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(De)coupling limit of DGP

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Abstract

We investigate the decoupling limit in the DGP model of gravity by studying its non-linear equations of motion. We show that, unlike 4D massive gravity, the limiting theory does not reduce to a sigma model of a single scalar field: Non-linear mixing terms of the scalar with a tensor also survive. Because of these terms physics of DGP is different from that of the scalar sigma model. We show that the static spherically-symmetric solution of the scalar model found in [A. Nicolis, R. Rattazzi, JHEP 0406 (2004) 059, hep-th/0404159], is not a solution of the full set of non-linear equations. As a consequence of this, the interesting result on hidden superluminality uncovered recently in the scalar model in [A. Adams, N. Arkani-Hamed, S. Dubovsky, A. Nicolis, R. Rattazzi, hep-th/0602178], is not applicable to the DGP model of gravity. While the sigma model violates positivity constraints imposed by analyticity and the Froissart bound, the latter cannot be applied here because of the long-range tensor interactions that survive in the decoupling limit. We discuss further the properties of the Schwarzschild solution that exhibits the gravitational mass-screening phenomenon.

1. Introduction and summary

In a simplest perturbative expansion non-linear interactions in the DGP model [1] become strong at a certain scale determined by the graviton mass, coupling constant and physical parameters of a problem at hand (e.g., mass/energy of the source) [2]. The strong interactions and its scale can be seen in a breakdown of the classical perturbation theory around a static source (in analogy with a similar phenomenon observed in Ref. [3] in the context of massive Fierz–Pauli gravity (FP) [4]). More generically, this can be understood in terms of non-linear Feynman diagrams that are enhanced by inverse powers on the graviton mass [2]. This perturbative breakdown, and classical resummation of the corresponding large diagrams, is what makes the model agreable with predictions of general relativity [2,5–9], and yields tiny but potentially measurable deviations from it [10–14].

It is instructive to consider the decoupling limit of the DGP model, as it was done in Refs. [15] and [16], motivated by the original studies of the decoupling limit of massive gravity in Ref. [17]. It was argued that in this limit the DGP model reduces to a non-linear scalar model, while the self-interactions of a tensor mode vanish [15,16]. The remaining equation for the scalar field reads:

$$3\Box\pi = \frac{(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2}{\Lambda^3}, \tag{1}$$

where $\Lambda \equiv M_{\rm Pl} m_c^2$ is held fixed while $M_{\rm Pl} \to \infty$, $m_c \to 0$, and m_c plays the role of the graviton mass (the above equation is given here in a source-free region). Although, the results of [15,16] are similar in spirit to those found in [17] in the context of massive gravity, there are also important differences between the scalar models of Refs. [17] and [15,16]. These differences were shown to reproduce [18,19] the known non-linear ghost-like instability in the FP theory [20,21], and were argued to be responsible for its absence in (1) [18,19].

In spite of all the above, the formal similarity between the two theories seems somewhat surprising because of fundamental differences in their Lagrangians (see more on this in Section 6). Most importantly, however, there exist an expression for the metric [13] such that: (a) It is an exact solution of the non-linear DGP equations on and near the brane; (b) It is a per-

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fectly regular solution in the decoupling limit (see Section 5); (c) Yet, it does not satisfy Eq. (1).

The above observation motivated us to re-examine the decoupling limit of the DGP model in the present work. By analyzing the full set of non-linear equations we will show below that one of the equations that survives in the decoupling limit is more general than (1), and looks as follows:

$$3\Box\pi + \tilde{R} = \frac{(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2}{\Lambda^3} + \frac{\tilde{R}^2 - 3\tilde{R}^2_{\mu\nu}}{3\Lambda^3} + \frac{\tilde{R}\Box\pi - 2\tilde{R}^{\mu\nu}\partial_\mu\partial_\nu\pi}{\Lambda^3}. \tag{2}$$

Here $\tilde{R} \equiv M_{\rm Pl} R$ and $\tilde{R}_{\mu\nu} \equiv M_{\rm Pl} R_{\mu\nu}$, where R and $R_{\mu\nu}$ are the Ricci scalar and tensor of a spin-2 field. Eq. (2) reduces to Eq. (1) if, e.g., both Ricci tensor and scalar vanish. However, this does not have to be the case, as will be discussed below. For instance, the Schwarzschild solution of [13,14] gives $\tilde{R}$ and components of $\tilde{R}_{\mu\nu}$ that are non-zero even in the decoupling limit, and satisfies (2), but not (1).

The issue of small fluctuations about classical backgrounds is different in (1) and (2). Eq. (1) implies that the fluctuations of the π field are decoupled from those of the tensor and carry an independent physical meaning. On the other hand, Eq. (2) shows that the fluctuations of the π field mix with those of a tensor. In fact, as we will see below, the π field can be completely absorbed into a tensor field, which survives as a non-linearly interacting mode and propagates five physical polarizations.

More importantly, however, there exist yet another non-linear equation that survives in the decoupling limit. The latter is somewhat involved to be presented here without lengthy explanations (see Eq. (18)). Consistent solutions should satisfy that equation too. We will show that the spherically-symmetric solution of Eq. (1) found in Ref. [16], although obeys (2), does not satisfy Eq. (18). Hence, it is not a solution of the DGP model.[1] On the other hand, the solution of [13] satisfies both (2) and (18).

As was recently shown in Ref. [22], the scalar model (1) presents a very interesting field-theoretic example: In spite of its appearance as a local, Lorentz-invariant effective field theory, it exhibits certain hidden non-locality [22]. The latter can manifest itself in superluminal propagation of perturbations on the background defined by the spherically-symmetric solution of [16]. If the DGP model were to reduce in the decoupling limit to the scalar theory (1), then, any experimental test of this model would present a window of opportunity for discovering small effects of an $\mathcal{O}(1)$ superluminality in gravitation [22].

While this intriguing possibility can exist in the scalar sigma model [22], our results show that it is not offered by DGP. The solution of [16], and its fluctuations that manifest superluminal propagation, are not consistent solutions of the DGP model. More generally, the scalar model violates positivity constraints [22] imposed by analyticity and the Froissart bound, however, the latter cannot be applied to the present model of gravity because of the long-range tensor interactions that survive in the decoupling limit.

The Letter is organized as follows. In Section 2 we discuss the ADM formalism for DGP and write down explicitly some of the key non-linear equations. In Section 3 we take the decoupling limit of those equations and derive Eq. (2). In Section 4 we deal with the remaining non-linear equations of the model and find their decoupling limit. We also show that the solution of Ref. [16] does not satisfy these equations. In Section 5 we discuss the physical interpretation of the Schwarzschild solution of Refs. [13,14] in the decoupling limit. We emphasize that this solution, although naively seems counterintuitive, actually has a clear physical interpretation in terms of the screening effects. In Section 6, we discuss some interesting open questions.

2. DGP in the ADM formalism

Let us start with the action of the DGP model [1] in the ADM formalism [23] (see, e.g., [24,25]):

$$S = \frac{M^2_{\rm Pl}}{2}\int d^4x\, dy\, \sqrt{g}\Big(R\delta(y) + \frac{m_c}{2}N\big(R + K^2 - K_{\mu\nu}K^{\mu\nu}\big)\Big). \tag{3}$$

Here, the $(4+1)$ coordinates are $x^M = (x^\mu, y)$, $\mu = 0,\ldots,3$; g and R are the determinant and 4D curvature for the 4D components $g_{\mu\nu}(x,y)$ of the 5D metric $g_{AB}(x,y)$. $K = g^{\mu\nu}K_{\mu\nu}$ is the trace of the extrinsic curvature tensor defined as follows:

$$K_{\mu\nu} = \frac{1}{2N}(\partial_y g_{\mu\nu} - \nabla_\mu N_\nu - \nabla_\nu N_\mu), \tag{4}$$

and ∇_μ is a covariant derivative w.r.t. the metric $g_{\mu\nu}$. We introduced the *lapse* scalar field N, and the *shift* vector field N_μ according to the standard rules:

$$g_{\mu y} \equiv N_\mu = g_{\mu\nu}N^\nu, \qquad g_{yy} \equiv N^2 + g_{\mu\nu}N^\mu N^\nu. \tag{5}$$

Equations of motion of the theory are obtained by varying the action (3) w.r.t. $g_{\mu\nu}$, N and N_μ. Here we start with a subset of two equations, the junction condition across the brane, and the $\{yy\}$ equation that can be obtained by varying the action w.r.t. N. The former reads as follows:

$$G_{\mu\nu} - m_c(K_{\mu\nu} - g_{\mu\nu}K) = T_{\mu\nu}/M^2_{\rm Pl}, \tag{6}$$

where $G_{\mu\nu}$ is the 4D Einstein tensor of the induced metric $g_{\mu\nu}$ and $T_{\mu\nu}$ is the matter stress tensor. The $\{yy\}$ equation takes the form

$$R = K^2 - K^2_{\mu\nu}. \tag{7}$$

Note that (6) is valid only at $y = 0^+$ while (7) should be fulfilled for arbitrary y. The terms with the extrinsic curvature contain derivatives w.r.t. the extra coordinate as well as the yy and μy components of the metric. Nevertheless, one can deduce a single equation that contains the 4D induced metric only! This is

[1] The solution of [16], in our nomenclature, is a solution of the equations of motion that are obtained from the sigma model Lagrangian in which the π field is decoupled from the tensor field (i.e., there are no mixings between them) [15,16]. In the solution discussed in [6] the π and tensor fields are sourcing each other.

done by expressing from (6) K and $K_{\mu\nu}$ in terms of R and $R_{\mu\nu}$ outside of the source

$$K_{\mu\nu} = \frac{R_{\mu\nu} - g_{\mu\nu}R/6}{m_c}, \qquad K = \frac{R}{3m_c}, \tag{8}$$

and substituting these expressions into (7). The result at $y = 0^+$ is:

$$R = \frac{R^2 - 3R_{\mu\nu}^2}{3m_c^2}. \tag{9}$$

We will study the decoupling limit of this equation below.

3. (De)coupling limit

The purpose of this section is to show that the non-linear interactions of the tensor field in the DGP model do not disappear in the decoupling limit.

The decoupling limit in DGP is defined as follows [15]:

$$M_{\rm Pl} \to \infty, \qquad m_c \to 0, \qquad M \to \infty, \tag{10}$$

where M is a mass of a source entering the stress-tensor T and the following quantities are held finite and fixed

$$\Lambda \equiv \left(M_{\rm Pl} m_c^2\right)^{1/3}, \qquad \frac{M}{M_{\rm Pl}}. \tag{11}$$

To take this limit in (9), we multiply both sides of that equation by $M_{\rm Pl}$ and obtain

$$\bar{R} = \frac{\bar{R}^2 - 3\bar{R}_{\mu\nu}^2}{3\Lambda^3}, \tag{12}$$

where we defined $\bar{R} \equiv M_{\rm Pl}R$ and $\bar{R}_{\mu\nu} \equiv M_{\rm Pl}R_{\mu\nu}$. Now we are ready to proceed to (10), (11). Before doing so, it is instructive to make parallels with FP gravity. The analog of the mass term in the DGP action (3) is the one containing squares of the extrinsic curvature. Looking at the expression for the extrinsic curvature (4) one can think of $g_{\mu\nu}$ in DGP to be an analog of the tensor field of FP gravity in which the Stückelberg field is manifestly exposed in the mass term [17]. Then, N_μ is an analog of the vector-like Stückelberg field as far as the reparametrizations with the gauge-function $\zeta_A(x, y) = (\zeta_\mu(x, y), 0)$ are concerned. The longitudinal component of N_μ should play the role similar to that of the longitudinal component of the vector-like Stückelberg field of FP gravity (note that there is an additional Stückelberg field N in DGP, the important role of which will be discussed in Section 6).

Eq. (12), on the other hand, does not contain any of the Stückelberg fields but the tensor $g_{\mu\nu}$ which in the decoupling limit has a conventional scaling.[2] That is why it is straightforward to take the limit directly in (12). For this we recall that

$$\bar{R} = \Box\bar{h} - \partial^\mu\partial^\nu\bar{h}_{\mu\nu} + \mathcal{O}\left(\frac{\bar{h}\Box\bar{h}}{M_{\rm Pl}}\right),$$

$$\bar{R}_{\mu\nu} = \frac{1}{2}\left(\Box\bar{h}_{\mu\nu} - \partial_\mu\partial^\alpha\bar{h}_{\alpha\nu} - \partial_\nu\partial^\alpha\bar{h}_{\alpha\mu} + \partial_\mu\partial_\nu\bar{h}\right) + \mathcal{O}\left(\frac{\bar{h}\Box\bar{h}}{M_{\rm Pl}}\right), \tag{13}$$

where we defined a field $\bar{h}_{\mu\nu} \equiv M_{\rm Pl}(g_{\mu\nu} - \eta_{\mu\nu})$, which has the canonical dimensionality and is held fixed in the limit.[3]

Note that we have not done any small field approximation, but merely expanded the non-linear expressions for $\bar{R}$ and $\bar{R}_{\mu\nu}$ in powers of $\bar{h}$ and took the decoupling limit. Because all the non-linear terms in $\bar{R}$ and $\bar{R}_{\mu\nu}$ are suppressed by extra powers of $M_{\rm Pl}$, the curvatures reduce to their linearized form.

The expressions in (13), should be substituted into (12). To make closer contact with the π language of Section 1 we perform the following shift of the $\bar{h}_{\mu\nu}$ field

$$\bar{h}_{\mu\nu} = \tilde{h}_{\mu\nu} + \eta_{\mu\nu}\pi, \tag{14}$$

where the π field has a canonical dimensionality and is held fixed in the decoupling limit. With this substitution Eq. (12) turns into (2) which we repeat here for convenience

$$3\Box\pi + \tilde{R} = \frac{(\Box\pi)^2 - (\partial_\mu\partial_\nu\pi)^2}{\Lambda^3} + \frac{\tilde{R}^2 - 3\tilde{R}_{\mu\nu}^2}{3\Lambda^3} + \frac{\tilde{R}\Box\pi - 2\tilde{R}^{\mu\nu}\partial_\mu\partial_\nu\pi}{\Lambda^3}. \tag{15}$$

$\tilde{R}$ and $\tilde{R}_{\mu\nu}$ denote the linear terms on the right-hand sides of (13) where $\bar{h}_{\mu\nu}$ is replaced by $\tilde{h}_{\mu\nu}$.

Eq. (15) shows that a tensor field would have non-linear interactions in the decoupling limit as long as there are no additional equations constraining all the components of $\tilde{R}_{\mu\nu}$ to be zero. We will turn to the remaining equations in the next section and show that they do not necessarily imply vanishing $\tilde{R}_{\mu\nu}$. Before that we would like to make a comment on small fluctuations about classical solutions. The π field in (15) can be reabsorbed back into the tensor field $\bar{h}_{\mu\nu}$ using (14). Small perturbations on any background in the decoupling limit are those of $\bar{h}_{\mu\nu}$. The given background is what defines the light-cone, and there are no additional degrees of freedom that could propagate outside of that light-cone.

4. More bulk equations

Some of the bulk equations have not been considered in our discussions so far. These are the $\{\mu y\}$ and bulk $\{\mu\nu\}$ equations (the $\{\mu\nu\}$ equation on the brane (6) has already been taken into account). We will discuss them in the present section.

We start with the $\{\mu y\}$ equation which for arbitrary y reads as follows:

$$\nabla_\mu K = \nabla^\nu K_{\mu\nu}. \tag{16}$$

The covariant derivative in the above equation is the one for $g_{\mu\nu}$. At $y = 0$ Eq. (16) is trivially satisfied due to (8). For $y \neq 0$,

[2] If we were to do perturbative calculations, this would imply the gauge choice for which the Stückelberg fields are manifestly present.

[3] The signs in the expressions (13) and the definition of $\bar{h}_{\mu\nu}$ given above determine our convention for the sign of the curvature tensor.

(16) sets the relation between N_μ, N and $g_{\mu\nu}$. Hence, (16) gives a relation between these quantities for both $y = 0$ and $y \neq 0$.

One can use the bulk $\{yy\}$ equation (7) to determine N in terms of N_μ and $g_{\mu\nu}$, and then use (16) in order to express N_μ in terms of $g_{\mu\nu}$. If so, there must exist one more equation which should allow to determine the bulk $g_{\mu\nu}$ itself. This is the bulk $\{\mu\nu\}$ equation, to which we turn now. The latter can be written in a few different ways. For the case at hand, the formalism by Shiromizu, Maeda and Sasaki [26] is most convenient. In this approach, the bulk $\{\mu\nu\}$ equation and the junction condition can be combined to yield a $\{\mu\nu\}$ equation at $y \to 0^+$. This gives a "projection" of the bulk $\{\mu\nu\}$ equation onto the brane. Since the bulk itself is empty in our case, this is the most restrictive form one can work with. The equation reads as follows [26]:

$$G_{\mu\nu} = K K_{\mu\nu} - K^\rho_\mu K_{\nu\rho} - \frac{1}{2} g_{\mu\nu}\left(K^2 - K^2_{\mu\nu}\right) - E_{\mu\nu}, \qquad (17)$$

where all the quantities are taken at $y = 0^+$. $G_{\mu\nu}$ denotes the 4D Einstein tensor of the metric $g_{\mu\nu}$, $E_{\mu\nu}$ denotes the electric part of the bulk Weyl tensor, projected onto the brane. An important property of the Weyl tensor is that it is invariant under the conformal transformations. Moreover, $E_{\mu\nu}$ is traceless.

We now turn to the decoupling limit in this equation. For this we multiply both sides of (17) by $M_{\rm Pl}$, exclude K and $K_{\mu\nu}$ using (8), and take the limit (10), (11) holding the canonically normalized components of $g_{\mu\nu}$ fixed. The resulting equation reads:

$$\bar{G}_{\mu\nu} = \frac{\bar{B}_{\mu\nu}}{\Lambda^3} - \bar{E}_{\mu\nu}, \qquad (18)$$

where

$$\bar{B}_{\mu\nu} \equiv -\bar{G}_{\mu\alpha}\bar{G}^\alpha_\nu + \frac{1}{3}\bar{G}^\alpha_\alpha \bar{G}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\bar{G}_{\alpha\beta}\bar{G}^{\alpha\beta} - \frac{1}{6}\eta_{\mu\nu}\left(\bar{G}^\alpha_\alpha\right)^2, \qquad (19)$$

and $\bar{G}_{\mu\nu} \equiv M_{\rm Pl} G_{\mu\nu}$, $\bar{E}_{\mu\nu} \equiv M_{\rm Pl} E_{\mu\nu}$. This is the equation that has to be satisfied by any consistent solution.

It is straightforward to check that the trace of the above equation is nothing but (12). Moreover, as in Eq. (12), there are non-linear interactions of the tensor field that survive in the decoupling limit in (18). This is true irrespective of the explicit form of $\bar{E}$, as long as it's traceless. Once the values of N and N_ν are determined, as described at the beginning of this section, one can calculate $E_{\mu\nu}$ in terms of $g_{\mu\nu}$. Thus, Eq. (18) turns into a single equation for the determination of $g_{\mu\nu}$.

The above considerations show that the DGP model in general does not reduce to the scalar theory (1) in the limit (10), (11). This can also be deduced by looking at the solutions of the scalar model (1) which are not solutions of (18). One example of this is the spherically-symmetric static solution of (1) found in [16]. We will discuss this solution in the remaining part of this section. The ansatz corresponding to the solution of [16] reads:

$$\tilde{R}_{\mu\nu}(\tilde{h}) = 0, \qquad \bar{G}_{\mu\nu}(\bar{h}) = \partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\Box\pi,$$
$$K_{\mu\nu} = \frac{m_c}{\Lambda^3}\partial_\mu\partial_\nu\pi. \qquad (20)$$

One can calculate the expression for $\bar{E}_{\mu\nu}$ on the ansatz (20) ($\bar{E}_{\mu\nu}$ comes out to be non-zero), and use that expression in (18). The resulting equation, after substitution (1), reduces to Eq. (24) presented below. These calculations are tedious, and instead of them, we present here relatively easier calculations that produce the same result (24) (this also serves as a self-consistency check of our result).

Since the solution (20) is written in terms of the extrinsic curvature, it is easier to check its compatibility with the bulk $\{\mu\nu\}$ equation also written in terms of this quantity. This equation takes the form [25][4]:

$$\begin{aligned} G_{\mu\nu} = {} & \frac{1}{2} g_{\mu\nu}\left(K^2 - K^2_{\alpha\beta}\right) + 2\left(K^\alpha_\mu K_{\nu\alpha} - K K_{\mu\nu}\right) \\ & - \frac{2}{N}\left(\nabla_\nu(N_\mu K) - \nabla_\alpha\left(K^\alpha_\nu N_\mu\right)\right. \\ & \left. - \frac{1}{2} g_{\mu\nu}\nabla^\alpha(K N_\alpha) + \frac{1}{2}\nabla^\alpha(N_\alpha K_{\mu\nu})\right) \\ & - \frac{\nabla_\mu\nabla_\nu N - g_{\mu\nu}\nabla^2 N}{N} \\ & + g_{\mu\alpha}g_{\nu\beta}\frac{\partial_y(\sqrt{g}(K^{\alpha\beta} - K g^{\alpha\beta}))}{N\sqrt{g}}. \end{aligned} \qquad (21)$$

To check whether the solution of [16] is consistent with (21), we take the decoupling limit in the above equation with the substitution [15,16]

$$h_{\mu y} = -\frac{m_c}{\Lambda^3}\partial_\mu\Pi, \qquad h_{yy} = -\frac{2m_c}{\Lambda^3}\partial_y\Pi,$$
$$\Pi(x, y) = e^{-y\sqrt{-\Box}}\pi(x). \qquad (22)$$

This can be expressed in terms of the lapse scalar and shift vector in the decoupling limit

$$N_\mu = h_{\mu y}, \qquad N^2 = g_{yy} + \frac{m_c^2}{\Lambda^6}(\partial_\mu\Pi)^2. \qquad (23)$$

The decoupling limit for g_{yy} has been specified in Ref. [16] only in the leading approximation $g_{yy} = 1 - 2m_c\partial_y\Pi/\Lambda^3$. However, for consistency, the subleading term in (21) is also needed, otherwise the ansatz itself would be inconsistent as it would violate some sacred properties of the bulk equation (21) (for instance, a non-linearly incomplete ansatz, after substitution into the bulk equation, does not respect the Bianchi identities and would not reproduce correctly the trace equation). On the other hand, in the next-to-leading order there is a unique non-linear completion for g_{yy} that is consistent with the Bianchi identities and trace equation. The latter reads $g_{yy} = 1 - 2m_c\partial_y\Pi/\Lambda^3 + m_c^2(\partial_y\Pi)^2/\Lambda^6$. For this ansatz, the expression for N in the decoupling limit reads as follows: $N \simeq 1 - m_c\partial_y\Pi/\Lambda^3 + m_c^2(\partial_\mu\Pi)^2/2\Lambda^6$. The latter combined with the other components above gives a consistent ansatz on

[4] Note that our curvature sign conventions are different from those of Ref. [25].

which Eq. (21) looks as follows:

$$\partial_\mu\partial_\nu\pi - \eta_{\mu\nu}\Box\pi = -\eta_{\mu\nu}\frac{(\Box\pi)^2 - (\partial_\alpha\partial_\beta\pi)^2}{2\Lambda^3} + \frac{(\Box\pi)\partial_\mu\partial_\nu\pi - \partial_\mu\partial_\alpha\pi\,\partial_\nu\partial^\alpha\pi}{\Lambda^3}. \quad (24)$$

As before, trace of (24) gives precisely (1), i.e., the former equation is more general than (1). Hence, not all the solutions of (1) satisfy (24). For instance, consider the {00} component of Eq. (24) and look at a static spherically-symmetric solution. Eq. (24) reduces to

$$\Delta\pi = \frac{(\Delta\pi)^2 - (\partial_i\partial_j\pi)^2}{2\Lambda^3}, \quad (25)$$

where Δ denotes a 3D Laplacian and $i, j = 1, 2, 3$. The above equations is incompatible with the time independent part of (1), unless both left and right-hand sides of (1) and (25) are zero. The solution found in [16] does not satisfy this constraint, and, therefore, it does not satisfy the {00} component of Eq. (24).

Likewise, one can show that the solution of [16] does not satisfy the $\{ij\}$ components of (24) either. To demonstrate this, we consider the $\mu \neq \nu$ part of Eq. (24) and look at a spherically-symmetric static solution. Using the notations of [16], $E_j = \partial_j\pi \equiv r_j E/r$, where $j = 1, 2, 3$, Eq. (24) with $i \neq j$ gives

$$\left(\frac{dE}{dr} - \frac{E}{r}\right)\left(1 - \frac{E}{r\Lambda^3}\right) = 0. \quad (26)$$

The only non-trivial solution of (26) reads: $E = cr$, where c can be an arbitrary constant (including $c = \Lambda^3$), which is not the solution of [16]. This is because the ansatz (22) is a pure gauge in the bulk only in the linearized approximation, but not in a full non-linear theory. In the next section we discuss a solution that exactly satisfies the above equations on and near the brane [13,14].

5. Schwarzschild solution

The Schwarzschild solution found in [13] satisfies all the above non-linear equations on and near the brane. Here, we will discuss some of the main features of this solution in the decoupling limit and give its further physical interpretation.

For a static point-like source, $T_{00} = -M\delta^3(\vec{x})$ with $T_{ij} = 0$. A new physical scale emerges in this problem as a combination of r_c and r_M [2] ($r_M \equiv 2G_N M$ is the Schwarzschild radius and G_N the Newton constant)

$$r_* \equiv \left(r_M r_c^2\right)^{1/3}. \quad (27)$$

(This is similar to the Vainshtein scale in massive gravity [3].) It is straightforward to check that this scale is finite and fixed in the decoupling limit (10), (11).

The metric on/near the brane was found exactly in Refs. [13, 14]. Here, for our purposes it suffices to concentrate on the Newton potential alone $\phi(r) = h_{00}/2$. In the notations adopted in the previous sections

$$g_{00} = -1 + h_{00} = -1 + \bar{h}_{00}/M_{\rm Pl}. \quad (28)$$

The exact expression for $\bar{h}_{00}$ [13] is a solution of (9). Furthermore, (17) can be used to determine the off-diagonal and $\{yy\}$ components of the metric near the brane, as was done in [13, 14]. Below we will check directly that the terms containing curvatures in (2) are not zero on this solution. Hence, it is physically different from the solution of [16].

For simplicity we concentrate on the $r \ll r_*$ region and do the following: we split $\bar{h}$ into two parts according to (14), and for comparison with [16] insist that π has the form obtained in [16]. Using the results of [13], for scales $r \ll r_*$, we get:

$$\frac{\tilde{h}_{00}}{M_{\rm Pl}} = \frac{r_M}{r} - \sqrt{2}m_c^2 r^2\left(\frac{r_*}{r}\right)^{3/2}\left[\frac{\alpha}{\sqrt{2}}\left(\frac{r}{r_*}\right)^{\beta} - 1\right], \quad (29)$$

where $\beta = 3/2 - 2(\sqrt{3} - 1) \simeq 0.04$, and $\alpha \simeq \pm 0.84$. While the π field takes the form:

$$\frac{\pi}{M_{\rm Pl}} = \sqrt{2}m_c^2 r^2\left(\frac{r_*}{r}\right)^{3/2}. \quad (30)$$

Note that both $\tilde{h}_{00}$ and π given above are finite in the limit (10), (11). It is the non-trivial part (29) that differentiates the solution of [13,14] from that of [16]. In particular, the curvatures $\tilde{R}$ in Eq. (2) are non-zero on this solution.

The solution of [13,14] has a number of interesting physical properties. Here we would like to emphasize the effect of the gravitational mass screening [13,14]. In the region $r \ll r_*$ the solution recovers the results of the Einstein theory with tiny, but potentially measurable deviations [10,11,14]. Gravitational screening, on the other hand, manifests itself for $r \gg r_*$. In this region, the solution (Newton's potential) scales as $\sim r_* r_M/r^2$. Naively this seems counterintuitive, since it contradicts the $1/r$ scaling expected from perturbation theory in the region $r_* \ll r \ll r_c$ [1]. However, as was explained in [13,14], there is no actual contradiction since the perturbative approach does not take into account the effect of gravitational mass screening of the source. Unlike in conventional GR, a static source in this theory produces a non-zero scalar curvature even outside of the source [13]. This curvature extends to scales $\sim r_*$. Hence, the source is surrounded by a huge halo of scalar curvature. This halo screens the bare mass of the source. It is easy to estimate that the screening mass should be of the order of the bare mass itself [14]: A deviation from the conventional GR metric at $r \ll r_*$ scales as $m_c\sqrt{r_M r}$ (we ignore small β here.) This can give rise to the scalar curvature that scales as $m_c\sqrt{r_M}r^{-3/2}$. The curvature extends approximately to distances $r \sim r_*$. Because of this the integrated curvature scales as $m_c\sqrt{r_M}r_*^{3/2} \sim r_M$, and the "effective mass" due to this curvature can be estimated to be $r_M M_{\rm Pl}^2 \sim M$, i.e., of the order of the bare mass M itself!

Given the above estimate, there could be either complete or almost complete "gravitational mass screening" for the problem at hand. Which one is realized in DGP? As we will argue below, the solution found in [13,14] suggests that the screening is complete from the 4D point of view and is incomplete from the 5D perspective. Let us discuss this in more detail. From a 4D perspective (i.e., from the point of view of the induced metric on the brane) the solution behaves as a spherically-symmetric distributions of mass/energy of radius $r \sim r_*$, with the bare mass

M placed in the center, and the screening halo surrounding it. The solution [13] shows that the potential at $r \geqslant r_*$ has no "monopole moment", i.e., it has no $1/r$ scaling. Instead, it has the "dipole moment" that scales as $\sim r_* r_M / r^2$, which fits the interpretation of a potential due to a 4D spherically-symmetric mass distribution of size $\sim r_*$ and zero net mass.[5] Hence, the 4D screening should be complete. The picture is slightly different from the 5D perspective. In the definition of the 5D ADM mass the off-diagonal component of the 5D metric of [13,14] is also entering (this component vanishes on the brane and that is why it does not contribute to the 4D ADM mass). Because of this component, there is no complete screening of the 5D ADM mass, and the potential has the 5D "monopole" component which scales as $\sim r_* r_M / r^2$ and yields the 5D ADM mass $\sim M(r_M/r_c)^{1/3}$ [13]. The potential due to an incompletely screened 5D "monopole" smoothly matches onto the 4D potential due to a 4D "dipole".

6. Discussions and outlook

It looks like the present model works in subtle ways. Yet, there are a number of issues that still need to be understood. We will outline some of them below.

(1) *DGP vs. massive gravity*. The results of the present work show that in the decoupling limit massive gravity and the DGP model behave very differently. It would be instructive to understand this in terms of the Lagrangians of the two theories. It is useful to start by recalling some details of the decoupling limit in 4D massive gravity [17]. The most convenient way is to use the Stükelberg method and complete the FP mass term to a reparametrization invariant form [17]. This can be done order-by-order in powers of the fields. Let us call the 4D massive gravitational field $f_{\mu\nu}$, the corresponding Stückelberg vector field A_μ and its longitudinal part ϕ. Due to the mass term the ϕ field acquires a kinetic mixing term with the f field. The quadratic part of the action can be diagonalized by a conformal transformation $f_{\mu\nu} = \tilde{f}_{\mu\nu} + \eta_{\mu\nu}\phi$. As a result, the ϕ field acquires its own kinetic term as well as coupling to the trace of the stress-tensor. The only non-linear interactions that survive in the decoupling limit are those of ϕ, all the non-linear terms containing $\tilde{f}$ vanish [17]. The analog of the $f_{\mu\nu}$ field in DGP is $g_{\mu\nu}$, and the analog of A_μ is N_μ, as it can be read off (3). Besides these fields there is an additional field N. In the linearized theory it enters the action (3) as a Lagrange multiplier and enforces a constraint that is consistent with the linearized Bianchi identities. A linear combination of this field with the longitudinal component of N_μ is an analog of the longitudinal component of the A_μ field of massive gravity—these modes acquire their own kinetic terms through mixing with the tensor fields. However, the similarities end here. At the non-linear level the N field ceases to be a Lagrange multiplier, but enters (3) algebraically. Hence, it can be integrated out explicitly. The resulting action is a functional of $g_{\mu\nu}$ and N_μ only, however, it is very different from the action of massive gravity expressed in terms of $f_{\mu\nu}$ and A_μ. The former contains non-local interactions between $g_{\mu\nu}$ and N_μ. Moreover, if expanded on a flat background, it stays non-local and does not produce quadratic mixing terms between them. Under the circumstances, a way to proceed is to solve first equations for a non-trivial background, then expand the non-local action in perturbations about the background, and only then take the limit. This is, essentially, what we have done in the present work, but in easier terms of the equations of motion.

(2) *Analyticity and the unitarity bound*. The tree-level one-particle exchange amplitude in DGP is non-local from 4D point of view as it contains terms $(\Box + m_c\sqrt{-\Box})^{-1}$ [27]. The amplitude has a branch-cut because of the square root. As a result, there is a way do define a contour in the complex plane so that the pole in the amplitude ends up being on the second Riemann sheet (see, e.g., [28]). The usual 4D dispersion relation can be written for this amplitude, and it does respect 4D analyticity. Having this established, one can consistently take $m_c \to 0$, as it is done in the decoupling limit. How about the non-linear amplitudes? Generically, those contain terms that are singular in the $m_c \to 0$ limit (e.g., different powers of $1/m_c\sqrt{-\Box}$) [2]. These are the terms that make the perturbative expansion to break down at the scale r_*. The results of the present work suggest that it is unlikely that the limiting theory below r_* can be thought of a local 4D theory of massless helicity-2, helicity-1 and helicity-0 states. However, in the regions and/or in the backgrounds where the conventional perturbative expansion is valid (i.e., where the above singular terms can be ignored), it is reasonable to expect that 4D analyticity is respected.
What is more certain, however, is that the Froissart bound cannot be assumed to hold even for the limiting theory. The reason being that in the decoupling limit there are long-range tensor interactions present, and the Froissart saturation can only be taking place if such interactions were absent. Clearly, the present theory of gravity has no mass-gap.

(3) (*Important*) *miscellanea*. The issue of the UV completion and quantum consistency of the model need more detailed studies (see, e.g., [29]). In this respect, it would be interesting to pursue furtherer the issue of the string theory realization of brane induced gravity, perhaps, along the lines of Refs. [30–32]. Finally, in the present work we have not discussed the self-accelerated universe [33,34]. On issues of consistency of this approach will be reported in [35].

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