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Polynomial function and derivative approximation of Sinc data

Frank Stenger

School of Computing, University of Utah, Salt Lake City, UT 84112, USA

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ABSTRACT

Sinc methods consist of a family of one dimensional approximation procedures for approximating nearly every operation of calculus. These approximation procedures are obtainable via operations on Sinc interpolation formulas. Nearly all of these approximations – except that of differentiation – yield exceptional accuracy. The exception: when differentiating a Sinc interpolation formula that gives an approximation over an interval with a finite end-point. In such cases, we obtain poor accuracy in the neighborhood of the finite end-point. In this paper we derive novel polynomial-like procedures for differentiating a function that is known at Sinc points, to obtain an approximation of the derivative of the function that is uniformly accurate on the whole interval, finite or infinite, in the case when the function itself has a derivative on the closed interval.

1. Introduction and summary

To date, all Sinc methods, i.e., one dimensional formulas for approximating operations of calculus on functions defined on an arc Γ, have been obtained by applying the operation of calculus to the Sinc interpolation formula,

$$f(x) \approx \sum_{k=-M}^{N} f(z_k)\omega_k(x), \quad x \in \Gamma, \tag{1.1}$$

where the z_k are Sinc points on Γ, and where the ω_k are basis functions to be described more explicitly in Section 2.

While such Sinc methods are generally accurate, they are inaccurate for approximating derivatives in the neighborhood of a finite Sinc end-point, where these derivative approximations are obtained

E-mail address: stenger@cs.utah.edu.

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by differentiating the right hand side of (1.1), i.e., via the use of the formula

$$f'(x) \approx \sum_{k=-M}^{N} f(z_k)(\omega_k)'(x). \tag{1.2}$$

Derivatives of Sinc approximations are nevertheless necessary at times, for many problems of applications.

It turns out that the formula (1.2) is uniformly accurate whenever the (open) interval Γ of approximation is fully infinite, i.e., if neither of the end-points of Γ is finite. Indeed, the formula (1.2) is known to converge uniformly on all compact subsets of the open interval Γ, even when Γ has a finite end-point. However, the approximation (1.2) does not converge uniformly on all of Γ if Γ has a finite end-point, i.e., the accuracy of (1.2) gets progressively worse as we approach a finite end-point of Γ, even when f' exists at such end-points.

For example, in his excellent thesis, Hohn (see [1]) recently applied directly, the formula (1.2), but in doing so, he had to go to considerable experimental efforts along the lines of parameter selections of M, N, and h (see below) to get his results.

In the hope of alleviating these approximation problems, we study in this paper the use of "polynomials" to approximate functions defined at Sinc points, and the resulting derivative approximations obtainable via differentiating such polynomials. This intuitive approach is based on the fact that polynomials as well as their derivatives converge rapidly for functions that are analytic in a region containing a closed interval, and that is the setting under which we derive our approximations.

Initially, we obtain our results for the interval $\Gamma = (0, 1)$, and we then also describe in detail how to extend the results to other intervals or arcs Γ.

2. Sinc notations

It is convenient to recall some Sinc notations (see [2,3]).

Let $\mathbb{Z}$ denote the set of all integers, let $\mathbb{R} = (-\infty, \infty)$ denote the real line, and let $\mathbb{C}$ denote the complex plane $\{x + iy : x \in \mathbb{R},\ y \in \mathbb{R}\}$. Let h denote a positive parameter, let $k \in \mathbb{Z}$, $x \in \mathbb{C}$, and let $\operatorname{sinc}(x)$ and $S(k, h)(x)$ be defined by

$$\begin{aligned} \operatorname{sinc}(x) &= \frac{\sin(\pi x)}{\pi x} \\ S(k,h)(x) &= \operatorname{sinc}\left(\frac{x}{h} - k\right). \end{aligned} \tag{2.1}$$

Let d denote a positive number, and let φ denote a conformal map of a simply connected region $\mathcal{D} \subset \mathbb{C}$ onto the strip

$$\mathcal{D}_d = \{z \in \mathbb{C} : |\Im z| < d\}, \tag{2.2}$$

let $\Gamma = \varphi^{-1}(\mathbb{R})$ and let $a = \varphi^{-1}(-\infty)$ and $b = \varphi^{-1}(\infty)$ denote the end-points of Γ. We consider Γ to be an open arc. We also define a set of Sinc points $x_k \in \Gamma$ by $x_k = \varphi^{-1}(kh)$, and we set $\rho = \exp(\varphi)$. For example, for the case of $\Gamma = (0, 1)$, it convenient to take $\varphi(x) = \log(x/(1-x))$, so that $x_k = \exp(kh)/(1 + \exp(kh))$, $\rho(x) = x/(1-x)$, and $\rho(x_k) = \exp(kh)$.

Corresponding to positive integers M and N, we can now define a set of Sinc basis functions $\{\omega_j\}_{j=-M}^{N}$ by

$$\begin{aligned} \gamma_j(x) &= S(j,h) \circ (\varphi(x)), \quad j = -M, \ldots, N, \\ \omega_j(x) &= \gamma_j(x), \quad j = -M+1, \ldots, N-1, \\ \omega_{-M}(x) &= \frac{1}{1+\rho(x)} - \sum_{j=-M+1}^{N} \frac{\gamma_j(x)}{1+e^{jh}}, \\ \omega_N(x) &= \frac{\rho(x)}{1+\rho(x)} - \sum_{j=-M}^{N-1} \frac{e^{jh}\gamma_j(x)}{1+e^{jh}}. \end{aligned} \tag{2.3}$$

Finally, let $\alpha \in (0, 1]$ and $\beta \in (0, 1]$ denote fixed positive numbers, and let us restrict the number d introduced above to the interval $(0, \pi)$. Let $\mathbf{L}_{\alpha,\beta}(\mathcal{D})$ denote the family of all functions that are analytic in $\mathcal{D}$, such that for all $z \in \mathcal{D}$, we have

$$|f(z)| \le c_1 \frac{|\rho(z)|^{\alpha}}{[1 + |\rho(z)|]^{\alpha+\beta}}. \tag{2.4}$$

The space of functions $\mathbf{M}_{\alpha,\beta}(\mathcal{D})$ denotes the set of all functions g defined on $\mathcal{D}$ that have finite limits $g(a) = \lim_{z\to a} g(z)$ and $g(b) = \lim_{z\to b} g(z)$, where the limits are taken from within $\mathcal{D}$, and such that $f \in \mathbf{L}_{\alpha,\beta}(\mathcal{D})$, where

$$f = g - \frac{g(a) + \rho g(b)}{1 + \rho}. \tag{2.5}$$

3. "Polynomial" approximation on $(0, 1)$

For the sake of simplicity, we take $\Gamma = \Gamma_2 = (0, 1)$ as our initial arc, which is mapped via φ_2 onto $\mathbb{R}$, where $\varphi_2(x) = \log(x/(1 - x))$. We wish to approximate derivatives of a function belonging to a space of functions $\mathbf{Y}(\varphi_2)$. Here we define $\mathbf{Y}(\varphi_2)$ as the family of all functions f defined on Γ_2, such that f not only belongs to $\mathbf{M}_{\alpha,\beta}(\mathcal{D})$, but such that f is also analytic and uniformly bounded by $\mathcal{M}(f)$ in the larger region

$$D_2 = \mathcal{D} \bigcup_{t\in(0,1)} B(t, r). \tag{3.1}$$

Here $r > 0$, and $B(t, r) = \{z \in \mathbb{C} : |z - t| < r\}$. We remark here, that while neither the region $\mathcal{D}$ nor the space $\mathbf{M}_{\alpha,\beta}(\mathcal{D})$ is necessary for our derivations of the formulas of this paper, we have included these notations here, since we assume that the most important applications of these formulas will occur during Sinc approximation procedures. The region D_2 is an open region containing the closed interval $[0, 1]$, and hence functions that are analytic on D_2 are differentiable at all points of $[0, 1]$. We shall approximate such functions via the use of polynomials in the variable x.

3.1. The Lagrange polynomial

We shall assume that we are given the $m = M + N + 1$ data values $\{(x_k, f(x_k))\}_{k=-M}^{N}$, where the x_k are Sinc points of Γ_2, i.e., the points $\varphi_2^{-1}(kh) = \exp(kh)/(1 + \exp(kh))$. Thus, setting

$$g(x) = \prod_{\ell=-M}^{N} (x - x_\ell) \tag{3.2}$$

we immediately get the Lagrange polynomial

$$p(x) = \sum_{k=-M}^{N} b_k(x) f(x_k), \tag{3.3}$$

where

$$b_k(x) = \frac{g(x)}{(x - x_k) g'(x_k)}. \tag{3.4}$$

This polynomial p is of degree at most $m - 1$ in x, and it interpolates the function $f \in \mathbf{Y}(\varphi_2)$ at the m Sinc points $\{x_k\}_{k=-M}^{N}$.

3.2. The derivative of $p(x)$

Let us differentiate the formula (3.3) with respect to x, and let us then form an $m \times m$ matrix $A = [a_{j,k}]$, $j, k = -M, \ldots, N$ with the property that

$$f'(x_j) \approx p'(x_j) = \sum_{k=1}^{m} a_{j,k} f(x_k). \tag{3.5}$$

Then (3.3) immediately yields for $k \neq j$, that $a_{j,k} = b_k'(x_j) = g'(x_j)/((x_j - x_k)g'(x_k))$, whereas for $k = j$, we find that $a_{j,j} = \sum_{\ell \neq j} 1/(x_j - x_\ell)$. That is,

$$a_{j,k} = \begin{cases} \dfrac{g'(x_j)}{(x_j - x_k)g'(x_k)} & \text{if } k \neq j \\ \displaystyle\sum_{\ell \in [-M,N], \ell \neq j} \frac{1}{x_j - x_\ell} & \text{if } k = j. \end{cases} \tag{3.6}$$

3.3. *Errors of the approximations*

We shall obtain an estimate of the error of approximation of $f(x)$ by $p(x)$ on $[0, 1]$, and we shall also obtain a bound on the error of approximation of $f'(x_j)$ by the sum on the right hand side of (3.5). We shall take $M = N$ for the sake of achieving simplicity of our results.

Theorem 3.1. *Let f, r, D_2, $\mathcal{M}(f)$, g, and $p(x)$ be defined as above, let $M = N$, take $h = c/\sqrt{N}$ with c a positive constant independent of N, and let $\{x_j\}_{-N}^{N}$ denote the Sinc points, as above. Then there exist constants A and B, independent of N, such that:*

(i) *An estimate of a bound on the error, $f(x) - p(x)$, for $x \in [0, 1]$ is given by*

$$|f(x) - p(x)| \leq A \frac{N^{1/2}}{(2r)^{2N}} \exp\left(-\frac{\pi^2 N^{1/2}}{2c}\right); \tag{3.7}$$

and

(ii) *The difference, $f'(x_j) - p'(x_j)$, is bounded by*

$$\left|f'(x_j) - p'(x_j)\right| \leq B \frac{N}{(2r)^{2N}} \exp\left(-\frac{\pi^2 N^{1/2}}{2c}\right). \tag{3.8}$$

We note, at the outset, that the error of approximation of f via p on $[0, 1]$ can be expressed as a contour integral,

$$E_m(f, x) = \frac{g(x)}{2\pi \mathrm{i}} \int_{\partial D_2} \frac{f(z)}{(z - x)g(z)} \mathrm{d}z. \tag{3.9}$$

Also, by differentiating this expression, we get

$$f'(x_j) - p'(x_j) = \frac{g'(x_j)}{2\pi \mathrm{i}} \int_{\partial D_2} \frac{f(z)}{(z - x)g(z)} \mathrm{d}z. \tag{3.10}$$

By our definition of D_2, we have $|z - x| \geq r$ for $x \in [0, 1]$, and also, $|z - x_j| \geq r$, for $z \in \partial D_2$ for $j = -N, \ldots, N$. Hence the denominator of the integrand in (3.9) is at least as large as r^{m+1}. (This is probably a gross overestimate for most problems of applications, but we shall stick to it for now, with a charge to the researcher, that if necessary, she/he may be able to get a much better estimate for most specific applications.) By assumption, f is uniformly bounded in D_2, by $\mathcal{M}(f)$. Hence

$$|E_m(f, x)| \leq \frac{\mathcal{M}(f)}{r^{m+1}} \max_{x \in [0,1]} |g(x)| \frac{L(\partial D_2)}{2\pi}, \tag{3.11}$$

where $L(\partial D_2) = 2 + 2\pi r$ is the length of the boundary of D_2.

Setting $x = z/(1+z)$, which transforms the interval $z \in (0, \infty)$ to the interval $x \in (0, 1)$, we get

$$\begin{aligned} \left|g\left(\frac{z}{z+1}\right)\right| &= \prod_{j=-N}^{N} \left|\frac{z - \mathrm{e}^{jh}}{(1+z)(1+\mathrm{e}^{jh})}\right| \\ &= \frac{z^{N+1}}{(1+z)^{2N+1}} \prod_{j=1}^{N} \left(\frac{1 - z\mathrm{e}^{-jh}}{1+\mathrm{e}^{jh}}\right)^2 \left|1 - z\mathrm{e}^{-Nh}\right| \\ &\le \frac{z^{N+1}}{(1+z)^{2N+1}} \prod_{j=1}^{N} \left(\frac{1 - z\mathrm{e}^{-jh}}{1+\mathrm{e}^{jh}}\right)^2 . \end{aligned} \tag{3.12}$$

In obtaining the right hand side of (3.7), we estimated the bound of $g(x)$ on $[0, 1]$ as occurring at $x = z/(1+z)$, with $z = \exp(h/2)$. This value of x is not the exact maximum value of $|g(x)|$ on $[0, 1]$, but is close to the maximum, and this is why we stated our result as being an estimate.

It is convenient for purposes of presenting the ideas of the proof of Theorem 3.1 to first prove the (ii)-Part of this theorem. To this end, it is furthermore convenient to split the proof into the proofs of some lemmas.

Lemma 3.2. *Let g be defined as in (3.2). Then*

$$\max_{j \in \{-N,\ldots,N\} \cap \mathbb{Z}} |g'(x_j)| = |g'(x_0)|. \tag{3.13}$$

Proof of Lemma 3.2. We skip our lengthy proof of Lemma 3.2, based, in part, on the easily verifiable fact that the function

$$w(\xi) = \int_{x_{-N-1/2}}^{x_{N+1/2}} \log|x - \xi| \frac{\mathrm{d}x}{x(1-x)} \tag{3.14}$$

with $x_{N+1/2} = \varphi^{-1}((N+1/2)h)$ and similarly for $x_{-N-1/2}$, and where $w(\xi)$ takes on its maximum value on $[0, 1]$ at $\xi = 1/2$, and also, because:

1. Our proof uses the methods of the proof of Lemmas 3.3 and 3.4 below, as well as the result of Lemma 3.4; and
2. The result of Lemma 3.2 can easily be verified directly, since all of the values of $g'(x_j)$ are required for the application of Theorem 3.1 above. ■

Lemma 3.3. *Let g be defined as in (3.2), and let $h > 0$. Then*

$$|g'(x_0)| = \frac{1}{2^{2N}} \prod_{k=1}^{N} \left(\frac{1 - \mathrm{e}^{-kh}}{1 + \mathrm{e}^{-kh}}\right)^2 . \tag{3.15}$$

Proof of Lemma 3.3. Since $x_k = \mathrm{e}^{kh}/(1+\mathrm{e}^{kh})$, since $x_0 = 1/2$, and since $x_{-k} = 1 - x_k$, we have

$$\begin{aligned} \left|g'(x_0)\right| &= \prod_{k=1}^{N} |x_0 - x_k|\,|x_0 - x_{-k}| \\ &= \prod_{k=1}^{N} \left|\left(\frac{1}{2} - \frac{\mathrm{e}^{kh}}{1+\mathrm{e}^{kh}}\right)\left(\frac{1}{2} - \frac{1}{1+\mathrm{e}^{kh}}\right)\right| \end{aligned} \tag{3.16}$$

from which (3.15) follows. ■

Lemma 3.4. *Let the conditions of Lemma 3.3 be satisfied. Then*

$$\sum_{k=1}^{N} \log\left(\frac{1-e^{-kh}}{1+e^{-kh}}\right) \leq -\frac{\pi^2}{4h} + 1/2 + (1/2)\left(\log\left(\frac{2}{h}\right) - \log(\tanh(h/4)) + \frac{2}{h\sinh(Nh)}\right). \tag{3.17}$$

Proof of Lemma 3.4. Since $\log((1-e^{-z})/(1+e^{-z}))$ is concave on $(0, \infty)$, we have, recalling the sign of the error of midordinate quadrature,

$$\begin{aligned} h\sum_{k=1}^{N} \log\left(\frac{1-e^{-kh}}{1+e^{-kh}}\right) &\leq \int_{h/2}^{(N+1/2)h} \log\left(\frac{1-e^{-x}}{1+e^{-x}}\right) dx \\ &\leq I_1 - I_2 - I_3 \end{aligned} \tag{3.18}$$

where

$$\begin{aligned} I_1 &= \int_0^\infty \log\left(\frac{1-e^{-z}}{1+e^{-z}}\right) dz \\ I_2 &= \int_0^{h/2} \log\left(\frac{1-e^{-z}}{1+e^{-z}}\right) dz \\ I_3 &= \int_{(N+1/2)h}^\infty \log\left(\frac{1-e^{-z}}{1+e^{-z}}\right) dz. \end{aligned} \tag{3.19}$$

Now, by expanding $\log\left((1-e^{-z})/(1+e^{-z})\right)$ in powers of e^{-z} and performing term-wise integration, we get

$$I_1 = -\sum_{k=1}^{\infty} \frac{2}{(2k+1)^2} = -\frac{\pi^2}{4}. \tag{3.20}$$

Next, since $(1-e^{-z})/(1+e^{-z}) \geq z\tanh(h/4)$ on $(0, h/2)$, we get

$$I_2 \geq \int_0^{h/2} \log(z\tanh(h/2))dz = h/2\left(\log(h/2) - 1 + \log(\tanh(h/4))\right). \tag{3.21}$$

Finally, by again expanding $\log\left((1-e^{-z})/(1+e^{-z})\right)$ in powers of e^{-z} and performing term-wise integration, we get

$$\begin{aligned} I_3 &= -\sum_{k=0}^{\infty} \frac{e^{-(2k+1)(N+1/2)}}{(2k+1)^2} \\ &\geq -\frac{e^{-(N+1/2)h}}{1-e^{-2(N+1/2)h}} \\ &= -\frac{2}{\sinh((N+1/2)h)} \\ &\geq -\frac{2}{\sinh(Nh)}. \end{aligned} \tag{3.22}$$

This completes the proof of Lemma 3.4. ■

Completion of Proof of Theorem 3.1. By multiplying the right hand sides of Lemma 3.4 by 2 and dividing by h, and then setting $h = c/\sqrt{N}$, we get the results of Theorem 3.1(ii). ■

The proof of the (i)-Part is similar, and we omit it.

Remark 3.5. We may note that our error bounds do not converge to zero as $N \to \infty$, unless $r \geq 1/2$. If however, we replace x by Ty and then perform Sinc approximation of the same function f but now on the interval $y \in [0, 1]$ rather than as before, on the interval $x \in [0, 1]$ (in effect, doing Sinc approximation for $x \in [0, T]$), we get an additional factor T^{4N+4} in the errors of the above approximations. Thus, convergence of the above approximations as $N \to \infty$ occurs whenever $2r/T^2 > 1$. On the other hand, shortening the length of the interval from $(0, 1)$ to $(0, T)$ does not enable accurate uniform approximation of the derivative when we use the derivative of a Sinc approximation based on using (1.2), even when T is very small, due to the fact that the derivative of the map $\varphi_2(x) = \log(x/(T - x))$ is unbounded at the end-points of $(0, T)$.

4. "Polynomial" approx. on $\Gamma = \varphi^{-1}(\mathbb{R})$

Examination of the transformation $\varphi_2(x) = \log(x/(1 - x))$ shows that $x = \rho_2/(1 + \rho_2)$, where $\rho_2(x) = \exp(\varphi_2(x)) = x/(1-x)$. It is easy to see that the space of functions $\mathbf{Y}(\varphi_2)$ defined in the above subsection is a subspace of every space $\mathbf{M}_{\alpha,\beta,d}(\varphi_2)$, with $0 < \alpha \leq 1$, $0 < \beta \leq 1$, and $0 < d < \pi$ (see [2]). It thus follows that if $f \in \mathbf{Y}(\varphi_2)$, and if φ is any other transformation that satisfies the conditions stated at the outset of this paper, then $F = f \circ \varphi_2^{-1} \circ \varphi \in \mathbf{Y}(\varphi)$, where this is the space of all functions F with the property that $F \circ \varphi^{-1} \circ \varphi_2 \in \mathbf{Y}(\varphi_2)$.

4.1. Approximating F via a polynomial in $\rho/(1 + \rho)$

Note, from above, that if $F \in \mathbf{Y}(\varphi)$, and if g is defined as in the above subsection, if we define G by $G(t) = g(\rho(t)/(1 + \rho(t)))$, where $\rho = e^{\varphi}$, and if we let $t_k = \varphi^{-1}(kh)$ denote the Sinc points of this map with respect to the arc $\Gamma = \varphi^{-1}(\mathbb{R})$, then we can achieve the exact same accuracy in approximating F via the "polynomial"

$$P(t) = \sum_{k=1}^{m} \frac{G(t)}{(t - t_k)G'(t_k)} F(t_k) \tag{4.1}$$

as we did above, in approximating f via the use of the polynomial p defined as in Section 3.1. The above formula for $p(x)$ is thus readily extended to any arc Γ. Note: if we set $x = \rho(t)/(1 + \rho(t))$ in (3.3), then

$$P(t) = p(x) = \sum_{k=1}^{m} b_k(x) F(t_k). \tag{4.2}$$

The transformations of Examples 2.1–2.6 of [4] thus immediately yield several "polynomial" approximation methods of functions defined on arcs. We illustrate properties of these approximations for the case of $\Gamma = (0, 1)$ in the next section.

4.2. "Polynomial" deriv. approx. on $\Gamma = \varphi^{-1}(\mathbb{R})$

Upon differentiating the formula (4.2) with respect to t, we can determine a matrix $B = [b_{j,k}]$ such that

$$P'(t_j) = \sum_{k=1}^{m} b_{j,k} F(t_k). \tag{4.3}$$

Let us now derive the expressions for $b_{j,k}$. To this end, we readily arrive at the formula

$$b_{j,k} = \begin{cases} \dfrac{G'(t_j)}{(t_j - t_k)G'(t_k)} & \text{if } k \neq j, \\ \dfrac{G''(t_j)}{2G'(t_j)} & \text{if } k = j. \end{cases} \tag{4.4}$$

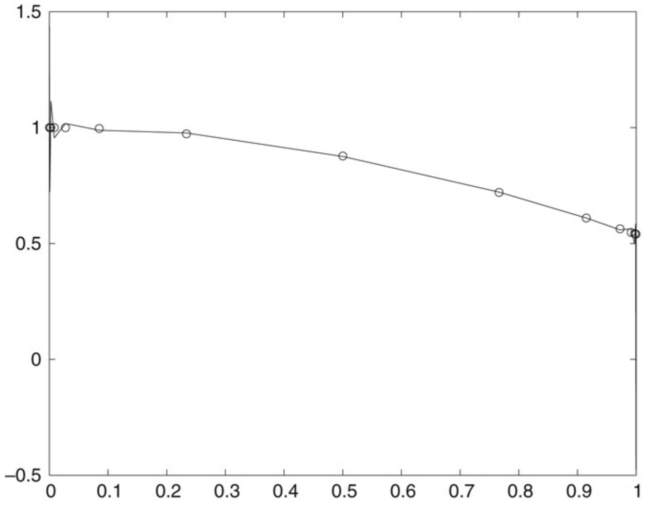

Fig. 1. Exact and (1.2)-approx. deriv. at Sinc pts.

Since

$$\begin{aligned} G'(t) &= g'(x)\frac{\mathrm{d}x}{\mathrm{d}t} \\ G''(t) &= g''(x)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + g'(x)\frac{\mathrm{d}^2x}{(\mathrm{d}t)^2}, \end{aligned} \tag{4.5}$$

we find that

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\rho(t)\varphi'(t)}{(1+\rho(t))^2} \\ \frac{\mathrm{d}^2x}{(\mathrm{d}t)^2} &= \frac{\rho(t)}{(1+\rho(t))^2}\left(\varphi''(t) - \frac{\rho(t)-1}{\rho(t)+1}\varphi'(t)^2\right), \end{aligned} \tag{4.6}$$

and these expressions readily enable us to compute values for $b_{k,j}$.

We thus again get a family of matrices for approximating derivatives on arcs Γ as defined in the transformations of Examples 2.1–2.6 of [4].

5. Examples

We illustrate here some examples of applications of the above results, for approximating the function $f(x) = \sin(x)$ and its derivative, $\cos(x)$, on the interval [0, 1]. Here we use $N = 7$, i.e., 15 point approximations, and $h = \pi/\sqrt{N}$.

It is interesting to note that the Sinc-polynomial basis method of this paper yields more accurate results for approximation at Sinc points than Sinc basis approximation, for approximation of functions that are analytic in an open region containing the interval [0, 1].

The following points are to be noted regarding the plots in this paper:

1. Fig. 1 contains plots of $\cos(x)$ and the derivative of $\sin(x)$ as obtained via the use of (1.2), at the Sinc points $\{z_j\}_{-N}^{N}$;
2. Fig. 2 is a plot of the difference of the two approximations in Fig. 1.
   Note the distortion in accuracy at the end-points of the interval, illustrated in Figs. 1 and 2. Since plots are at Sinc points, these plots do not show that the derivative obtained via the series (1.2) does, in fact, become unbounded at the end-points of the interval;

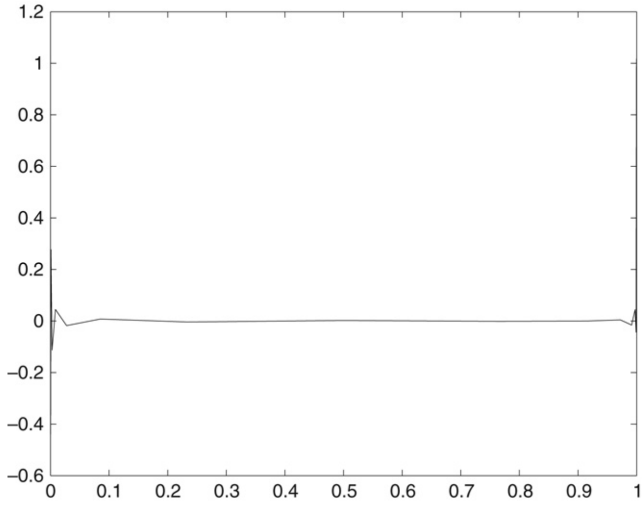

Fig. 2. Exact minus (1.2)-approx. deriv. at Sinc pts.

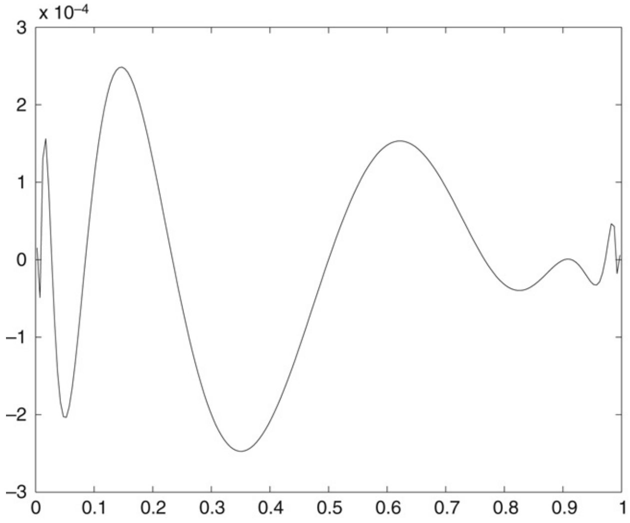

Fig. 3. Exact minus Sinc (1.1)-approx. of $\sin(x)$ at fine mesh.

3. Fig. 3 is a plot of the difference between $\sin(x)$ and its interpolation, obtained via the use of (1.1), at 200 equi-spaced points on $(0, 1)$;
4. Fig. 4 is a plot of the difference between $\sin(x)$ and its polynomial interpolation, obtained via the use of (3.3), at 200 equi-spaced points on $(0, 1)$;

   Note with reference to Figs. 3 and 4, that interpolation via the use of the polynomial (3.3) is considerably more accurate than interpolation via the use of (1.1). Note, in particular, the maximum error of (1.1) is about 2.5×10^{-4}, whereas the maximum error of (3.3) is less than 6×10^{-6}.

   However, the Sinc interpolant (1.1) has other advantages not shared by (3.3) (see [3, Chapter 4]);

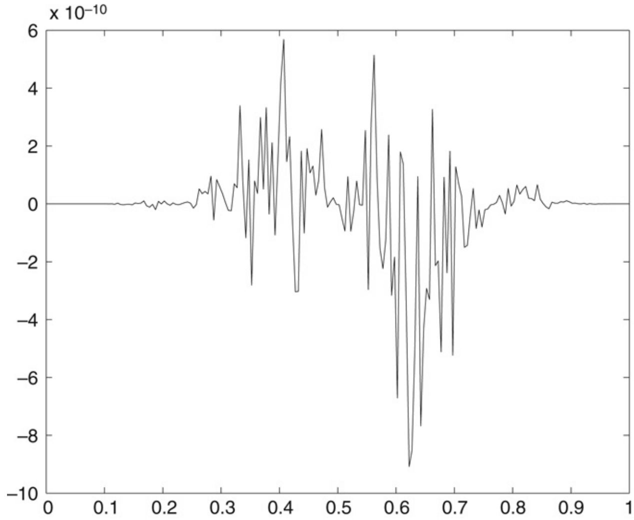

Fig. 4. Exact minus (3.3)-poly approx. of $\sin(x)$ at fine mesh.

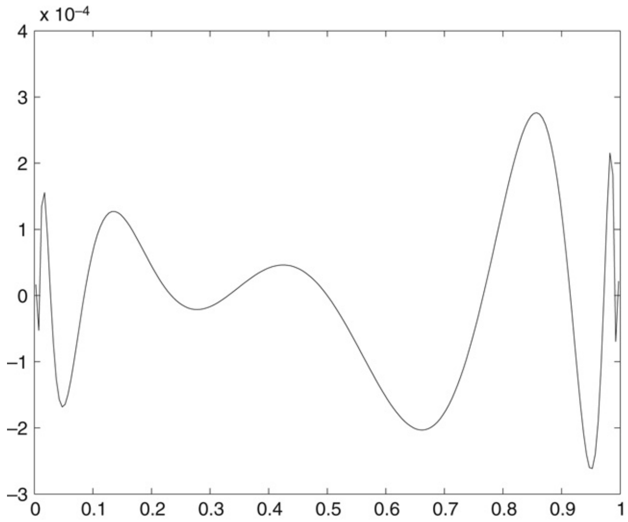

Fig. 5. Exact minus (1.2) formula deriv. of $\sin(x)$ at fine mesh.

5. Fig. 5 is a plot of the difference between $\cos(x)$ and its Sinc derivative interpolation, where the latter is obtained via the use of evaluation of the derivative formula (1.2) at 200 equi-spaced points on $(0, 1)$;
6. Fig. 6 is a plot of the difference between $\cos(x)$ and its polynomial approximation, obtained via the use of the polynomial $p(x)$ in (3.3).

   Figs. 5 and 6 also illustrate the remarkable accuracy of the formulas derived in this paper.

   Note also, the large errors in Figs. 3 and 5 due to roundoff: The actual errors may be shown to be considerably smaller, via the use of quadruple precision.

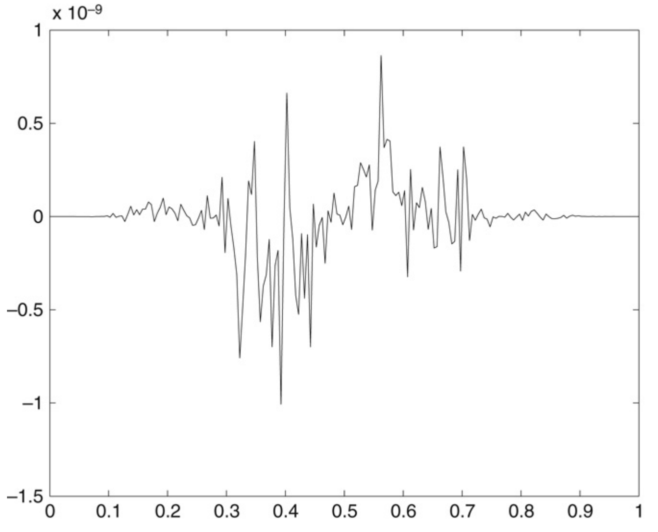

Fig. 6. Exact minus Sinc_ poly derivative of $\sin(x)$ at fine mesh.

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