lT is contained in \mathcal{X}_0 . In particular, we have that $\text{Ext}^1(T, lT) = 0$ and $\text{Ext}^1(lT, lT) = 0$ follows.

By Corollary 6.3 properties (a)–(e) of the assumptions hold respectively for $\text{Hom}(T, S)$ in $\text{mod}(A)$.

In particular, the embedding $j': \text{Hom}(T, S)^\perp \rightarrow \text{mod}(A)$ has a left adjoint $l': \text{mod}(A) \rightarrow \text{Hom}(T, S)^\perp$. Property (f) translates to the property $\text{Hom}(\text{Hom}(T, S), A) = 0$. Hence there is an exact sequence

$$0 \rightarrow A \rightarrow l'A \rightarrow \text{Hom}(T, S)^n \rightarrow 0$$

and $\text{proj dim}_A l'A \leq 1$ follows. Moreover, $\text{Hom}(T, lT) \cong l'A$ and by Corollary 3.9, Corollary 4.10, and Theorem 4.4 we have that $\text{End}(lT) \cong \text{End}(l'A)$ is right noetherian of finite global dimension.

It remains to prove that lT generates $D^b(S^\perp)$. For this it is sufficient to show that S^\perp is the smallest subcategory \mathcal{C} of S^\perp containing all direct factors of finite direct sums of lT and which is closed under kernels of epimorphisms, cokernels of monomorphisms, and extensions.

Suppose first that X is contained in $S^\perp \cap \mathcal{X}_0$. Then $M = \text{Hom}(T, X) \in \mathcal{Y}_0$. Since $\text{End}(l'A)$ is of finite global dimension the module M has a finite projective resolution

$$0 \rightarrow P_n \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$$

in $\text{Hom}(T, S)^\perp$. Since all P_i are also contained in \mathcal{Y}_0 , application of $-\otimes_A T$ yields an exact sequence

$$0 \rightarrow lT_n \rightarrow \dots \rightarrow lT_1 \rightarrow lT_0 \rightarrow X \rightarrow 0,$$

where all lT_i are direct factors of finite direct sums of lT . Hence $X \in \mathcal{C}$.

Now, let X be in $S^\perp \cap \mathcal{X}_1$, $M = \text{Ext}^1(T, X)$, and

$$0 \rightarrow K \rightarrow P \rightarrow M \rightarrow 0$$

be exact, where P is projective in $\text{Hom}(T, S)^\perp$. Then K is contained in \mathcal{U}_0 and application of $-\otimes_A T$ yields the exact sequence

$$0 \rightarrow X \rightarrow K \otimes T \rightarrow P \otimes T \rightarrow 0;$$

hence X is contained in \mathcal{C} .

Now Lemma 6.1 implies $\mathcal{C} = S^\perp$. This finishes the proof of the proposition. ■

Remark 6.6. We note that the exact sequence

$$0 \rightarrow T \rightarrow IT \rightarrow S^n \rightarrow 0$$

in the proof of Proposition 6.5 implies $\text{proj dim}_{\mathcal{A}} IT \leq 1$. If $\mathcal{A} = \text{mod}(R)$ is the category of finitely presented modules over a right noetherian ring R this yields $\text{proj dim}_{S^\perp} IT \leq 1$ by Theorem 4.4 since $A = \text{Hom}(T, T) \rightarrow \text{Hom}(IT, IT)$ is a homological epimorphism. Hence, in this case the additional assumption $\text{proj dim}_{S^\perp} IT \leq 1$ is superfluous.

Also if S is a simple sheaf on a weighted projective line (see Section 9) we know that S^\perp has global dimension ≤ 1 . Hence again the additional assumption $\text{proj dim}_{S^\perp} IT \leq 1$ is superfluous.

Remark 6.7. We recall that a Grothendieck category \mathcal{A} is locally noetherian if \mathcal{A} has a set of generators consisting of noetherian objects.

In the case where \mathcal{A} is a locally noetherian category a noetherian object $\bar{T} \in \mathcal{A}$ is called a tilting object in \mathcal{A} if the following conditions are satisfied:

- (1) $\text{proj dim } \bar{T} < \infty$,
- (2) $\text{Ext}^i(\bar{T}, \bar{T}) = 0$ for all $i > 0$,
- (3) the class of all (possibly infinite) direct sums of copies of \bar{T} generates $D^b(\mathcal{A})$ as a triangulated category,
- (4) $\bar{A} = \text{End}(\bar{T})$ is a ring of finite global dimension.

Similar to the noetherian case, where a tilting object induces an equivalence $D^b(\mathcal{A}) \cong D^b(\text{mod}(A))$, a tilting object \bar{T} in a locally noetherian category \mathcal{A} induces an equivalence

$$D^b(\mathcal{A}) \rightarrow D^b(\text{Mod}(\bar{A})),$$

$\bar{A} = \text{End}(\bar{T})$. With the obvious modifications, all the above assertions remain valid in this modified context.

7. SHEAFIFICATION

In this section we study the passage from graded modules to coherent sheaves over projective varieties or schemes.

Let H be an abelian group and R be a commutative H -graded ring. Thus, R has a decomposition

$$R = \bigoplus_{h \in H} R_h$$

with $1 \in R_0$ and $R_h \cdot R_l \subset R_{h+l}$ for all $h, l \in H$. An H -graded R -module M is an R -module with a decomposition $M = \bigoplus_{l \in H} M_l$, where we assume $R_h \cdot M_l \subset M_{h+l}$. If M is an H -graded R -module we denote by $M(h) = \bigoplus_{l \in H} N_l$ its h -shift, where $N_l = M_{l+h}$.

In the context of graded modules, $\text{Hom}_R(M, N)$ always means the set of all homomorphisms of graded modules of degree zero; also the notion of isomorphism of graded modules always refers to degree zero maps. By $\text{Mod}^H(R)$ and $\text{mod}^H(R)$ we denote the categories of all H -graded and of all finitely presented H -graded R -modules, respectively.

Additionally we consider the H -graded homomorphism groups $\text{HOM}_R(M, N)$ defined by

$$\text{HOM}_R(M, N) := \bigoplus_{h \in H} \text{Hom}_R(M, N(h)).$$

$\text{HOM}_R(M, N)$ is again an H -graded R -module and obviously $\text{HOM}_R(R, N) \cong N$.

LEMMA 7.1. *Let H be an abelian group and R be a commutative noetherian H -graded ring. Further let r_1, \dots, r_n be an R -sequence of homogeneous elements, $h_i = \text{deg } r_i$, $E = R/(r_1, \dots, r_n)$, $n \geq 2$, and M be a finitely generated graded R -module.*

(1) $\text{Ext}^i(E, R(h)) \neq 0$ if and only if $i = n$ and $E_{h+\sum h_i} \neq 0$.

(2) If additionally $E_h \neq 0$ only for finitely many $h \in H$, then $\text{Ext}^1(E(h), M) \neq 0$ only for finitely many $h \in H$.

Proof. Let

$$K_\bullet : 0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \dots \rightarrow P_1 \rightarrow P_0 \rightarrow 0$$

be the Koszul complex induced by (r_1, \dots, r_n) . Thus $P_k = \bigwedge^k R^n$ is the k th exterior power of R^n and if e_1, \dots, e_n is a basis of R^n , P_k has a basis consisting of the system $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid 1 \leq i_1 < \dots < i_k \leq n\}$. By setting $\text{deg}(e_{i_1} \wedge \dots \wedge e_{i_k}) = h_{i_1} + \dots + h_{i_k}$, P_k becomes a graded R -module, $P_k = \bigoplus_{i_k < \dots < i_1} R(-h_{i_1} - \dots - h_{i_k})$, and the boundary maps are homomorphisms of graded modules. Since (r_1, \dots, r_n) is an R -sequence the Koszul complex defines a projective resolution of E viewed as a graded module.

Since $\text{HOM}_R(K_\bullet, R) \cong K_\bullet(\sum h_i)$ is isomorphic to the $\sum h_i$ -shift of the Koszul complex K_\bullet (with the i th part of $\text{HOM}_R(K_\bullet, R)$ corresponding to the $(n - i)$ th part of $K_\bullet(\sum h_i)$) we have

$$\text{Ext}_R^i(E, R(h)) = H^{n-i}K_\bullet\left(\sum h_i\right)_h.$$

The fact that (r_1, \dots, r_n) is an R -sequence implies $H^{n-i}K_\bullet(\sum h_i) = 0$ for $i \neq n$ and $H^0K_\bullet(\sum h_i) = E(\sum h_i)$. Hence (1) follows.

In order to prove (2) let $M \in \text{mod}^H(R)$ and $Q_n \rightarrow Q_{n-1} \rightarrow \dots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0$ be a graded projective resolution of M by finitely generated projective modules. Further let K be the kernel of the map $Q_{n-2} \rightarrow Q_{n-3}$. Suppose $\text{Ext}_R^1(E(h), M) \neq 0$. Since $\text{Ext}_R^i(E(h), Q_j) = 0$ for all $i \neq n$ we obtain $\text{Ext}_R^n(E(h), K) \neq 0$ by dimension shift. Then $\text{Ext}_R^n(E(h), Q_{n-1}) \neq 0$ because $\text{Ext}_R^n(E(h), -)$ is right exact. By assumption this holds true only for finitely many $h \in H$. ■

Let H be an abelian group. R is called *graded local* if R has a unique maximal graded ideal. An H -graded local commutative noetherian ring R is called *graded Cohen-Macaulay* if there exists a regular sequence $f_1, \dots, f_n \in R$ of homogeneous elements such that the R -module $R/(f_1, \dots, f_n)$ is of finite graded length. The natural number n occurring equals the (graded Krull) dimension of R .

In the following we assume that H is an ordered group with the additional property that for every positive element h there exist only finitely many positive elements h' with $h' \leq h$.

An H -graded ring R (resp. H -graded R -module M) is called *positively graded* if $R_h \neq 0$ (resp. $M_h \neq 0$) only if $h \geq 0$. We denote by $\text{Mod}^{H^+}(R)$ (resp. $\text{mod}^{H^+}(R)$) the category of all (resp. all finitely presented) positively graded R -modules.

PROPOSITION 7.2. *Let R be a positively H -graded Cohen-Macaulay ring of dimension $n \geq 2$ with R_0 a field.*

Then the category $\text{mod}_0^{H^+}(R)$ of all positively H -graded R -modules of finite length is a localizing subcategory in $\text{mod}^{H^+}(R)$.

Proof. Since R_0 is a field, R is graded local with unique maximal graded ideal $\mathfrak{m} = \bigoplus_{h>0} R_h$. Let $S[h] = (R/\mathfrak{m})(-h)$ be the simple graded module concentrated in $h \geq 0$. We show that the system $\{S[h] \mid h \geq 0\}$ satisfies conditions (a)–(e) of Theorem 3.6.

Obviously $\text{End}(S[h])$ is a field and $\text{Hom}(S[h], S[h']) = 0$ for all $h \neq h'$. Hence, conditions (a) and (c) are satisfied.

By applying the functor $\text{Hom}(-, S[h'])$ to $0 \rightarrow K \rightarrow R(-h) \rightarrow S[h] \rightarrow 0$, we obtain an epimorphism $\text{Hom}(K, S[h']) \rightarrow \text{Ext}^1(S[h], S[h'])$. Hence,

$\text{Ext}^1(S[h], S[h']) \neq 0$ implies $\text{Hom}(K, S[h']) \neq 0$. Let $f: K \rightarrow S[h']$ be a non-zero homomorphism which is necessarily an epimorphism. Thus the natural homomorphism $R(-h') \rightarrow S[h']$ can be lifted to K and we obtain a non-zero morphism $R(-h') \rightarrow S[h']$ can be lifted to K and we obtain a non-zero morphism $R(-h') \rightarrow R(-h)$ which is not an isomorphism. Since R is positively graded, this yields $h < h'$. This proves conditions (b) and (d).

Let f_1, \dots, f_n be a regular sequence of homogeneous elements such that $E[h] = R/(f_1, \dots, f_n)(-h)$ has finite length. Since each homogeneous component of $E[h]$ is finite dimensional over $R_0 \cong \text{End}(S[h])$, the argument of Lemma 7.1 shows that for all modules $M \in \text{mod}^{H^+}(R)$ the module $\text{Ext}^1(E[h], M)$ has finite length over $\text{End}(S[h])$ and is non-zero only for finitely many $h \in H_+$. By means of the exact sequence

$$0 \rightarrow K[h] \rightarrow E[h] \rightarrow S[h] \rightarrow 0$$

the same assertion holds true for $\text{Ext}^1(S[h], M)$ for all those modules $M \in \text{mod}^{H^+}(R)$ such that $\text{Hom}(S[h], M) = 0$ for all $h \in H_+$. ■

Let (H, \leq) be a finitely generated ordered abelian group and R a positively H -graded local Cohen–Macaulay ring with R_0 a field. The corresponding *projective scheme* \mathbf{X} is the set of all homogeneous prime ideals \mathfrak{p} strictly contained in $\mathfrak{m} := \bigoplus_{h>0} R_h$. For $f \in \mathfrak{m}$ homogeneous we define $D(f) := \{\mathfrak{p} \in \mathbf{X} \mid f \notin \mathfrak{p}\}$. The sets $D(f)$, $f \in \mathfrak{m}$, form a basis of open sets (called *principal open sets*) for a topology on \mathbf{X} , called the *Zariski topology*. We note that \mathbf{X} has an interpretation as the orbit space of the action of the diagonalizable algebraic group $\mathbf{Spec}(\mathbf{Z}[H])$ on the affine spectrum $\mathbf{Spec}(R)$ of R . Here $\mathbf{Z}[H]$ denotes the group algebra of H with its natural structure as an Hopf algebra. For the following discussion we refer the reader mainly to [21], where a similar but more restricted context is assumed. Basically, everything that follows is modeled after Serre's treatment of projective varieties in [42]; see also [24] with the only distinction being that here we use graded localization in order to define the structure sheaf as well as coherent and quasicohherent sheaves, whereas the traditional treatment avoids the grading by passing directly to the zero component of the graded case. (Note that in the case where one deals with a \mathbf{Z} -graded affine k -algebra which is generated by homogeneous elements of *degree one*, both the graded and the non-graded theory lead to equivalent categories of coherent sheaves.) The *graded* sheaf theory also marks the essential difference from the treatment of weighted projective varieties in [13, 15, 8].

We define on \mathbf{X} a *graded structure sheaf* $\mathcal{O}_{\mathbf{X}}$ setting $\mathcal{O}_{\mathbf{X}}(D(f)) = R_f$ on principal open sets $D(f)$. Note that R_f is again H -graded, and thus $\mathcal{O}_{\mathbf{X}}$ is a sheaf of H -graded algebras. Similarly an $\mathcal{O}_{\mathbf{X}}$ -*module* is a sheaf of H -graded $\mathcal{O}_{\mathbf{X}}$ -modules. The category of all $\mathcal{O}_{\mathbf{X}}$ -modules is denoted by $\text{Mod}(\mathcal{O}_{\mathbf{X}})$. The

full subcategories of *quasi-coherent* and *coherent* (graded) \mathcal{O}_X -modules are denoted by $\text{Qcoh}(X)$ and $\text{coh}(X)$, respectively.

Let M be a graded R -module. Then M induces an \mathcal{O}_X -module \tilde{M} defined by $\tilde{M}(D(f)) = M_f$ on the principal open sets $D(f)$, $f \in R_+$. \tilde{M} is a quasi-coherent sheaf and in the case where M is finitely generated, \tilde{M} is coherent. For a proof of the following theorem modeled after Serre [42] we refer to [21].

THEOREM 7.3 (Serre). *The sheafification functor*

$$\sim : \text{Mod}^H(R) \rightarrow \text{Qcoh}(X), \quad M \mapsto \tilde{M},$$

is exact and representative and induces an equivalence of categories

$$\text{Mod}^H(R)/\text{Mod}_0^H(R) \simeq \text{Qcoh}(X),$$

where $\text{Mod}_0^H(R)$ is the localizing subcategory of $\text{Mod}^H(R)$ generated by all simple graded modules.

The restriction $\sim : \text{mod}^H(R) \rightarrow \text{coh}(X)$ induces an equivalence

$$\text{mod}^H(R)/\text{mod}_0^H(R) \simeq \text{coh}(X),$$

where $\text{mod}_0^H(R)$ is the Serre subcategory of $\text{mod}^H(R)$ generated by all simple graded modules.

Since $\text{Mod}_0^H(R)$ is a localizing subcategory we have a section functor $\Gamma_H : \text{Qcoh}(X) \rightarrow \text{Mod}^H(R)$ given by $\Gamma_H(\mathcal{F}) = \bigoplus_{h \in H} \text{Hom}_X(\mathcal{O}_X, \mathcal{F}(h))$. In general, for a coherent sheaf \mathcal{F} , $\Gamma_H(\mathcal{F})$ will not be a finitely generated R -module. In particular $\text{mod}_0^H(R)$ usually is not a localizing subcategory of $\text{mod}^H(R)$.

Remark 7.4. Suppose H has rank 1 and let $\text{res} : \text{mod}^H(R) \rightarrow \text{mod}^{H^+}(R)$ be the restriction functor given by $\text{res}(M) = \bigoplus_{h \in H_+} M_h$. Since M is of finite length if and only if $\text{res}(M)$ is of finite length, res induces an equivalence

$$\text{mod}^H(R)/\text{mod}_0^H(R) \simeq \text{mod}^{H^+}(R)/\text{mod}_0^{H^+}(R).$$

Thus Serre's theorem also holds true for the sheafification functor $\sim : \text{mod}^{H^+}(R) \rightarrow \text{coh}(X)$.

In view of the above remark Proposition 7.2 yields the following in the case of a grading group of rank 1:

THEOREM 7.5. *Let (H, \leq) be a finitely generated ordered abelian group of rank 1 and R be a positively H -graded local Cohen–Macaulay ring of dimension ≥ 2 with R_0 a field.*

Then $\text{mod}_0^{H_+}(R)$ is a localizing subcategory of $\text{mod}^{H_+}(R)$. In particular, for any coherent sheaf \mathcal{F} the module

$$\Gamma_+(\mathcal{F}) = \bigoplus_{h \in H_+} \text{Hom}(\mathcal{O}, \mathcal{F}(h))$$

is finitely generated over R . Moreover the section functor

$$\Gamma_+ : \text{coh}(\mathbf{X}) \rightarrow \text{mod}^{H_+}(R), \quad \mathcal{F} \mapsto \Gamma_+(\mathcal{F}),$$

induces an equivalence of $\text{coh}(\mathbf{X})$ with the perpendicular category $\text{mod}_0^{H_+}(R)^\perp$ viewed as a full subcategory of $\text{mod}^{H_+}(R)$.

COROLLARY 7.6. *Under the assumptions of Theorem 7.5, the module $\Gamma_H(\mathcal{F})$ is contained in $\text{mod}^H(R)$ and therefore in the perpendicular category $\text{mod}_0^H(R)^\perp$ formed in $\text{mod}^H(R)$ if and only if there exists $h_0 \in H$ such that $\Gamma_H(\mathcal{F})_h \neq 0$ implies $h \geq h_0$.*

Proof. The condition is obviously necessary. To prove sufficiency we may assume that $h_0 = 0$. Thus $\Gamma_H(\mathcal{F}) = \Gamma_+(\mathcal{F})$ is a finitely generated R -module by Theorem 7.5. ■

COROLLARY 7.7. *Let H be a finitely generated abelian group of rank 1 and $R = k[X_1, \dots, X_n]$ be the polynomial algebra in $n \geq 2$ indeterminates endowed with an H -grading such that all X_i are homogeneous of strictly positive degree. Then $\text{mod}_0^{H_+}(R)$ is a localizing subcategory of $\text{mod}^{H_+}(R)$.*

Let $\mathbf{p} = (p_0, \dots, p_n)$ be a sequence of non-zero natural numbers and let $L(\mathbf{p})$ be the abelian group with generators $\bar{x}_0, \dots, \bar{x}_n$ and relations $p_0 \bar{x}_0 = \dots = p_n \bar{x}_n := \bar{c}$. The string group $L(\mathbf{p})$ is an abelian group of rank 1 and is ordered by defining $\sum_{i=0}^n \mathbf{N} \bar{x}_i$ as the set of its positive elements. Further let $\underline{\lambda} = (\lambda_0, \dots, \lambda_n)$ be a sequence of pairwise distinct elements of $\mathbf{P}_1(k)$ normalized such that $\lambda_0 = \infty$, $\lambda_1 = 0$, and $\lambda_i = 1$, and consider the k -algebra

$$R(\mathbf{p}, \underline{\lambda}) = k[X_0, \dots, X_n] / (X_i^{p_i} - X_1^{p_i} + \lambda_i X_0^{p_0}, i = 2, \dots, n)$$

$L(\mathbf{p})$ -graded by virtue of $\text{deg } X_i = \bar{x}_i$. $R(\mathbf{p}, \underline{\lambda})$ is called the string singularity of type $(\mathbf{p}, \underline{\lambda})$. The projective scheme $\mathbf{C}(\mathbf{p}, \underline{\lambda})$ corresponding to $R(\mathbf{p}, \underline{\lambda})$ is one dimensional and called a weighted projective line of weight \mathbf{p} ; see [21].

COROLLARY 7.8. $\text{mod}_0^{L(\mathbf{p})_+}(R(\mathbf{p}, \underline{\lambda}))$ is a localizing subcategory of the category $\text{mod}^{L(\mathbf{p})_+}(R(\mathbf{p}, \underline{\lambda}))$.

Proof. (X_0, X_1) is a homogeneous $R(\mathbf{p}, \underline{\lambda})$ -sequence and $R(\mathbf{p}, \underline{\lambda}) / (X_0, X_1)$ is of finite length. Hence $R(\mathbf{p}, \underline{\lambda})$ is an $L(\mathbf{p})$ -graded Cohen-Macaulay algebra of dimension 2. ■

8. VECTOR BUNDLES AND COHEN–MACAULAY MODULES

We assume that H is a finitely generated abelian group of rank 1 and R is a positively H -graded local ring of dimension d , where dimension refers to the (graded) Krull dimension, defined in terms of chains of homogeneous prime ideals. We recall that the *graded depth* of a finitely generated H -graded R -module M is defined as the maximal length n of a sequence (x_1, \dots, x_n) of homogeneous elements of the graded maximal ideal \mathfrak{m} that form a regular sequence for M . (Notation: $\text{depth}(M) = n$.) $\text{depth}(M)$ is always bounded by the dimension of R , and in case $\text{depth}(M) = d$ we call M a graded *maximal Cohen–Macaulay module*. By $\text{CM}^H(R)$ we denote the full subcategory of $\text{mod}^H(R)$ consisting of all these modules. Also R is a graded Cohen–Macaulay ring if and only if—viewed as a graded R -module— R is Cohen–Macaulay.

PROPOSITION 8.1. *We assume that R is an H -graded local noetherian ring of dimension ≥ 2 and $\mathbf{X} = \mathbf{Proj}(R)$. For a finitely generated graded R -module M the following assertions are equivalent:*

- (i) M has graded depth ≥ 2 .
- (ii) M is a section module; i.e., for some coherent graded sheaf \mathcal{F} on \mathbf{X} we have $M = \Gamma_H(\mathcal{F})$.
- (iii) M belongs to the full subcategory $(\text{mod}_0^H(R))^\perp$ of $\text{mod}^H(R)$ right perpendicular to the family of simple graded R -modules.

Proof. The equivalence (i) \Leftrightarrow (iii) is the graded analogue of a well-known characterization of depth (cf. [41]).

(ii) \Leftrightarrow (iii): Sheafification $M \mapsto \tilde{M}$ represents $\text{Qcoh}(\mathbf{X})$ as the quotient category of $\text{Mod}^H(R)$ with respect to the localizing subcategory $\text{Mod}_0^H(R)$ generated by the simple graded R -modules. Since $\Gamma_H: \text{Qcoh}(\mathbf{X}) \rightarrow \text{Mod}^H(R)$ serves in this context as a section functor, it induces an equivalence

$$\text{Qcoh}(\mathbf{X}) \rightarrow (\text{Mod}_0^H(R))^\perp$$

which by Proposition 2.2 implies the assertion. \blacksquare

Note that in general for a coherent sheaf \mathcal{F} on \mathbf{X} the section module $\Gamma_H(\mathcal{F})$ is not a finitely generated R -module. This will, however, be the case if we assume that \mathcal{F} is additionally locally free, i.e., all stalks \mathcal{F}_x , where $x \in \mathbf{X}$, are graded free over $\mathcal{O}_{\mathbf{X},x}$. As usual, the locally free coherent sheaves on \mathbf{X} are called *vector bundles* and $\text{vect}(\mathbf{X})$ denotes the full subcategory of $\text{coh}(\mathbf{X})$ consisting of all vector bundles on \mathbf{X} .

LEMMA 8.2. *We assume that $\text{depth}(R) \geq 2$; for instance, R is graded Cohen–Macaulay of dimension at least 2. If \mathcal{F} is a vector bundle on \mathbf{X} , then the module $\Gamma_H(\mathcal{F})$ is a finitely generated section module.*

Proof. In an obvious way, the functor

$$\vee : \text{vect}(\mathbf{X}) \rightarrow \text{vect}(\mathbf{X}), \quad \mathcal{F} \mapsto \mathcal{F}^\vee = \mathcal{H}em_{\mathcal{O}_{\mathbf{X}}}(\mathcal{F}, \mathcal{O}_{\mathbf{X}}),$$

induces a duality for $\text{vect}(\mathbf{X})$. With \mathcal{F}^\vee presented by means of Serre's theorem as the quotient of a finite direct sum $\bigoplus \mathcal{O}_{\mathbf{X}}(h_i)$ of twisted structure sheaves, we conclude that \mathcal{F} embeds into $\bigoplus \mathcal{O}_{\mathbf{X}}(-h_i)$. Now left exactness of $\Gamma(X, -)$ shows that $\Gamma(X, \mathcal{F})$ becomes a submodule of $\bigoplus R(-h_i)$ and hence is finitely generated by the noetherianness of R . ■

R is called an *isolated singularity* if the homogeneous quotient ring $R_{\mathfrak{p}}$ is graded regular local, i.e., of finite (graded) global dimension, for any non-maximal graded prime ideal \mathfrak{p} of R . It is an equivalent assertion that \mathbf{X} is non-singular.

PROPOSITION 8.3. *If R is a graded isolated Cohen–Macaulay singularity of dimension 2, the category $(\text{mod}_0^H(R))^\perp$ of all finitely generated graded section modules coincides with the category $\text{CM}^H(R)$ of graded maximal Cohen–Macaulay modules over R ; moreover, by means of the correspondences $M \mapsto \tilde{M}$, $\mathcal{F} \mapsto \Gamma(\mathbf{X}, \mathcal{F})$, this category is equivalent to the category $\text{vect}(\mathbf{X})$ of vector bundles over \mathbf{X} .*

Proof. The first assertion follows from Proposition 8.1. With regard to the last assertion we observe that any localization $M_{\mathfrak{p}}$ of a maximal Cohen–Macaulay module with respect to a non-maximal graded prime ideal is maximal Cohen–Macaulay over $R_{\mathfrak{p}}$ and hence—due to graded regularity of \mathbf{X} —is $R_{\mathfrak{p}}$ free. This proves that \tilde{M} is, in fact, a vector bundle over \mathbf{X} . ■

This setting in particular applies to the string singularities defined in [21]. Let $\mathbf{p} = (p_0, \dots, p_n)$ with $p_i \geq 1$, $\underline{\lambda} = (\lambda_0, \dots, \lambda_n)$ be pairwise distinct elements of $\mathbf{P}_1(k)$ normalized such that $\lambda_0 = \infty$, $\lambda_1 = 0$, and $\lambda_n = 1$, and $R(\mathbf{p}, \underline{\lambda})$ be corresponding $L(\mathbf{p})$ -graded algebra ([21]; see also Section 7).

If $\sum_{i=0}^n 1/p_i > n - 1$, we say that $R(\mathbf{p}, \underline{\lambda})$ is of *Dynkin type*. Apart from the process of inserting additional ones in the weight sequence this condition singles out exactly the weight types (p, q) , $(2, 2, n)$, $(2, 3, 3)$, $(2, 3, 4)$, and $(2, 3, 5)$ describing the Dynkin diagrams $\mathbf{A}_{p,q}$, \mathbf{D}_{n-2} , \mathbf{E}_6 , \mathbf{E}_7 , and \mathbf{E}_8 . Note that no parameters are necessary here; thus a string singularity of Dynkin type will depend only on the weight sequence (p_0, p_1, p_2) . We hence use the notation $R(p_0, p_1, p_2)$. We further recall from [21] that the Dynkin case is characterized by the condition that $\mathbf{C} = \text{Proj}(R(\mathbf{p}, \underline{\lambda}))$ has (virtual)

genus < 1 . Equivalently the degree of the dualizing element $\bar{\omega} = (n - 1)\bar{c} - \sum_{i=0}^n \bar{x}_i$ is strictly negative; in particular $\bar{\omega}$ is torsion free. To each string singularity $R(\mathbf{p})$, \mathbf{p} of Dynkin type, we attach a \mathbf{Z} -graded algebra $R' = R'(\mathbf{p})$ by restriction to the subgroup $\mathbf{Z}\bar{\omega}$ of $L(\mathbf{p})$; thus

$$R'_n = R_{-n\bar{\omega}}.$$

The occurring algebras are—in the case of the base field of complex numbers—the *rational surface singularities* that are well known from the invariant theory of the binary polyhedral groups; see for instance F. Klein [29, 44]. We emphasize that, here, R' appears equipped with a \mathbf{Z} -grading.

PROPOSITION 8.4. *For any Dynkin type $\Delta = (p_0, p_1, p_2)$ the \mathbf{Z} -graded algebra $R'(p_0, p_1, p_2)$ has the form*

$$k[x, y, z] = k[X, Y, Z]/(f_\Delta(X, Y, Z)),$$

where the homogeneous generators (x, y, z) , their degrees under the identification $\mathbf{Z}\bar{\omega} = \mathbf{Z}$, $-\bar{\omega} = 1$, and the relation $f_\Delta(X, Y, Z)$ are displayed by the following list:

Dynkin type	Generators (x, y, z)	\mathbf{Z} -degrees	Relations f_Δ
(p, q)	$(x_0x_1, x_1^{p+q}, x_0^{p+q})$	$(1, p, q)$	$X^{p+q} - YZ$
$(2, 2, 2l)$	$(x_2^2, x_0^2, x_0x_1x_2)$	$(2, 2l, 2l+1)$	$Z^2 + X(Y^2 + YX^l)$
$(2, 2, 2l+1)$	$(x_2^2, x_0x_1, x_0^2x_2)$	$(2, 2l+1, 2l+2)$	$Z^2 + X(Y^2 + ZX^l)$
$(2, 3, 3)$	(x_0, x_1x_2, x_1^3)	$(3, 4, 6)$	$Z^2 + Y^3 + X^2Z$
$(2, 3, 4)$	(x_1, x_2^2, x_0x_2)	$(4, 6, 9)$	$Z^2 + Y^3 + X^3Y$
$(2, 3, 5)$	(x_2, x_1, x_0)	$(6, 10, 15)$	$Z^2 + Y^3 + X^5$

In characteristic $\neq 2$ —by an easy parameter change—we obtain the equations of the rational double points in the form in which they are usually listed (see F. Klein [29, 44]):

$$(2, 3, 3) \quad X^4 + Y^3 + Z^2$$

$$(2, 2, n) \quad X(Y^2 - X^n) + Z^2.$$

Proof. The proof is straightforward using the explicit form of the homogeneous components of $R(p_0, p_1, p_2)$ given in [21]. ■

Note also that each algebra $R'(\mathbf{p})$ is a \mathbf{Z} -graded isolated Gorenstein singularity and hence in particular graded Cohen–Macaulay but—apart from the case $(2, 3, 5)$ —not graded factorial. Here, R' is called *graded Gorenstein* if R' is injective as a graded R' -module.

PROPOSITION 8.5. For any weight sequence $\mathbf{p} = (p_0, p_1, p_2)$ of Dynkin type let $R' = R'(\mathbf{p})$ denote the restriction of the string singularity $R = R(\mathbf{p})$ to the subgroup $\mathbf{Z}\bar{\omega}$. If $\mathbf{C} = \mathbf{Proj}(R)$, $\mathbf{C}' = \mathbf{Proj}(R')$ there are natural equivalences

$$\text{coh}(\mathbf{C}) \rightarrow \text{coh}(\mathbf{C}')$$

and

$$\text{vect}(\mathbf{C}) \rightarrow \text{vect}(\mathbf{C}')$$

induced by restriction.

Proof. Restriction to $U = \mathbf{Z}\bar{\omega}$ defines an exact functor

$$\varphi: \text{mod}^L(R) \rightarrow \text{mod}^U(R'), \quad M \mapsto M' = M|_U.$$

This functor is clearly exact and, by means of Kan extension, easily seen to be representative. Moreover, M' has finite graded length over R' if and only if M has finite graded length over R : By noetherianness it is sufficient to deal with the case $M = (R/\mathfrak{p})(\bar{x})$, where \mathfrak{p} is a homogeneous non-maximal prime ideal of R . It follows from [21, Proposition 1.3] that the $L(\mathbf{p})$ -support $\{h \in L(\mathbf{p}) \mid M_h \neq 0\}$ of M is thus of the form $L^+(\bar{x})$ or $L_i^+(\bar{x})$, where

$$L^+ = \sum_{j=0}^n \mathbf{N}\bar{x}_j \quad \text{and} \quad L_i^+ = \sum_{j \neq i}^n \mathbf{N}\bar{x}_j.$$

It is easy to check whether any of the sets $L^+(\bar{x})$ or $L_i^+(\bar{x})$ has an infinite intersection with $\mathbf{Z}\bar{\omega}$.

If we pass to the quotient categories, φ thus induces an equivalence

$$\text{mod}^L(R)/\text{mod}_0^L(R) \rightarrow \text{mod}^U(R')/\text{mod}_0^U(R');$$

thus by Serre's theorem an equivalence $\text{coh}(\mathbf{C}) \rightarrow \text{coh}(\mathbf{C}')$. ■

For any of the $R'(\mathbf{p})$'s we may pass to the completion $S(\mathbf{p})$ with respect to the \mathfrak{m} -adic topology. Clearly, $S(\mathbf{p}) = [[X, Y, Z]]/(f(X, Y, Z))$, where $f(X, Y, Z)$ is the polynomial figuring in Proposition 8.4.

In particular, $S(\mathbf{p})$ is graded local. Also each $M \in \text{CM}^{\mathbf{Z}}(R'(\mathbf{p}))$ leads, by \mathfrak{m} -adic completion, to a Cohen–Macaulay module \hat{M} ; i.e., $\hat{M} \in \text{CM}(S)$. Our next theorem uses basic properties of the completion functor for Cohen–Macaulay modules, recently established by Auslander and Reiten [2]:

THEOREM 8.6. Let $R = R(\mathbf{p})$ and $S = S(\mathbf{p})$ be the graded, resp. complete

rational, double point singularities for some Dynkin type \mathfrak{p} . The completion functor

$$\Phi: \text{CM}^{\mathbb{Z}}(R) \rightarrow \text{CM}(S), \quad M \mapsto \hat{M},$$

has the following properties:

- (i) Φ preserves indecomposability and almost-split sequences.
- (ii) If M_1, M_2 are indecomposable in $\text{CM}^{\mathbb{Z}}(R)$ we have $\Phi(M_1) \cong \Phi(M_2)$ if and only if $M_1 \cong M_2(n)$ for some $n \in \mathbb{Z}$.
- (iii) Φ is representative.

In particular, the rational double point $S(\mathfrak{p})$ —viewed as an ungraded algebra—has finite Cohen–Macaulay type.

Proof. Trivially any graded CM module is also a CM module in the ungraded sense. If we further view $M \in \text{CM}^{\mathbb{Z}}(R)$ as a vector bundle on $\mathbf{C} = \text{Proj}(R)$, it follows from [21] that for any indecomposable graded CM module M its graded endomorphism ring

$$E = \text{END}_R(M) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}(M, M(n))$$

is graded local, and moreover that E is bounded from below; i.e., $E_n = 0$ for $n \ll 0$. Its completion $\hat{E} = \text{End}(\Phi(M))$ is therefore also local [2], and hence $\Phi(M)$ is indecomposable. A similar argument proves that Φ preserves almost-split sequences (cf. [22, 2]). From the classification of indecomposable vector bundles on $\mathbf{C}(\mathfrak{p})$, for \mathfrak{p} of Dynkin type, we know [21] that the number of orbits of indecomposable vector bundles under the Auslander–Reiten translation is in one–one correspondence with the vertices of the extended Dynkin diagram corresponding to type \mathfrak{p} and hence is finite. Therefore the set (of isomorphism classes) of all $\Phi(M)$ —for M indecomposable in $\text{CM}^{\mathbb{Z}}(R)$ —is in view of (i) a finite connected component of the Auslander–Reiten quiver of $\text{CM}(S)$, containing S . By a Brauer–Thrall type result of Auslander and Reiten [3] this implies (iii) and also proves the last assertion. ■

This shows in particular that—in arbitrary characteristic—any of the rational double point singularities (8.4) is of finite Cohen–Macaulay type [17].

In terms of covering theory [36, 19, 10, 23]:

COROLLARY 8.7. *Let $\mathbf{C}(\mathfrak{p}) = \text{Proj}(R(\mathfrak{p}))$ and $\mathbf{Y}(\mathfrak{p})$ be the punctured spectrum of $S = S(\mathfrak{p})$ for some Dynkin type \mathfrak{p} . The category $\text{vect}(\mathbf{C}(\mathfrak{p}))$ serves as*

a Galois covering of $\text{CM}(S(\mathbf{p})) = \text{vect}(\mathbf{Y})$ with covering group \mathbf{Z} by means of the functor given as the composition

$$\text{vect}(\mathbf{C}(\mathbf{p})) \xrightarrow{r} \text{CM}^{\mathbf{Z}}(R'(\mathbf{p})) \xrightarrow{\phi} \text{CM}(S(\mathbf{p})).$$

The Auslander–Reiten quiver of the rational double point of Dynkin type \mathbf{p} arises from the Auslander–Reiten quiver of indecomposable vector bundles on the weighted projective line $\mathbf{C}(\mathbf{p})$ as the quotient with respect to the \mathbf{Z} -action given by the Auslander–Reiten translation.

Let $A(\mathbf{p})$ be the path algebra of an extended Dynkin quiver of type \mathbf{p} . We recall that the derived categories $D^b(\text{coh}(\mathbf{C}(\mathbf{p})))$ and $D^b(\text{mod}(A(\mathbf{p})))$ are equivalent; moreover the corresponding comparison theorem [21, 30] for $\text{coh}(\mathbf{C}(\mathbf{p}))$ and $\text{mod}(A(\mathbf{p}))$ establishes a one–one correspondence between the set of (isomorphism classes of) indecomposable vector bundles on $\mathbf{C}(\mathbf{p})$ and the union of the sets (of isomorphism classes) of preprojective and preinjective indecomposable $A(\mathbf{p})$ -modules, respectively. In this way the classification of indecomposable CM modules over $S(\mathbf{p})$ may also be derived from the classification of indecomposable $A(\mathbf{p})$ -modules. For the latter classification we refer to Nazarova [33], Donovan and Freislich [16], and Dlab and Ringel [14].

9. REDUCTION OF WEIGHT FOR WEIGHTED PROJECTIVE LINES

This section deals with the comparison of weighted projective lines $\mathbf{C}(\mathbf{p}, \underline{\lambda})$ of different weight type $(\mathbf{p}, \underline{\lambda})$. For the present discussion it is convenient to change the notation slightly. If $\mathbf{P}_1(k)$ denotes the projective line over k —viewed as a k -variety—(k assumed to be algebraically closed) then a function $\mathbf{w}: \mathbf{P}_1(k) \rightarrow \mathbf{Z}$ is called a *weight function* if $\mathbf{w}(\lambda) \geq 1$ for all $\lambda \in \mathbf{P}_1(k)$ and, moreover, $\mathbf{w}(\lambda) = 1$ for all $\lambda \in \mathbf{P}_1(k)$ outside a finite set $\{\lambda_0, \dots, \lambda_n\}$. Here we may assume that the sequence $\underline{\lambda} = (\lambda_0, \dots, \lambda_n)$ is *normalized*, i.e., that the λ_i are pairwise distinct and further $\lambda_0 = \infty$, $\lambda_1 = 0$, $\lambda_2 = 1$. By means of the resulting weight sequence $\mathbf{p} = (p_0, \dots, p_n)$, where $p_i = \mathbf{w}(\lambda_i)$, we put

$$R_{\mathbf{w}} = R(\mathbf{p}, \underline{\lambda}), \quad L_{\mathbf{w}} = L(\mathbf{p}), \quad \text{and} \quad \mathbf{C}_{\mathbf{w}} = \mathbf{C}(\mathbf{p}, \underline{\lambda}).$$

Hence $L_{\mathbf{w}}$ is an ordered group and $R_{\mathbf{w}}$ is a positively $L_{\mathbf{w}}$ -graded algebra. Note that by means of the canonical bijection

$$\mathbf{C}_{\mathbf{w}} \rightarrow \mathbf{P}_1(k), \quad (x_0, \dots, x_n) \mapsto (x_0^{p_0}, x_1^{p_1})$$

(see [21]), we may view \mathbf{w} also as a weight function on $\mathbf{C}_{\mathbf{w}}$.

We further recall that a morphism of an H -graded algebra R into

an H' -graded algebra R' consists of a pair (φ, u) , where $\varphi: R \rightarrow R'$ is a k -algebra homomorphism and $u: H \rightarrow H'$ is a morphism of abelian groups such that $\varphi(R_h) \subset R'_{u(h)}$ holds for each $h \in H$. Accordingly, R and R' are called isomorphic as graded algebras if there exists such a morphism $(\varphi, u): (R, H) \rightarrow (R', H')$ with φ and u being isomorphisms.

Further we need the concept of the *companion category* $[H, R]$ of an H -graded algebra R .

- The objects of $[H; R]$ are the elements of H .
- If $h_1, h_2 \in H$, then $\text{Hom}(h_1, h_2) := R_{h_2-h_1}$.
- Composition of morphisms is given by the multiplication in R .

The companion category $[H; R]$ of R allows us to identify H -graded R -modules with additive covariant functors from $[H; R]$ to the category \mathbf{Ab} of abelian groups by means of the obvious correspondence

$$([H, R], \mathbf{Ab}) \rightarrow \text{Mod}^H(R), \quad F \mapsto \bigoplus_{h \in H} F(h).$$

With respect to this correspondence, in particular, the module $R(-h)$ corresponds to representable functor $\text{Hom}(h, -)$, sometimes abbreviated to $(h, -)$.

To each weight function \mathbf{w} on $\mathbf{P}_1(k)$ we attach a finite dimensional algebra $A_{\mathbf{w}}$, called the *canonical algebra* of weight type \mathbf{w} . Strictly speaking $A_{\mathbf{w}}$ is defined as the full subcategory of $[L(\mathfrak{p}); R_{\mathbf{w}}]$, whose objects form the so-called *canonical configuration* $\Sigma_{\mathbf{w}}$ of $L_{\mathbf{w}}$. By definition $\Sigma_{\mathbf{w}}$ is the finite subset of elements $\vec{x} \in L_{\mathbf{w}}$ satisfying $0 \leq \vec{x} \leq \vec{c}$. Note that in view of the formula $\text{Hom}(\mathcal{O}_{\mathbf{C}}(\vec{x}), \mathcal{O}_{\mathbf{C}}(\vec{y})) = (R_{\mathbf{w}})_{\vec{y}-\vec{x}}$ the algebra $A_{\mathbf{w}}$ is equivalent to the full subcategory of $\text{coh}(\mathbf{C}_{\mathbf{w}})$ consisting of all line bundles $\mathcal{O}(\vec{x})$, with $\vec{x} \in \Sigma_{\mathbf{w}}$.

Two weight functions \mathbf{w} and \mathbf{v} are called *equivalent* (notation $\mathbf{w} \approx \mathbf{v}$) if for some linear transformation $\sigma \in \mathbf{PSL}(2, k)$ we have $\mathbf{v} = \mathbf{w} \circ \sigma$. More generally, we say that \mathbf{w} *dominates* \mathbf{v} (notation $\mathbf{w} \succcurlyeq \mathbf{v}$) if for some $\sigma \in \mathbf{PSL}(2, k)$ the relation $\mathbf{w} \succcurlyeq \mathbf{v} \circ \sigma$ holds, where $\mathbf{w} \succcurlyeq \mathbf{v}$ means that $\mathbf{w}(\lambda) \succcurlyeq \mathbf{v}(\lambda)$ for each $\lambda \in \mathbf{P}_1(k)$. As is easily seen \mathbf{w} and \mathbf{v} are equivalent if and only if $\mathbf{w} \succcurlyeq \mathbf{v}$ and $\mathbf{v} \succcurlyeq \mathbf{w}$ hold true.

The next proposition illustrates the notion of the equivalence of weight functions.

PROPOSITION 9.1. *For two weight functions \mathbf{w} and \mathbf{v} on $\mathbf{P}_1(k)$ the following assertions are equivalent:*

- (i) \mathbf{w} and \mathbf{v} are equivalent, i.e., differ only by some linear transformation $\sigma \in \mathbf{PSL}(2, k)$.
- (ii) $R_{\mathbf{w}}$ and $R_{\mathbf{v}}$ are isomorphic as graded algebras.

- (iii) $\text{coh}(\mathbf{C}_w)$ and $\text{coh}(\mathbf{C}_v)$ are equivalent as abelian categories.
- (iv) The canonical algebras A_w and A_v are isomorphic.

Proof. Implications (i) \Rightarrow (ii), (ii) \Rightarrow (iii) are obvious.

(iii) \Rightarrow (iv): It suffices to show that, up to equivalence, A_w can be recovered from the abelian category $\text{coh}(\mathbf{C}_w)$. To show this, we first observe that the category $\text{vect}(\mathbf{C}_w)$ of vector bundles on \mathbf{C}_w consists exactly of those coherent sheaves on \mathbf{C}_w which do not have any simple subsheaf. Since, moreover, the quotient category $\text{coh}(\mathbf{C}_w)/\text{coh}_0(\mathbf{C}_w)$, where $\text{coh}_0(\mathbf{C}_w)$ denotes the category of all finite length coherent sheaves, is equivalent to the category of finite dimensional vector spaces over the rational function field $k(X)$, the rank of a coherent sheaf \mathcal{F} is a categorical invariant given as the length of \mathcal{F} viewed as an object in $\text{coh}(\mathbf{C}_w)/\text{coh}_0(\mathbf{C}_w) = \text{mod}(k(X))$.

Now we select a rank one bundle L in $\text{coh}(\mathbf{C}_w)$. For each $\lambda \in \mathbf{C}_w$ there is a unique way to arrange the (isomorphism classes of) simple sheaves concentrated at λ into a sequence $\mathcal{S}_1, \dots, \mathcal{S}_p$, $p = \mathbf{w}(\lambda)$, such that there exist line bundles $L = L_0(\lambda), \dots, L_p(\lambda)$ and for any such i a non-split exact sequence

$$0 \rightarrow L_{i-1}(\lambda) \rightarrow L_i(\lambda) \rightarrow \mathcal{S}_i \rightarrow 0.$$

Moreover, up to isomorphism the bundles $L_0(\lambda), \dots, L_p(\lambda)$ are uniquely determined by L and λ . As is shown in [21] the collection of all $L_i(\lambda)$, $\lambda \in \mathbf{C}_w$, $i = 0, \dots, \mathbf{w}(\lambda)$, defines a full subcategory of $\text{coh}(\mathbf{C}_w)$ which has only finitely many non-isomorphic objects and is equivalent to the canonical algebra A_w .

(iv) \Rightarrow (i): It suffices to show that it is possible to recover from the canonical algebra A_w the weight function $\mathbf{w}: \mathbf{P}_1(k) \rightarrow \mathbf{Z}$ up to a linear transformation $\sigma \in \text{PSL}(2, k)$. The morphism space $A_w(\vec{0}, \vec{c})$ is a two-dimensional vector space over k spanned by $x_0^{p_0}, x_1^{p_1}$ and has a system of $n + 1$ distinguished one-dimensional subspaces $V_i = kx_i^{p_i}$, $i = 0, \dots, n$. Passing to k^* -orbits hence allows us to define $\mathbf{w}: \mathbf{P}_1(k) \rightarrow \mathbf{Z}$ as the function which takes value one except at points $[V_i]$, where the value equals p_i , $i = 0, \dots, n$. ■

Next, we give a characterization of the localizing subcategories of $\text{coh}(\mathbf{C}_w)$.

PROPOSITION 9.2. *Let \mathcal{C} be a Serre subcategory of $\text{coh}(\mathbf{C}_w)$ for some weight function \mathbf{w} on $\mathbf{P}_1(k)$ and assume that \mathcal{C} is properly contained in $\text{coh}(\mathbf{C}_w)$. Then \mathcal{C} is the Serre subcategory generated by a set \mathcal{C}' of simple sheaves on \mathbf{C}_w .*

Moreover, \mathcal{C} is localizing in $\text{coh}(\mathbf{C}_w)$ if and only if for each $\lambda \in \mathbf{P}_1(k)$ the

set \mathcal{C}' contains at most $w(\lambda) - 1$ non-isomorphic simple sheaves concentrated at λ .

Proof. First, assume that \mathcal{C} is a Serre subcategory of $\text{coh}(C_w)$ which contains a non-zero vector bundle F . From a line bundle filtration of F we conclude that \mathcal{C} contains a line bundle, necessarily of the form $\mathcal{O}(\bar{x})$ [21], together with all its subbundles $\mathcal{O}(\bar{y})$ for $\bar{y} \leq \bar{x}$ in L_w . Passing to cokernels this fact implies that \mathcal{C} contains all simple sheaves and thus (taking extensions) all line bundles and hence all vector bundles on C_w ; therefore $\mathcal{C} = \text{coh}(C_w)$.

Therefore any proper Serre subcategory of $\text{coh}(C_w)$ is contained in $\text{coh}_0(C_w)$, the category of finite length coherent sheaves; hence \mathcal{C} is generated—as a Serre subcategory—by the system \mathcal{C}' of all simple sheaves belonging to \mathcal{C} . This proves the first assertion.

Next assume that \mathcal{C} is a localizing subcategory contained in $\text{coh}_0(C_w)$ and that for some $\lambda \in P_1(k)$ every simple sheaf concentrated at λ belongs to \mathcal{C} . Thus \mathcal{C} contains the Serre subcategory $\text{coh}_\lambda(C_w)$ of all finite length coherent sheaves concentrated at λ .

Note that for a fixed $\lambda \in C_w$ and for any coherent sheaf \mathcal{F} of rank ≥ 1 there exists some simple sheaf \mathcal{S} concentrated at λ with the property $\text{Ext}^1(\mathcal{S}, \mathcal{F}) \neq 0$. In fact, by right exactness of $\text{Ext}^1(\mathcal{S}, -)$ it suffices to prove the assertion for the case of a line bundle \mathcal{F} ; this allows us to reduce the question to the case $\mathcal{F} = \mathcal{O}_C$, where it is obvious.

In particular there does not exist an exact sequence $0 \rightarrow \mathcal{O}_C \rightarrow \mathcal{F} \rightarrow \mathcal{S} \rightarrow 0$ with $\mathcal{F} \in \mathcal{C}^\perp$ (and $\mathcal{S} \in \mathcal{C}$); hence by Proposition 2.2, \mathcal{C} is not localizing in $\text{coh}(C_w)$. ■

COROLLARY 9.3. *Any localizing subcategory of $\text{coh}(C_w)$ which is properly contained in $\text{coh}(C_w)$ is generated (as a Serre subcategory) by simple sheaves, which are concentrated in points of weight > 1 . In particular $\text{coh}(C_w)$ admits only finitely many localizing subcategories.*

As we show now the passage to the quotient category with respect to a proper localizing subcategory \mathcal{C} of $\text{coh}(C_w)$, equivalently the passage to the full exact subcategory \mathcal{C}^\perp right perpendicular to \mathcal{C} , leads again to a category of type $\text{coh}(C_v)$, where the weight function v is dominated by w . First, we study the degeneration and embedding functors attached to such a situation. For this purpose let $\mathbf{p} = (p_0, p_1, \dots, p_n)$ be a weight sequence and $\underline{\lambda} = (\lambda_0, \lambda_1, \dots, \lambda_n)$ be a normalized sequence elements of $P_1(k)$. We suppose that $p_j > 1$ for some $0 \leq j \leq n$ and put

$$\mathbf{p}' = (p'_0, \dots, p'_n) := (p_0, \dots, p_{j-1}, p_j - 1, p_{j+1}, \dots, p_n).$$

Let $L(\mathbf{p})$ and $L(\mathbf{p}')$ be the corresponding abelian groups of rank 1. We use

the symbols $\bar{x}_0, \dots, \bar{x}_n, \bar{c}$, for the generators and the canonical element of $L(\mathbf{p})$ while those of $L(\mathbf{p}')$ are denoted by $\bar{x}'_0, \dots, \bar{x}'_n, \bar{c}'$. We also use the abbreviations $R = R(\mathbf{p}, \underline{\lambda}), C = C(\mathbf{p}, \underline{\lambda}), R' = R(\mathbf{p}', \underline{\lambda}'),$ and $C' = C(\mathbf{p}', \underline{\lambda}')$.

We define a full embedding $\varphi: [L(\mathbf{p}'); R'] \rightarrow [L(\mathbf{p}), R]$ for the companion categories. Let $\bar{x}' = \sum_{i=0}^n l_i \bar{x}'_i + l \bar{c}' \in L(\mathbf{p}')$ be in *normal form*; i.e., we assume $0 \leq l_i < p'_i$. Then by means of

$$\varphi \left(\sum_{i=0}^n l_i \bar{x}'_i + l \bar{c}' \right) = \sum_{i=0}^n l_i \bar{x}_i + l \bar{c}$$

we define a mapping

$$\varphi: L(\mathbf{p}') \rightarrow L(\mathbf{p}).$$

Note that φ is not a homomorphism of abelian groups. Further let $\bar{y}' = \sum r_i \bar{x}'_i + s \bar{c}' \in L(\mathbf{p}')$ be represented in normal form and

$$f' = \left(\prod_{i=0}^n (x'_i)^{r_i} \right) \cdot g((X'_0)^{p'_0}, \dots, (X'_n)^{p'_n}) \in R'$$

be a polynomial of degree \bar{y}' , where g is a homogeneous polynomial in $n + 1$ indeterminates of *total degree* s . With f' viewed as a morphism $\bar{x}' \rightarrow \bar{x}' + \bar{y}'$ in the companion category $[L(\mathbf{p}'); R']$, the correspondence

$$f' \mapsto f = \left(\prod_{i=0}^n x_i^{k_i} \right) \cdot g(x_0^{p_0}, \dots, x_n^{p_n}): \varphi(\bar{x}') \rightarrow \varphi(\bar{x}' + \bar{y}'),$$

where

$$k_i = \begin{cases} r_i & \text{if } i \neq j \text{ or } (i = j \text{ and } l_j + r_j \leq p_j - 2) \\ r_i + 1 & \text{if } i = j \text{ and } l_j + r_j > p_j - 2, \end{cases}$$

defines the wanted full embedding. Clearly the embedding φ induces an exact functor

$$\varphi_*: \text{Mod}^{L(\mathbf{p})}(R) \rightarrow \text{Mod}^{L(\mathbf{p}')}(R')$$

by restriction. Finally we define a mapping $\varphi': L(\mathbf{p}) \rightarrow L(\mathbf{p}')$ by the formula

$$\varphi' \left(\sum_{i=0}^n l_i \bar{x}_i + l \bar{c} \right) = \begin{cases} \sum_{i=0}^n l_i \bar{x}'_i + l \bar{c}' & \text{if } l_j < p_j - 1 \\ \sum_{i \neq j} l_i \bar{x}'_i + (l + 1) \bar{c}' & \text{otherwise.} \end{cases}$$

PROPOSITION 9.4. *The functor $\varphi_*: \text{Mod}^{L(\mathfrak{p})}(R) \rightarrow \text{Mod}^{L(\mathfrak{p}')} (R')$ has the following properties:*

- (1) $\varphi_*(R(\bar{x})) \cong R'(-\varphi'(-\bar{x}))$.
- (2) *If $M \in \text{Mod}^{L(\mathfrak{p})}(R)$ is finitely presented, $\varphi_*(M)$ is finitely presented in $\text{Mod}^{L(\mathfrak{p}')} (R')$.*
- (3) *If $L \in \text{Mod}^{L(\mathfrak{p})}(R)$ is of finite length, $\varphi_*(L)$ is of finite length in $\text{Mod}^{L(\mathfrak{p}')} (R')$.*

Proof. Property (2) follows from (1) and the fact that φ_* is exact; (3) holds true by construction. Thus, it remains to prove (1). By means of the identification of $R(\bar{x})$ with the representable functor $(-\bar{x}, -)$, property (1) is equivalent to the assertion $\varphi_*(\bar{x}, -) \cong (\varphi'(\bar{x}), -)$. In the case where $\bar{x} = \varphi(\bar{x}')$ with $\bar{x}' \in L(\mathfrak{p}')$ we have $\varphi'(\bar{x}) = \bar{x}'$ and $\varphi_*(\bar{x}, -) \cong (\bar{x}', -)$ since φ is a full functor. If $\bar{x} = \sum_{i=0}^n l_i \bar{x}_i + l\bar{c}$ with $l_j = p_j - 1$, the morphism $x_j: \bar{x} \rightarrow \bar{x} + \bar{x}_j$ in the companion category of R induces functorial isomorphisms $x_j: (\bar{x} + \bar{x}_j, \varphi(\bar{x}')) \rightarrow (\bar{x}, \varphi(\bar{x}'))$ for every $\bar{x} \in L(\mathfrak{p}')$; hence $\varphi_*(\bar{x}, -) \cong \varphi_*(\bar{x} + \bar{x}_j, -) \cong (\varphi'(\bar{x} + \bar{x}_j), -)$. ■

We note that φ_* has a left and a right adjoint given by left and right Kan extension. In particular, the adjoints are induced from functors $[L(\mathfrak{p}); R] \rightarrow [L(\mathfrak{p}'); R]$ and commute with the respective shift operations; moreover these functors map finitely presented modules to finitely presented modules and modules of finite length to modules of finite length. By passing to the respective quotient categories modulo the Serre subcategory of all locally finite modules (resp. finite length modules) we obtain:

THEOREM 9.5. $\varphi_*: \text{Mod}^{L(\mathfrak{p})}(R) \rightarrow \text{Mod}^{L(\mathfrak{p}')} (R')$ induces an exact functor

$$\varphi_*: \text{Qcoh}(\mathbf{C}) \rightarrow \text{Qcoh}(\mathbf{C}'), \quad \widetilde{M} \mapsto \varphi_*(\widetilde{M}),$$

with the following properties:

- (a) $\varphi_*(\mathcal{O}_{\mathbf{C}}(\bar{x})) \cong \mathcal{O}_{\mathbf{C}'}(-\varphi'(-\bar{x}))$.
- (b) *If $\mathcal{F} \in \text{Qcoh}(\mathbf{C})$ is coherent, $\varphi_*(\mathcal{F})$ is coherent.*
- (c) *If $\mathcal{F} \in \text{coh}(\mathbf{C})$ is a vector bundle, then $\varphi_*(\mathcal{F})$ is a vector bundle and $\text{rank}(\varphi_*(\mathcal{F})) = \text{rank}(\mathcal{F})$.*
- (d) *If \mathcal{S} is a simple sheaf, then $\varphi_*\mathcal{S} = 0$ if and only if \mathcal{S} is concentrated at λ_j and $\text{Ext}^1(\mathcal{O}_{\mathbf{C}}, \mathcal{S}) \neq 0$. Otherwise $\varphi_*\mathcal{S}$ is simple. In particular, for a sheaf \mathcal{F} of finite length, $\varphi_*(\mathcal{F})$ is again of finite length.*
- (e) φ_* induces equivalences of categories

$$\text{Qcoh}(\mathbf{C})/\text{Cl}(\mathcal{S}_{j,0}) \rightarrow \text{Qcoh}(\mathbf{C}') \quad \text{and} \quad \text{coh}(\mathbf{C})/\text{cl}(\mathcal{S}_{j,0}) \rightarrow \text{coh}(\mathbf{C}').$$

Moreover, $\text{cl}(\mathcal{S}_{j,0})$ is a localizing subcategory of $\text{coh}(\mathbf{C})$ and $\mathcal{S}_{j,0}^\perp$ is equivalent to $\text{coh}(\mathbf{C}')$.

(f) φ_* induces an epimorphism $K_0(\mathbf{C}) \rightarrow K_0(\mathbf{C}')$, $[M] \mapsto [\varphi_*(M)]$, for the Grothendieck groups of $\text{coh}(\mathbf{C})$ and $\text{coh}(\mathbf{C}')$ with kernel $\mathbf{Z}[\mathcal{S}_{j,0}]$.

Proof. The existence of φ_* with properties (a) and (b) follows from Proposition 9.4. Property (c) follows from (a) using line bundle filtrations.

Let \mathcal{S} be a simple sheaf. If \mathcal{S} is simple concentrated at an ordinary point, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbf{C}} \xrightarrow{x_1^{p_1} \dots \lambda x_0^{p_0}} \mathcal{O}_{\mathbf{C}}(\vec{c}) \longrightarrow \mathcal{S} \longrightarrow 0$$

with $\lambda \neq \lambda_i$ for all i . Application of φ_* gives the exact sequence

$$0 \longrightarrow \mathcal{O}'_{\mathbf{C}} \xrightarrow{(x'_i)^{p_i} - \lambda(x'_0)^{p_0}} \mathcal{O}'_{\mathbf{C}}(\vec{c}') \longrightarrow \varphi_* \mathcal{S} \longrightarrow 0;$$

thus $\varphi_*(\mathcal{S})$ is again simple and concentrated at an ordinary point of \mathbf{C}' , i.e., is an *ordinary simple* sheaf.

If \mathcal{S} is *exceptional simple*, i.e., concentrated at a point λ_i of weight > 1 , then \mathcal{S} is one of the sheaves $\mathcal{S}_{i,k}$ given by the exact sequences

$$0 \longrightarrow \mathcal{O}_{\mathbf{C}}(k\vec{x}_i) \xrightarrow{x_i} \mathcal{O}_{\mathbf{C}}((k+1)\vec{x}_i) \longrightarrow \mathcal{S}_{i,k} \longrightarrow 0,$$

where $k = 0, \dots, p_i - 1$. Thus—up to an appropriate shift of $\text{coh}(\mathbf{C})$ —we may assume that in this situation \mathcal{S} equals to $\mathcal{S}_{i,0}$.

By applying φ_* we get the exactness of

$$0 \longrightarrow \mathcal{O}_{\mathbf{C}}(-\varphi'(-k\vec{x}_i)) \xrightarrow{x'_i} \mathcal{O}_{\mathbf{C}}(-\varphi'(-(k+1)\vec{x}_i)) \longrightarrow \varphi_* \mathcal{S}_{i,k} \longrightarrow 0$$

and x'_i induces an isomorphism if and only if $k = 0$ and $i = j$. Otherwise $\varphi_* \mathcal{S}_{i,k}$ is again simple.

$\varphi_* : \text{Qcoh}(\mathbf{C}) \rightarrow \text{Qcoh}(\mathbf{C}')$ (resp. $\varphi_* : \text{coh}(\mathbf{C}) \rightarrow \text{coh}(\mathbf{C}')$) is exact and representative and has a right adjoint and the kernel is generated by $\mathcal{S}_{j,0}$; therefore (e) holds.

Finally $\varphi_* : \text{coh}(\mathbf{C}) \rightarrow \text{coh}(\mathbf{C}')$ induces an epimorphism $K_0(\mathbf{C}) \rightarrow K_0(\mathbf{C}')$; hence $\ker(K_0(\varphi_*))$ is a direct factor of $K_0(\mathbf{C})$ of rank one. Since $\mathbf{Z}[\mathcal{S}_{j,0}]$ is contained in $\ker(K_0(\varphi_*))$ and is itself a direct factor in $K_0(\mathbf{C})$, assertion (f) follows. ■

For the next proposition the notations of Theorem 9.5 remain in force. By $\varphi^* : \text{coh}(\mathbf{C}') \rightarrow \text{coh}(\mathbf{C})$ we denote the functor which is right adjoint to φ_* . The following assertions are obvious consequences of Theorem 9.5 and Proposition 9.4:

PROPOSITION 9.6. *The functor $\varphi^*: \text{coh}(\mathbf{C}') \rightarrow \text{coh}(\mathbf{C})$ is a full exact embedding whose image is closed under extensions. Moreover φ^* has the following properties:*

(a) φ^* is rank preserving and maps bundles (resp. finite length sheaves) to bundles (resp. finite length sheaves).

(b) $\varphi^*(\mathcal{O}_{\mathbf{C}'}) = \mathcal{O}_{\mathbf{C}}(\varphi(\vec{x}'))$.

(c) $\varphi^*(\mathcal{F}(\vec{c}')) = \varphi^*(\mathcal{F})(\vec{c})$ for any coherent sheaf \mathcal{F} on \mathbf{C}' .

(d) If we identify \mathbf{C} and \mathbf{C}' as point sets, $\varphi^*(\text{coh}_\lambda(\mathbf{C}')) \subset \text{coh}_\lambda(\mathbf{C})$ with equality for $\lambda \neq \lambda_j$. In the case where $\lambda = \lambda_j$, $\varphi^*(\text{coh}_\lambda(\mathbf{C}'))$ becomes the Serre subcategory of $\text{coh}(\mathbf{C}')$ generated by $\{\mathcal{S}_{j,1}, \dots, \mathcal{S}_{j,p_j-1}\}$.

Remark 9.7. (1) Let $\mathcal{S}_{j,k}$ be an arbitrary simple sheaf concentrated in λ_j . Then by slight modification of the maps φ and φ' we obtain an exact functor $\varphi_*: \text{Qcoh}(\mathbf{C}) \rightarrow \text{Qcoh}(\mathbf{C}')$ (resp. $\varphi_*: \text{coh}(\mathbf{C}) \rightarrow \text{coh}(\mathbf{C}')$) with the kernel being the localizing subcategory generated by $\mathcal{S}_{j,k}$. Theorem 9.5 and Proposition 9.6 hold respectively.

(2) Let $\mathbf{p} = (p_0, \dots, p_n)$ and $\mathbf{q} = (q_0, \dots, q_n)$ be weight sequences with $p_i \geq q_i$ for all $i = 0, \dots, n$ and let $\mathbf{C}_1 = \mathbf{C}(\mathbf{p}, \underline{\lambda})$ and $\mathbf{C}_2 = \mathbf{C}(\mathbf{q}, \underline{\lambda})$. Then successive application of the above construction yields an exact functor $\psi_*: \text{Qcoh}(\mathbf{C}_1) \rightarrow \text{Qcoh}(\mathbf{C}_2)$ (resp. $\psi_*: \text{coh}(\mathbf{C}_1) \rightarrow \text{coh}(\mathbf{C}_2)$) and a full embedding $\psi^*: \text{Qcoh}(\mathbf{C}_2) \rightarrow \text{Qcoh}(\mathbf{C}_1)$ (resp. $\psi^*: \text{coh}(\mathbf{C}_2) \rightarrow \text{coh}(\mathbf{C}_1)$). Theorem 9.5 and Proposition 9.6 hold respectively. The kernel of ψ_* is the localizing subcategory of $\text{Qcoh}(\mathbf{C}_1)$ (resp. $\text{coh}(\mathbf{C}_1)$) generated by $p_i - q_i$ simple sheaves over each exceptional point λ_i of \mathbf{C}_1 . In particular there are such functors for any choice of $p_i - q_i$ simple sheaves over each λ_i .

(3) In the case where $\mathbf{q} = (1, \dots, 1)$ we have $\mathbf{C}_2 = \mathbf{P}_1(k)$. Hence, any choice of $p_i - 1$ simple sheaves over each $\lambda_i \in \mathbf{C}_1$ leads to an exact functor $\psi_*: \text{Qcoh}(\mathbf{C}_1) \rightarrow \text{Qcoh}(\mathbf{P}_1(k))$ (resp. $\psi_*: \text{coh}(\mathbf{C}_1) \rightarrow \text{coh}(\mathbf{P}_1(k))$) with the kernel being the localizing subcategory generated by these sheafs and to full exact embeddings $\psi^*: \text{Qcoh}(\mathbf{P}_1(k)) \rightarrow \text{Qcoh}(\mathbf{C}_1)$ (resp. $\psi^*: \text{coh}(\mathbf{P}_1(k)) \rightarrow \text{coh}(\mathbf{C}_1)$). In particular there are p_0, \dots, p_n such pairs of functors.

THEOREM 9.8. *Let \mathbf{v} and \mathbf{w} be weight functions on $\mathbf{P}_1(k)$. Then the following assertions are equivalent:*

(i) \mathbf{w} dominates \mathbf{v} ; i.e., $\mathbf{w} \geq \mathbf{v}$ up to composition with some linear transformation $\sigma \in \text{PSL}(2, k)$.

(ii) $\text{coh}(\mathbf{C}_\mathbf{v})$ is equivalent to a quotient category $\text{coh}(\mathbf{C}_\mathbf{w})/\mathcal{C}$ with respect to a Serre (resp. a localizing) subcategory \mathcal{C} of $\text{coh}(\mathbf{C}_\mathbf{w})$.

(iii) $\text{coh}(\mathbf{C}_\mathbf{v})$ is equivalent to a full (exact) subcategory of $\text{coh}(\mathbf{C}_\mathbf{w})$ which is closed under extensions.

Moreover, each exact functor $\Psi: \text{coh}(\mathbf{C}_w) \rightarrow \text{coh}(\mathbf{C}_v)$ inducing an equivalence $\text{coh}(\mathbf{C}_w)/\mathcal{C} \cong \text{coh}(\mathbf{C}_v)$ for a Serre (resp. a localizing) subcategory \mathcal{C} of $\text{coh}(\mathbf{C}_w)$ (resp. each full exact embedding $\text{coh}(\mathbf{C}_v) \rightarrow \text{coh}(\mathbf{C}_w)$) has the form ψ_* (resp. ψ^*) described in 9.7(2).

Proof. Implication (i) \Rightarrow (ii) is covered by Theorem 9.5, while (i) \Rightarrow (iii) follows from Proposition 9.6.

(ii) \Rightarrow (i): From Proposition 9.2 it follows that a localizing Serre subcategory \mathcal{C} of $\text{coh}(\mathbf{C}_w)$ is generated by a finite number of simple exceptional sheaves; hence in virtue of Theorem 9.5 and Proposition 9.1, $\text{coh}(\mathbf{C}_v) \cong \text{coh}(\mathbf{C}_w)/\mathcal{C}$ is equivalent to a category $\text{coh}(\mathbf{C}_{w'})$ of weight $w' \approx v$ dominated by w . Moreover, if \mathcal{C} is a Serre subcategory of $\text{coh}(\mathbf{C}_w)$ and $\text{coh}(\mathbf{C}_w)/\mathcal{C}$ is again a category of coherent sheaves on a weighted projective line, it follows that \mathcal{C} is localizing in $\text{coh}(\mathbf{C}_w)$: By Proposition 9.2 it suffices to show that there does not exist an element $\lambda \in \mathbf{C}_w$ such that \mathcal{C} contains the category \mathcal{C}_λ of finite length sheaves concentrated at λ . If we anticipate results from the last section it follows that the endomorphism ring of $\mathcal{O}_{\mathbf{C}}$, viewed as an object in $\text{coh}(\mathbf{C}_w)/\mathcal{C}_\lambda$, has infinite k -dimension. This allows us to deduce the corresponding assertion for $\text{coh}(\mathbf{C}_w)/\mathcal{C}$, contradicting the fact that $\text{coh}(\mathbf{C}_v)$ has finite dimensional Hom-spaces.

(iii) \Rightarrow (i): Let $\Phi: \text{coh}(\mathbf{C}_v) \rightarrow \text{coh}(\mathbf{C}_w)$ be a full exact embedding whose image is closed under extensions. By virtue of Proposition 9.1 the claim immediately follows from the following properties:

(a) Φ is rank preserving and maps bundles on \mathbf{C}_v to bundles on \mathbf{C}_w .

(b) Up to an equivalence of $\text{coh}(\mathbf{C}_w)$, the functor Φ maps the canonical configuration A_v into A_w with $\mathcal{O}_{\mathbf{C}_v}$ (resp. $\mathcal{O}_{\mathbf{C}_v}(\vec{\tau})$) going to $\mathcal{O}_{\mathbf{C}_w}$ (resp. $\mathcal{O}_{\mathbf{C}_w}(\vec{\tau})$).

To prove (a) we first show that Φ maps vector bundles to vector bundles: If for some line bundle L on \mathbf{C}_v the (indecomposable) sheaf $\Phi(L)$ has finite length, each $\Phi(L')$ with $\text{Hom}(L, L') \neq 0$ will also have finite length and the support of $\Phi(L')$ will agree with the support λ of $\Phi(L)$. Since there are only finitely many non-isomorphic indecomposable sheaves on \mathbf{C}_w with endomorphism ring k and support λ this leads to a contradiction.

If \mathcal{S} is an ordinary simple sheaf on \mathbf{C}_v there exists a sequence of line bundles $L_i, i \in \mathbf{Z}$, together with short exact sequences $0 \rightarrow L_i \rightarrow L_{i+1} \rightarrow \mathcal{S} \rightarrow 0$. Since the rank of a coherent sheaf must be an integer ≥ 0 , we conclude from the exactness of Φ that $\Phi(\mathcal{S})$ has rank zero and hence is an indecomposable sheaf of finite length with endomorphism ring k . Because there are only finitely many finite length sheaves which are not ordinary simple having that property, we deduce that for some ordinary simple sheaf

10. TAME HEREDITARY AND CANONICAL ALGEBRAS

In this section we study homological epimorphisms $\varphi: A \rightarrow A'$, A a tame hereditary or a canonical algebra, where φ is induced by the selection of non-homogeneous simple regular modules.

First, let A be a tame hereditary algebra. For the representation theory of these algebras and the notions involved we refer to [14, 6]. For finite dimensional A -modules we use the notion of rank, defined by

$$\text{rk} = -\dim_k \text{Hom}(R, -) + \dim_k \text{Ext}^1(R, -),$$

where R is a homogeneous simple regular A -module. The rank does not depend on the choice of R ; moreover, it is invariant under the Auslander–Reiten transformation. We note that $\delta = -\text{rk}$ is usually [14] called the defect.

If S is a non-homogeneous simple regular module then $\text{End}(S)$ is a skew field, $\text{Ext}^1(S, S) \cong 0$, and $\text{Hom}(S, A) = 0$. Thus Theorem 4.16 applies and, by forming the subcategory of $\text{mod}(A)$ right perpendicular to S , we obtain a finite dimensional algebra A' together with a homological epimorphism $\varphi: A \rightarrow A'$. We recall that for the case of an algebraically closed base field any tame hereditary connected algebra A is given as the path algebra of a quiver, whose underlying graph is an extended Dynkin diagram $\tilde{\Delta}$. Contrary to usual practice, we call Δ the Dynkin type of A . Further, $\text{reg}(A)$ denotes the category of all finite dimensional regular A -modules. In the hereditary case it is possible to provide the following additional information:

THEOREM 10.1. *Let k be a field, A be a finite dimensional tame hereditary k -algebra, S be a non-homogeneous simple regular A -module, and $\varphi: A \rightarrow A'$ be the corresponding homological epimorphism. Then:*

- (1) A' is tame hereditary.
- (2) If A is connected, the same holds true for A' .
- (3) Suppose k is algebraically closed and A is Morita equivalent to the path algebra of an extended Dynkin quiver $\tilde{\Delta}$, where $\Delta = (p, q, r)$. Further let $p > 1$ and S belong to a tube of rank p . Then A' is Morita equivalent to the path algebra of an extended Dynkin quiver of Dynkin type $(p - 1, q, r)$.
- (4) The induced functors $\varphi_*: \text{mod}(A') \rightarrow \text{mod}(A)$ and $\varphi^*: \text{mod}(A) \rightarrow \text{mod}(A')$ map preprojective (resp. regular, preinjective) modules to preprojective (resp. regular, preinjective) modules; moreover φ_* preserves the rank, while $\text{rk}(\varphi^*M) = \text{rk}(M)$ ($\text{rk}(\varphi^*M) \geq \text{rk}(M)$) holds for each preprojective or regular (resp. each preinjective module).

- (5) $\varphi^*: \text{reg}(A) \rightarrow \text{reg}(A')$ is an exact functor inducing an equivalence $\text{reg}(A)/\mathcal{L} \rightarrow \text{reg}(A')$,

where \mathcal{L} is the localizing subcategory of $\text{reg}(A)$ generated by S .

Proof. (1) According to Theorem 4.16 we have only to show that A' is of infinite representation type. This follows from the fact that for each regular module R not belonging to the component containing S we have $\text{Hom}(S, R) = 0 = \text{Ext}^1(S, R)$, and thus $R \in S^\perp$.

(2) Let A be connected and R be a simple regular homogeneous A -module. Then $\text{Hom}_A(P, R) \neq 0$ for all preprojective A -modules. Since $R \in S^\perp$ and every projective A' -module is a preprojective A -module via φ_* (see (4)), A' is connected.

(3) If A is Morita equivalent to a path algebra of extended Dynkin type (p, q, r) , $\text{mod}(A)$ has exceptional tubes of rank p, q , and r , respectively. Then A' has exceptional tubes of rank $p - 1, q, r$ and the assertion follows.

(4) There exists a homogeneous simple regular A' -module R such that $\varphi_*(R)$ is homogeneous simple regular. Since by means of φ_* , $\text{mod}(A')$ becomes a full subcategory of $\text{mod}(A)$, closed under extensions, we obtain

$$\begin{aligned} \text{rk}(\varphi_* M) &= -\dim_k \text{Hom}_A(\varphi_* R, \varphi_* M) + \dim_k \text{Ext}_A^1(\varphi_* R, \varphi_* M) \\ &= -\dim_k \text{Hom}_A(R, M) + \dim_k \text{Ext}_A^1(R, M) = \text{rk}(M). \end{aligned}$$

Since φ_* preserves indecomposability, the assertion follows.

Let P be a preprojective A -module. Then there exists an exact sequence

$$0 \rightarrow P \rightarrow \varphi_* \varphi^* P \rightarrow S^n \rightarrow 0.$$

It follows from [6, Lemma 2.2] that $\varphi_* \varphi^* P$ is preprojective and $\text{rk} \varphi^* P = \text{rk} \varphi_* \varphi^* P = \text{rk} P$. The cases of regular and preinjective modules are similar.

- (5) For each regular A -module R there is an exact sequence

$$0 \rightarrow S^n \rightarrow R \rightarrow \varphi_* \varphi^* R \rightarrow S^m \rightarrow 0.$$

This proves that the Serre subcategory $\mathcal{L} \subseteq \text{reg}(A)$ generated by S is localizing. Since $S^\perp \subseteq \text{reg}(A)$ is equivalent to $\text{reg}(A')$, (5) follows. ■

Remark. Let S be an indecomposable regular module with $\text{Ext}^1(S, S) = 0$, let $0 = S_0 \subset S_1 \subset \dots \subset S_t = S$ be a finite filtration of S , whose factors $T_i = S_i/S_{i-1}$ are simple regular, and put $\mathcal{S} = \{T_1, \dots, T_t\}$. Then \mathcal{S} consists of non-homogeneous simple regular modules and \mathcal{S}^\perp can

be obtained by the successive formation of the perpendicular categories with respect to T_1, T_2, \dots, T_l (in that order). This means that Theorem 10.1 also applies to this more general situation, with virtually no changes necessary. Note further that according to [46] the two categories S^\perp and \mathcal{S}^\perp differ only by a module category over a representation-finite hereditary algebra Σ . More precisely Σ has type \mathbb{A}_{l-1} and $S^\perp = \mathcal{S}^\perp \amalg \text{mod}(\Sigma)$.

PROPOSITION 10.2. *Let A and A' be tame hereditary k -algebras Morita equivalent to path algebras of extended Dynkin type $\mathbf{p} = (p_0, p_1, p_2)$ and $\mathbf{q} = (q_0, q_1, q_2)$, respectively. If there exists an epimorphism $\varphi: A \rightarrow A'$, then \mathbf{p} dominates \mathbf{q} , i.e. the weight function corresponding to \mathbf{p} dominates the weight function corresponding to \mathbf{q} .*

Proof. φ induces a full exact embedding $\varphi_*: \text{mod}(A') \rightarrow \text{mod}(A)$. Let R be an indecomposable regular A' -module. Then the functors $\text{Hom}_{A'}(R, -)$ and $\text{Hom}_A(-, R)$ are non-zero on infinitely many pairwise non-isomorphic indecomposable A -modules.

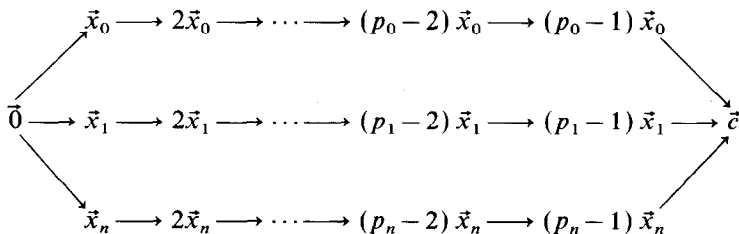
Clearly, the functors $\text{Hom}_A(\varphi_* R, -)$ and $\text{Hom}_{A'}(-, \varphi_* R)$ have the corresponding properties, and consequently $\varphi_* R$ is a regular A -module. In particular, different regular components are mapped to different regular components by means of φ_* .

Let \mathcal{R}' be a regular tube of rank p in $\text{mod}(A')$ and \mathcal{R} be the regular component in $\text{mod}(A)$ such that $\varphi_* R \in \mathcal{R}$ for all $R \in \mathcal{R}'$.

If $R \in \mathcal{R}'$ is indecomposable of regular length p , then $\text{End}(R) = k$. Thus $\varphi_* R$ is indecomposable, $\text{End}(\varphi_* R) = k$, and $\varphi_* R$ has regular length $\geq p$. Hence the rank of the tube \mathcal{R} is $\geq p$ and the assertion follows. ■

As specified by Theorem 10.1 and Proposition 10.2, the existence of homological epimorphisms between algebras, Morita equivalent to path algebras of extended Dynkin type, is therefore given by the Fig. 1. Note that Fig. 1 agrees with the degeneration scheme for the simple singularities of differentiable maps (cf., for instance, [1, p. 76]).

The situation for the canonical algebras is similar but no longer restricted to weight sequences of Dynkin type. Recall that for given sequences $\mathbf{p} = (p_0, \dots, p_n)$, $\underline{\lambda} = (\lambda_0, \dots, \lambda_n)$, the canonical algebra $A(\mathbf{p}, \underline{\lambda})$ in terms of quivers and relations is given by the quiver



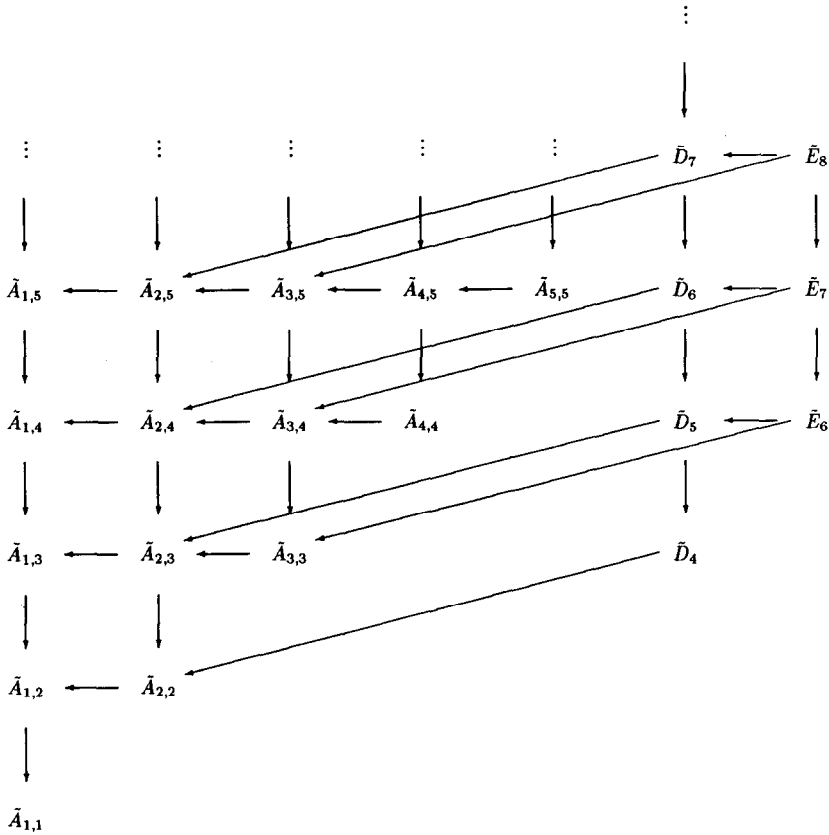


FIG. 1. Domination for algebras of extended Dynkin type.

with relations

$$X_i^{p_i} = X_0^{p_0} - \lambda_i X_1^{p_1}, \quad \text{for } i = 2, \dots, n.$$

For the properties of modules over canonical algebras we refer to [38, 21]. Note that also in this case we have a rank function, defined in the same way as is the hereditary case but allowing the more accessible alternative definition

$$rk_k M = \dim_k (M_{\bar{c}}) - \dim_k (M_{\bar{0}}).$$

A A -module R is called *regular* if R is a direct sum of indecomposable

A -modules of rank 0. The category $\text{reg}(A)$ of regular modules is an abelian length category and decomposes into a coproduct

$$\text{reg}(A) = \coprod_{\lambda \in \mathbf{C}(\mathbf{p}, \underline{\lambda})} \mathcal{R}_\lambda,$$

where each \mathcal{R}_λ is a uniserial length category with $w(\lambda)$ simple modules. Here w denotes the weight function corresponding to $(\mathbf{p}, \underline{\lambda})$. If $p_i > 1$ and $S \in \mathcal{R}_{\lambda_i}$ is simple, we have $\text{End}(S) = k$, $\text{Ext}^1(S, S) = 0$, $\text{proj dim } S = 1$, and $\text{Hom}(S, A) = 0$. Again, due to Theorem 4.16, we obtain a finite dimensional algebra A' and a homological epimorphism $\varphi: A \rightarrow A'$.

Extending the terminology used for the hereditary case, we call a finite dimensional A -module M *preprojective* (*preinjective*, *regular*) if any indecomposable direct factor has rank > 0 (< 0 , respectively, $= 0$).

THEOREM 10.3. *Let $A = A(\mathbf{p}, \underline{\lambda})$ be a canonical algebra. We fix some $p_i > 1$ and some $S \in \mathcal{R}_{\lambda_i}$ and denote by $\varphi: A \rightarrow A'$ the corresponding homological epimorphism. Then the following assertions hold true:*

- (1) A' is Morita equivalent to the canonical algebra $A(\mathbf{p}', \underline{\lambda})$, where $\mathbf{p}' = (p_0, \dots, p_{i-1}, p_i - 1, p_{i+1}, \dots, p_n)$.
- (2) The induced functors $\varphi_*: \text{mod}(A') \rightarrow \text{mod}(A)$ and $\varphi^*: \text{mod}(A) \rightarrow \text{mod}(A')$ map preprojective (preinjective, resp., regular) modules to modules with the same property. In particular $\varphi^*: \text{reg}(A) \rightarrow \text{reg}(A')$ is an exact functor, inducing an equivalence

$$\text{reg}(A)/\mathcal{L} \rightarrow \text{reg}(A'),$$

where \mathcal{L} denotes the localizing subcategory of $\text{mod}(A)$ generated by S .

Proof. (1) As was shown in [21] the indecomposable A -modules of rank 1 may be parametrized by the elements of $L(\mathbf{p})^+$, notation $P(\vec{x})$. The modules $P(k\vec{x}_j)$ with $0 \leq j \leq n$, $0 \leq k \leq p_j$, are just the projective A -modules. We have $\text{Ext}^1(S, P(k\vec{x}_i)) \neq 0$ for exactly one $k \in \{0, \dots, p_i - 1\}$ and in this case the dimension is 1. If $k \neq 0$, $\text{Ext}^1(S, P(l\vec{x}_{j\ell})) = 0$ for all $0 \leq j \leq n$, $i \neq j$, and $0 \leq l \leq p_j$. Thus $\varphi^*(A)$ has the form

$$P(\vec{0}) \oplus \dots \oplus P((k-1)\vec{x}_i) \oplus P((k+1)\vec{x}_i)^2 \\ \oplus \dots \oplus P(\vec{c}) \oplus \bigoplus_{\substack{j=0 \\ i \neq j}}^n \bigoplus_{l=1}^{p_j-1} P(l\vec{x}_j).$$

If $k = 0$, $\dim \text{Ext}^1(S, P(l\tilde{x}_j)) = 1$ for all $0 \leq j \leq n$, $i \neq j$, and $0 \leq l \leq p_j$ and $\varphi^*(A)$ has the form

$$P(\tilde{x}_i)^2 \oplus P(2\tilde{x}_i) \oplus \dots \oplus P(\tilde{c}) \oplus P(\tilde{c} + \tilde{x}_i) \oplus \bigoplus_{\substack{j=0 \\ i \neq j}}^n \bigoplus_{l=1}^{p_j-1} P(l\tilde{x}_j + \tilde{x}_i).$$

In both cases A' is Morita equivalent to $A(\mathbf{p}', \underline{\lambda})$.

The proof of assertion (2) is identical to the proof of assertions (4) and (5) of Theorem 10.1. ■

11. AFFINE AND LOCAL ALGEBRAS FOR WEIGHTED PROJECTIVE LINES AND CANONICAL AND TAME HEREDITARY ALGEBRAS

Let $\mathbf{C} = \mathbf{C}(\mathbf{p}, \underline{\lambda})$ be a weighted projective line and $U \subset \mathbf{C}$ be a subset. We denote by \mathcal{S}_U the system of all simple sheaves concentrated in a point of $\mathbf{C} \setminus U$ and by \mathcal{L}_U the localizing subcategory of $\text{Qcoh}(\mathbf{C})$ generated by \mathcal{S}_U . Further let R_U be the homogeneous quotient ring of R with respect to the multiplicative subset generated by all elements f_λ , $\lambda \in \mathbf{C} \setminus U$, where

$$f_\lambda = \begin{cases} X_1^{p_1} - \lambda X_0^{p_0} & \text{if } \lambda \neq \lambda_i, i = 0, \dots, n \\ X_i & \text{if } \lambda = \lambda_i. \end{cases}$$

Then the functor

$$\Phi_U: \text{Qcoh}(\mathbf{C}) \rightarrow \text{Mod}^{L(\mathbf{p})}(R_U), \quad \mathcal{G} \mapsto \lim_{V \supseteq U} \mathcal{G}(V),$$

where $V \supseteq U$ is open in \mathbf{C} , has kernel \mathcal{L}_U and induces an equivalence

$$\text{Qcoh}(\mathbf{C})/\mathcal{L}_U \rightarrow \text{Mod}^{L(\mathbf{p})}(R_U).$$

Thus, the perpendicular category \mathcal{S}_U^\perp is equivalent to the category of all $L(\mathbf{p})$ -graded modules over R_U . Since the module $P_U = \bigoplus_{0 \leq \tilde{x} \leq \tilde{c}} R_U(\tilde{x})$ is always a small projective generator in $\text{Mod}^{L(\mathbf{p})}(R_U)$, \mathbf{S}_U^\perp is equivalent to $\text{Mod}(A_U)$, where $A_U = \text{End}(P_U)$. Note that A_U , in general, is non-commutative and not Morita equivalent to any commutative algebra.

If $U = \mathbf{C} \setminus \{\mu_1, \dots, \mu_n\}$ is an affine open subset, $R_U = R_{f_{\mu_1}, \dots, f_{\mu_n}}$ is an affine algebra and Φ_U becomes the restriction to the affine open subset U , while in the case $U = \{\lambda\}$, $R_U = \mathcal{O}_{\mathbf{C}, \lambda}$ and Φ_U becomes the passage to the stalk at λ .

In the following we give a more explicit description of an algebra A with $\text{Mod}(A) \simeq \text{Mod}^{L(\mathbf{p})}(R_U)$ in the cases $U = \mathbf{C} \setminus \{\lambda\}$ and $U = \{\lambda\}$.

We start with the affine case. Thus let $\lambda \in \mathbf{P}_1(k)$ and $U = \mathbf{C} \setminus \{\lambda\}$. By

applying a transformation $\sigma = \mathbf{SL}(2, \mathbf{C})$ with $\sigma(\lambda) = \infty$ we may assume that $\lambda = \infty$; hence $f_\lambda = X_0$ and R_U is the $L(\mathbf{p})$ -graded algebra

$$R_U = k[X_0, X_0^{-1}, X_1, \dots, X_n]/(X_i^{p_i} - X_1^{p_1} + \lambda_i X_0^{p_0}, i = 2, \dots, n).$$

Let \bar{R}_U denote the algebra

$$\bar{R}_U = k[X_1, \dots, X_n]/(X_i^{p_i} - X_1^{p_1} + \lambda_i, i = 2, \dots, n).$$

Further let the abelian group $H = H(p_1, \dots, p_n)$ be defined by generators $\hat{x}_1, \dots, \hat{x}_n$ and relations $p_1 \hat{x}_1 = \dots = p_n \hat{x}_n = 0$. We note that H is isomorphic to $\mathbf{Z}_{p_1} \times \dots \times \mathbf{Z}_{p_n}$ and \bar{R}_U is H -graded by $\deg X_i = \hat{x}_i$ for $i = 1, \dots, n$.

We define a homomorphism of abelian groups $\varphi: L(\mathbf{p}) \rightarrow H$ by $\bar{x}_0 \mapsto 0$ and $\bar{x}_i \mapsto \hat{x}_i$ for $i = 1, \dots, n$ and a homomorphism of algebras $u: R_U \rightarrow \bar{R}_U$ by $X_0 \mapsto 1$ and $X_i \mapsto X_i$ for $i = 1, \dots, n$.

One easily checks that $u: (R_U)_{\bar{\tau}} \rightarrow (\bar{R}_U)_{\varphi(\bar{\tau})}$ is an isomorphism for all $\bar{\tau} \in L(\mathbf{p})$. The morphism of graded algebras $(u, \varphi): R_U \rightarrow \bar{R}_U$ induces an equivalence of categories

$$(u, \varphi)_*: \text{Mod}^H(\bar{R}_U) \rightarrow \text{Mod}^{L(\mathbf{p})}(R_U), \quad \bigoplus_{h \in H} \bar{M}_h \mapsto \bigoplus_{\bar{\tau} \in L(\mathbf{p})} M_{\bar{\tau}},$$

where $M_{\bar{\tau}}$ is defined by $M_{\bar{\tau}} = \bar{M}_{\varphi(\bar{\tau})}$.

R_U is a $L(\mathbf{p})$ -graded factorial, where the complete list of primes is given by $f_{\lambda'}$, $\lambda' \in k$. The elements $u(f_{\lambda'})$, $\lambda' \in k$, form a complete list of primes in \bar{R}_U . Moreover \bar{R}_U is an H -graded principal ideal domain.

The modules $\bar{R}_U(h)$ ($h \in H$) are projective and form a system of generators for $\text{Mod}^H(\bar{R}_U)$. Hence there is $(p_1 \dots p_n \times p_1 \dots p_n)$ -matrix algebra A such that $\text{Mod } A$ is equivalent to $\text{Mod}^{L(\mathbf{p})}(R_U)$ and the indecomposable projective A -modules correspond to the elements of H .

Now we deal with the local case. Let $U = \{\lambda\}$ and without loss of generality we assume that $\lambda = 0$. Now $R_U = R_{(f_0)} = S^{-1}R$ is the localization with respect to the multiplicative subset S generated by all f_λ with $\lambda \in \mathbf{P}_1(k)$ and $\lambda \neq 0$.

Let $\bar{R}_{(f_0)} = k[Y]_{(Y)}$ be the localization of $k[Y]$ in the prime ideal (Y) . If $H = \mathbf{Z}_{p_1}$ with generator \hat{x}_1 , $\bar{R}_{(f_0)}$ is H -graded by $\deg Y = \hat{x}_1$.

Let $\varphi: L(\mathbf{p}) \rightarrow H$ be defined by $\bar{x}_1 \mapsto \hat{x}_1$ and $\bar{x}_i \mapsto 0$ for all $i \neq 1$. Further we define $v: R \rightarrow \bar{R}_{(f_0)}$ by $X_1 \mapsto Y$ and $X_i \mapsto 1$ for all $i \neq 1$. Since for all $\lambda \neq 0$ the element $v(f_\lambda)$ is invertible in $\bar{R}_{(f_0)}$, the morphism v induces a homomorphism of algebras $u: R_{(f_0)} \rightarrow \bar{R}_{(f_0)}$. Again, the morphism of graded algebras $(u, \varphi): R_{(f_0)} \rightarrow \bar{R}_{(f_0)}$ induces an equivalence of categories

$$(u, \varphi)_*: \text{Mod}^H(\bar{R}_{(f_0)}) \rightarrow \text{Mod}^{L(\mathbf{p})}(R_{(f_0)}).$$

The modules $\bar{R}_{(f_0)}(h)$ ($h \in H$) are projective and form a system of generators for $\text{Mod}^H(\bar{R}_{(f_0)})$.

Let A denote the matrix algebra

$$\begin{pmatrix} \bar{R}_0 & \bar{R}_1 & \cdots & \bar{R}_{p-2} & \bar{R}_{p-1} \\ \bar{R}_{p-1} & \bar{R}_0 & \cdots & & \bar{R}_{p-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \bar{R}_2 & & \cdots & \bar{R}_0 & \bar{R}_1 \\ \bar{R}_1 & \bar{R}_2 & \cdots & \bar{R}_{p-1} & \bar{R}_0 \end{pmatrix},$$

where $\bar{R}_k = Y^k k[Y^p]_{(Y^p)}$. Then $\text{Mod}(A)$ is equivalent to $\text{Mod}^{L(p)}(R_{(f_0)})$.

Now let $U \subset C$ be arbitrary. Since $\text{proj dim } \mathcal{S}_U = 1$, \mathcal{S}_U^\perp is an exact subcategory and the section functor $\Sigma_U: \text{Qcoh}(C)/\mathcal{L}_U \rightarrow \text{Qcoh}(C)$ is exact. Further Σ_U commutes with arbitrary direct sums because $\text{Hom}(S, -)$ has this property for all $S \in \mathcal{S}_U$ (Lemma 2.6). Since \mathcal{S}_U is closed under twists, the same holds true for \mathcal{S}_U^\perp .

The following describes the sheaves belonging to \mathcal{S}_U^\perp .

PROPOSITION 11.1. *Let \mathcal{G} be a quasi-coherent sheaf, $\lambda \in C$ and \mathcal{S} the direct sum of all simple sheaves concentrated in λ . Then $\mathcal{G} \in \mathcal{S}^\perp$ if and only if the stalk \mathcal{G}_λ is injective and $\text{Hom}(\mathcal{S}_\lambda, \mathcal{G}_\lambda) = 0$.*

Proof. Since \mathcal{S} is concentrated in λ , $\text{Hom}_C(\mathcal{S}, \mathcal{G}) = 0$ if and only if $\text{Hom}_{\mathcal{O}_U}(\mathcal{S}_U, \mathcal{G}_U) = 0$ for all affine open neighborhoods U of λ and this is equivalent to $\text{Hom}_{\mathcal{O}_{C,\lambda}}(\mathcal{S}_\lambda, \mathcal{G}_\lambda) = 0$. Analogously, $\text{Ext}_\theta^1(\mathcal{S}, \mathcal{G}) = 0$ if and only if $\text{Ext}_{\mathcal{O}_{C,\lambda}}^1(\mathcal{S}_\lambda, \mathcal{G}_\lambda) = 0$.

Using reduction of weight, we may assume that λ is an ordinary point. Then $\mathcal{O}_{C,\lambda}$ is a graded valuation ring and $\text{Ext}_{\mathcal{O}_{C,\lambda}}^1(\mathcal{S}_\lambda, \mathcal{G}_\lambda) = 0$ if and only if \mathcal{G} is injective. ■

Let $A = A(\mathbf{p}, \underline{\lambda})$ be a canonical algebra and \mathcal{S} be a system of simple objects in \mathcal{R} . We compute the perpendicular category \mathcal{S}^\perp using, via tilting, the corresponding results for the categories of coherent sheaves on weighted projective lines and the results from Section 6.

In [21], a coherent sheaf \mathcal{F} over $C = C(\mathbf{p}, \underline{\lambda})$ was called a *tilting sheaf* if

- (1) $\text{Ext}^1(\mathcal{F}, \mathcal{F}) = 0$,
- (2) \mathcal{F} generates $D^b(\text{coh}(C))$, and
- (3) $\text{gl dim}(\text{End}(\mathcal{F})) < \infty$.

Note that with the definition given in Section 6 the sheaf \mathcal{F} is just a tilting object in $\text{coh}(C)$.

A tilting sheaf \mathcal{F} in $\text{Qcoh}(C)$ is by definition a tilting object in $\text{Qcoh}(C)$.

THEOREM 11.2. *Let $\mathbf{C} = \mathbf{C}(\mathbf{p}, \underline{\lambda})$ be a weighted projective line and $\mathcal{F} \in \text{coh}(\mathbf{C})$ be a tilting sheaf. Then \mathcal{F} is a tilting sheaf in $\text{Qcoh}(\mathbf{C})$.*

Proof. Since \mathcal{F} is a tilting sheaf in $\text{coh}(\mathbf{C})$ it remains to prove that \mathcal{F} generates $D^b(\text{Qcoh}(\mathbf{C}))$. For this it is sufficient to show that $\text{Qcoh}(\mathbf{C})$ is the smallest subcategory \mathcal{A} of $\text{Qcoh}(\mathbf{C})$ which contains all direct factors of \mathcal{F} and is closed under arbitrary direct sums, kernels of epimorphisms, cokernels of monomorphisms, and extensions.

Since \mathcal{F} is a tilting sheaf in $\text{coh}(\mathbf{C})$, we have $\text{coh}(\mathbf{C}) \subset \mathcal{A}$; hence also arbitrary direct sums of coherent sheaves are contained in \mathcal{A} . We prove by induction on $p = \prod_{i=0}^n p_i$ that this implies $\mathcal{A} = \text{Qcoh}(\mathbf{C})$.

If $p = 1$, $\mathbf{C}(\mathbf{p}, \underline{\lambda}) = \mathbf{P}_1(k)$ and since $k[X, Y]$ is of finite global dimension, every quasi-coherent sheaf has a finite resolution by direct sums of line bundles. Now, let $p > 1$. Then by reduction of weight there exists a full exact embedding

$$\text{Qcoh}(\mathbf{C}(\mathbf{p}', \underline{\lambda})) \rightarrow \text{Qcoh}(\mathbf{C}(\mathbf{p}, \underline{\lambda}))$$

with $p' = \prod_{i=0}^n p'_i < p$. Since this embedding maps coherent sheaves to coherent sheaves and commutes with arbitrary direct sums we have $\text{Qcoh}(\mathbf{C}(\mathbf{p}', \underline{\lambda})) \subset \mathcal{A}$ by the induction hypothesis. Moreover, for $\mathcal{G} \in \text{Qcoh}(\mathbf{C})$ there is an exact sequence

$$0 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{G} \rightarrow \bar{\mathcal{G}} \rightarrow \mathcal{F}_1 \rightarrow 0,$$

where $\bar{\mathcal{G}} \in \text{Qcoh}(\mathbf{C}(\mathbf{p}', \underline{\lambda}))$ and $\mathcal{F}_0, \mathcal{F}_1$ are contained in the localizing subcategory generated by a simple sheaf \mathcal{S} concentrated in an exceptional point of \mathbf{C} . Since $\text{Ext}^1(\mathcal{S}, \mathcal{S}) = 0$, $\mathcal{F}_0, \mathcal{F}_1$ are semi simple and thus contained in \mathcal{A} . Hence $\mathcal{G} \in \mathcal{A}$ and $\mathcal{A} = \text{Qcoh}(\mathbf{C})$ follows. ■

LEMMA 11.3. *Let $\mathcal{F} \in \text{coh}(\mathbf{C})$ be a vector bundle and $\mathcal{G} \in \mathcal{S}^\perp$. Then we have $\text{Ext}^1(\mathcal{F}, \mathcal{G}) = 0$.*

Proof. By means of a line filtration for \mathcal{F} we have only to show that $\text{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{G}) = 0$ for all $\vec{x} \in L(\mathbf{p})$. Let $\bigoplus_{i \in I} \mathcal{O}(\vec{y}_i) \rightarrow \mathcal{G}$ be an epimorphism. Since the section functor $\Sigma_U: \text{Qcoh}(\mathbf{C})/\mathcal{L}_U \rightarrow \text{Qcoh}(\mathbf{C})$ is exact and commutes with arbitrary direct sums, we obtain an epimorphism $\bigoplus_{i \in IA} \Sigma_U T_U \mathcal{O}(\vec{y}_i) \rightarrow \Sigma_U T_U \mathcal{G} \cong \mathcal{G}$, where $T_U: \text{Qcoh}(\mathbf{C}) \rightarrow \text{Qcoh}(\mathbf{C})/\mathcal{L}_U$ denotes the quotient functor. Since the category $\text{Qcoh}(\mathbf{C})$ has global dimension 1, $\text{Ext}^1(\mathcal{O}(\vec{x}), \mathcal{G}) = 0$ follows from

$$\text{Ext}^1 \left(\mathcal{O}(\vec{x}), \bigoplus_{i \in I} \Sigma_U T_U \mathcal{O}(\vec{y}_i) \right) \cong \bigoplus_{i \in I} \text{Ext}^1(\mathcal{O}(\vec{x}), \Sigma_U T_U \mathcal{O}(\vec{y}_i)) = 0.$$

Thus, it remains to show that $\text{Ext}^1(\mathcal{O}(\vec{x}), \Sigma_U T_U \mathcal{O}(\vec{y})) = 0$ for all $\vec{x}, \vec{y} \in L(\mathbf{p})$.

Let $0 \rightarrow \mathcal{O}(\bar{y}) \rightarrow \Sigma_U T_U \mathcal{O}(\bar{y}) \rightarrow \mathcal{F} \rightarrow 0$ be exact with $\mathcal{F} \in \mathcal{L}_U$. If $\mathcal{S} \subset \mathcal{F}$ is a simple subsheaf, the inverse image of \mathcal{S} in $\Sigma_U T_U \mathcal{O}(\bar{y})$ has the form $\mathcal{O}(\bar{y} + \bar{y}')$ with $y' = \bar{x}_i$ or $y' = \bar{c}$ and $\Sigma_U T_U \mathcal{O}(\bar{y})/\mathcal{O}(\bar{y} + \bar{y}') \in \mathcal{L}_U$. This shows that $\Sigma_U T_U \mathcal{O}(\bar{y}) \cong \Sigma_U T_U \mathcal{O}(\bar{y} + n\bar{c})$ for all integers n and we may assume that $\bar{x} + \bar{\omega} < \bar{y}$, where $\bar{\omega}$ denotes the dualizing element. Then $\text{Ext}^1(\mathcal{O}(\bar{x}), \mathcal{O}(\bar{y})) = 0$ by Serre duality and $\text{Ext}^1(\mathcal{O}(\bar{x}), \Sigma_U T_U \mathcal{O}(\bar{y})) = 0$ follows. ■

THEOREM 11.4. *Let $\mathcal{F} \in \text{coh}(\mathbf{C})$ be a tilting sheaf and a vector bundle and let $A = \text{End}(\mathcal{F})$. Further let \mathcal{S}'_U be the system of all A -modules of the form $\text{Hom}_\theta(\mathcal{F}, S)$ with $S \in \mathcal{L}_U$.*

Then $\text{Hom}(\mathcal{F}, -): \mathcal{S}'_U \rightarrow (\mathcal{S}'_U)^\perp$ is an equivalence of categories and the embedding $(\mathcal{S}'_U)^\perp \rightarrow \text{Mod}(A)$ has a left adjoint. In particular, there exists an algebra A Morita equivalent to A_U and a homological epimorphism $\varphi: A \rightarrow A$ inducing this embedding.

Proof. By Lemma 11.3, \mathcal{S}'_U is contained in \mathcal{X}_0 ; hence by Corollary 6.3, $\text{Hom}(\mathcal{F}, -): \mathcal{S}'_U \rightarrow (\mathcal{S}'_U)^\perp$ is an equivalence.

Let $G_0 = - \otimes_A \mathcal{F}$ and $F_0 = \text{Hom}(\mathcal{F}, -)$. We define $l: \text{Mod}(A) \rightarrow (\mathcal{S}'_U)^\perp$ as the composition $l = F_0 \Sigma_U T_U G_0$. Let $M \in \text{Mod}(A)$, $M = M_0 \oplus M_1$ with $M_0 \in \mathcal{Y}_0$, $M_1 \in \mathcal{Y}_1$, and $N \in (S')^\perp$. Then we have functorial isomorphisms

$$\begin{aligned} \text{Hom}_A(M, N) &\cong \text{Hom}_A(M_0, N) \cong \text{Hom}_\theta(G_0 M, G_0 N) \\ &\cong \text{Hom}_\theta(\Sigma_U T_U G_0 M, G_0 N) \\ &\cong \text{Hom}_A(F_0 \Sigma_U T_U G_0 M, F_0 G_0 N) \cong \text{Hom}(lM, N). \end{aligned}$$

Thus l is left adjoint to the embedding $(S')^\perp \rightarrow \text{Mod}(A)$. Since \mathcal{F} is coherent, lA is a small projective generator of $\text{Mod}^{L(\mathbf{p})}(R_U)$; thus $\text{Mod}^{L(\mathbf{p})}(R_U) \cong \text{Mod}(\text{End}_A(lA))$ and the embedding is induced by a homological epimorphism $\varphi: A \rightarrow \text{End}_A(lA)$. ■

By applying this theorem to canonical algebras we obtain:

COROLLARY 11.5. *Let $A = A(\mathbf{p}, \lambda)$ be a canonical algebra, $U \subset \mathbf{C} = \mathbf{C}(\mathbf{p}, \lambda)$ a subset, and \mathcal{S}'_U the system of all simple objects in $\mathcal{R}_U = \coprod_{\mu \in \mathbf{C} \setminus U} \mathcal{R}_\mu$. Then $(\mathcal{S}'_U)^\perp$ is a full exact subcategory closed under arbitrary direct sums and equivalent to $\text{Mod}(A_U)$. In particular, there exists a homological epimorphism $\varphi: A \rightarrow A_U$ inducing this embedding.*

Proof. $\mathcal{F} = \bigoplus_{\bar{0} \leq \bar{x} \leq \bar{c}} \mathcal{O}(\bar{x})$ is a tilting sheaf consisting of line bundles with $\text{End}(\mathcal{F}) = A$ and \mathcal{S}'_U corresponds to \mathcal{S}_U by means of the functor $\text{Hom}_\theta(\mathcal{F}, -)$. Further $\text{End}(lA) \cong \text{End}(T_U G_0 A) \cong \text{End}(T_U \mathcal{F}) \cong \text{End}(P_U) = A_U$. ■

A similar result holds true for tame hereditary algebras over algebraically closed fields:

COROLLARY 11.6. *Let A be tame hereditary algebra over an algebraically closed field of Dynkin type \mathfrak{p} , let U be a subset of $\mathbf{C} = \mathbf{C}(\mathfrak{p}, \underline{\lambda})$, and let \mathcal{S}'_U be the system of all simple regular modules in $\mathcal{R}_U = \coprod_{\lambda \in \mathbf{C} \setminus U} \mathcal{R}_\lambda$.*

Then $(\mathcal{S}'_U)^\perp$ is a full exact subcategory of $\text{Mod}(A)$, closed under arbitrary direct sums and equivalent to $\text{Mod}(A_U)$. In particular, there exists an algebra A , Morita equivalent to A_U and a homological epimorphism $\varphi: A \rightarrow A$, inducing this embedding.

Proof. There exists a tilting sheaf $\mathcal{T} \in \text{coh}(\mathbf{C})$ consisting of vector bundles with $\text{End}(\mathcal{T}) \cong A$.

So far, we computed the perpendicular category $(\mathcal{S}'_U)^\perp \subset \text{Mod}(A)$, A tame hereditary, only in the case where A is an algebra over an algebraically closed field. In order to extend this result to arbitrary tame hereditary algebras, we use the same strategy, replacing the category $\text{Qcoh}(\mathbf{C})$ by the category $\mathcal{G} = (\mathcal{P}, \mathbf{Ab})/(\mathcal{P}, \mathbf{Ab})_0$. Here, \mathcal{P} denotes the category of all preprojective right A -modules of finite length, $(\mathcal{P}, \mathbf{Ab})$ the category of all abelian group valued additive functors on \mathcal{P} , and $(\mathcal{P}, \mathbf{Ab})_0$ the localizing subcategory generated by all simple functors. If $F: \mathcal{P} \rightarrow \mathbf{Ab}$ is a covariant functor, its image in \mathcal{G} is denoted by \tilde{F} .

\mathcal{G} is a locally noetherian Grothendieck category with the objects $\mathcal{P}(P, -)^\sim$, P preprojective, forming a set of small noetherian generators. The structure of the category \mathcal{F} of noetherian objects in \mathcal{G} was determined in [30]; see also [21].

If $T = \text{Hom}_A(A_A, -)$, then \tilde{T} is a tilting object in \mathcal{G} , and hence:

- (1) $\text{Ext}^i(\tilde{T}, \tilde{T}) = 0$ for all $i > 0$,
- (2) \tilde{T} generates $D^b(\mathcal{G})$,
- (3) $\text{gl dim}(\text{End}(\tilde{T})) < \infty$.

(1) follows from $\text{gl dim } \mathcal{F} = 1$ and $\text{Ext}^1(\tilde{T}, \tilde{T}) = 0$; (2) follows from the fact that $(\mathcal{P}, \mathbf{Ab})$ has global dimension 2, with Auslander–Reiten theory and $\text{End}(\tilde{T}) \cong A$ invoked. Now, Theorem 11.2 applies. In particular, the category of finite length objects in \mathcal{G} and the category of regular right A -modules are equivalent.

Let \mathbf{C} be the set of all regular Auslander–Reiten components and $U \subset \mathbf{C}$ a subset. Further let \mathcal{S}'_U be the system of all simple objects in $\mathcal{R}_U = \coprod_{\lambda \in \mathbf{C} \setminus U} \mathcal{R}_\lambda$, \mathcal{S}_U the corresponding system of simple objects in \mathcal{G} , and \mathcal{L}_U the localizing subcategory generated by \mathcal{S}_U . Analogously to Lemma 11.3, $\text{Ext}_{\mathcal{G}}^1(\tilde{T}, G) = 0$ for all $G \in \mathcal{S}'_U$ and Theorem 11.4 applies. Thus $\text{Hom}(\tilde{T}, -): \mathcal{S}'_U \rightarrow (\mathcal{S}'_U)^\perp$ is an equivalence and the embedding $(\mathcal{S}'_U)^\perp \rightarrow \text{Mod}(A)$ has a left adjoint.

It remains to describe the structure of $\mathcal{G}/\mathcal{L}_U$. For that purpose, let Σ_U denote the set of all monomorphisms in \mathcal{P} with cokernel in \mathcal{R}_U and $\Sigma_U^{-1}\mathcal{P}$ be the category of all (left) fractions of \mathcal{P} with respect to Σ_U ; see [6] for details. Then the kernel of the localizing functor

$$\Sigma_U^{-1}: (\mathcal{P}, \mathbf{Ab}) \rightarrow (\Sigma_U^{-1}\mathcal{P}, \mathbf{Ab})$$

is the localizing subcategory of $\text{Mod}(A)$ generated by \mathcal{S}_U and the simple functors [6, Lemma 6.3]. Thus the perpendicular subcategory \mathcal{S}_U^\perp in \mathcal{G} is equivalent to $(\Sigma_U^{-1}\mathcal{P}, \mathbf{Ab})$. Finally, if $\Pi(A)$ is the preprojective algebra of preprojective right A -modules [6, Lemma 6.3] and Σ'_U the set of all monomorphisms $f: A \rightarrow (\text{Tr } D)^n A$ with cokernel in \mathcal{R}_U , $(\Sigma_U^{-1}\mathcal{P}, \mathbf{Ab})$ is equivalent to the category $\text{Mod}^{\mathbf{Z}^+}((\Sigma'_U)^{-1}\Pi(A)^{\text{op}})$ of \mathbf{Z}_+ -graded left $\Sigma_U^{-1}\Pi(A)$ -modules.

In summarizing the preceding we obtain

THEOREM 11.7. *Let A be a tame hereditary Artin algebra, U be a set of regular Auslander–Reiten components of $\text{mod}(A)$, and \mathcal{P}'_U be the system of all simple regular right A -modules in $\mathcal{R}_U = \coprod_{\lambda \in C \setminus U} \mathcal{R}_\lambda$. Then $(\mathcal{P}'_U)^\perp$ is a full exact subcategory of $\text{Mod}(A)$ closed under arbitrary direct sums and equivalent to the category $\text{Mod}^{\mathbf{Z}^+}((\Sigma'_U)^{-1}\Pi(A)^{\text{op}})$ of all \mathbf{Z}_+ -graded $\Pi(A)$ -modules. In particular, there is a noetherian algebra A_U Morita equivalent to the \mathbf{Z}_+ -graded algebra $(\Sigma_U)^{-1}\Pi(A)^{\text{op}}$ and a homological epimorphism $\varphi: A \rightarrow A_U$ inducing this embedding (up to equivalence).*

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