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The Shilov Boundary of an Operator Space and the Characterization Theorems[1]

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We study operator spaces, operator algebras, and operator modules from the

properties, which we studied earlier in the operator space framework, replace certain topological tools. We introduce certain multiplier operator algebras and C^*-algebras of an operator space, which generalize the algebras of adjointable operators on a C^*-module and the imprimitivity C^*-algebra. It also generalizes a classical Banach space notion. This multiplier algebra plays a key role here. As applications of this perspective, we unify and strengthen several theorems characterizing operator algebras and modules. We also include some general notes on the commutative case of some of the topics we discuss, coming in part from joint work with Christian Le Merdy, about function modules.

1. INTRODUCTION

One basic idea in modern analysis is that C^*-algebras are noncommutative $C(K)$ spaces. The basic idea of noncommutative functional analysis (see [32, 51]) is to study operator spaces (i.e., subspaces of C^*-algebras) as a generalization of Banach spaces. The point is that every Banach space is linearly isometric to a function space, i.e., a subspace of some $C(K)$. A natural idea, therefore, and this was the beginning of the subject of operator spaces, was Arveson's introduction of appropriate noncommutative generalizations of the Choquet and Shilov boundaries. This was done in the foundational papers [2, 3], which gave birth to several subfields of mathematics.

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Ten years after [2, 3], Hamana continued Arveson's approach to operator spaces in a series of deep and important papers. In [36] he defined the *triple envelope* $\mathcal{T}(X)$ of an operator space X, which we shall think of and refer to as the (noncommutative) Shilov boundary of X here. Unfortunately this latter work seems to have been completely overlooked. In the course of time, the subject of operator spaces took a different turn and has grown in many directions. Our main purpose here is to show how this Arveson–Hamana boundary approach can be used to strengthen and unify several important results in the theory.

We prefer to use Hilbert C^*-modules instead of the equivalent theory of triple systems (or ternary rings of operators (TROs)) which Hamana used. The text [41] is a good introduction to C^*-modules. We began to explore the connections between operator space theory and C^*-modules, and the companion theory of strong Morita equivalence, in [10, 11, 15].

If H, K are Hilbert spaces, then any concrete subspace $X \subset B(K, H)$ clearly generates[2] a C^*-subalgebra $\mathcal{L}$ of $B(H \oplus K)$. It is easily checked from the definitions (see, for example, [55, p. 288] or [20]) that this subalgebra $\mathcal{L}$ is the linking C^*-algebra for a strong Morita equivalence. In this paper we view $\mathcal{L}$, or rather the Hilbert C^*-module Z which is its 1–2-corner, as the space on which X is represented.

More generally, suppose that Z is a C^*-module containing a completely isometric copy of X, whose Morita linking C^*-algebra $\mathcal{L}(Z)$ is generated by this copy of X. Then we say that Z is a *Hilbert C^*-extension* of X. The embedding $i: X \to Z$ is taken to be a noncommutative analogue of the statement "$X \subset C(K)$, X separates points of K."

For any abstract operator space X, Hamana's triple envelope $\mathcal{T}(X)$ of X [36, 37] is the *smallest* Hilbert C^*-extension of X. We also write this envelope as ∂X. A little later in this introduction we show how $\mathcal{T}(X)$ may be constructed.

We first describe Part B of the paper. In Section 4 which is perhaps the central section of the paper, we define in terms of $\mathcal{T}(X)$ certain multiplier operator algebras associated with X. We then give alternative characterizations of these multiplier algebras as the sets of adjointable and order bounded operators on X. Another way to think of these left multipliers is as the linear maps $T: X \to X$ which are restrictions to X of the operation of left multiplication by a fixed $S \in B(H)$, for the various $B(H)$ containing X completely isometrically. See Appendix B for other descriptions

These multiplier algebras simultaneously generalize the common C^*-algebras associated with C^*-modules, and the multiplier and centralizer algebras

[2] As is usual in mathematics, to say that a subset X of an object Y, generates Y as an object, means that there exists no proper subobject of Y containing X. In this paper the word *object* should be replaced by *C^*-algebra* or *operator module* etc., as will be clear from the context.

of a Banach space, developed by Alfsen and Effros [1] and later by Behrends [4] and others. The paper [26] is also an important historical source for some of these ideas. In Section 4 we also look at several important examples. We also study a related notion of finiteness for operator spaces which we call extremely nonvanishing, or e.n.v. for short.

Using these multiplier algebras, some basic theory of Hilbert C^*-modules, and some by-now-classical operator space methods, we obtain in Section 5 our main result. This result, loosely speaking, shows how one operator space can act upon another. It is a characterization theorem which unifies and contains, as one-liner special cases, several key characterization type results in operator space theory. For example, it contains the BRS theorem [18], characterizing operator algebras (i.e., norm closed, possibly non-self-adjoint, subalgebras of a C^*-algebra). BRS states that A is an operator algebra if and only if A is an algebra which is an operator space such that the multiplication is completely contractive, i.e., $\|xy\| \leqslant \|x\| \, \|y\|$, for all matrices x, y with entries in the algebra. This has been a useful result in the recent program of developing a completely abstract theory of (not-necessarily-self-adjoint) operator algebras (as one has for C^*-algebras).

Moreover, the approach given here to theorems such as the BRS theorem gives much more precise information and in addition allows one to relax the hypotheses.

Most of the consequences of our main result listed here are to *operator modules*. An operator module is defined to be an operator space with a nondegenerate module action which obeys a condition like that of a Banach module, that is $\|ax\| \leqslant \|a\| \, \|x\|$, except that we allow a, x to be matrices and ax means multiplication of matrices. For example, we shall see that the operator modules over C^*-algebras are simply the B-submodules of Rieffel's A-rigged B-modules (sometimes called C^*-*correspondences*). The latter modules play a significant role in noncommutative geometry [25]. Also, if X is a given operator space, then operator module actions on X are in 1–1 correspondence with completely contractive homomorphisms into the multiplier algebra of X.

All the above is contained in Part B. It is not strictly necessary to read Part A first; this can be skipped if the reader is solely interested in operator spaces. In Part A, and in a companion paper [14], we study the commutative version of a few of the topics from Part B. In particular we study a class of Banach modules over a function algebra A, which we call *function modules*. This work on function modules suggested, and led to, everything else here. We must emphasize though that the most important modules over function algebras, such as those coming from representations on Hilbert space, are not function modules but operator modules. We spend some time in Sections 3 and 6 studying singly generated modules, ending Section 6 with an application to "automatically associative" BRS theorems.

In Appendix A, we give an alternative development of Hamana's universal property of $\mathcal{T}(X)$. We also give several interesting consequences which were not explicitly pointed out in [36, 37], and some other applications. Since the proofs are not lengthy, and since these results are called upon throughout Part B, it seems worthwhile to include this. In addition, we develop these results from the C^*-module viewpoint using Theorem 1.1 as the main tool, as opposed to Hamana's approach via triple systems using results of Harris, etc.

In Appendix B, we state a few results from our recent paper [17]. We then use one of these results and some other facts to prove a Banach–Stone theorem for operator algebras with contractive approximate identities. Finally we mention briefly some very recent progress.

We return now to the noncommutative Shilov boundary, which we will describe in a little more detail. This will also serve the purpose of introducing notation we will need later.

In classical functional analysis, a common trick for studying a Banach space X is to consider it as a function space by embedding X linearly isometrically as a subspace of $C(K)$, where K is a compact Hausdorff space. Often K is taken to $X_1^* = Ball(X^*)$ with the weak* topology. The question arises of finding the smallest K which works; i.e., the minimal or essential compact topological space on which X can be supported in this way. A minimal representation may often be found by looking at the set $ext(S)$ of extreme points of S, where $S = X_1^*$ or a suitable set derived from X_1^*. Function spaces which separate points of K, and which contain constant functions, have a canonical extremal representation, namely by restricting the functions to the Shilov boundary ∂X of X.

As we said earlier, one of the purposes of Arveson's foundational papers [2, 3] was to construct a good candidate for the extremal noncommutative $C(K)$ containing X. His setting was that of unital operator spaces, by which we mean a pair (X, e) consisting of an operator space X with distinguished element e such that there exists a linear complete isometry T of X onto a subspace of a unital C^*-algebra A, with $T(e) = 1_A$. This notion turns out to be quite independent of the particular A; indeed Arveson showed that if $S: X \to B$ is another such complete isometry with $S(e) = 1_B$, then there exists a unique complete order isomorphism between the operator systems $T(X) + T(X)^*$ and $S(X) + S(X)^*$ which extends the map $S \circ T^{-1}$ from $T(X) \to S(X)$. Let us recall what these terms mean: An operator system is a self-adjoint unital operator space. The appropriate morphisms between operator systems are unital completely positive maps. Such a map is called a complete order injection (resp. complete order isomorphism) if it is 1–1 (resp. and onto) and its inverse is completely positive. We will also frequently use the fact that a unital linear map S between operator systems is completely contractive iff it is completely positive, and then it is *-linear; i.e., $S(v^*) = S(v)^*$. See [2, 47] for proofs.

The noncommutative version of separation of points is, by the Stone–Weierstrass theorem, that the operator subspace of A generates A as a C^*-algebra. Thus, in perfect analogy with the function space case, given a unital operator space X, Arveson was interested in the minimal C^*-algebra A containing and generated by a completely isometric unital copy of X. It is a highly nontrivial fact that such a minimal C^*-algebra exists. Arveson gave various such boundary theorems; for example, in [3] he showed that this C^*-*envelope* exists for an irreducible linear space S of operators such that S contains a nonzero compact operator. His methods center around a powerful use of completely positive maps, their iterates, and their fixed points. In addition he built up a formidable array of machinery, including the theory of boundary representations, multivariable dilation theory, and much more.

In [34] Hamana continued this work, adding the new tool of the *injective envelope*. Just as in the Banach space setting, given an operator space X and a completely isometric embedding i of X into an injective operator space Z, Hamana shows (see [34, 36, 37, 58]) that there exists a minimal X-projection (i.e., completely contractive idempotent map whose range contains $i(X)$) φ on Z. The pair $I(X) = (\varphi(Z), i)$ is the *injective envelope* of X, and it is unique as an operator superspace of X (i.e., it is independent of which Z we started with). Any X-projection on $I(X)$ is the identity map. See [32] for an account of this.

From these facts, and a well known theorem of Choi and Effros, one immediately derives the existence of the minimal generated C^*-algebra of a unital operator space. We include Hamana's proof because of its importance in what follows.

THEOREM 1.1 (The Arveson–Hamana theorem [34, 2, 3]). *If V is a unital operator algebra, or unital operator space, then there exists a C^*-algebra $C^*_e(V)$, and a unital complete isometry $J: V \to C^*_e(V)$, such that $J(V)$ generates $C^*_e(V)$ as a C^*-algebra, and such that:*

> *for any other unital complete isometry $i: V \to B$ to a C^*-algebra B whose range generates B, there is a (necessary unique and surjective) $*$-homomorphism $\pi: B \to C^*_e(V)$, such that $\pi \circ i = J$.*

Proof. Suppose that $B \subset B(H)$, as a nondegenerate C^*-subalgebra. Let φ be a minimal $i(V)$-projection on $B(H)$, and let $R = Im\, \varphi$. As noted earlier, φ is completely positive and *-linear. By a theorem of Choi–Effros ([21, Theorem 3.1], R is a unital C^*-algebra with respect to the old linear and involutive structure, but with product $\varphi(x) \circ \varphi(y) = \varphi(\varphi(x)\, \varphi(y))$. Also, (R, i) is, by the note above the statement of the theorem, a copy of the injective envelope of V. Let $C_e^*(V)$ be the C^*-subalgebra of R generated by $i(V)$, with respect to the new product. By the universality of the injective envelope and [32, 5.2.3], it is clear that as a C^*-algebra generated by a copy of V, $(C_e^*(V), i)$ only really depends on V and its identity element. With respect to the *usual* product on $B(H)$, the C^*-subalgebra of $B(H)$ generated by R contains B, the C^*-subalgebra of $B(H)$ generated by $i(V)$. A key part of the Choi–Effros theorem is the relation $\varphi(rb) = \varphi(r\varphi(b))$, for $r \in R$, $b \in B(H)$. Hence by induction it follows that $\varphi(r_1 r_2 \cdots r_n) = r_1 \circ r_2 \circ \cdots \circ r_n$, for $r_1, ..., r_n \in R$. Hence $\pi = \varphi|_B$ is a $*$-homomorphism $B \to R$, with respect to the new product on R. Since π extends the identity map on V it clearly also maps into $C_e^*(V)$. Since π has dense range, it is necessarily surjective. ∎

Hamana dubbed this minimal C^*-algebra $C_e^*(V)$ the C^*-*envelope* of V. As we saw in the proof, if V is a unital operator space, then $I(V)$ is a unital C^*-algebra; and $C_e^*(V)$ is defined to be the C^*-subalgebra of $I(V)$ generated by $J(V)$.

Under the conditions of this theorem, there exists an ideal I of B such that $B/I \cong C_e^*(V)$ as C^*-algebras. Clearly J is also a homomorphism if V is an operator algebra. If A is a function algebra or function space containing constants, then it is easy to deduce from the Arveson–Hamana theorem above that $C_e^*(A)$ equals the space of continuous functions on the ordinary Shilov boundary ∂A of A.

Then in [35, 58] Hamana and Ruan (independently) combined Hamana's results for injective envelopes of operator systems with a famous method of Paulsen [47] which embeds any operator subspace X of $B(H)$ in an operator system

$$\mathcal{S}(X) = \begin{bmatrix} \mathbb{C} & X \\ X^* & \mathbb{C} \end{bmatrix}$$

in $M_2(B(H))$. This system only really depends on the operator space structure of X and not on H. This follows from the following fact (see [47] Lemma 7.1 for a proof), which we will state separately since we will invoke it frequently:

Lemma 1.2 (Paulsen's lemma). *Suppose that for $i = 1, 2$, we are given Hilbert spaces H_i, K_i, and linear subspaces $X_i \subset B(K_i, H_i)$. Let $\mathcal{S}_i$ be the following operator system inside $B(H_i \oplus K_i)$:*

$$\mathcal{S}_i = \begin{bmatrix} \mathbb{C}I_{H_i} & X_i \\ X_i^* & \mathbb{C}I_{K_i} \end{bmatrix}.$$

If $T: X_1 \rightarrow X_2$ is completely contractive (resp. completely isometric), then the map

$$\begin{bmatrix} \lambda & x \\ y^* & \mu \end{bmatrix} \mapsto \begin{bmatrix} \lambda & T(x) \\ T(y)^* & \mu \end{bmatrix}$$

taking $\mathcal{S}_1$ to $\mathcal{S}_2$ is completely positive (resp. a complete order injection).

Hamana and Ruan considered the injective envelope $I(\mathcal{S}(X))$ of $\mathcal{S}(X)$; with a little thought one can see that the two diagonal idempotents in $\mathcal{S}(X)$ become mutually orthogonal projections in $I(\mathcal{S}(X))$. With respect to these one may write $I(\mathcal{S}(X))$ as a 2×2 matrix algebra (this point is explained more carefully later in the introduction). Hamana shows in [36, 37] that the 1–2-corner I_{12} is simply $I(X)$. In [17] we characterize the other corners (but we do not need this characterization here). From this it is clear (see also [65]) that $I(X)$ is a Hilbert C^*-module (being a corner of a C^*-algebra).

At the same time (around 1984) Hamana constructed $\mathcal{T}(X)$. One simply forms $C^*_e(\mathcal{S}(X))$; that is, one considers the closed *-subalgebra of $I(\mathcal{S}(X))$ generated by $\mathcal{S}(X)$. It is clear from the last paragraph that this C^*-algebra may be viewed as a 2×2 matrix algebra, in which X sits within the 1–2-corner. We define $C^*(\partial X)$ to be the C^*-subalgebra generated by this copy of X. Clearly we may write

$$C^*(\partial X) = \begin{bmatrix} \mathcal{E} & W \\ Z & \mathcal{F} \end{bmatrix},$$

where $\mathcal{E}$, $\mathcal{F}$ are C^*-algebras and W and $Z = \bar{W}$ are Hilbert C^*-bimodules. We write W as $\mathcal{T}(X)$ and call this the *triple envelope*, *Hilbert C^*-envelope* or (noncommutative) *Shilov boundary* of X. We also write ∂X for the pair (W, J), where $J: X \rightarrow W$ is the canonical embedding. Clearly $\mathcal{T}(X) \subset I_{12} = I(X)$; indeed one may restate the construction as saying that $\mathcal{T}(X)$ is the subTRO of $I(X)$ generated by X (see [37]).

If X is an operator system, unital operator algebra, or more generally, a unital operator space, then $\mathcal{T}(X)$ is the usual thing. That is, in these cases $\mathcal{T}(X) = C^*_e(X)$, the C^*-envelope of X. Proof of this may be found in 4.3 or Appendix A.12.

It is very instructive to apply this noncommutative extremal space construction to a Banach space X. The Shilov boundary emerges in this setting as a Hermitian line bundle. We write down the details in Section 4, following earlier work [66].

We now list some more of the notation we will use.

If S is a subset of a C^*-algebra A, then we shall write $C^*_A(S)$, or $C^*(S)$ if the context is clear, for the C^*-subalgebra of A generated by S. That is,

$C^*(S)$ is the smallest C^*-subalgebra of A containing S. We write E_1 for the set $\{x \in E : \|x\| \leqslant 1\}$.

If X is a left module over an algebra with identity 1, then we shall assume that $1x = x$ for all $x \in X$. We often write a.s.g. and t.s.g. for algebraically singly generated and topologically singly generated, respectively. The latter term means, for a left A-module X, that there is an $x_0 \in X$ such that Ax_0 is norm-dense in X. We recall that a module X is said to be *faithful*, if whenever $a \cdot x = 0$ for all $x \in X$, then $a = 0$. We will say that an A-module is *λ-faithful* if there exists a $\lambda > 0$ such that $\|a\| \leqslant \lambda \sup\{\|ax\| : x \in X_1\}$, for all $a \in A$. We are most interested in the case $\lambda = 1$, that is, the 1-faithful case.

If X is an A-module, and if $\rho: B \to A$ is a contractive (or completely contractive) unital homomorphism, then X becomes a B-module in a canonical way, namely $m'(b, x) = m(\rho(b), x)$, where m is the A-action. We shall call this a *prolongation* of the action m. Of course this is just a change of rings in the sense of algebra. Two modules X and Y are *A-isometrically isomorphic* if they are isometrically isomorphic via an A-module map. We write $X \cong Y$ *A-isometrically*. Similar notations apply with the word "completely" inserted.

We will freely use without ceremony standard terminology associated with operator spaces and completely bounded maps, see [32, 51, 47, 5, 16, 9] for example. We recall that operator spaces may be considered concretely as closed linear subspaces of $B(H)$, or abstractly via Ruan's axioms [32]. In this paper $CB(X)$ is the space of completely bounded maps on X, with the cb-norm $\|\cdot\|_{cb}$. In fact $CB(X)$ is also an operator space with matrix norms coming from the canonical identification $M_n(CB(X)) \cong CB(X, M_n(X))$. For some of this paper, issues of complete boundedness do not arise. This is because for a linear operator T between operator spaces whose range lies in a minimal operator space (i.e., a subspace of a commutative C^*-algebra) we have $\|T\| = \|T\|_{cb}$. The same is true for maps between MAX spaces, or for bounded module maps between C^*-modules [64].

If Y is an operator space, and I is a cardinal number, we write $\mathbb{K}_I$ for the compact operators $\ell_2(I)$, and $\mathbb{K}_I(Y)$ for $\mathbb{K}_I \otimes_{spatial} Y$. We write C_I for the column Hilbert space of dimension I, which may be identified with a column in $\mathbb{K}_I$, and which may also be viewed as a right C^*-module over $\mathbb{C}$. We write $C_I(Y)$ for $C_I \otimes_{spatial} Y$. A similar notation holds for the row space or more generally for nonsquare matrices $\mathbb{K}_{I,J}(Y) \cong R_j(C_I(Y)) \cong C_I(Y) \otimes_{spatial} R_J$.

We remind the reader that a contractive homomorphism from a C^*-algebra into an operator algebra B is a *-homomorphism into $B \cap B^*$ and is completely contractive.

For an operator algebra B with contractive approximate identity, we write $LM(B)$ for the left multiplier algebra. This may be described as

$\{G \in B^{**}: GB \subset B\}$ or as a similar subspace of any $B(H)$ on which B is nondegenerately represented (see, e.g., [50, 52]). Similarly assertions hold for right multipliers. The multiplier algebra is written as $M(B)$. It follows from the previous paragraph that a contractive homomorphism from a C^*-algebra into $LM(B)$, maps into $M(B)$, if B is a C^*-algebra.

We will use the following common ideas frequently. Suppose that A is a C^*-algebra and that p and $q = 1 - p$ are orthogonal projections in A or $M(A)$. Then A may be written as a 2×2 matrix C^*-algebra with corners pAp, pAq, qAp, qAq. Suppose that $\pi: A \to B$ is a nondegenerate $*$-homomorphism into a C^*-algebra B (for example, if A and B are unital and $\pi(1) = 1$). Then p and q correspond to two projections $p', q' = 1 - p'$, with respect to which B also decomposes as a 2×2 matrix C^*-algebra, and π maps each corner of A into the corresponding corner of B. Hence π may be written as a 2×2 matrix of maps between these corners, and we will refer to the "1–2-corner of π" for example. The above also works if A, B are unital, π is completely positive, and $\pi(p)$ and $\pi(q)$ are complementary orthogonal projections $p', q' = 1 - p'$ in B. For we may first write B as a 2×2 matrix C^*-algebra w.r.t. p', q'; then π restricted to the linear span of p and q is a unital $*$-homomorphism and it follows by Choi's multiplicative domain lemma (cf. [47, Example 4.3]) that $\pi(pa) = p'\pi(a)$ and $\pi(ap) = \pi(a)\, p'$ for $a \in A$. From this it follows again that π maps each corner of A into the corresponding corner of B and that π may be written as a 2×2 matrix of maps between these corners as before. We will apply these tricks frequently to C^*-algebras A generated by a copy of the Paulsen system $\mathcal{S}(X)$, where the two complementary main diagonal projections in $\mathcal{S}(X)$ are complementary orthogonal projections in A (such as is the case for the C^*-algebra $A = C_e^*(\mathcal{S}(X))$).

In this paper we will usually allow an operator module to be over an algebra A which is not necessarily an operator algebra, but instead is an operator space and an algebra. We will usually assume that the algebra has an identity of norm 1, although in most cases a contractive approximate identity (c.a.i.) will suffice. We also do not insist that our bimodules have the property $(ax)\, b = a(xb)$, unless we explicitly say so. We show that this is automatic. A completely 1-faithful module is an operator A-module X for which the canonical homomorphism $A \to CB(X)$ is a complete isometry.

If A and B are C^*-algebras, then a *B-rigged A-module* or *$A - B$-C^*-correspondence*, is a right C^*-module Z over B for which there exists a nondegenerate contractive homomorphism $\rho: A \to B_{\mathscr{B}}(Z)$. Here $B_{\mathscr{B}}(Z)$ is the space of bounded B-module maps on Z. By a result of Lin [43], we also have $B_{\mathscr{B}}(Z) = LM(\mathbb{K}(Z))$, where $\mathbb{K}(Z) = \mathbb{K}_{\mathscr{B}}(Z)$ is the imprimitivity C^*-algebra of Z (that is, the C^*-algebra of compact right module maps on Z) [41]. Since A is a C^*-algebra, it is clear that θ automatically has range within the adjointable operators on Z, so that this coincides with the usual definition. Actually we shall prove in Section 5 that a B-rigged A-module is the same as a right C^*-module Z over B, which is a left operator A-module. We write $\mathbb{B}(Z)$ or $\mathbb{B}_{\mathscr{B}}(Z)$ for the algebra of adjointable maps on Z. As is well known, $\mathbb{B}(Z) = M(\mathbb{K}(Z))$.

We will also use the following C^*-module notations. The reader unfamiliar with C^*-Morita theory might skip these for now or might consult [41] and the cited papers of Rieffel for further details. Suppose that Z is a right C^*-module over a C^*-algebra B. Let $\mathscr{D}$ be the closed span in B of the range of the B-valued inner product. We say that Z is *full* if $\mathscr{D} = B$. Of course Z is a full right C^*-module over $\mathscr{D}$. Also, Z is a $\mathscr{C} - \mathscr{D}$-*imprimitivity bimodule* or *strong Morita equivalence bimodule*, in Rieffel's sense, where $\mathscr{C} = \mathbb{K}_{\mathscr{D}}(Z)$.

Conversely, given any strong Morita equivalence $\mathscr{C} - \mathscr{D}$-bimodule Z, we have $\mathscr{C} \cong \mathbb{K}_{\mathscr{D}}(Z)$. An important construction for us will be the linking C^*-algebra $\mathscr{L}(Z)$, which we have already mentioned. This is a C^*-algebra which may be written as a 2×2 matrix algebra

$$\mathscr{L}(Z) = \begin{bmatrix} \mathscr{C} & Z \\ \bar{Z} & \mathscr{D} \end{bmatrix}$$

in such a way that the usual product of 2×2 matrices encodes all the module structure and inner products. See [20] and [55, p. 288] for more details or [15] for non-self-adjoint generalization of most of the linking algebra facts below (although this is much more than is necessary the statements below are really at the level of an extended exercise suitable for someone familiar with the basic aspects of strong Morita equivalence, although we advise the frequent use of Cohen's factorization theorem to simplify the calculations). We will sometimes write c for the corner map that takes $z \in Z$ to its image in $\mathscr{L}(Z)$. We may take the operator space structure on Z to be the one coming from its identification with $c(Z)$. We also identify $\mathscr{C}$ and $\mathscr{D}$ with their images in $\mathscr{L}(Z)$. The important reason for this is that now we have replaced all the module actions and inner products by natural operations in a C^*-algebra. For example, the $\mathscr{D}$-valued inner product $\langle z_1, z_2 \rangle$ of $z_1, z_2 \in Z$ is the product $c(z_1)^* c(z_2)$ in the $\mathscr{L}(Z)$. This may all be expressed in terms of concrete operators between Hilbert spaces, as follows. Consider a faithful nondegenerate *-representation of $\mathscr{L}(Z)$ on a Hilbert space L. It is quite standard to show that $L = H \oplus K$ for Hilbert spaces H and K on which $\mathscr{C}$ and $\mathscr{D}$ respectively act nondegenerately. We may identify Z completely isometrically with a subspace of $B(K, H)$. In this way, all the module actions and inner products get replaced by products and involutions of operators between these Hilbert spaces. Thus we may (and will) interpret expressions such as $z_1^* z_2 z_3^* z_4$ for example, for $z_i \in Z$, as a product of concrete operators between H and K, landing us back in one of the spaces $\mathcal{C}$, $\mathcal{D}$, Z or $\bar{Z}$ (in the example it would be $\mathcal{D}$). Of course the construction we have just outlined is completely standard to workers in this field and gives one direction of the well-known equivalence between Hilbert C^*-modules and TROs.

From this perspective it is clear that Z is an operator $\mathcal{C}-\mathcal{D}$-bimodule (see also [64]). The following fact will be of great importance for us. We sketch one proof.

PROPOSITION 1.3. *A strong Morita equivalence $\mathcal{C}-\mathcal{D}$-bimodule Z is a left operator $LM(\mathcal{C})$-module, or equivalently, is a left operator $B_{\mathcal{D}}(Z)$-module.*

Proof. Here is one proof using the construction above. Since $\mathcal{C}$ acts nondegenerately on H, we may view $LM(\mathcal{C}) \subset B(H)$. Then if $S \in B(H)$ corresponds to such a left multiplier, and if $T \in B(K, H)$ corresponds to an element in Z, we may write (by Cohen's factorization theorem) $T = RT'$ for an operator $R \in B(H)$ (resp. $T' \in B(K, H)$) corresponding to an element in $\mathcal{C}$ (resp. Z). Hence $ST = SRT' \in \mathcal{C}Z \subset Z \subset B(K, H)$. Thus it is clear that Z is a left operator $LM(\mathcal{C})$-module. ∎

Suppose that Z_1 is a $\mathcal{C}-\mathcal{D}$-imprimitivity bimodule and that Z_2 is a $\mathcal{C}_2-\mathcal{D}_2$-imprimitivity bimodule. We will say that Z_1 and Z_2 are *isomorphic as imprimitivity bimodules* if there is a linear bijection $\phi: Z_1 \to Z_2$, and bijective $*$-isomorphisms $\theta: \mathcal{C}_1 \to \mathcal{C}_2$ and $\pi: \mathcal{D}_1 \to \mathcal{D}_2$, such that $\phi(c_1 z d_1) = \theta(c_1)\, \phi(z)\, \pi(d_1)$ for all $c_1 \in \mathcal{C}_1$, $d_1 \in \mathcal{D}_1$, $z \in Z_1$, and such that $\langle \phi(z) \mid \phi(w) \rangle_{\mathcal{C}_2} = \theta(\langle z \mid w \rangle_{\mathcal{C}_1})$ for all $z, w \in Z_1$, and similarly for the $\mathcal{D}_i$-valued inner products. This is equivalent to the linking C^*-algebras of Z_1 and Z_2 being $*$-isomorphic, with the isomorphism mapping each of the four corners of the linking C^*-algebra into the matching corner. We call ϕ an *imprimitivity bimodule isomorphism*. This is, in other language, the same as a triple isomorphism. It is clear that such ϕ is completely isometric.

The relationships between C^*-modules and operator spaces are deeper than perhaps suspected. First, our main theme here is the consideration of operator spaces and modules as subspaces of C^*-modules. Second, in [10, 11, 15] we showed that the theory of Hilbert C^*-modules fits well with operator spaces (it does not with Banach spaces). The modules and bimodules dealt with in that theory are operator modules and bimodules, and one can describe basic constructions of that theory as operator space constructions. Third, whereas in [10] we showed that the Banach *module* (or operator module) structure on a C^*-module Z completely specifies all other information (e.g., the inner product), Hamana's approach [36, 37] shows that the *operator space* structure of Z is essentially enough. From this operator space structure one may obtain all the essential data. For example, the algebra $\mathbb{B}(Z)$ of adjointable right module traps on a right

C^*-module Z may now be described as the left self-adjoint multiplier C^*-algebra of Z, where here Z is considered merely as an operator space. Similarly for the imprimitivity C^*-algebra $\mathbb{K}(Z)$ of Z, and similarly, therefore, by Morita equivalence symmetry, we may obtain (a copy of) the C^*-algebra acting on the right of Z. See Appendix A for details.

We should point out that there seems to have been in the past year a rapid growth of interest in the study of C^*-modules (= TROs) and their duals and preduals, from an operator space perspective, and applications of this to important classes of operator spaces. Much of this grew out of the important 1999 preprint [29] of Effros, Ozawa, and Ruan, whose recent revision contains many interesting facts about TROs. (Our paper is quite independent of [29] and does not overlap.) It is clear that the use of TRO/C^*-modules in operator space theory is an idea whose time has come.

We thank Krzysztof Jarosz for much helpful information on function spaces and Christian Le Merdy for many important insights which are included here. We also thank Vern Paulsen for important insights and for many good questions, which facilitated progress and which also led to the work [17]. We also thank N. Ozawa and A. Torok for help with the ideas around Lemma 4.20. Finally, we thank M. Hamana for a letter pointing out some theorems in [36] which we had not seen (although we were familiar with part of this paper), in particular Theorem 4.3 in his preprint from 1991, which is Theorem 3.2 in the Pitman volume [36]. Because of this oversight we had attributed, in an earlier version of this paper, the construction of $\mathcal{T}(X)$ to C. Zhang (who was unaware of Hamana's work), and we had thought that almost all of the results in Appendix A were new. One positive byproduct of this oversight was that it seems to have resulted in the publication of [37].

The present paper is a revised and slightly expanded version of a preprint that has been circulating since the first half of 1999.

PART A

2. *Function Modules*

In this section we summarize some classical definitions of multiplier algebras of Banach spaces, restate some results from [14], and make some related observations. Again we begin with some notation used in this part. In this section A will be a unital Banach algebra, unless stated to the contrary. Sometimes A will be a function algebra, that is a uniformly closed, point separating, unital subalgebra of $C(\Omega)$, for a compact Hausdorff space Ω. A *unital function space* is a closed subspace X of $C(\Omega)$ which contains constant functions. For a set of scalar valued functions $\mathscr{E}$ we will write $\mathscr{E}^+$ for the strictly positive functions in $\mathscr{E}$. For a closed subspace H of $C(\Omega)$, the Choquet boundary $Ch(H)$ of H may be defined to be the set of points $w \in \Omega$ such that if μ is a probability measure on Ω such that $\mu(f) = f(w)$ for all $f \in H$, then $\mu = \delta_w$. Here δ_w is the Dirac point mass. The Shilov boundary is the closure in Ω of $Ch(H)$. Unfortunately, unless we assume some extra property on H, these boundaries are not independent of Ω, as one is used to in the function algebra case. We write ∂H for the Shilov boundary of H and M_A for the maximal ideal space of a Banach algebra A, considered as characters in A^* in the usual way.

We will try to reserve the letters K, K' for certain special topological spaces. In fact, if X is a Banach space, usually we will employ the letters j, K for the canonical isometric embedding $j: X \to C_0(K)$, where K is the weak*-closure of the extreme points of X_1^*, with the zero functional taken out (if it was ever in). We will refer to this as the *extremal embedding* of X. We use $C_b(K)$ for the bounded continuous functions.

The following definitions are classical (see [1, 4, 38]). The *multiplier function algebra* of a Banach space is the closed unital algebra $\mathscr{M}(X) = \{f \in C_b(K) : fj(X) \subset j(X)\}$ Here j and K are as above. The *centralizer algebra* of X is $Z(X) = \{f \in C_b(K) : f, \bar{f} \in \mathscr{M}(X)\}$. Note that $Z(X)$ is a commutative C^*-algebra, and $\mathscr{M}(X)$ is a function algebra.

Every Banach space X is an $\mathscr{M}(X)$-module. This will be important. The canonical map $\mathscr{M}(X) \to B(X)$ is easily seen to be an isometric homomorphism, and we will often identify $\mathscr{M}(X)$ as the range of this homomorphism.

THEOREM 2.1. *Let X be a Banach space and $T \in B(X)$. The following are equivalent*:

(i) $T \in \mathscr{M}(X)$.

(ii) *There is a constant M s.t. $|\psi(T(x))| \leqslant M\,|\psi(x)|$ for all $x \in X$ and $\psi \in ext(X_1^*)$.*

(iii) *ψ is an eigenvector for T^* for all $\psi \in$ ext(X_1^*).*

(iv) *There is a compact space Ω, a linear isometry $\sigma: X \to C(\Omega)$, and an $f \in C(\Omega)$, such that $\sigma(Tx) = f\sigma(x)$, for all $x \in X$.*

The least M in (ii), and least $\|f\|_\infty$ in (iv), coincides with the usual norm of T. See [1, 4] and [38, Sect. I.3], where other important equivalent statements such as M-boundedness are added. Actually we are not aware of (ii) and (iv) explicitly in the literature, so perhaps we should say a word about the proof of these. It is obvious that (iii) implies (ii), and the reverse follows from a well-known fact about containment of kernels of functionals.

It is obvious that (i) together with the fact that $C_b(K)$ is a $C(\Omega)$ for compact Ω, gives (iv). Finally, (iv) implies (iii) by the well-known consequence of Krein–Milman that extreme points of $\sigma(X)_1^*$ extend to extreme points of $C(\Omega)_1^*$, and the latter extreme points are the obvious ones.

THEOREM 2.2 (See [14]). *Let X be a Banach A-module. The following are equivalent*:

(i) *There is a compact Hausdorff space Ω, a contractive unital homomorphism θ: $A \to C(\Omega)$, and an isometric linear map Φ: $X \to C(\Omega)$ such that $\Phi(a \cdot x) = \theta(a)\, \Phi(x)$ for all $a \in A$, $x \in X$.*

(ii) *Same as* (i), *but, with Ω replaced by K, the* (*possibly locally compact*) *weak*-closure in X^* of the extreme points of X_1^*, with the zero functional removed. We take Φ to be the canonical isometry j: $X \to C_0(K)$ given by $j(x)(\phi) = \phi(x)$. The homomorphism θ has range inside $\mathscr{M}(X) \subset C_b(K)$.*

(iii) *$MIN(X)$ is an operator module over $MIN(A)$.*

(iv) *The module action considered as a map $A \otimes_\lambda X \to X$ is contractive. Here λ is the injective Banach space tensor product.*

This last theorem was inspired by Tonge's characterization of function algebras [62]. In [14] we prove a stronger result than 2.2. It is clear from (iii) that one can replace λ in (iv) by a bigger tensor norm. A still larger norm is given in [14].

One of the most useful points of the above is that in (ii) we have $\phi(ax) = \theta(a)(\phi)\, \phi(x)$ for all $\phi \in ext(X_1^*)$, and $a \in A$, $x \in X$.

DEFINITION 2.3. A module X satisfying one of the equivalent conditions of Theorem 2.2 will be called a *function module* (or an abstract function module). We shall call a tuple (θ, Φ, Ω) as in (i) or (ii) a *representation* of the function module. The representation found in (ii) above will be called the *extremal representation*.

This is not the classical usage of the term function module [4], but it will serve our purpose.

The best known examples of function modules are ideals in a function algebra, or modules $(AX)^-$, where X is a (often finite) subset of $C(\Omega)$, where A is a subalgebra of $C(\Omega)$. Any Banach space is a $\mathscr{M}(X)$-function module and a $Z(X)$-function module. Conversely, by 2.2, every function module action on a Banach space X is a prolongation of the $\mathscr{M}(X)$-action. Thus if X is a Banach space and A a unital Banach algebra, then there is a 1–1 correspondence between function A-module actions on X, and contractive unital homomorphisms θ: $A \to \mathscr{M}(X)$. In particular, if A is a function algebra, then the function A-module actions on X which extend to function $C(\Omega)$-module actions correspond to the contractive unital homomorphisms $\theta: A \to Z(X)$. Here Ω is a compact Hausdorff space on which A sits as a function algebra, such as $\Omega = M_A$.

Definition 2.4. We shall say that a Banach space X with the property that $0 \notin \overline{ext}(X_1^*)$ is *extremely nonvanishing* (or e.n.v.).

For an e.n.v. function module, its extremal representation is on a compact space. For example, any function algebra is e.n.v. More generally a unital function space is e.n.v. If A is a function algebra on compact Ω, and if f_0 is a nonvanishing continuous function on Ω, then the submodule Af_0 of $C(\Omega)$ is e.n.v. These last three facts follow from the fact (used in the proof of (iv) of 2.1 above) that if X is a closed linear subspace of $C(\Omega)$, then $ext(X_1^*) \subset \{\alpha\delta_w : w \in \Omega, \alpha \in \mathbb{T}\}$. The function $C([0, 1])$-module $C_0((0, 1])$ is not e.n.v. Any finite dimensional Banach space is e.n.v.; in fact we have:

Proposition 2.5. *If X is an algebraically finitely generated function module, then X is e.n.v.*

Proof. Suppose X has generators $x_1, ..., x_n$. The map $A^{(n)} \to X$ which takes $(a_i) \mapsto \sum a_i x_i$ is onto. By the open mapping theorem there is a constant $C > 0$ such that for any $x \in X_1$ there exists $(a_i) \in A^{(n)}$ with $\sum_i \|a_i\|^2 \leqslant C^2$ such that $\sum a_i x_i = x$.

Given any $\phi \in ext(X_1^*)$ and $x \in X_1$, choose (a_i) as above. We have

$$|\phi(x)| = \left|\sum_i \phi(a_i x_i)\right| = \left|\sum_i \theta(a_i)(\phi)\, \phi(x_i)\right|$$
$$\leqslant \left(\sum_i \|a_i\|^2\right)^{1/2} \left(\sum_i |\phi(x_i)|^2\right)^{1/2} \leqslant C \left(\sum_i |\phi(x_i)|^2\right)^{1/2}.$$

Here θ is as in the remark after Theorem 2.2. Thus we see that $(\sum_i |\phi(x_i)|^2)^{1/2} \geqslant C^{-1}$. Thus X is e.n.v. ∎

Remarks. (1) By looking at elementary examples of function modules X, one sees that if (θ, Φ, Ω) is a general representation of X one cannot hope in general that $\theta(A)$, or its closure, separates points of Ω. Thus $\theta(A)$, or its closure, is not necessarily a function algebra on Ω in the strict sense. However, we shall see that in certain cases one can find a separating representation.

(2) Theorem 2.2 shows that we can always find a representation (θ, Φ, K) of a function module X such that $\Phi(X)$ separates points of K and also such that for any $w \in K$ there exists $x \in X$ such that $\Phi(x)(w) \neq 0$. Indeed in the extremal representation, $K \subset X^*$, so that it is clear that there is a much stronger point separation property here, and this is exploited in

the next section. It is also always possible (see Corollary 2.10 in [14]) to find a representation (θ, Φ, K) of X, with θ an isometry. However we cannot hope in general to simultaneously have such separation properties and also to have θ be an isometry. Nonetheless see Section 3 for a class of function modules for which this is possible.

PROPOSITION 2.6. *Let X be a function module, with representation (θ, Φ, Ω).*

(1) *If X is a faithful (resp. λ-faithful) function module, then θ is 1–1 (resp., isometric).*

(2) *Suppose that (θ, Φ, Ω) is a representation with the property that for any $w \in \Omega$ there exists $x \in X$ such that $\Phi(x)(w) \neq 0$ (resp. for any $\varepsilon > 0$ there exists $x \in X_1$ with $|\Phi(x)(w)| > 1 - \varepsilon$). Then if θ is 1–1 (respectively, isometric) then X is faithful (resp. 1-faithful).*

(3) *The extremal representation (θ, j, K) has the properties in the first sentence of (2). Hence (2) applies to this representation.*

Proof. Most of these follow simply from the fact that $\|ax\| = \|\theta(a)\, \Phi(x)\|$ for $a \in A, x \in X$. For example, if X is λ-faithful then $\|a\| \leqslant \lambda \sup\{\|ax\| : x \in X_1\} \leqslant \lambda\, \|\theta(a)\|$, so that θ is bicontinuous. However, since norm equal spectral radius on function algebras, θ is isometric. We leave the remaining assertions as exercises. ∎

It is not true in general that if θ is 1–1 then it is isometric. A good example to bear in mind is the following:

EXAMPLE 2.7. Let $A = A(\mathbb{D})$ be the disk algebra considered as a function module over itself as follows: $m(f, g)(z) = f(z/2)\, g(z)$. It is easy to see that this module is faithful, t.s.g., and e.n.v. Also, it is not a.s.g., and in any of the obvious representations of this function module, the associated θ is 1–1 but not isometric. Indeed $\theta(f)(z) = f(\frac{z}{2})$ maps A onto a dense subalgebra of A and also separates points of $\overline{\mathbb{D}}$.

For $A = A(\mathbb{D})$ and any Banach space X, the function module A-actions on X are in an obvious correspondence with elements of $Ball(\mathscr{M}(X))$, whereas function module $C(\overline{\mathbb{D}})$-actions on X are in a correspondence with $Ball(Z(X))$.

Let T be any function on a compact space Ω with $\|T\|_\infty \leqslant 1$. Let X be any subspace of $C(\Omega)$ for which $TX \subset X$. Then X is an $A(\mathbb{D})$-function module, with action $fx = f(T)\, x$, for all $f \in A(\mathbb{D}), x \in X$. For example, X could be the smallest closed subalgebra of $C(\Omega)$ containing T or containing T and the identity, and these are clearly t.s.g. modules. The latter module is faithful if the range of the function T has a limit point in the open disk; on the other hand if $\|T\|_\infty < 1$ then $\theta\colon f \to f(T)$ is not isometric on $A(\mathbb{D})$.

We will consider two other examples of function modules over commutative C^*-algebras which, together with the one above, will show that our theorems in the next section are best possible. We leave it to the reader to check the assertions made:

EXAMPLE 2.8. The $C([0, 1])$-module $C_0((0, 1])$ is 1-faithful, topologically singly generated, but not e.n.v. or algebraically singly generated.

EXAMPLE 2.9. Consider the closure X of Bx in $C([0, 1])$, where $x(t) = t$ and $B = \{f \in C([0, 1]) : f(0) = f(\frac{1}{2}) = f(1)\}$. Then X is a topologically singly generated and 1-faithful B-module. The reader might compute the extremal representation of X; it is not hard to see that $ext(X_1^*)$ is the product of $\mathbb{T}$ and $E = (0, \frac{1}{2}) \cup (\frac{1}{2}, 1]$, where the latter set is identified with the functionals $f \mapsto f(s)$ on X, for any $s \in E$.

We end this section with the following remark which we will not use later, but include for motivational purposes, and for contrast with some later results. Suppose that B is a unital function space, and as usual let $j: B \to C(K)$ be the extremal embedding. As we said earlier, such B is e.n.v., so K is compact. We also note that all functions in the multiplier algebra of B are of the form $\phi \mapsto \frac{\phi(b)}{\phi(1)}$, for $b \in B$, $\phi \in K$. By restricting to a representation (J, K_0) on the subset $K_0 = \{\phi \in K : \phi(1) = 1\}$ one may eliminate the action of the circle on K (see the next section for more on this). One obtains an analogous multiplier algebra $\mathcal{M}'(B) \subset C(K_0)$. From the above one sees that $\mathcal{M}'(B) \subset J(B)$. Thus B contains a function algebra $C = J^{-1}(\mathcal{M}'(B))$, and B is a C-module. It also is easy to see that if m is a function A-module action on B, then $m(a, b) = m(a, 1)\, b$ for all $a \in A$, $b \in B$, and it follows that $a \mapsto m(a, 1)$ is a unital homomorphism into C. Therefore there is a 1–1 correspondence between function A-module actions m on B and contractive unital homomorphisms $\pi: A \to C$. If B is a function algebra then $C = B$.

3. *Singly Generated Function Modules and Nonvanishing Elements*

This section, again, may be skipped by those primarily interested in operator spaces. The main point here is to apply the function multiplier algebra, and the results summarized in Section 2, to characterize a large class of singly generated function modules. In Section 6 we will see which of the results below have noncommutative versions.

For simplicity, in this section we will assume that A, B are function algebras, although this is only needed in a few places. We will also always regard $\mathcal{M}(X)$ as a concrete subalgebra of $C(K)$, where K is the extremal space of the previous section.

We begin by looking at the multiplier algebra of an a.s.g. function module. Recall that in this case the extremal space K is compact.

PROPOSITION 3.1. *Let X be a Banach space, and let j: $X \to C_0(K)$ be as usual.*

(i) *Suppose that X is an a.s.g. function A-module, with single generator x_0. Then $g_0 = j(x_0)$ is a nonvanishing function in $C(K)$, and X is an a.s.g. as an $\mathcal{M}(X)$-module. If θ is the associated extremal representation for the A-action, then $\mathcal{M}(X) = g_0^{-1} j(X) = \theta(A)$. In this case, the a.s.g. function B-module actions on X are in 1–1 correspondence with the contractive surjective homomorphisms θ: $B \to \mathcal{M}(X)$.*

(ii) *X is an a.s.g. and faithful function A-module, if and only if X is a.s.g. as a function $\mathcal{M}(X)$-module and $A \cong \mathcal{M}(X)$ isometrically isomorphically, via the homomorphism θ in (i). In this case the a.s.g. and faithful function B-module actions on X are in 1–1 correspondence with the (necessarily isometric) bijective unital homomorphisms $B \to \mathcal{M}(X)$.*

Proof. (i) If x_0 is any single generator of X, then $g_0 = j(x_0)$ is nonvanishing (since $\phi(ax_0) = \theta(a)(\phi)\,\phi(x_0)$ for all $\phi \in K$). If $f \in \mathcal{M}(X)$ then $fg_0 \in j(X)$, so that $f = j(x)\,g_0^{-1}$ for some $x \in X$, and if $x = ax_0$ then $f = \theta(a) \in \theta(A)$. Conversely, if $f = j(x)\,g_0^{-1}$ then $fj(ax_0) = \theta(a)\,j(x) \in j(X)$. The last part is clear.

From (i), the open mapping theorem, and Proposition 2.6 (i), we get (ii). That θ is isometric in this case is because norm equals spectral radius on function algebras. ∎

Thus, X possesses an a.s.g. function module action if and only if the natural $\mathcal{M}(X)$ action is a.s.g. It may be interesting to characterize this as a Banach space property of X.

DEFINITION 3.2. For any contractive unital homomorphism θ: $A \to A$, and any function A-module X, we define X_θ to be X with the new module action $m(a, x) = \theta(a)\,x$.

COROLLARY 3.3. *If X and Y are two a.s.g. faithful function modules over function algebras A and B, respectively, and if $X \cong Y$ linearly isometrically, then there exists an (isometric) isomorphism α: $A \to B$ such that $Y_\alpha \cong X$ A-isometrically. If $Y = A$, then $X \cong A$ A-isometrically.*

We omit the proof of this, which is generalized later in 6.7.

Suppose that X is a Banach space and that $j\colon X \to C_0(K)$ is the extremal representation of X, as usual. By a *nonvanishing element* we mean an element $x_0 \in X$ such that $j(x_0)(\phi) \neq 0$ for all $\phi \in K$. We let $K' = \{\phi \in K : \phi(x_0) \geqslant 0\}$. This is a nonempty weak*-closed convex subset of K. By restricting to K' we obtain a new representation (π, J, K'). The map J is still an isometry. Suppose that X is e.n.v.; then K' is compact. Also, the Choquet boundary of $H = J(X)$ in K' contains $ext(K')$ (see 29.5 in [22]), which in turn clearly contains the set $E = \{\phi \in ext(X_1^*) : \phi(x_0) \geqslant 0\}$. We claim that the weak*-closure $\bar{E} = K'$. To see this, pick $\psi \in K'$. Since $K' \subset K$, there exists a net $\phi_\lambda \in ext(X_1^*)$ converging weak* to ψ. Choose $\alpha_\lambda \in \mathbb{T}$ such that $\alpha_\lambda \phi_\lambda \in E$. A subnet of the α_λ converges to $\alpha \in \mathbb{T}$ say. Replace the net with this subnet, so that $\alpha_\lambda \phi_\lambda \to \alpha\psi$ weak*. Thus $\alpha_\lambda \phi_\lambda(x_0) \to \alpha\psi(x_0) \geqslant 0$, which implies that $\alpha \geqslant 0$, so that $\alpha = 1$. Thus E is indeed dense in K'. We have also proved:

LEMMA 3.4. *If a Banach space X is e.n.v. and contains a nonvanishing element x_0, then K' is the Shilov boundary of $J(X)$ in $C(K')$.*

It is clear that if A is a function algebra on a compact space Ω, and if $f \in C(\Omega)$ is a nonvanishing function, then the submodule $X = Af$ of $C(\Omega)$ (which is A-isometric to $A\,|f|$) is a.s.g. and faithful, and as we remarked earlier X is e.n.v. We now move toward proving the converse to this assertion.

Suppose that X is a t.s.g. and e.n.v. A-module, with single generator x_0. Again let (θ, j, K) be the extremal representation of X. If $j(x_0)(\phi) = 0$ for some $\phi \in K$, then $j(ax_0)(\phi) = \theta(a)(\phi)\, j(x_0)(\phi) = 0$ for all $a \in A$. Thus $\phi(x) = 0$ for all $x \in X$, which is impossible. Therefore $j(x_0)$ is nonvanishing on K. We then define K' and J as above, and let π be the restriction of θ to K', and let $B = (\pi(A))^-$ and $g_0 = J(x_0) \in C(K')$. Again g_0 is nonvanishing. Notice $J(X)$ and $\pi(A)$ separate points of K' in the following strong sense. If $\phi_1, \phi_2 \in K'$ are distinct, then $\ker\,\phi_1 \neq \ker\,\phi_2$, so there exists an $x \in X$ such that $\phi_1(x) = 0 \neq \phi_2(x)$. Thus if $x = \lim a_n x_0$ then $J(x)(\phi_1) = 0 = \lim_n \pi(a_n)(\phi_1)$, but $J(x)(\phi_2)$ and $\lim_n \pi(a_n)(\phi_2)$ are nonzero.

Clearly $X \cong H$ A-isometrically, where H is the closed, point-separating submodule $B\,|g_0|$ of $C(K')$. We have proved (i) of the following characterization of function modules which are t.s.g. and e.n.v.:

THEOREM 3.5. *Let X be a function A-module.*

(i) *If X is t.s.g. and e.n.v. then there exists a representation (π, Φ, K') of X with the following properties: K' is the Shilov boundary of $\Phi(X)$ in K'; $\pi(A)$ and $\Phi(X)$ each separate points of K'; and there is an $f_0 \in C(K')^+$ such that $X \cong (\pi(A))^-\, f_0$ A-isometrically.*

(ii) *Conversely the existence of* π *and* f_0 *satisfying the last of these three properties implies that* X *is t.s.g. and e.n.v.*

(iii) *If* X *is t.s.g.,* λ*-faithful, and e.n.v., then it is a.s.g.*

Item (ii) above follows from a remark after Definition 2.4. Item (iii) follows from (i) and Proposition 2.6 (1).

See Example 2.7 for an explicit exhibit of the situation of (i) above. This example, and 2.8, also shows that (iii) of the theorem is sharp.

COROLLARY 3.6. *If* Ω *is any compact space, and if* X *is a function module over* $C(\Omega)$*, then the following are equivalent:*

(i) X *is a. s. g.,*

(ii) X *is t.s.g. and e.n.v.,*

(iii) X *is the quotient of* $C(\Omega)$ *by a closed ideal.*

The only one of these which is also faithful, of course, is $C(\Omega)$.

Proof. Clearly (iii) implies (i). By Proposition 2.5, (i) implies (ii). If (ii) holds then by the previous theorem we obtain a unital *-homomorphism $\pi: C(\Omega) \to C(K')$, whose range is a C^*-subalgebra which separates points. Thus π is onto, so again by the previous theorem $X \cong C(K') f_0 = C(K')$, giving (iii). ∎

An obvious question which arises in light of the last result is whether every t.s.g. function module over $C(\Omega)$ is a commutative C^*-algebra. Example 2.9 gives the lie to this. This may be seen perhaps most easily from the fact that for any commutative C^*-algebra A, the map θ: $A \to C(E)$, where $E \subset ext(A_1^*)$ and $\theta(a)(g) = |g(a)|$ for $g \in E$, has range which is closed w.r.t. multiplication. However in 2.9 it is easy to see that $\theta(x)^2 \neq \theta(f)$ for any $f \in X$.

COROLLARY 3.7. *Let* A *be a function algebra. A function* A*-module* X *is faithful and a.s.g. if and only if there is a compact space* Ω *such that* A *is a function algebra on* Ω *(that is,* A *is represented isometrically homomorphically as a unital point separating subalgebra of* $C(\Omega)$*) and there is an* $f_0 \in C(\Omega)^+$ *such that* $X \cong Af_0$ A*-isometrically.*

Proof. The ($\Leftarrow$) direction is easy. The ($\Rightarrow$) direction follows from the proof of the theorem as follows. If X is a.s.g. then $\Phi(X) = \pi(A)\, g_0$. Since g_0 is bounded away from 0, it follows that $\pi(A)$ is uniformly closed. If in addition X is faithful then it follows from Proposition 2.6 (i) and the open mapping theorem that π is isometric. The rest is clear. ∎

PART B

4. *The Noncommutative Shilov Boundary and Multiplier Algebras*

The main purpose of this section is to define the multiplier algebras of an operator space, give several alternative definitions the reader may prefer, and compute them in situations of particular interest. En route we will also develop some other concepts.

We come back to some ideas which were described in the Introduction. With the notation there, we have the following C^*-subalgebras

$$C^*(\partial X) \subset C_e^*(\mathcal{S}(X)) \subset I(\mathcal{S}(X)),$$

where we write $I(\cdot)$ for the injective envelope [34, 37, 58]. Indeed $C_e^*(\mathcal{S})$ is defined to be the C^*-subalgebra of $I(\mathcal{S})$ generated by $\mathcal{S}$. As we saw, $I(\mathcal{S}(X))$ may be written as a 2×2 matrix algebra whose 1–2-corner is $I(X)$. Similarly, as in the Introduction, we write $C^*(\partial X)$ as:

$$C^*(\partial X) = \begin{bmatrix} \mathcal{E} & \mathcal{T}(X) \\ \mathcal{T}(X)^* & \mathcal{F} \end{bmatrix}.$$

We also wrote $\mathcal{T}(X)$, together with the canonical embedding $J\colon X \to \mathcal{T}(X)$, as ∂X. Sometimes, however, we shall suppress mention of J and write x for $J(x)$.

Unlike in Part A, the spaces above are not at the present time defined canonically—the injective envelope $I(\mathcal{S}(X))$, for example, is only defined up to a $*$-isomorphism (which is fixed on the copy of $\mathcal{S}(X)$). Nonetheless, up to appropriate isomorphisms, these objects, and the multiplier algebras discussed below, are unique. See Appendix A for more on this. However, this lack of canonicity is always a potential source of blunders in this area, if one is not careful about various identifications.

Hamana wrote $K_l(X)$ or $K_l(\mathcal{T}(X))$ for $\mathcal{E}$, whereas Zhang [66] wrote $C_0^*(XX^*)$ for this space. Instead, to be consistent with C^*-module notation and our multiplier terminology we will write $\mathcal{E}$ as $\mathbb{K}_l(\mathcal{T}(X))$ or $\mathbb{K}_{\mathcal{F}}(\mathcal{T}(X))$. Also, we will continue to reserve the symbol $\mathcal{E}$ for this C^*-algebra, or $\mathcal{E}(X)$ when we wish to emphasize the dependence on X. Similarly, $\mathcal{F}(X) = \mathbb{K}_r(\mathcal{T}(X))$ and so on. We now make several important observations which are clear if one takes the time to write out some sample products of the matrices in $\mathcal{S}(X)$. First, $\mathcal{E}$ and $\mathcal{T}(X)$ have dense subsets consisting of sums of products, the terms in each product alternating between $J(X)$ and $J(X)^*$. An important principle, which we will refer to as the *first term principle*, is that these products always begin with an element from $J(X)$. Similarly for $\mathcal{T}(X)^*$ and $\mathcal{F}$, the corresponding products begin with a term from $J(X)^*$.

Also $\mathcal{T}(X) = \overline{(\mathcal{E}J(X))} = \overline{(J(X)\,\mathcal{F})}$. It is also clear from these facts that $\mathcal{T}(X)$ is a strong Morita equivalence $\mathcal{E}$–$\mathcal{F}$-bimodule and that $C^*(\partial X)$ is its linking C^*-algebra.

We now introduce two technical terms which are not central to our discussion—the more casual reader may skip to the material after 4.1 if desired. We shall say that an operator space X is C^*-*generating* if $\mathcal{T}(X)$ is (completely isometrically isomorphic to) a C^*-algebra. This is equivalent, by the universal property in Appendix A, to X having *some* Hilbert C^*-extension Z which is a C^*-algebra. We do not need this here, but it follows by a more or less well-known result Appendix A.5 that this is also the same as $\mathcal{T}(X)$ being imprimitivity bimodule isomorphic to the C^*-algebra. We shall see that examples of C^*-generating operator spaces include all unital operator spaces and operator algebras with c.a.i., and we shall find more in Section 6.

We will say that an operator space X is e.n.v. if $C^*(\partial X)$ is a unital C^*-algebra. Again, by the universal property in Appendix A, this is equivalent to X having *some* Hilbert C^*-extension Z whose linking C^*-algebra is unital. From this it is easy to see, for example, that any subspace $X \subset M_n$ is e.n.v. In this case view X as within the 1–2-corner of M_{2n}, and consider the C^*-subalgebra of M_{2n} generated by this copy of X. This is a unital C^*-algebra, so that X is e.n.v.

We will use left/right e.n.v. to mean that just one of the main diagonal corners of this C^*-algebra is unital. Thus X is left e.n.v. if and only if $\mathbb{K}_l(\mathcal{T}(X))$ is unital, which happens exact when $\mathcal{T}(X)$ is an algebraically finitely generated right Hilbert C^*-module [63, Sect. 15.4]. Thus e.n.v. is some kind of finiteness condition on X. Later we will justify the new use of the term e.n.v. by showing that it is a genuine noncommutative analogue: a Banach space X is e.n.v. in the sense of Part A if and only if $MIN(X)$ is e.n.v. in the new sense.

Lemma 4.1. *An operator space X is e.n.v. if and only if $C^*(\partial X) = C^*_e(\mathcal{S}(X))$. Similarly, X is left (resp. right) e.n.v. if and only if $\mathcal{E}(X)$ (resp. $\mathcal{F}(X)$) equals the 1–1 (resp. 2–2) corner of $C^*_e(\mathcal{S}(X))$ as sets.*

Proof. Suppose that X is left e.n.v. The image of $J(X)$ in the corner of $C^*(\partial X)$, together with the identity of the 2–2 corner of $C^*_e(\mathcal{S}(X))$, generates a unital C^*-algebra $\mathcal{B}$ inside $C^*_e(\mathcal{S}(X))$. We do not assert yet that the identity of $\mathcal{B}$ is the identity of $C^*_e(\mathcal{S}(X))$. Inside $\mathcal{B}$, the image of $J(X)$ and $J(X)^*$, and the two idempotents on the diagonal of $C^*_e(\mathcal{S}(X))$ corresponding to $1_{\mathcal{E}}$ and to the identity of the 2–2 corner of $C^*_e(\mathcal{S}(X))$, form an operator system $\mathcal{S}_1$. By Paulsen's lemma, the obvious map $\Phi: \mathcal{S}_1 \to \mathcal{S}(X)$ is a complete order isomorphism. By the Arveson–Hamana theorem (1.1 above), Φ extends to a surjective *-homomorphism $\theta: C^*(\mathcal{S}_1) = \mathcal{B} \to C^*_e(\mathcal{S}(X))$. If $c: \mathcal{T}(X) \to C^*_e(\mathcal{S}(X))$ is the embedding into the

1–2-corner, and if i is the embedding of the 1–1-corner of $C_e^*(\mathcal{S}(X))$ inside $C_e^*(\mathcal{S}(X))$, then it is easy to see that $\theta(xy^*) = xy^*$ for $x, y \in c(J(X))$. Hence the restriction of θ to $i(\mathcal{E}(X))$ is the identity map on $i(\mathcal{E}(X))$. If e is the idempotent in the 1–1 corner of $\mathcal{S}(X)$ then $i(e) = \theta(i(1_{\mathcal{E}})) = i(1_{\mathcal{E}})$. So $e = 1_{\mathcal{E}}$. ∎

We now give a first definition of the multiplier algebras of an operator space, in terms of $\mathcal{T}(X)$. We have retained this as our basic definition for historical reasons because it fits well into the theoretical framework of this paper and also because for some purposes it does seem to have decided advantages. The reader who is unfamiliar with C^*-modules will probably prefer the equivalent characterizations given later (for example in Theorem 4.10 or Appendix B), in which case we beg for their patience for now.

Our first definition, then, of the *left multiplier algebra of X* is:

$$\mathcal{M}_l(X) = \{S \in B_{\mathcal{F}}(\mathcal{T}(X)) : SJ(X) \subset J(X)\}.$$

An important fact, which has been somewhat overlooked, is that $B_{\mathcal{F}}(W)$ is a not-necessarily-self-adjoint operator algebra, for any right C^*-module W over $\mathcal{F}$. By the result of Lin cited in the Introduction, $B_{\mathcal{F}}(\mathcal{T}(X))$ may be identified with the left multiplier algebra $LM(\mathcal{E})$ of $\mathcal{E}$, and $\mathcal{E} \cong \mathbb{K}(\mathcal{T}(X))$, in the language of C^*-modules. By 1.3, $\mathcal{T}(X)$ is a left operator $B_{\mathcal{F}}(\mathcal{T}(X))$-module. From all of this it is obvious that:

Proposition 4.2. *For any operator space X, we have that $\mathcal{M}_l(X)$ is an operator algebra with identity of norm* 1, *and X is a left operator $\mathcal{M}_l(X)$-module.*

We define the self-adjoint left multiplier C^*-algebra $\mathbb{B}_l(X) = \{S\colon S$ and $S^* \in \mathcal{M}_l(X)\}$. The last adjoint S^* is taken with respect to a Hilbert space which $\mathcal{M}_l(X)$ is nondegenerately represented on (completely isometrically). Alternatively, we may define $\mathbb{B}_l(X) = \{S \in \mathbb{B}_{\mathcal{F}}(\mathcal{T}(X)) : SX \subset X, S^*X \subset X\}$, where the last adjoint is the one in $\mathbb{B}_{\mathcal{F}}(\mathcal{T}(X))$.

We can also define a (left) imprimitivity operator algebra and imprimitivity C^*-algebra of X, generalizing the imprimitivity C^*-algebra $\mathbb{K}(Z)$ of a C^*-module Z. Namely, $\mathbb{K}_l(X) = \{S \in \mathbb{K}_{\mathcal{F}}(\mathcal{T}(X)) : SX \subset X\}$ and $\mathbb{K}_l^*(X) = \{S \in \mathbb{K}_l(X) : S^*X \subset X\}$.

If X is left e.n.v., then $\mathcal{E}(X) \cong \mathbb{K}_{\mathcal{F}}(\mathcal{T}(X))$ is unital, so that $\mathbb{K}_l(X) = \mathcal{M}_l(X) \subset \mathcal{E}(X)$.

Similarly one may define right multiplier algebras of X. For an operator space X which has the property that $\mathcal{E}(X) = \mathcal{F}(X)$, we can define two-sided multiplier algebras, analogously to the above; for example, $M(X) = \{T \in M(\mathcal{E}(X)) : TX \subset X, XT \subset X\}$.

These left and right multiplier algebras play a key role later. Clearly X is an operator $\mathcal{M}_l(X) - \mathcal{M}_r(X)$-bimodule, and hence X is also an operator module over the other multiplier algebras we defined above.

The following results identify the Shilov boundaries and multiplier algebras in some useful cases. In (i) below, V is a unital operator space, that is, a subspace of a unital C^*-algebra which contains the unit. In this case, the minimal V-projection of [34, 58] is unital and completely positive, so that as mentioned in the Introduction (by the result of Choi–Effros), $I(V)$ is a unital C^*-algebra.

Proposition 4.3. (i) (*cf.* [66, *Theorem* 2 *and Propositions* 3 *and* 4). *If V is a unital operator space then $C^*(\partial V) = M_2(C_e^*(V))$, where $C_e^*(V)$ is the C^*-envelope of V (i.e., the unital C^*-subalgebra of $I(V)$ generated by V). Thus $\mathcal{T}(V) = C_e^*(V)$, and so V is e.n.v. and C^*-generating. As subsets of $C_e^*(V)$ we have $\mathcal{M}_l(V) \subset V$ and $\mathcal{M}_r(V) \subset V$. If A is a unital operator algebra then $\mathcal{M}_l(A) = \mathcal{M}_r(A) = A$. If V is an operator system then $\mathbb{B}_l(V) = \mathbb{B}_r(V)$ as subsets of $C_e^*(V)$.*

(ii) *If Y is a right Hilbert C^*-module, then $\mathcal{T}(Y) = Y$, $\mathcal{M}_l(Y) = LM(\mathbb{K}(Y))$, $\mathbb{B}_l(Y) = M(\mathbb{K}(Y)) = \mathbb{B}(Y)$, whereas $\mathbb{K}_l(Y) = \mathbb{K}_l^*(Y) = \mathbb{K}(Y)$.*

Proof. (i) That $\mathcal{T}(V) = C_e^*(V)$ and $C^*(\partial V) = M_2(C_e^*(V))$ follows from A.12, or by 4.18 below. Hence $\mathcal{E}(V) = C_e^*(V)$, and so within $C_e^*(V)$, we have $\mathcal{M}_l(V) \subset J(V)$. If A is a unital operator algebra then we remarked earlier that one can take $J\colon A \to C_e^*(A)$ to be a unital homomorphism, so that $J(A)\,J(A) \subset J(A)$, implying that $J(A) \subset \mathcal{M}_l(A)$. If V is a system, then $aJ(V) \subset J(V)$ implies that $J(V)\,a^* \subset J(V)$ (since $J(V)^* = J(V)$). This yields the last assertion.

(ii) See A.4 for example.

It is clear that for any operator space X we have a canonical completely contractive homomorphism $\mathcal{M}_l(X) \to CB(X)$. Let us call this map ρ. By the first term principle, ρ is 1–1, and thus $\mathcal{M}_l(X)$ may be viewed as a unital subalgebra of $CB(X)$. In many cases ρ is a complete isometry, for example if X is a unital operator space, a C^*-module, or of the form $C_n(A)$ for an operator algebra A with c.a.i. We shall see shortly that it is also true if X is a minimal operator space. However, the following example shows that, unlike in the classical theory, $\mathcal{M}_l(X)$ is not in general isometrically contained in $CB(X)$ (or $B(X)$).

Example 4.4. Let E be the matrices in M_3 which are supported on the second and third entries of the first row only. Let $A = \mathbb{C}I_3 + E$, a subalgebra of M_3. Let Q be the 3×3 matrix which is the sum of $2I_3$ and the matrix of all 1's; and let $P = Q^{1/2}$. Let $X = AP$. Notice that $XX^* = AQA^*$

contains $e_{12}Q$, $e_{13}Q$, and $e_{12}Qe_{21}$. Hence XX^* contains all the matrices in M_3 supported on the first row. Thus $C^*(XX^*) = M_3$, and also $C^*(XX^*)\,X = M_3 X = M_3$. Thus the copy of X in the 1–2-corner of M_6 generates M_6 as a C^*-algebra. Since M_6 has no ideals, the natural representation of X is its Shilov representation (see A.3). That is $\mathscr{T}(X) = M_3$, and J is the natural inclusion. Hence $\mathscr{M}_l(X) = A$. We shall show that the canonical map ρ: $\mathscr{M}_l(X) \to CB(X)$ (or into $B(X)$) in this case is not an isometry by showing that its restriction to E is not an isometry. Notice that if $x, y \in E$, and $\mu \in \mathbb{C}$, then $x(y + \mu I_3)\,P = \mu x P$. Thus $\|\rho(x)\|_{B(X)} = \kappa\,\|xP\|$, where κ is the constant $\sup\{|\mu| : \|(y + \mu I_3)\,P\| \leqslant 1\}$. Similarly, $\|\rho(x)\|_{cb}$ is $\|xP\|$ times a constant. If ρ was an isometry, then $\|xQx^*\| = \kappa^{-2}\,\|xx^*\|$ for all $x \in E$, which immediately implies the contradiction that the upper left corner of Q is $\kappa^{-2} I_2$.

On the other hand, for any operator space X, the map π given by restricting ρ to $\mathbb{B}_l(X)$ *is* isometric as a map into $CB(X)$ or $B(X)$. Indeed it was shown in [17] that π is completely isometric as a map into $CB_l(X)$ or $B_l(X)$, where the latter spaces are defined as follows: Namely $B_l(X) = B(X)$ but with matrix norms

$$\|[T_{ij}]\|_n^l = \sup\left\{\left\|\left[\sum_{k=1}^{n} T_{ik}(x_k)\right]\right\|_{C_n(X)} : x \in BALL(C_n(X))\right\}.$$

That is, we identify $M_n(B_l(X))$ with $B(C_n(X))$ via the natural correspondence of left matrix multiplication. Similarly one defines $CB_l(X)$ so that $M_n(CB_l(X)) \cong CB(C_n(X))$ via the same natural correspondence. We will give a different proof of this result from [17], which at the same time gives a new characterization of $\mathbb{B}_l(X)$. We should add that at this point in time we do not know whether $\mathbb{B}_l(X) \subset CB(X)$ completely isometrically.

Definition 4.5. We will say that a function f: $X \to X$ is (left) *adjointable* if there exists a linear complete isometry σ from X into a C^*-algebra, and a function g: $X \to X$, such that $\sigma(f(x))^*\,\sigma(y) = \sigma(x)^*\,\sigma(g(y))$ for all $x, y \in X$. We write $\mathscr{A}_l(X)$ for the set of such adjointables on X and $\mathscr{A}_l^\sigma(X)$ for the set of functions f satisfying the above condition, but for a fixed σ.

Thus $\mathscr{A}_l(X) = \bigcup_\sigma \mathscr{A}_l^\sigma(X)$. We will see that any adjointable function on X is linear and completely bounded. It is pretty clear that without any real change, we can replace C^*-algebra in the definition by C^*-module if we replace $\sigma(x)^*\,\sigma(g)$ by $\langle \sigma(x) \mid \sigma(y)\rangle$ (by the remarks at the end of the Introduction). Then it is clear that w.l.o.g. one can replace C^*-module by Hilbert C^*-extension of X. With this and Hamana's universal property in mind (see A.3), it is immediate that $\mathscr{A}_l(X) = \mathscr{A}_l^J(X)$, where J is the canonical embedding of X into $\mathscr{T}(X)$, into $C^*(\partial X)$, or into $I(X)$.

THEOREM 4.6. *Let Z be a right Hilbert C^*-module, and let X be a closed linear subspace of Z. Define the set $\mathscr{A}^Z(X)$ to be*

$$\{T: X \to X: \text{there exists } S: X \to X \text{ such that } \langle T(x) \mid y \rangle = \langle x \mid S(y) \rangle \text{ for all } x, y \in X\}.$$

Then

(1) *$T \in \mathscr{A}^Z(X)$ implies that T is linear, bounded, completely bounded, and $\|T\| = \|T\|_{cb}$.*

(2) *$\mathscr{A}^Z(X)$ is a C^*-algebra with the norm from (1).*

(3) *$\mathscr{A}^Z(X)$ is completely isometrically isomorphic to a unital subalgebra of $B_l(X)$ or $CB_l(X)$;*

(4) *$M_n(\mathscr{A}^Z(X)) \cong \mathscr{A}^{C_n(Z)}(C_n(X))$ as C^*-algebras;*

(5) *X is a left $\mathscr{A}^Z(X)$-operator module, with respect to the natural action.*

Proof. It is easy to check that any $\mathscr{A}^Z(X)$ is linear and bounded by the closed graph theorem. We leave it to the reader to check that $\mathscr{A}^Z(X)$ is a C^*-algebra with the $B(X)$ norm and involution $T^* = S$, where S is as in the definition of $\mathscr{A}^Z(X)$. For example, for $x \in X_1$ and with $S = T^*$ we have:

$$\|Sx\|^2 = \|\langle Sx \mid Sx \rangle\| = \|\langle TSx \mid x \rangle\| \leqslant \|TS\|.$$

(4) It is an easy exercise to check that $M_n(\mathscr{A}^Z(X)) \cong \mathscr{A}^{C_n(Z)}(X))$ as *-algebras. Hence the *-isomorphism must be isometric.

(1) By the above, $\mathscr{A}^Z(X) \subset B(X)$ isometrically. On the other hand, if $T \in \mathscr{A}^Z(X)$, with $\|T\| < 1$, then T is a finite convex combination of unitaries in $\mathscr{A}^Z(X)$ [50]. For such a unitary U, $\|[U(x_{ij})]\| = \|\sum_k \langle U(x_{ki}) \mid U(x_{kj}) \rangle\|^{1/2} = \|[x_{ij}]\|$, for $x_{ij} \in X$. Hence $\|U\|_{cb} = 1$, so that $\|T\|_{cb} \leqslant 1$. Hence $\|T\|_{cb} = \|T\|$. Thus we have proved (2) also.

(3) This follows from (1) and the above, with X replaced by $C_n(X)$.

(5) Let $R \in M_n(\mathscr{A}^Z(X))$, and let $\underline{x_1}, ..., \underline{x_n} \in C_n(X)$ be such that the row $[x_1; ...; x_n] \in R_n(C_n(X))$ has norm $\leqslant 1$. Then from (1) and (4), we have that $\|[(R\underline{x_1}); ...; (R\underline{x_n})]\| \leqslant \|R\|$. ∎

Thus for any operator space X, and any completely isometric linear σ from X into a C^*-algebra or C^*-module, the five parts of the previous theorem hold with $\mathscr{A}^Z$ replaced by $\mathscr{A}^\sigma(X)$.

COROLLARY 4.7. *For any operator space X, the five assertions of the previous theorem hold with $\mathscr{A}^Z$ replaced by $\mathscr{A}_l$. Hence X is a left operator module over the C^*-algebra $\mathscr{A}_l(X)$. The canonical map $\pi: \mathbb{B}_l(X) \to CB(X)$ is* *a *-isomorphism onto $\mathscr{A}_l(X)$. Thus $\mathbb{B}_l(X)$ may be regarded as a closed subalgebra of $B_l(X)$ or of $CB_l(X)$ (up to completely isometric isomorphism).*

Proof. We need only prove the assertions about $\mathbb{B}_l(X)$. The map π above clearly maps $\mathbb{B}_l(X)$ into $\mathscr{A}_l(X)$, since any $T \in \mathbb{B}_l(X)$ is adjointable in the usual sense on $\mathscr{T}(X)$. This map $\mathbb{B}_l(X) \to \mathscr{A}_l(X)$ is clearly a 1–1 *-homomorphism. If U is a unitary in $\mathscr{A}_l^J(X)$, then $\langle Ux \mid Uy \rangle = \langle x \mid y \rangle$ for all $x, y \in X$. Since $X\mathscr{F}$ is dense, there is one possible extension of U to a $\tilde{U} \in B_{\mathscr{F}}(\mathscr{T}(X))$, and it is easy to see that $\tilde{U}$ is well-defined and isometric (cf. proof of 4.10 (1)). Clearly $\pi(\tilde{U}) = U$. Since the unitaries span a C^*-algebra, π is onto. ∎

We now prove some similar results for $\mathscr{M}_l(X)$.

LEMMA 4.8. *Let X be an operator space. Then $M_n(\mathscr{M}_l(X)) \cong \mathscr{M}_l(C_n(X)) \cong \mathscr{M}_l(M_n(X))$ isometrically as Banach algebras, for every $n \in \mathbb{N}$.*

Proof. We just prove the first result, the second being similar. Recall $\mathscr{T}(C_n(X)) \cong C_n(\mathscr{T}(X))$ (see A.13) and the facts

$$LM(\mathbb{K}(C_n(Z))) \cong LM(M_n(\mathbb{K}(Z)) \cong M_n(LM(\mathbb{K}(Z))),$$

for any right Hilbert C^*-module Z [41, 50, 63]. Putting $Z = \mathscr{T}(X)$ and appealing to the definition of $\mathscr{M}_l$ gives the result. ∎

DEFINITION 4.9. We will say that a linear map $S\colon X \to X$ is (left) *order bounded* if there exists a linear complete isometry σ of X into a C^*-algebra and a constant $M \geqslant 0$ such that $[\sigma(S(x_i))^* \, \sigma(S(x_j))] \leqslant M^2[\sigma(x_i)^* \, \sigma(x_j)]$, for all $x_1, ..., x_n \in X$. We write $LOB(X)$ for the set of left order bounded maps on X. The least such M defines the order bounded norm $|||S|||$. Similarly, we define $LOB^\sigma(X)$ to be the operators S which satisfy the above condition, but with a fixed σ; the least M will be written as $|||S|||^\sigma$.

There is a similar definition for the right order bounded operators $ROB(X)$, but we shall not need to refer to these again. However we do note that what we do below will show that any operator T on X which is left order bounded commutes with any right order bounded operator on X.

The same remarks as for the adjointables above show that we may replace C^*-algebra by C^*-module in the last definition and that $LOB(X) = \bigcup_\sigma LOB^\sigma(X) = LOB^J(X)$. To see the last statement, apply the canonical *-homomorphism $C^*(\sigma(X)\,\sigma(X)^*) \to \mathscr{E}(X)$ coming from the universal property of $\mathscr{T}(X)$ to the inequality in the third line of 4.9. Thus S is left order bounded iff we have $[\langle S(x_i) \mid S(x_j) \rangle] \leqslant M^2[(\langle x_i \mid x_j \rangle]$, for all $x_1, ..., x_n \in X$. The inner product here is the $\mathscr{F}(X)$-valued one on $\mathscr{T}(X)$ (or, if you like, $\langle x \mid y \rangle = J(x)^* J(y)$, where the product is taking place in $I(\mathcal{S}(X))$ or $C^*(\partial X)$).

The following summarizes some connections between the definitions above:

THEOREM 4.10. *Let X be an operator space and T: $X \to X$ a linear map. Then*:

(1) *The following are equivalent*:

(i) $T \in \mathcal{M}_l(X)$ *(regarding $\mathcal{M}_l(X)$ as maps on X).*

(ii) $T \in LOB(X)$.

(iii) *There exists a Hilbert space H, an $S \in B(H)$, and a completely isometric linear embedding σ: $X \to B(H)$ such that $\sigma(Tx) = S\sigma(x)$ for all $x \in X$.*

(2) *The following are equivalent*:

(i) $T \in \mathcal{A}_l(X)_{sa}$ *(resp. $T \in \mathcal{A}^Z(X)_+$, T is a projection in $\mathcal{A}_l(X)$);*

(ii) *There exist H, S, σ satisfying all the conditions of* (1) (iii), *but also $S = S^*$ (resp. $S \geqslant 0$, S an orthogonal projection)*;

(iii) *for any $x \in X$, there is a complete isometric linear map σ: $X \to B(H)$ say, such that $\sigma(Tx)^* \sigma(x)$ is self-adjoint (resp. $\geqslant 0$, satisfies $\sigma(Tx)^* \sigma(x) = \sigma(Tx)^* \sigma(Tx)$).*

If these hold one may take σ to be the Shilov embedding J.

(3) $T \in \mathcal{A}_l(X)$ *if and only if there exist H, S, σ satisfying all the conditions of* (1) (iii) *and also $S^*\sigma(X) \subset \sigma(X)$.*

We have that the norm of $\mathcal{M}_l(X)$ coincides with the LOB norm $|||\cdot|||$ and also with the least value of $\|S\|$ possible in (1)(iii). *This least value is achieved.*

Proof. (1) (iii) $\Rightarrow$ (ii): Any T satisfying (iii) is clearly in $LOB^{\sigma}(X)$, and moreover we have $|||T||| \leqslant \|S\|$. Thus $|||T||| \leqslant \inf \|S\|$ over all S as in (iii).

(i) $\Rightarrow$ (iii): There are many ways to see this. For example, we saw at the end of the Introduction that there is a complete isometric injection σ of $\mathcal{T}(X)$ into $B(K, H)$ such that for every $T \in LM(\mathcal{E}(X))$ there is an $S_0 \in B(H)$ such that $S_0\sigma(z) = \sigma(Tz)$ for every $z \in \mathcal{T}(X)$. Let S' be the 2×2 matrix in $B(H \oplus K)$ with S_0 in the 1–1 corner and zero elsewhere. This gives (iii). Since $\|S'\|$ equals the $\mathcal{M}_l(X)$-norm of T, the latter dominates the infimum of $\|S\|$ possible in (iii).

(ii) $\Rightarrow$ (i): if $T \in LOB(X)$, we define $\tilde{T}: \mathcal{T}(X) \to \mathcal{T}(X)$ by $\tilde{T}(\sum_k x_k f_k) = \sum_k T(x_k) f_k$, for $x_1, ..., x_n \in X$ and $f_1, ..., f_k \in \mathcal{F}(X)$. We see that $\tilde{T}$ is well defined and bounded, since if $w = \sum_k x_k f_k$ then

$$\langle \tilde{T}(w) \mid \tilde{T}(w) \rangle = \sum_{i,j} f_i^* \langle T(x_i) \mid T(x_j) \rangle f_j$$
$$\leqslant M^2 \sum_{i,j} f_i^* \langle x_i \mid x_j \rangle f_j = M^2 \langle w \mid w \rangle$$

Clearly $\tilde{T} \in B_{\mathcal{F}}(\mathcal{T}(X))$ and $\|\tilde{T}\| \leqslant |||T|||$.

We have also, by the way, established the final assertion of our theorem. The least value of $\|S\|$ possible in (iii) is achieved by the S' above.

The ($\Rightarrow$) direction in (3), and the fact that (i) $\Rightarrow$ (ii) in (2), may be proved almost identically to the ((i) $\Rightarrow$ (iii)) direction in (1) after using the fact that $\mathcal{A}_l(X) = \mathbb{B}_l(X)$ (established in 4.7).

Next we prove ($\Leftarrow$) in (3). Notice the hypothesis $S^* \sigma(X) \subset \sigma(X)$ defines a function $R: X \to X$ given by $R(x) = \sigma^{-1}(S^* \sigma(x))$. Also we have

$$\sigma(T(x))^* \sigma(y) = \sigma(x)^* S^* \sigma(y) = \sigma(x)^* \sigma(R(y))$$

for $x, y \in X$.

It is very clear that (ii) $\Rightarrow$ (iii) in (2). To check that (iii) $\Rightarrow$ (i), note that in (iii) we may as well assume that σ is the embedding of X into its injective envelope, or triple envelope, by Hamana's universal property A.3. Then if $\sigma(Tx)^* \sigma(x)$ is self-adjoint, we have that

$$\sigma(Tx)^* \sigma(x) = (\sigma(Tx)^* \sigma(x))^* = \sigma(x)^* \sigma(Tx),$$

so that by polarization, we see that T is left adjointable on X, with $T^* = T$. If $\sigma(Tx)^* \sigma(x) \geqslant 0$ we may obtain $T \in \mathcal{A}_l(X)_+$ exactly as in [63] 15.2.5. And if $\sigma(Tx)^* \sigma(x) = \sigma(Tx)^* \sigma(Tx)$ for all x, then one can say that since $\sigma((T^2 - T)x)^* \sigma(x) = 0$, we have $T^2 - T$ both positive and negative in $\mathcal{A}_l(X)$. Consequently $T^2 = T$, and since $T \geqslant 0$ we have that T is a (orthogonal) projection in $\mathcal{A}_l(X)$. ∎

We leave it to the interested reader to supply the simple proofs of adaptions of (2) characterizing unitaries or normal elements of $\mathcal{A}_l(X)$. Also in (1)(iii), we can replace $B(H)$ by $B(H, K)$, or a C^*-algebra, or a C^*-module, with no loss.

Corollary 4.11. *Let T be a left multiplier of an operator space X. If Y is a closed subspace of X with $T(Y) \subset Y$, then $T|_Y$ is a left multiplier of Y, with a multiplier norm smaller than or equal to that of T. If in addition, T is adjointable on X and $T^*(Y) \subset Y$, then $T|_Y$ is adjointable on Y.*

The following result is a "Banach–Stone" type result. Many known "Banach–Stone" type theorems may be deduced from it.

COROLLARY 4.12. *If $v: X \to Y$ is a linear surjective complete isometry between operator spaces, then the map $T \mapsto vTv^{-1}$ is a completely isometric isomorphism $\mathcal{M}_l(X) \to \mathcal{M}_l(Y)$.*

Proof. This can be proved directly from the definition of $\mathcal{M}_l(X)$, and the universal property of $\mathcal{T}(X)$. Or it follows immediately from 4.10 1(iii). ∎

We put a natural matrix norm on $LOB(X)$ by identifying $M_n(LOB(X))$ with $LOB(C_n(X))$. With this structure we have from Lemma 4.8 that:

COROLLARY 4.13. *For any operator space X, we have that $\mathcal{M}_l(X)$ is completely isometrically isomorphic to the algebra $LOB(X)$ of left order bounded operators on X.*

We will not need the following, but it may be proved similarly to 4.13.

COROLLARY 4.14. *For any operator space X, and any linear complete isometry $\sigma: X \to Z$ into a C^*-module, let $\mathcal{M}_l^\sigma(X)$ be the operator algebra of bounded right module maps on the Hilbert C^*-extension inside Z generated by $\sigma(X)$, which leave $\sigma(X)$ invariant. Then $LOB^\sigma(X)$ is completely isometrically isomorphic to $\mathcal{M}_l^\sigma(X)$. Thus X is an operator $LOB^\sigma(X)$-module, for any σ as above.*

We now leave the general multiplier theory and turn to some more examples of interest.

EXAMPLE 4.15. We consider the operator spaces $MAX(\ell_n^1)$ $(=(\ell_n^\infty)^*)$ and M_n^*. The $*$ here means the operator space or standard dual (see [5], for example). In fact $V = MAX(\ell_n^1)$ is a unital operator space, which is canonically identifiable with the linear span of the generators 1, g_1, $g_2, ..., g_{n-1}$ in the free group C^*-algebra $C^*(F_{n-1})$ (see [48]). Hence V is e.n.v., for example. In fact Zhang showed in his thesis [67] that $C^*(F_{n-1}) = C_e^*(MAX(\ell_n^1))$, from which it is easy to see that $\mathcal{M}_l(V) = \mathbb{B}_l(V) = \mathcal{M}_r(V) = \mathbb{B}_r(V) = \mathbb{C}$. It will follow from Section 5 that there are no interesting operator module actions on $MAX(\ell_n^1)$.

Turning to M_n^*, it has a canonical completely isometric representation as the subspace spanned by the generators of $\mathcal{U}_n^{nc}$, Brown's noncommutative unitary C^*-algebra (see [19]). We are not sure if this is the Shilov representation, but this does show that $X = M_n^*$ is e.n.v., since by the definition of Brown's C^*-algebra, the C^*-subalgebras generated by XX^* and X^*X, respectively, are unital, and $C^*(\partial X)$ is thus a quotient of a unital C^*-algebra.

EXAMPLE 4.16. We now extend part of Proposition 4.3(i) by calculating some multiplier algebras, and the injective envelope, of an operator algebra A with c.a.i. but no identity. In the following, we write A^1 for the unitization of A if A is nonunital; otherwise set $A^1 = A$. We define $C_e^*(A)$ to be the C^*-subalgebra of $C_e^*(A^1)$ generated by A. Note that $C_e^*(A)^1 = C_e^*(A^1)$. To see this, let e be the identity of A^1, which will be the identity of $C_e^*(A^1)$. If $e \in C_e^*(A)$ then that space is unital, so we are done. If $e \notin C_e^*(A)$ then $span\{C_e^*(A), e\}$ is a unital $*$-subalgebra of $C_e^*(A^1)$, which is easy to see is closed, and contains A^1. Hence this span equals $C_e^*(A^1)$; consequently we obtain the claimed statement.

THEOREM 4.17. *Suppose that A is an operator algebra with c.a.i. Then $\mathcal{T}(A) = C_e^*(A)$. Also $\mathbb{K}_l(A) = A$, $\mathcal{M}_l(A) = LM(A)$, $\mathcal{M}_r(A) = RM(A)$, and the 2-sided multiplier algebra defined above 4.3 coincides with its multiplier algebra $M(A)$ (as defined in [52] for example). Moreover, $I(A) = I(A^1) = I(C_e^*(A)) = I(C_e^*(A^1))$.*

Proof. Note that $C_e^*(A)$ is a Hilbert C^*-extension of A, since by [9, Lemma 8.1], any c.a.i. for A is also one for $C_e^*(A)$. Thus by the universal property of $\mathcal{T}(X)$ (see A.3), we see that $\mathcal{T}(X)$ is the quotient of $C_e^*(A)$ by a closed ideal $\mathcal{I}$ say. Therefore $B = C_e^*(A)/\mathcal{I}$ is a C^*-algebra generated by A. Therefore B^1 is a C^*-algebra generated by A^1, and by the Arveson–Hamana theorem, there exists a $*$-homomorphism $B^1 \to C_e^*(A^1)$, which restricts to a $*$-homomorphism $B \to C_e^*(A)$, extending Id_A. Thus it is clear that $\mathcal{I} = (0)$, so that $\mathcal{T}(X) = C_e^*(A)$.

It is easy to see from the fact mentioned above that any c.a.i. for A is also one for $C_e^*(A)$, that $\mathbb{K}_l(A) = \{b \in C_e^*(A) : bA \subset A\} = A$. Similarly, $\mathcal{M}_l(A) = \{T \in LM(C_e^*(A)) : TA \subset A\} = LM(A)$, since we may represent $A \subset C_e^*(A)$ nondegenerately on the same Hilbert space. The 2-sided multiplier algebra of an operator space which we defined earlier (above 4.3), is thus equal to the usual multiplier algebra of A [52].

Now $I(A) = I(\mathcal{T}(A))$, since any minimal $\mathcal{T}(A)$-projection on $I(A)$ is an A-projection and is consequently the identity. Thus $I(A) = I(C_e^*(A)) = I(C_e^*(A^1))$, since for any C^*-algebra B we have $I(B) = I(B^1)$ (see, for example, the end of [17]). Since $A \subset A^1 \subset I(C_e^*(A^1)) = I(A)$, we see that any minimal A^1-projection on $I(A)$ is an A-projection and is consequently the identity. Thus $I(A) = I(A^1)$. ∎

We do not see right off how to obtain the last line directly from the method used in [17]. We will continue with this example in Appendix B.

The last example, and many other common operator spaces, fall within the scope of the following simple result, which we will have cause to use later:

THEOREM 4.18. *Suppose that X is an operator space such that $I(X)$ is a (necessarily unital—see [17], for example) C^*-algebra $\mathscr{C}$. Let $\mathscr{D}$ be the C^*-subalgebra of $\mathscr{C}$ generated by X, let $\mathscr{E}$ be the C^*-subalgebra of $I(X)$ generated by XX^*, and let $\mathscr{F}$ be the C^*-subalgebra generated by X^*X. Set $W = \overline{(\mathscr{E}X)}$. All the products here are taken in $\mathscr{D}$. Then*

(i) $I(\mathscr{S}(X)) = M_2(\mathscr{C})$.

(ii)

$$C^*(\partial X) = \begin{bmatrix} \mathscr{E} & W \\ W^* & \mathscr{F} \end{bmatrix} \subset M_2(\mathscr{D}).$$

Proof. We will write 1 for the identity in $\mathscr{C}$. The C^*-algebra $M_2(\mathscr{C})$ is injective and has as subsystems

$$\mathscr{S}(X) \subset \mathscr{S}(\mathscr{C}) \subset M_2(\mathscr{C}),$$

where we have identified the diagonal idempotents in $\mathscr{S}(X)$ and $\mathscr{S}(\mathscr{C})$ with 1. As we said in the Introduction, Hamana's results imply that there is a minimal $\mathscr{S}(X)$-projection Φ on $M_2(\mathscr{C})$. As in [36, 37, 58], we may write

$$\Phi = \begin{bmatrix} \psi_1 & \phi \\ \phi^* & \psi_2 \end{bmatrix}.$$

Now $\phi: \mathscr{C} \to \mathscr{C}$ fixes X, so by rigidity of $I(X)$, $\phi = Id_{\mathscr{C}}$. Thus Φ is a $\mathscr{S}(\mathscr{C})$-projection. Now one sees that Φ fixes $1 \otimes M_2$, which is a unital C^*-algebra. Hence by Choi's multiplicative domain lemma (see, for example [47, Example 4.3]) Φ is a M_2-bimodule map. Thus $\psi_1 = \psi_2 = \phi = Id_{\mathscr{C}}$. Hence $\Phi = Id$, which proves (i). Part (ii) is straightforward to check. ∎

COROLLARY 4.19. *If V is a Banach space and $X = MIN(V)$ then $I(\mathscr{S}(X)) = M_2 \otimes_\lambda C(\Omega)$, where $C(\Omega)$ is the (Stonean) Banach space injective envelope of V. Also, $\mathscr{E} = \mathscr{F}$, and these are commutative C^*-algebras. Thus $\mathscr{M}_l(X) \subset LM(\mathscr{E}) = M(\mathscr{E})$, so that $\mathscr{M}_l(X)$ $(= \mathscr{M}_r(X))$ is a function algebra.*

We will improve on this result next. Indeed for most of the remainder of this section we investigate the connections between the commutative (i.e., Banach space) version of some of the notions we have discussed and the noncommutative version.

In [66] it is shown that for a finite dimensional Banach space X, the spaces $\mathscr{E}(X) = \mathscr{F}(X)$ and $\mathscr{T}(X)$ have a simple description, in terms of the space $ext(X_1^*)$. In fact it is not hard to show that Zhang's proof works to give a similar representation for a general Banach space X. Namely, define S to be the weak*-closure of $ext(X_1^*)$, and define

$$\mathcal{G} = \{a \in C(S) : a(\alpha\phi) = a(\phi) \text{ for all } \alpha \in \mathbb{T},\ \phi \in S\}$$
$$\mathcal{H} = \mathcal{H}(X) = \{h \in C(S) : h(\alpha\phi) = \alpha h(\phi) \text{ for all } \alpha \in \mathbb{T},\ \phi \in S\}$$

and let $\mathcal{C}$ be the following C^*-subalgebra of $M_2(C(S))$:

$$\begin{bmatrix} \mathcal{G} & \mathcal{H} \\ \mathcal{H}^* & \mathcal{G} \end{bmatrix}.$$

It is clear that $\mathcal{H}$ is a C^*-module over $\mathcal{G}$ and that $\mathcal{G} \cong C(S/\equiv)$, where $\equiv$ is the equivalence relation on X_1^* given by the circle action. Let $U = (S \backslash \{0\})/\equiv$. Topologically, this is the same as $(S/\equiv)\backslash\{0\}$. We leave it as an exercise for the reader that U is locally compact and Hausdorff, and the quotient map $S\backslash\{0\} \to U$ is continuous and open. It is easy to see that X is e.n.v. as a Banach space (that is, $0 \notin \overline{ext}(X_1^*)$) if and only if U is compact.

If $h \in \mathcal{H}$, then clearly $h(0) = 0$ if $0 \in S$, so that the inner product on $\mathcal{H}$ takes values in $C_0(U)$. There is a canonical complete isometry j: $MIN(X) \to \mathcal{H}$ and a canonical copy of $\mathcal{S}(j(X))$ inside $\mathcal{C}$. We define $\mathcal{D}$ to be the C^*-subalgebra of $\mathcal{C}$ generated by this system. By Stone–Weierstrass, $\mathcal{D}$ has $\mathcal{G}$ as its 1–1 or 2–2 corner. In fact by using the following Stone–Weierstrass theorem for line bundles (which is no doubt well known), we will be able to say a little more.

THEOREM 4.20. *Suppose that Z is a $C_0(U) - C_0(U)$-bimodule, with a symmetric action (so that $az = za$ for all $a \in C_0(U), z \in Z$), where U is a locally compact Hausdorff space. Suppose further that Z is a full left C^*-module over $C_0(U)$ and that we also have the relation $\langle v, w\rangle z = \langle z, w\rangle v$, for all $v, w, z \in Z$. Suppose that X is a subset of Z, and let $\mathcal{S} = \{\langle x, y\rangle : x, y \in X\}$. This is a subset of $C_0(U)$. The following are equivalent*:

(i) *$\mathcal{S}$ separates points of U, and $\mathcal{S}$ does not vanish identically at any fixed point in U.*

(ii) *$\mathcal{S}$ generates $C_0(U)$ as a C^*-algebra, and X generates Z as a $C_0(U)$-module (that is, there is no nontrivial closed $C_0(U)$-submodule between X and Z).*

(iii) *the copy of X within the 1–2 corner of the linking C^*-algebra $\mathcal{L}(Z)$ of Z generates $\mathcal{L}(Z)$.*

Proof. It is not hard to see by the ordinary Stone–Weierstrass theorem that (i) is equivalent to the first condition in (ii). It is also not hard to see that (ii) $\Leftrightarrow$ (iii). We will therefore be done if we can show that the first condition in (ii) implies the second.

To that end, note that if M is a closed $C_0(U)$-submodule containing X, then $\mathscr{S} \subset \langle M, M \rangle$. The latter set is a *-subalgebra and is therefore dense in $C_0(U)$ by the first condition in (ii). Choose (by basic C^*-module theory) an approximate identity $\{e_i\}$ for $C_0(U)$ such that each e_i is of the form $\sum_{k=1}^{m} \langle m_k, m_k \rangle$ for $m_k \in M$. For any such e_i and $f \in Z$, we have $e_i f = \sum_{k=1}^{m} m_k \langle f, m_k \rangle \in M$, since M is a module. Thus $f = \lim e_i f \in M$. ▮

THEOREM 4.21. *For an operator space $X = MIN(V)$, where V is a (not necessarily finite dimensional) Banach space, we have $C_e^*(\mathscr{S}(X)) \cong \mathscr{C}$, where $\mathscr{C}$ is as just defined. Also, $\mathscr{T}(X) = \mathscr{H}(X)$ and $\mathscr{E}(X) = C_0(U)$.*

Proof. To see that $C_e^*(\mathscr{S}(X)) \cong \mathscr{D}$ one may follow the proof in [66], except for one detail. By the argument of [66] Proposition 8, it is enough to show that $\mathscr{G}$ is isomorphic to the 1–1 corner $C(Y)$ of $C_e^*(\mathscr{S}(X))$. As in [66] Theorem 4, there is a 1–1 continuous map η: $Y \to S/\equiv$, which we need to show is onto. However, if it were not onto, then there would exist a $\phi \in ext(X_1^*)$ such that $[\phi] \notin \eta(Y)$. Then $\phi \notin F$, where F is the compact preimage in X_1^* of $\eta(Y)$ under the quotient map. If ϕ is not in the closed convex hull of F, then the rest of Zhang's proof works to give a contradiction. However, if ϕ is in this hull, then by elementary Choquet theory there is a probability measure μ supported on F (and therefore also on X_1^*) which represents ϕ. Since ϕ is an extreme point, $\mu = \delta_\phi$, which forces $\phi \in F$, which is a contradiction.

The last two assertions now follow from 4.20; clearly (i) of 4.20 holds since the conditions there hold for the collection of functions $\langle x, x \rangle([\psi]) = |\psi(x)|^2$ for all $x \in X$, $\psi \in U$. ▮

COROLLARY 4.22. *If X is a Banach space, then $\mathscr{M}_l(MIN(X)) = \mathscr{M}_r(MIN(X)) = \mathscr{M}(X)$ completely isometrically isomorphically, where the latter is the Banach space multiplier algebra (see Part A) of X. Similarly, $\mathscr{A}_l(MIN(X)) = \mathscr{A}_r(MIN(X)) = Z(X)$, the Banach space centralizer algebra. In this case, $\mathscr{M}(X) \subset CB(MIN(X))$ completely isometrically and as a subalgebra.*

Proof. If $S \in LOB(MIN(X))$, then it follows from the definition of the inner product on $\mathscr{H}(X) = \mathscr{T}(X)$, that

$$|\phi(S(x))| \leqslant |||S||| \ |\phi(x)|$$

for all $x \in X$, $\phi \in ext(X_1^*)$. By Theorem 2.1, $S \in \mathscr{M}(X)$. We also get $\|S\| \leqslant |||S|||$.

Conversely, if $S \in \mathscr{M}(X)$, then by (iv) of 2.1 and 1(iii) of 4.10, we see that $S \in \mathscr{M}_l(MIN(X))$, with norm in that latter space $\leqslant \|S\|$. This proves that $\mathscr{M}_l(MIN(X)) = \mathscr{M}(X)$ isometrically. By 4.19 this will be a complete isometry (since both are MIN spaces).

To see the last assertion, suppose that $[S_{ij}] \in M_n(CB(MIN(X)))$, with $S_{ij} \in \mathscr{M}(X)$. We let K be the weak*-closure of $ext(X_1^*)$, with 0 removed, as in Part A. There exist $g_{ij} \in C_b(K)$ such that $\phi(S_{ij}(x)) = g_{ij}(\phi)\,\phi(x)$, for any $x \in X$. Thus

$$\begin{aligned}\|[S_{ij}]\| &\geqslant \sup\{\|[S_{ij}(x)]\| : x \in X_1\}\\ &\geqslant \sup\{\|[g_{ij}(\phi)\,\phi(x)]\| : x \in X_1, \phi \in K\}\\ &= \|[g_{ij}]\|_{M_n(C(K))}.\end{aligned}$$

This gives the complete isometry needed. ∎

We recall that an operator space X is e.n.v. if $C^*(\partial X)$ (or equivalently, $\mathscr{E}(X)$ and $\mathscr{F}(X)$) is a unital C^*-algebra. The following justifies this notation:

THEOREM 4.23. *If V is a Banach space, then V is e.n.v. in the Banach space sense if and only if $MIN(V)$ is e.n.v. as an operator space.*

Proof. We observed above 4.20 that X is e.n.v. iff U is compact, i.e., iff $C_0(U)$ is unital. However, $C_0(U) = \mathscr{E}(X)$ by Theorem 4.21. ∎

COROLLARY 4.24. *If a Banach space X is e.n.v., then (identifying X and $MIN(X)$) we have $\mathscr{T}(X) = \mathscr{H}$, $\mathscr{E}(X) = \mathscr{G}$, and $C^*(\partial X) = \mathscr{C}$, where $\mathscr{G}, \mathscr{H}, \mathscr{C}$ are as defined above.*

It is interesting to interpret Theorem 4.21 in the language of vector bundles. There is a well-known 1–1 correspondence between the space of sections of a locally trivial Hermitian vector bundle with locally compact base space U and certain Hilbert C^*-modules over $C_0(U)$ (see [27], for example). Via this correspondence, locally trivial Hermitian line bundles correspond to C^*-modules over $C_0(U)$ satisfying the hypotheses of the first few lines of Theorem 4.20. The correspondence of course, is $E \mapsto \Gamma_0(E)$, where E is a locally trivial Hermitian line bundle and $\Gamma_0(E)$ is the Banach space of continuous sections of E which vanish at infinity on U. This is a full C^*-module over $C_0(U)$ satisfying those hypotheses in the first few lines of Theorem 4.20. The space $\mathscr{H}(X)$ defined above satisfies these same hypotheses; hence there is a line bundle E such that $\mathscr{H}(X) = \Gamma_0(E)$. With a little thought one can write down this bundle explicitly. Namely, if X is a Banach space let $K = \overline{ext}(X_1^*)\setminus\{0\}$, as in Part A. Let $\equiv$ and U be as defined above 4.20. We let E be the quotient of $K \times \mathbb{C}$ by the equivalence of pairs $(\alpha\psi, \alpha\lambda)$ and (ψ, λ), for $\alpha \in \mathbb{T}$, $\psi \in K$, $\lambda \in \mathbb{C}$. We let $p\colon E \to U$ be the map $p([(\psi, \lambda)]) = [\psi]$. Then E is a locally trivial line bundle over U. To see that E is locally trivial, pick $[\varphi] \in U$ and fix $x \in X$ with $\varphi(x) > 2$. Let

$V = \{[\psi] \in U : |\psi(x)| > 1\}$. We define Φ: $p^{-1}(V) \to V \times \mathbb{C}$ by $\Phi([(\psi, \lambda)]) = ([\psi], \alpha\lambda)$, if $\alpha \in \mathbb{T}$ with $\bar{\alpha}\psi(x) > 0$. This is well-defined on $p^{-1}(V)$, linear on fibers, 1–1, and onto. We leave it to the reader to check that it is continuous and open.

We take the inner product on E to be the obvious one. This is clearly continuous on E by local triviality. The induced norm on a fiber is $|[(\psi, \lambda)]| = |\lambda|$.

We claim that $\Gamma_0(E) = \mathscr{H}(X)$. Clearly if $g \in \mathscr{H}$ then defining $f([\psi]) = [(\psi, g(\psi))]$ gives a well-defined continuous section vanishing at infinity. The converse, too, is not hard to see. Indeed this correspondence is an "imprimitivity bimodule isomorphism" $\Gamma_0(E) \cong \mathscr{H}(X)$.

This bundle E we shall call the Shilov boundary bundle of X, and we shall write ∂X for this E. The canonical linearly isometric map $X \to \Gamma_0(E)$ will be written as J. Clearly $J(X)$ satisfies the equivalent conditions of 4.20.

Theorem 4.21 now translates as:

THEOREM 4.25 (The boundary theorem for Banach spaces). *Let X be a Banach space. Then there exists a line bundle ∂X and a linear isometry J: $X \to \Gamma_0(\partial X)$ such that $J(X)$ generates $\Gamma_0(\partial X)$ in the sense of the equivalent conditions of* 4.20 *and which has the following universal property: If F is any locally trivial Hermitian line bundle with locally compact base space, and if i: $X \to \Gamma_0(F)$ a linear isometry whose range also satisfies the equivalent conditions of* 4.20, *then there exists a unique unitary injection of bundles θ: $\partial X \to F$ such that $i(x) \circ \theta = j(x)$ for all $x \in X$.*

The term unitary injection of bundles means that E is fiberwise-unitarily bundle isomorphic to a sub-line-bundle of F. This theorem is a complete justification of the term Shilov boundary bundle used above. We believe this to be a new result. We have not tried to find a classical proof of it, but imagine that it would not be difficult.

Actually, this approach via Hamana's results immediately gives alternative proofs of many interesting facts concerning line bundles. For example, one can use it to prove that every C^*-module over $C_0(U)$ satisfying the hypotheses at the beginning of 4.20 is the space of sections of a locally trivial Hermitian line bundle E over U or that any such bundle F may be retrieved, up to unitary bundle isomorphism, from the Banach space structure of $\Gamma_0(F)$. Thus two such bundles are unitary bundle isomorphic if and only if their spaces of sections Γ_0 are linearly isometric to each other.

The following example illustrates what is going on. We leave it to the reader to work out the details. Let S be the vertical strip $\{x + iy \mid 0 \leqslant x \leqslant \ln\ R\}$, $R > 1$. We view S as the universal covering space of the annulus centered at zero with inner radius 1 and outer radius R, where the covering map is the exponential map $z \to e^z$. Fix an $\alpha \in \mathbb{T}$, and let X be the Banach space of functions that are continuous on S and analytic on the interior of S and satisfy the equation $x(z+2\pi i) = x(z)\,\alpha$. The norm on X is the usual supremum norm. It can be shown that X is not a unital function space, so the usual Shilov boundary approach does not work. However, one may show that X is isometrically linearly isomorphic to a closed subspace of the space of continuous sections of a nice line bundle over the annulus (which we leave to the reader to write down explicitly). Indeed, by the maximum modulus theorem, X is isometrically linearly isomorphic to a subspace of the space of continuous sections of a line bundle F over two circles, one which is easy to describe. However, X is not isometrically supported on any subbundle of this F. Thus F is the Shilov boundary bundle of X.

An interesting complement to the above general setup is the following. Suppose that K is compact and that $j: X \to C(K)$ is a linear isometry such that $j(X)$ *strongly* separates points. Or, more generally, suppose that $j(X)$ does not vanish identically at any fixed point in K and that the functions in $j(X)\,\overline{j(X)}$ separate points of K. Then $j(X) \subset C(K)$ satisfies the equivalent hypotheses of 4.20. Hence, by the boundary theorem for Banach spaces there is a surjective imprimitivity bimodule isomorphism $C(K) \to \Gamma_0(\partial X)$. Hence in this case, the bundle ∂X is trivial, and $\Gamma_0(\partial X)$ may be identified with the continuous functions on a closed subspace of K. Thus we are not really in a bundle situation at all, we are back in the classical Shilov boundary situation.

Further descriptions of the multiplier algebras may be found in [17, 69] (see Appendix B). Using [17] one obtains for $X = MIN(V)$, we have $\mathcal{M}_l(X) \cong \{f \in I(V) : fV \subset V\}$, and we know that the Banach space injective envelope $I(V)$ is a Stonean commutative C^*-algebra. This is interesting in the light of 4.22.

In [40], Kirchberg also studies some multiplier algebras of certain systems. No doubt there is a connection.

5. *Oplications*

We now consider a very general type of representation of an operator space. It will allow us to consider, under one large umbrella, operator algebras, operator modules, and C^*-correspondences. Namely, we consider linear representations $Y \to CB(Z)$ of an operator space Y such that the associated bilinear map $Y \times Z \to Z$ is completely contractive. Here Z is a C^*-module or, more generally, an operator space.

Definition 5.1. If X and Y are operator spaces, then a (left) *oplication* (of Y on X) is a bilinear map $\circ: Y \times X \to X$, such that

(1) $\|[\sum_{k=1}^{n} y_{ik} \circ x_{kj}]\| \leqslant \|[y_{ij}]\| \; \|[x_{ij}]\|$, for all $n \in \mathbb{N}$, $x_{ij} \in X$, $y_{ij} \in Y$, and

(2) there is an element $e \in Y_1$ such that $e \circ x = x$ for all $x \in X$.

The word oplication is intended to be a contraction of the phrase operator multiplication. Condition (1) may of course be rephrased as saying that m is *completely contractive* as a bilinear map. We shall refer to the following as the oplication theorem:

THEOREM 5.2. *Suppose that Y, X are operator spaces, and suppose that $\circ: Y \times X \to X$ is an oplication, with "identity" $e \in Y$. Then there exists a unique completely contractive linear map $\theta: Y \to \mathcal{M}_l(X)$ such that $y \circ x = \theta(y)\, x$, for all $y \in Y, x \in X$. Also $\theta(e) = 1$. Moreover, if Y is, in addition, an algebra with identity e, then θ is a homomorphism if and only if m is a module action. On the other hand, if Y is a C^*-algebra (or operator system) with identity e, then θ has range inside $\mathcal{A}_l(X)$ and is completely positive and *-linear.*

Thus left oplications on X are in a 1–1 correspondence with completely contractive maps $\theta: Y \to \mathcal{M}_l(X)$ with $\theta(e) = 1$. Similarly for right oplications.

The original proof of this theorem in our paper was considerably more difficult. We will sketch it at the end of this section because it contained some ideas which we believe will be important in the future. Subsequently however, we found other shorter proofs, see for example the end of Appendix B. The new proof we give here illustrates the usefulness of Theorem 4.10 (1).

Proof. Suppose that $\circ: Y \times X \to X$ is an oplication, with identity $e \in Y$. If one looks at Christian Le Merdy's proof of the BRS theorem [42, 3.3] it is clear that the same idea works in our case to show that there are Hilbert spaces H and K, a linear complete isometry $\sigma: X \to B(K, H)$, and a linear complete contraction $\Phi: Y \to B(H)$, such that $\Phi(y)\, \sigma(x) = \sigma(y \circ x)$ for all $y \in Y, x \in X$, and such that $\Phi(e) = I_H$. Indeed to get this one need only use part of Le Merdy's argument (see the first proof in Section 2 of [17] for details if the reader needs them).

By 4.10 (1)(iii), for any $y \in Y$, the map $\theta(y) = y \circ -$ on X is in $\mathcal{M}_l(X)$, with multiplier norm dominated by $\|\Phi(y)\|$. Thus θ is a linear unital contractive map $Y \to \mathcal{M}_l(X)$.

That θ is completely contractive follows easily from Lemma 4.8. For example, if $[y_{ij}] \in M_n(Y)$ then $[\theta(y_{ij})]$ may be identified, by the last isomorphism, with a $T \in \mathcal{M}_l(C_n(X))$. For σ as above, let $\sigma': C_n(X) \to C_n(B(K, H)) \cong B(K, H^{(n)})$ be the usual amplification. Then $[\Phi(y_{ij})] \in M_n(B(H)) \cong B(H^{(n)})$, and via these identifications, we have $[\Phi(y_{ij})]\, \sigma'(z) = \sigma'(T(z))$ for any $z \in C_n(X)$. Hence the $\mathcal{M}_l(C_n(X))$ norm of T is dominated by $\|[\Phi(y_{ij})]\| \leqslant \|[y_{ij}]\|_n$. Thus θ is completely contractive.

The uniqueness of θ is clear. If Y is an algebra, then θ is an algebra homomorphism if and only if $\theta(ab)\,x = \theta(a)\,\theta(b)\,x$, for all $a, b \in Y$, $x \in X$, which is obviously equivalent to $\circ$ being a module action.

Now we appeal to Theorem 4.13 to obtain all the results of the theorem except for those in the last line. To see these, we note that if Y is an operator system, then since θ is unital and completely contractive, it is completely positive. Hence θ maps into $\mathscr{A}_l(X)$. ∎

An immediate corollary of the previous theorem is the following refinement and generalization of the Christensen–Effros–Sinclair characterization of operator modules [23]. We will state it for left modules, but the bimodule version is similar. In fact it is clear from (iii) below that we can treat left and right actions on a bimodule quite separately and that the $(ax)\,b = a(xb)$ property is automatic.

Corollary 5.3. *Let A be an operator space which is also an algebra with identity of norm* 1. *Let X be an operator space which is also a non-degenerate A-module with respect to a module action m: $A \times X \to X$. The following are equivalent*:

(i) *X is a left operator module* (*that is, m is completely contractive*).

(ii) *There exist Hilbert spaces H and K, and a linear complete isometry Φ: $X \to B(K, H)$ and a completely contractive unital homomorphism π: $A \to B(H)$, such that $\Phi(m(a, x)) = \pi(a)\,\Phi(x)$, for all $a \in A$, $x \in X$.*

(iii) *There exists a completely contractive unital homomorphism θ: $A \to \mathscr{M}_l(X)$ such that $\theta(a)\,x = m(a, x)$ for all $a \in A$, $x \in X$.*

Moreover, it can be arranged so that H, K, and Φ in (ii) *only depend on X and not on M or the particular action.*

The implication (iii) $\Rightarrow$ (ii) here follows, for example, by taking Φ to be an embedding of the C^*-module $\mathscr{T}(X) \subset B(K, H)$ as in the proof of 1.3, composed with J: $X \to \mathscr{T}(X)$. The π is obtained by composing the embedding $\mathscr{M}_l(X) \subset LM(\mathscr{E}) \subset B(H)$ in that proof, with the θ in (iii).

Recall from the previous section that every operator space X is an operator $\mathscr{M}_l(X) - \mathscr{M}_r(X)$-bimodule. We shall call this the *extremal multiplier actions on X*. The above gives a converse:

Corollary 5.4. *Every operator space X is an operator $\mathscr{M}_l(X) - \mathscr{M}_r(X)$-bimodule. Conversely, any action of two unital operator algebras* (*resp. C^*-algebras*) *A and B on X, making X an operator A–B-bimodule, is a prolongation of the extremal multiplier actions* (*resp. self-adjoint extremal multiplier actions*). *Moreover, the $a(xb) = (ax)\,b$ condition in the definition of an operator bimodule is automatic.*

Thus the left operator A-module actions on an operator space X correspond in a 1–1 fashion to completely contractive prolongations of the $\mathcal{M}_l(X)$ action. Left operator A-module actions on X which extend to an operator module action of a C^*-algebra generated by A correspond to completely contractive prolongations of the $\mathcal{A}_l(X)$-action.

We also see that operator modules are exactly the A-submodules of B-rigged A-modules. The following result complements Lemma 4.1 in [10]:

Corollary 5.5. *Suppose that A is a C^*-algebra.*

(i) *If Z is a right C^*-module over a C^*-algebra $\mathcal{B}$, then Z is a $\mathcal{B}$-rigged A-module if and only if Z is a left operator A-module.*

(ii) *Every left operator A-module X is a closed A-submodule of a $\mathcal{B}$-rigged A-module Z, for some C^*-algebra $\mathcal{B}$. One may take $Z = \mathcal{T}(X)$.*

Proof. We will assume A is unital and leave the general case to the reader, who may want to use 5.7 and the first term principle too.

(ii) This follows from 5.2, since if X is a left operator A-module, then there is an associated unital homomorphism $\theta: A \to \mathbb{B}_l(X) \subset \mathbb{B}_{\mathcal{F}}(\mathcal{T}(X))$. Hence $\mathcal{T}(X)$ is a left operator A-module, and a $\mathcal{F}$-rigged A-module, containing X as an A-submodule.

(i) The ($\Rightarrow$) direction is clear (see Lemma 4.1 in [10]). The converse follows from the proof of (ii), for if Z is a left operator A-module, then by that proof, the associated homomorphism maps $A \to \mathbb{B}(\mathcal{T}(Z)) = \mathbb{B}_{\mathcal{B}}(Z)$. ∎

The proof above generalizes to the case when A is an operator algebra with c.a.i., but then we have to replace B-rigged A-modules in the statement of the corollary by right C^*-modules Z over B, for which there exists a completely contractive nondegenerate homomorphism $\theta: A \to B_B(Z)$. We omit the easy details.

Theorems 5.2 or 5.3 (and its right or bimodule version) unifies, and has as one line consequences, the following special cases:

Corollary 5.6. *The Blecher–Ruan–Sinclair (BRS) characterization of operator algebras [18], the Christensen–Effros–Sinclair characterization of operator modules over C^*-algebras [23], the generalization of the last characterization to non-self-adjoint algebras, the characterization of operator bimodules in [30], the Tonge characterization of uniform algebras mentioned in Section 2 (at least in the 2-summing case) [62], and a strengthening of BRS. This latter result allows one to relax one hypothesis of BRS, allowing a completely contractive multiplication $m: A' \otimes A \to A$ such that $m(1, a) = m(a, 1) = a$, where $A' = A$ isometrically, but with a different operator space* *structure. The conclusion is that there exists a third operator space structure on A between the two described, with respect to which A is an operator algebra.*

Proof (of Corollary 5.6). Here are the one-line proofs: To obtain BRS, take $X=A$; θ is a complete isometric homomorphism into the operator algebra $\mathcal{M}_l(A)$, since $\|\theta(a)\| \geqslant \|\theta(a)\, 1\| = \|a.1\| = \|a\|$. The strengthening of BRS referred to above is obtained by the same idea: take $X=A$ again, apply Theorem 5.2, and take the third operator space structure on A to be $\|[\theta(\cdot)]\|_n$. The Christensen–Effros–Sinclair result is evident, as is its non-self-adjoint version. It is well known that the characterization of operator bimodules in [30] follows in a few lines from the Christensen–Effros–Sinclair result, by a well-known 2×2 matrix trick. Finally, the Tonge hypothesis implies that $MIN(A)$ is a $MAX(A)$-operator module, so that by Theorem 5.2, there is an isometric homomorphism θ of A into $\mathcal{M}_l(MIN(A)) \subset M(\mathcal{E}(MIN(A)))$, and by [66] Theorem 4 (or our Theorem 4.19), $\mathcal{E}(MIN(A))$ is a commutative C^*-algebra. ∎

Remarks on the characterization theorems. If in addition to the hypotheses of BRS, one assumes that A is an operator system, the proof of BRS above shows that θ maps into $\mathcal{A}_l(A)$, since it is completely positive, so that A is a C^*-algebra. This characterization of C^*-algebras may also be proved directly from [34].

Note that 4.3(i) shows that a unital operator space (X, e) is an operator algebra with identity e if and only if $\mathcal{M}_l(X)=X$ or, equivalently, if the map $T\mapsto T(e)$ maps $LOB(X)$ (or $\mathcal{M}_l(X)$) onto X.

Christian Le Merdy showed me that the stronger version of BRS mentioned in Corollary 5.6 also follows directly from his method to prove BRS [42].

Concerning this strengthened version of BRS, it is a natural question as to whether the third operator space structure on A is necessary; perhaps A itself is completely isometrically isomorphic to an operator algebra. However, there are easy counterexamples to this. One such is $V = C_n \otimes_h MIN(\ell^2_n)$, which is isometrically isomorphic (as a Banach algebra) to M_n and therefore also has an identity of norm 1. By associativity of the Haagerup tensor product, V is a (interesting example of a) left operator M_n-module. However, V cannot be completely isometric to M_n.

Concerning BRS, we recall that its original proof in [18] did not use associativity of the product m. Here we obtain the same, since

$$m(a, m(b, c)) = \theta(a)\, m(b, c) = \theta(a)\, b\pi(c) = m(m(a, b), c)$$

for $a, b, c \in A$. In the next section we shall see that this automatic associativity is true under weaker hypotheses also.

By Corollary 5.3, any A as in that result, for which there exists a completely 1-faithful operator A-module, is an operator algebra (see 6.1). This is another strengthening of BRS.

It is interesting that until now, there was no operator space proof of the version of Tonge's result mentioned in Corollary 5.6 (namely that a "2-summing" Banach algebra is a function algebra).

More generally, it seems that our minimal representation technique gives more information than was hitherto available.

Next we generalize the oplication theorem from the unital case to that of a c.a.i. Thus if Y, X are operator spaces, we will generalize Definition 5.1 by allowing condition (2) to be replaced by the existence of a net $e_\beta \in Y_1$ such that $e_\beta \circ x \to x$ for all $x \in X$. In this case we say that $\circ$ is an oplication with c.a.i. $\{e_\beta\}$.

THEOREM 5.7. *Let Y, X be operator spaces, and let $\circ$ be an oplication of Y on X with c.a.i. Then there exists a unique complete contraction θ: $Y \to \mathcal{M}_l(X)$ such that $\theta(y)\, x = y \circ x$ for all $y \in Y$, $x \in X$. Moreover, if Y is an algebra then $\circ$ is associative if and only if θ is a homomorphism. In the latter case, and if Y is a C^*-algebra, then θ is a *-homomorphism into $\mathcal{A}_l(X)$.*

Proof. We define $Y^1 = Y \oplus \mathbb{C}$ and identify $e = (0, 1)$. Define matrix norms on Y^1 by

$$\|[y_{ij} + \lambda_{ij} e]\| = \sup_\beta \|[y_{ij} + \lambda_{ij} e_\beta]\|.$$

It is easy to check that this makes Y^1 an operator space. Extend $\circ$ to $Y^1 \times X \to X$ by letting e act as the identity on X. The first conclusion will follow from the unital case of the theorem, if we can show that the extended oplication satisfies (1) of 5.1. To that end, let $[x_{ij}] \in M_n(X)_1$ and $[a_{ij} + \lambda_{ij} e] \in M_n(Y^1)$ be given. Then

$$\left\| \left[\sum (a_{ik} + \lambda_{ik} e) \circ x_{kj} \right] \right\| = \lim_\beta \left\| \left[\sum (a_{ik} + \lambda_{ik} e_\beta) \circ x_{kj} \right] \right\|$$

$$\leqslant \sup_\beta \|[a_{ij} + \lambda_{ij} e_\beta]\|.$$

The last two assertions follow from the first term principle as before and the fact that a contractive representation of a C^*-algebra is a *-homomorphism. ∎

As a corollary one obtains, with essentially the same proofs, module characterization theorems, i.e., the analogues of 5.3 and 5.4, for algebras with c.a.i.. The c.a.i. version of BRS is immediate also from 5.7 and the earlier 1 line proof we gave of BRS. The "automatic associativity" version of BRS with c.a.i also goes through exactly as in the Remarks above. See also, e.g., [52] or [15, p. 8].

It may be interesting to study oplications in this sense, of $L^1(G)$ on X, where G is a locally compact group. The previous theorem suggests that these correspond to group homomorphism from G into a W^*-algebra containing $\mathscr{A}_l(X)$.

We note in passing that if m is an oplication of Y on X, and if Z is a closed subspace of X such that $m(Y, Z) \subset Z$, then we get a quotient oplication of Y on X/Z. Presumably there are other canonical constructions with oplications.

We end this section with a sketch of the original proof of Theorem 5.2, which was much more circuitous. It used a 3-step strategy. Step 1: prove the result first in the easy case that X is a C^*-algebra. Step 2 is also simple: For a general X, an oplication of Y on X clearly extends to an oplication of Y on the injective envelope $I(X)$ (by the injectivity of the Haagerup tensor norm, and the rigidity property of $I(X)$). As we said in the Introduction, the injective envelope is a C^*-module. Step 3 is to prove the oplication theorem for C^*-modules. This may be done by a variant of the stable isomorphism theorem, which states that every C^*-module is effectively a direct sum of copies of a C^*-algebra—thus placing us back into the situation of Step 1. There are, however, some technical points in the implementation of Step 3 concerning self-dual modules.

We believe that these principles will be important in the development of a noncommutative Choquet theory since they give a method to deduce results about general operator spaces from analogous results on C^*-algebras via the Hamana boundary/injective envelope techniques and passage to the second dual. This is very similar in spirit to the classical probability measure arguments on the Choquet boundary.

If Z is a dual operator space we write $C_I^w(Z)$ and $M_I(Z)$ for the weak*-versions of the spaces $C_I(Z)$ and $\mathbb{K}_I(Z)$ mentioned in the Introduction (see [30, 31, 11]). Since we cannot give specific references in the literature to the following results, we will list them here. We suppose them to be well known in some quarters:

LEMMA 5.8. 1. *If X is any full right C^*-module over a C^*-algebra D, then X^{**} is a self-dual W^*-algebra Morita equivalence $\mathbb{K}(X)^{**}$–D^{**}-bimodule, in the sense of* [56, *Definition* 7.5].

2. *If $\mathscr{M}$ is a W^*-algebra, and if Z is a self-dual W^*-algebra Morita equivalence $\mathscr{N}$–$\mathscr{M}$-bimodule, then there exists a cardinal I such that $C_I^w(Z) \cong C_I^w(\mathscr{M})$ $\mathscr{M}$-completely isometrically.*

3. *If X is any full right C^*-module over a C^*-algebra D, then $C_I^w(X^{**}) \cong C_I^w(D^{**})$ completely D^{**}-isometrically. Thus $M_I(X^{**})$ is D^{**}-completely isometric to the W^*-algebra $M_I(D^{**})$.*

Proof (Sketch). 1. This can be seen by following the idea of the proof of [44, Theorem 4.2], but working on the Hilbert space of the universal representation of the linking algebra $\mathscr{L}(X)$ of X. The key first step in his proof is to carefully compute the commutants $\mathscr{L}(X)'$ and $\mathscr{L}(X)''$. One obtains an explicit formula for the weak*-closure Z of X in $\mathscr{L}(X)''$ and also that Z is a self-dual W^*-algebra Morita equivalence bimodule over D''. The latter follows from [56, Theorem 6.5] (see also [44, Theorem 4.1]). Then one needs to check that via the usual identification of $\mathscr{L}(X)''$ and $\mathscr{L}(X)^{**}$ we have $D^{**} \cong D''$ and also that $Z \cong X^{**}$, "as bimodules," and it is easy to show that X^{**} is self-dual as a D^{**}-module. Similar assertions follow by symmetry for the left action.

2. This may be proved using basic facts about self-dual modules from [46] analogously to the proof of the stable isomorphism theorem or Kasparov's stabilization theorem (see for example [41] Proposition 7.4, or the proof of [8, Theorem 8.6]). Indeed this is one way to show the folklore fact that W^*-Morita equivalent W^*-algebras are w^*-stably isomorphic.

3. Follows from (1) and (2). ∎

We now give a sketch of Step 3, namely the proof of the oplication theorem in the case that X is a right C^*-module. So suppose that X is a full right C^*-module over a C^*-algebra D. It is enough to show that if $\circ$ is an oplication of an operator space Y on X, then $y \circ (xd) = (y \circ x)\, d$ for all $y \in Y, x \in X, d \in D$. We begin by dualizing the oplication to get an oplication m: $Y \otimes_h X^{**} \to X^{**}$. However, by 5.8 (3), there exists a cardinal I such that $C_I^w(X^{**}) \cong C_I^w(D^{**})$, as W^*-modules over D^{**}. Hence there exists a completely isometric D^{**}-module map: ϕ: $M_I(X^{**}) \cong M_I(D^{**})$, completely isometrically and as right W^*-modules over D^{**}.

It is easy to see, using the multilinear Stinespring representation of m (see [49]), that we get an oplication $\tilde{m}$: $Y \otimes_h M_I(X^{**}) \to M_I(X^{**})$, given by $\tilde{m}(y, [b_{ij}]) = [m(y, b_{ij})]$. We therefore obtain a transferred oplication $\tilde{\tilde{m}}$: $Y \otimes_h M_I(D^{**}) \to M_I(D^{**})$. By the case of the oplication theorem when X is a C^*-algebra, there exists a ρ such that $\tilde{\tilde{m}}(y, x) = \rho(y)\, x$, for all $x \in M_I(D^{**})$. Hence $\tilde{m}(y, [x_{ij}]) = \phi^{-1}(\rho(y)\, \phi([x_{ij}]))$, where $x_{ij} \in X$. Since ϕ is a D-module map, the result follows.

In summary, one sees that the crux of our technique above is to show that every operator space X is contained in $M_I(I(X)^{**})$, which is completely isometric to a W^*-algebra.

In addition, we have recently established the weak*-versions of most of the results in this section [68].

6. *Further Applications to Operator Modules*

In this section again, unless we say to the contrary, A is an operator space which is also an algebra with c.a.i. $\{e_\alpha\}$, and X will be a left operator A-module. We remark in passing though that almost all of the results of this section do not use the multiplicative structure of A; thus they are valid with minor modifications for oplications. As usual we write $J: X \to \mathcal{T}(X)$ for the Shilov representation of an operator space X. We will regard $\mathcal{M}_l(X)$ as a subalgebra of $LM(\mathcal{E}(X))$ in this section (as opposed to regarding it as a subalgebra of $CB(X)$). We shall write $\theta: A \to \mathcal{M}_l(X)$ for the canonical completely contractive homomorphism guaranteed by the oplication theorem; thus $\theta(a)\, J(x) = J(ax)$ for each $a \in A$, $x \in X$. In the c.a.i. case we have that $\theta(e_\alpha) \to 1$ strongly, but in fact under the hypotheses we will assume later we will find that $\theta(e_\alpha) \to 1$ in norm. A map θ with the latter property will be called unital in this section. If A is a C^*-algebra then we saw that θ is a $*$-homomorphism into $\mathbb{B}_l(X)$, whereas if X is e.n.v. then θ maps into $\mathcal{E}(X)$.

Recall that a left operator A-module X is *completely* 1-*faithful* if the canonical map $A \to CB(X)$ is a complete isometry. We will say that a left operator A-module X is *completely faithful* if the canonical homomorphism $\theta: A \to \mathcal{M}_l(X)$ above is completely isometric. This of course forces A to be an operator algebra. This observation, together with the proof of (1) below, gives another characterization of operator algebras, namely as the A which have a completely 1-faithful operator module action on some operator space.

PROPOSITION 6.1. *Suppose that X is a left operator A-module.*

(1) *If X is completely* 1-*faithful then X is completely faithful.*

(2) *If A is a C^*-algebra, then X is faithful if and only if it is completely faithful.*

Proof. (1) For any such X, and for $[x_{kl}] \in M_m(X)_1$ and $a_{ij} \in A$, we have:

$$\|[a_{ij}x_{kl}]\| = \|[\theta(a_{ij})\, J(x_{kl})]\| \leqslant \|[\theta(a_{ij})]\| \leqslant \|[a_{ij}]\|.$$

Taking the supremum over such $[x_{kl}]$ gives $\|[a_{ij}]\| = \|[\theta(a_{ij})]\|$.

(2) In this case the canonical θ is a $*$-homomorphism into $\mathbb{B}_l(X)$. It is easy to see that X is faithful if and only if θ is 1–1, and it is well known that the latter happens for a $*$-homomorphism if and only if it is completely isometric. ∎

The converse implication of (1) above is not valid, as may be seen from Example 4.4. However, it may yet be true if A is a C^*-algebra. Indeed this question is clearly equivalent to a question we were unable to settle earlier, namely whether $\mathscr{A}_l(X) \subset CB(X)$ completely isometrically.

We can add a little more information in the case that A is a C^*-algebra using Corollary 4.7. That result, together with (2), immediately shows that if A is a C^*-algebra, then any faithful left operator A-module has the property that the natural action of $M_n(A)$ on $C_n(X)$ is 1-faithful, for all $n \in \mathbb{N}$. We shall say that an X with the latter property is *column 1-faithful.*

We now concentrate on singly generated left operator modules in an attempt to find the correct generalizations of the results in Section 3 (and the remaining relevant ones from Section 2).

Recall that a left module X is (left) a.s.g. if there is an $x_0 \in X$ such that $Ax_0 = X$. (Left) t.s.g. means that $\overline{(Ax_0)} = X$. It is clear how to modify these ideas for right modules and bimodules. We say that x_0 is a *bigenerator* if it is both a left and a right generator. Of course, any unital algebra has an algebraic bigenerator.

The first comment to be made perhaps, is that we are not at all sure when algebraically singly generated implies e.n.v. (However, see 6.6 for one such result). Thus we will usually tack an e.n.v. hypothesis onto our results. We remark in passing that for any algebraically finitely generated, or even topologically countably generated, left operator module X, we have that $\mathscr{F}(X)$ is σ-unital or equivalently has a strictly positive element. Indeed a modification of the argument of 6.2 below shows that if $x_1, x_2, \ldots$ topologically generate X, then w.l.o.g. $\sum_i \|J(x_i)\|^2 < \infty$, and $\sum_i J(x_i)^* J(x_i)$ is a strictly positive element of $\mathscr{F}(X)$. (See also the proof of [15, Theorem 7.13]; this part of the argument only requires that the C^*-module be countably generated.)

By analogy to Section 3, we define a *right nonvanishing* element of X to be an element $x_0 \in X$ such that $J(x_0)^* J(x_0)$ is strictly positive in the 2–2 corner C^*-algebra $\mathscr{F}(X)$ of $C^*(\partial X)$. Similarly for left nonvanishing. Equivalently, x_0 is right nonvanishing if and only if $i(x_0)^* i(x_0)$ is strictly positive for *some* Hilbert C^*-extension (Z, i) of X in the sense of Appendix A.

LEMMA 6.2. *If x_0 is a left topological single generator of an operator A-module X, then x_0 is a right nonvanishing element in the sense above.*

Proof. Suppose that ϕ is a state on $\mathscr{F}(X)$ with $\phi(J(x_0)^* J(x_0)) = 0$. Then we have that $\phi(J(x_0)^* \theta(a)^* \theta(a) J(x_0)) = 0$ for all $a \in A$. By the polarization identity we see that $\phi(J(x)^* J(y)) = 0$ for all $x, y \in X$. Thus $\phi(J(y)^* J(x) J(x)^* J(y)) = 0$. Suppose that $\phi(z) = \langle \pi(z) \zeta, \zeta \rangle$ is a GNS representation of the state. Then if $a_1, ..., a_n$ are each of the form $J(x)^* J(y)$, for some $x, y \in X_1$, then

$$|\phi(a_1 \cdots a_n)|^2 \leqslant \|\pi(a_n)\zeta\|^2 = \phi(a_n^* a_n) = 0,$$

from which it is clear that $\phi = 0$. ∎

The following begins to show what is going on with singly generated operator modules:

THEOREM 6.3. *Let X be a left operator A-module with a topological single generator x_0. Suppose also that X is left e.n.v. (i.e., $\mathscr{E}(X)$ is unital). The following are equivalent:*

(i) *x_0 is left-nonvanishing (or equivalently, the element $E = (J(x_0)J(x_0)^*)^{1/2}$ is invertible in $\mathscr{E}(X)$).*

(ii) *X is also a right operator module over some operator algebra with c.a.i. such that x_0 is a topological single generator for this right action (i.e., x_0 is a topological bigenerator).*

Assume that these conditions hold. Then X is e.n.v. and C^*-generating (that is, $\mathscr{T}(X)$ is a C^*-algebra), and x_0 is also right nonvanishing. Also x_0 is an algebraic single bigenerator for the $\mathscr{M}_l(X)$- and $\mathscr{M}_r(X)$-actions. If θ is the canonical map $A \to \mathscr{M}_l(X)$, then θ is unital and has dense range, and $\mathscr{E}(X)$ is generated as a C^*-algebra by E and $\theta(A)$. The Shilov representation of X may be taken to be $(\mathscr{E}(X), J')$, where $J'(ax_0) = \theta(a)E$. We also have $\mathscr{M}_r(X) \cong E^{-1}\mathscr{M}_l(X)E$. Finally, x_0 is an algebraic single generator for the left A-action if and only if $\theta(A)$ is norm closed, and if and only if $\theta(A) = \mathscr{M}_l(X)$.

Proof. That (ii) implies (i) follows from Lemma 6.2.

If (i) holds, then we have the following completely isometric A-isomorphisms:

$$X \cong J(X) = \overline{\theta(A)J(x_0)} \cong \overline{\theta(A)E} = \overline{\theta(A)}E \subset \mathscr{E}(X).$$

If the generation is algebraic, then the closures in this string of equalities are unnecessary, and thus we see that $\theta(A)$ is closed. In any case if A has c.a.i. $\{e_\alpha\}$ then $\theta(e_\alpha)J(x_0) \to J(x_0)$, which implies that $\theta(e_\alpha)E \to E$. Hence θ is unital in the sense of the introduction to Section 6. Notice that $\overline{\theta(A)}E$ is also a right $E^{-1}\overline{\theta(A)}E$-module, and E is an algebraic single bigenerator. Thus we have (ii). Notice that $J(X)J(X)^* \subset C^*(\theta(A), E)$, so that $\mathscr{E}(X) = C^*(\theta(A), E)$. If $J(x_0) = EV$ is the right polar decomposition of $J(x_0)$, then $V = E^{-1}J(x_0) \in \mathscr{T}(X)$. The map $T: e \mapsto eV$ is a completely isometric $\mathscr{E}$-module map from $\mathscr{E}(X)$ onto $\mathscr{T}(X)$; hence, it is an imprimitivity bimodule isomorphism or triple isomorphism. Thus we may take $\mathscr{E}(X)$ to be the noncommutative Shilov boundary of X. That is, we may replace the

Shilov representation $(\mathcal{T}(X), J)$ by $(\mathcal{E}(X), J')$ where $J' = T^{-1} \circ J$. However, $T^{-1} \circ J(ax_0) = \theta(a)\, E$, for any $a \in A$, and hence $J'(X) = \overline{\theta(A)}\, E$. It is easy to see from this that $\mathcal{M}_l(X) = \overline{\theta(A)}$ and $\mathcal{M}_r(X) = E^{-1}\overline{\theta(A)}\, E$. Since $\mathcal{T}(X) \cong \mathcal{E}(X)$ is a unital C^*-algebra, X is e.n.v. By symmetry, x_0 is right-nonvanishing.

The last assertion of the theorem is clear: if $\theta(A)$ is norm closed then $X \cong \theta(A)\, E$, so X is a.s.g. The converse assertion was noted earlier. ∎

Remark. It seems that one cannot drop the condition in (i) and still expect to get the other powerful conclusions of the theorem. For example C_n is a faithful, a.s.g., e.n.v. M_n-module, which is not C^*-generating.

Corollary 6.4. *Suppose that X is a left e.n.v. left operator A-module, with a topological single generator x_0 which is left nonvanishing.*

(1) *If X is λ-faithful, or if A is a C^*-algebra, then x_0 is an algebraic single generator for the A-action.*

(2) *If x_0 is an algebraic single generator, and the action on X is faithful, then A is necessarily unital.*

Proof. Both follow from the last assertion of the theorem. Also (1) uses the easy fact that for a λ-faithful operator module, the θ is bicontinuous and consequently has closed range, whereas for A is a C^*-algebra we have the fact that the range of a $*$-homomorphism is closed. As for (2), if the A-action is faithful, then θ is 1–1 and onto $\mathcal{M}_l(X)$, so that A is unital. ∎

Remark. This last corollary is fairly sharp. It is easy to see that one cannot drop the e.n.v. hypothesis (consider $C([0, 1])$ acting on $C_0((0, 1]))$. A good example showing that we cannot replace *λ-faithful* with *faithful* here is to consider a 1–1 completely contractive unital homomorphism $\theta: A \to B$ with dense range, such as in Example 2.7. Here A, B are unital operator algebras. Then B considered as an A–A-bimodule via the θ action is e.n.v., has a topological bigenerator 1_B which is left and right nonvanishing, and is faithful, but B need not be algebraically singly generated over A.

Definition 6.5. We shall say that a left operator A-module X satisfying the equivalent hypotheses of the previous theorem is *invertibly topologically singly generated*, or i.t.s.g. (We are assuming such X is also left e.n.v). If, in addition, the generator x_0 an algebraic left generator of X, then we shall say that X is i.a.s.g.

The previous theorem shows that these are left–right symmetric properties. The reasoning behind the *invertibly* in the name the following tidy characterization of such modules:

THEOREM 6.6. *A left operator A-module X is i.a.s.g. (resp. i.t.s.g) if and only if X is completely A-isomorphic to a module of the form $\theta(A)\,P$ (resp. $\overline{\theta(A)}\,P$), where $\theta\colon A \to \mathscr{C}$ is a unital completely contractive homomorphism into a unital C^*-algebra $\mathscr{C}$ and P is a positive invertible element of $\mathscr{C}$.*

Proof. The ($\Rightarrow$) direction was proved in Theorem 6.3.

If P is as above, and if $X = \overline{\theta(A)}\,P$, then the subsets XX^* and X^*X of $\mathscr{C}$ contain P^2, which implies by spectral theory (since we can uniformly approximate the functions $\sqrt{t}$ and $\frac{1}{t}$ on any closed interval not containing 0 by polynomials with no constant term) that the C^*-subalgebra of $M_2(\mathscr{C})$ generated by the copy of X in the 1–2-corner is unital. Hence X is e.n.v., since $C^*(\partial X)$ is a quotient of this unital C^*-subalgebra. Moreover, X is a right $P^{-1}\overline{\theta(A)}\,P$-module, and P is a bigenerator for X considered as a bimodule. ∎

As in Proposition 3.1, we see that any X which is i.t.s.g. is an a.s.g. $\mathscr{M}_l(X)$-module and that the left i.a.s.g. A-module actions on such an X are in 1–1 correspondence with completely contractive surjective homomorphisms $\pi\colon A \to \mathscr{M}_l(X)$. Condition (iii) of Proposition 3.1 may he generalized to i.t.s.g operator modules by replacing *faithful* with *completely faithful*, and *isometrically* with *completely isometrically*.

The following result matches Corollary 3.3. As in that section, if Y is a left operator B-module, and $\theta\colon A \to B$ is a completely contractive unital homomorphism, then we define ${}_\alpha Y$ to be the operator module of Y with A-action $\alpha(a)\,y$. Namely, this is the prolongation of the B-action by α.

COROLLARY 6.7. *Let A and B be unital operator algebras and let X and Y be two completely faithful operator modules which are i.t.s.g.; X is an A-module and Y is a B-module. If $X \cong Y$ completely isometrically, then $A \cong B$ as operator algebras. Indeed there exists a completely isometric unital surjective homomorphism $\alpha\colon A \to B$ such that $X \cong {}_\alpha Y$, completely A-isometrically. Consequently, if $X \cong A$ linearly completely isometrically, then $X \cong A$ completely A-isometrically.*

Proof. Suppose that $v\colon X \to Y$ is the linear completely isometric isomorphism. By the above, there exist completely isometric unital surjective homomorphisms $\theta\colon A \to \mathscr{M}_l(X)$ and $\rho\colon \mathscr{M}_l(Y) \to B$. Define $\alpha(a) = \rho(v\theta(a)\,v^{-1})$. Then α is a completely isometric unital surjective homomorphism $A \to B$ by 4.12, and $\alpha(a)\,v(x) = v(\theta(a)(x)) = v(ax)$ for all $a \in A$, $x \in X$. So v is an A-isomorphism from X onto ${}_\alpha Y$.

The last assertion follows since ${}_\alpha A \cong A$ via the map α^{-1}. ∎

Some of the results in Section 3 have no valid generalization to operator modules. An example is Corollary 3.6, which implies that a t.s.g. and e.n.v.

function module over $C(\Omega)$ is a quotient of $C(\Omega)$ by a closed ideal. We shall see in the next example that this does not generalize.

The best result obtained in Section 3, was that the algebraically singly generated faithful left function A-modules, for a function algebra A, are exactly the Banach A-modules which are A-isometric, to one of the form Af_0, where f_0 is a strictly positive (thus invertible) function on a compact space Ω, on which A sits as a function algebra (in particular, A separates points of Ω). Thus if A is self-adjoint, then $A = C(\Omega)$, so that $Af_0 = C(\Omega) = A$. The full noncommutative version of this result is false, as may be seen by the example below, and another example we have with Roger Smith, which shows that not every i.t.s.g. completely 1-faithful operator module over a unital C^*-algebra is completely isometric to a C^*-algebra. However, there is a partial generalization of the later results in Section 3 to operator modules: by Theorem 6.3 (and 6.6) a completely faithful i.a.s.g. module "is" an AP, for an invertible P. However, the copy of A need not generate a C^*-algebra containing P, as happened in the commutative case. It is clear that one needs to have something like the i.a.s.g. condition in any generalization of 3.7 to operator modules (consider C_n as an M_n-module). We thank V. Paulsen for input concerning the next example.

EXAMPLE 6.8. Suppose that $A = D_n$, the diagonal C^*-algebra inside M_n, and let P be a positive invertible matrix in M_n, which is not in D_n. Let $X = AP \subset M_n$. This is an e.n.v., i.a.s.g. (by 6.6, for example), and faithful A-module. By an observation early in this section, X is in fact completely faithful, and column 1-faithful. By analogy with the function module case, one might expect that $X \cong D_n$ D_n-completely isometrically for any such P. However, this is false, for if $f: D_n \to X$ were a completely isometric D_n-module map, then $f(a) = abP$, for some fixed $b \in D_n$. If $f(a_0) = P$ then $a_0 b = I$, so b is invertible in D_n. Now

$$\|a\| = \|abP\| = \|abP^2b^*a^*\|^{1/2} = \|aQ\|,$$

where $Q = (bP^2b^*)^{1/2}$. Clearly $\|Q\| = 1$. By putting $a = e_i$ we see that the rows of Q have norm 1. Hence Q^2 has only 1's on its main diagonal. Thus the squares of the eigenvalues of Q add up to n, so that each eigenvalue of Q is 1. Hence $Q = I$. Thus P^2 and consequently P is a diagonal matrix, which is a contradiction.

Let us continue a little further with this example, but now suppose that P^2 has no nonzero entries. Then notice that the subset XX^* of M_n contains each matrix unit of M_n. Also $M_n X = M_n$. Thus the C^*-subalgebra of $M_2(M_n)$ generated by the copy of X in the 1–2-corner is all of $M_2(M_n)$. Since this is a simple C^*-algebra, we see that $C^*(\partial X) = M_2(M_n)$ and that

$\mathcal{T}(X) = M_n$. That is, the embedding $X \subset M_n$ that we started with is the Shilov representation of X. From this (or from Theorem 6.3) we see immediately that $\mathcal{M}_l(X) = D_n = \mathbb{B}_l(X)$, and that $\mathcal{M}_r(X) = P^{-1} D_n P$ and $\mathbb{B}_r(X) = \mathbb{C} I$.

Notice that in this example the natural right regular representation $\mathcal{M}_r(X) = P^{-1} D_n P \to CB(X)$ is not isometric. This is because for $d \in D_n$ and $a = P^{-1} dP$, we have that $(eP)\, a = edP = d(eP)$ for any $e \in D_n$. Thus the norm of right multiplying by a on X is the norm of left multiplying by d on X, which is $\|d\|$. However, $\|P^{-1} dP\| \neq \|d\|$ in general.

This example falls within the scope of the following theorem, which follows from previous results, and which sums up most of what we know about singly generated operator modules over C^*-algebras.

THEOREM 6.9. *Let A be a C^*-algebra and X a left operator A-module. Then the following are equivalent*:

(i) *X is completely isometrically A-isomorphic to a module of the form $\theta(A)\, P$, where $\theta\colon A \to \mathcal{C}$ is a unital $*$-homomorphism (respectively, faithful $*$-homomorphism) into a unital C^*-algebra $\mathcal{C}$ and P is a positive invertible element of $\mathcal{C}$.*

(ii) *X is an i.a.s.g. (respectively, and faithful) left A-module. (See Definition* 6.5.)

(iii) *X is i.t.s.g. (respectively, and faithful).*

Also all the conclusions of Theorem 6.3 *hold. Moreover, $\mathcal{M}_l(X) = \mathcal{A}_l(X)$, and this is also $*$-isomorphic to A if X is a faithful A-module.*

This completes our extension of the results in Section 3 (and Section 2) to operator modules.

We end this section by pointing out an application of these principles to nonassociative characterizations of operator algebras.

We first prove a strengthening of an earlier result:

LEMMA 6.10. *Let X be an operator space and let $g \in X$. Consider the following conditions on g*:

(1) *If $T \in \mathcal{M}_l(X)$ and $Tg = 0$, then $T = 0$.*

(2) *g is a left-nonvanishing element of X.*

(3) $\overline{g\mathcal{M}_r(X)} = X$.

(4) *There exists an operator space Y, and a right oplication $n\colon X \times Y \to X$, such that $\overline{\{n(g, y)\colon y \in Y\}} = X$.*

Then (4) $\Leftrightarrow$ (3) $\Rightarrow$ (2) $\Rightarrow$ (1). *Hence if X is a left operator A-module with algebraic single generator g satisfying any one of these four conditions, then $\mathcal{M}_l(X) = \theta(A)$, where θ is the canonical map $A \to \mathcal{M}_l(X)$.*

Proof. Taking $Y = \mathcal{M}_r(X)$ shows that (3) $\Rightarrow$ (4). Conversely, the oplication theorem shows that (4) $\Rightarrow$ (3). Lemma 6.2 shows that (3) $\Rightarrow$ (2). Finally, if we have (2), and if T is as in (1), then $P = J(g)\,J(g)^*$ is a strictly positive element of $\mathscr{E}(X)$, and $TP = 0$, where T is regarded as an element of $LM(\mathscr{E}(X))$. This implies that $\phi(TPT^*) = 0$ for every state on $\mathscr{E}(X)$. Since P is strictly positive, we see that $\phi(T \cdot T^*) = 0$ on $\mathscr{E}$, for every state ϕ on $\mathscr{E}(X)$. Hence, $\phi(Taa^*T^*) = 0$ for every such ϕ and $a \in \mathscr{E}(X)$, which implies that $Ta = 0$ for every $a \in \mathscr{E}(X)$. Thus $T = 0$.

To see the last part, note that if $T \in \mathcal{M}_l(X)$ then $TJ(g) = \theta(a)\,J(g)$ for some $a \in A$. Hence $T = \theta(a)$ by (1). ∎

Corollary 6.11 (A nonassociative BRS theorem). *Suppose that A is an operator space with an element "1" of norm 1, and suppose that $m: A \times A \to A$ is a bilinear map which satisfies $m(a, 1) = m(1, a) = a$ for all $a \in A$. Suppose further that there exists another operator space structure on A, such that if A with this structure is written as A', then m considered as a map $A' \otimes A \to A$ is completely contractive.* (*We are assuming the* norm *on A' is the same as A.*) *We will further assume that $g = 1$ satisfies any one of the four conditions of the previous lemma. Then m is an associative product on A, and with this product A is isometrically isomorphic to an operator algebra. Indeed there is a third operator space structure on A between A and A', with respect to which A is completely isometrically isomorphic to an operator algebra (namely $\mathcal{M}_l(A)$).*

Proof. By the oplication theorem, there exists a completely contractive linear map $\theta: A' \to \mathcal{M}_l(A)$, such that $J(m(a, x)) = \theta(a)\,J(x)$ for all $x, a \in A$. Moreover, θ is isometric: indeed we have for any $a \in M_n(A)$ that

$$\|\theta_n(a)\| \geqslant \|[\theta(a_{ij})\,J(1)]\| = \|J_n(a)\| = \|a\|_{M_n(A)}.$$

Any $T \in \mathcal{M}_l(A)$ has $TJ(1) = J(a) = J(m(a, 1)) = \theta(a)\,J(1)$ for some $a \in A$, so that $T = \theta(a)$. Thus $\theta(A) = \mathcal{M}_l(A)$.

For $a, b \in A$ we have $\theta(a)\,\theta(b)\,J(1) = \theta(a)\,J(b) = J(m(a, b)) = \theta(m(a, b))\,J(1)$. By (1) of the previous lemma, we have that $\theta(a)\,\theta(b) = \theta(m(a, b))$. Hence m is associative.

The rest is clear. ∎

Of course the last corollary might be particularly interesting if we are assuming (4) of the Lemma with $Y = A$ but with some possibly different operator space structure, such as A' or $Y = MAX(A)$. We will not take the time to explicitly write out the theorem in these cases.

APPENDIX A: HILBERT C^*-EXTENSIONS AND ENVELOPES

For the readers convenience, we give a brief and self-contained treatment, in the language of C^*-modules, of Hamana's results on the triple envelope $\mathcal{T}(X)$ and its universal property [37]. We also give several simple and interesting consequences.

We recall that a right C^*-module Z over a C^*-algebra A is the equivalence bimodule for a canonical strong Morita equivalence between $\mathbb{K}(Z)$ and the C^*-subalgebra of A generated by the range of the inner product.

Suppose that $\mathscr{C}$ and $\mathscr{D}$ are C^*-algebras and that W is a $\mathscr{C}$–$\mathscr{D}$-imprimitivity bimodule (that is, a strong Morita equivalence $\mathscr{C}$–$\mathscr{D}$-bimodule). There is associated with W a linking C^*-algebra $\mathscr{L}(W)$ whose corners are $\mathscr{C}$, W, $\bar{W}$, and $\mathscr{D}$. We will write $\mathscr{L}^1(W)$ for the unitized linking algebra with corners $\mathscr{C}^1$, W, $\bar{W}$, and $\mathscr{D}^1$. Here $\mathscr{C}^1$ is the unitization of $\mathscr{C}$ if $\mathscr{C}$ is not already unital, otherwise $\mathscr{C}^1 = \mathscr{C}$. We think of W as sitting in the 1–2-corner of $\mathscr{L}(W)$ and write c for the corner map $c\colon W \to \mathscr{L}(W)$.

Definition A.1. If X is an operator space, then a Hilbert C^*-extension of X is a pair (W, i) consisting of a Hilbert C^*-module W and a linear complete isometry $i\colon X \to W$ such that the image of $i(X)$ within $\mathscr{L}(W)$ generates $\mathscr{L}(W)$ as a C^*-algebra. A linear complete contraction (resp. surjective complete isometry) $R\colon (W_1, i_1) \to (W_2, i_2)$ between Hilbert C^*-extensions of X is called an X-complete contraction (resp. X-isomorphism) if $R \circ i_1 = i_2$. We say that an X-complete contraction R is a Hilbert X-epimorphism (resp. Hilbert X-isomorphism) if there exists a $*$-homomorphism (resp. *-isomorphism) θ from $\mathscr{L}(W_1)$ to $\mathscr{L}(W_2)$ such that $\theta \circ c = c \circ R$ on W_1 (or, equivalently, on $i_1(X)$).

By elementary C^*-algebra, the $*$-homomorphism θ above is necessarily unique and necessarily surjective, if it exists. Similarly it is easy to check from the definitions of strong Morita equivalence and the linking C^*-algebra that a Hilbert X-isomorphism is automatically a imprimitivity bimodule isomorphism (see Section 1 for the definition).

We shall see in a little while that Hilbert X-isomorphisms between Hilbert C^*-extensions are the same thing as X-isomorphisms. Note that a Hilbert X-epimorphism is a complete quotient map in the sense of operator space theory (since the associated θ is).

Clearly a Hilbert X-epimorphism $R\colon (W_1, i_1) \to (W_2, i_2)$ is unique, if it exists, because it is completely determined by the formula $R \circ i_1 = i_2$. Indeed we have $R(i_1(x_1)\, i_1(x_2)^* \cdots i_1(x_m)) = i_2(x_1)\, i_2(x_2)^* \cdots i_2(x_m)$, for any $x_1, ..., x_m \in X$. Thus we can define an ordering $(W_2, i_2) \leqslant (W_1, i_1)$ if and only if there exists such a Hilbert X-epimorphism.

Here are three examples of Hilbert C^*-extensions: Clearly, the Hilbert C^*-envelope $\partial X = (\mathcal{T}(X), J)$ discussed earlier is a Hilbert C^*-extension in this sense. We will see that this is the minimum element in the ordering just defined.

Early in the Introduction we explained how any concrete operator subspace of $B(K, H)$ has a natural Hilbert C^*-extension inside $B(K, H)$.

It is fairly easy to see that there is a maximum Hilbert C^*-extension of X which was essentially constructed in the first part of the last section of [12]. It was called the maximal C^*-correspondence of X there. We shall not discuss this further here.

THEOREM A.2. *For any operator space X we have:*

(i) *$\mathcal{T}(X)$ is rigid as an operator superspace of X. That is, if $R: \mathcal{T}(X) \to \mathcal{T}(X)$ is a complete contraction such that $R \circ J = J$, then $R = Id$.*

(ii) *$\mathcal{T}(X)$ is essential as an operator superspace of X. That is, if $R: \mathcal{T}(X) \to Z$ is a complete contraction into an operator space Z, and if $R \circ J$ is a complete isometry, then so is R.*

(iii) *$\mathcal{T}(X)$ is the unique Hilbert C^*-extension of X, which is essential as an operator superspace of X. The uniqueness is up to X-isomorphism (or up to Hilbert X-isomorphism).*

Proof. (i) This is easy: extend R to a map $I(X) \to I(X)$, and use rigidity of $I(X)$.

(ii) Similar to (i): extend R to a map $I(X) \to I(Z)$, and use the essentiality of $I(X)$.

(iii) Suppose that (Z, i) is any Hilbert C^*-extension of X. Inside $\mathcal{L}^1(Z)$ consider the operator system $\mathcal{S}_1$ given by the image of $i(X)$ in $\mathcal{L}(Z)$, together with the two idempotents on the diagonal of $\mathcal{L}^1(Z)$. So $\mathcal{S}_1$ generates $\mathcal{L}^1(Z)$ as a C^*-algebra. By Paulsen's lemma, the canonical complete isometry $i(X) \to J(X)$ is the 1–2-corner of a complete order isomorphism $\Phi: \mathcal{S}_1 \to \mathcal{S}(J(X))$. By the Arveson–Hamana theorem (1.1 above) Φ extends to a surjective *-homomorphism $\theta: \mathcal{L}^1(Z) = C^*(\mathcal{S}_1) \to C^*_e(\mathcal{S}(X))$. Let π be the restriction of θ to $\mathcal{L}(Z)$. Let R (resp. π_{11}) be the 1–2 corner (resp. 1–1-corner) of π. If Z is an essential operator superspace of X, we see that R is a complete isometry onto $\mathcal{T}(X)$. We claim that R is a Hilbert X-isomorphism of Z onto $\mathcal{T}(X)$; that is, π is 1–1. To see that π_{11} is 1–1, suppose that $\pi_{11}(c) = 0$. W.l.o.g., $c \geqslant 0$. We can then approximate c by sums of the form $\sum_i h_i h_i^*$ with $h_i \in Z$. We have $\|\pi_{11}(\sum_i h_i h_i^*)\| = \|\sum_i R(h_i) R(h_i)^*\| = \|\sum_i h_i h_i^*\|$, since R is a complete isometry. Thus $c = 0$. Similarly π_{22}, and consequently π, is 1–1 on $\mathcal{L}(Z)$. This gives the result. ∎

The first few lines of the proof of part (iii) of Theorem A.2 also shows the following result, which is referred to in earlier sections of our paper as the universal property of $\mathcal{T}(X)$:

THEOREM A.3 (Hamana). *If X is an operator space, then $\mathcal{T}(X)$ is the smallest Hilbert C^*-extension of X. That is, if (W, i) is any Hilbert C^*-extension of X, then there exists a (unique) Hilbert X-epimorphism from W onto $\mathcal{T}(X)$. Moreover, $\mathcal{T}(X)$ is the unique (in the sense of* A.2 (iii)) *Hilbert C^*-extension of X with this property.*

The uniqueness follows from the fact that a *-automorphism, which is the identity on a dense set, is the identity.

Either of the last two theorems shows that for any operator space X, the extremal space ∂X is essentially unique, as are the algebras $C^*(\partial X)$, $\mathcal{M}_l(X)$, and $\mathcal{M}_r(X)$ and the other multiplier algebras.

COROLLARY A.4. *If $\mathcal{G}$ is a C^*-algebra, or right Hilbert C^*-module, then $\mathcal{T}(\mathcal{G}) = \mathcal{G}$, and $C^*(\partial\mathcal{G})$ is the linking C^*-algebra for the strong Morita equivalence canonically associated with $\mathcal{G}$. Moreover, $\mathcal{M}_l(\mathcal{G}) = LM(\mathbb{K}(\mathcal{G}))$ which coincides with the space of bounded right module maps on $\mathcal{G}$; and $\mathcal{A}_l(\mathcal{G}) = M(\mathbb{K}(\mathcal{G})) = \mathbb{B}(\mathcal{G})$, and $\mathbb{K}_l(\mathcal{G}) = \mathbb{K}(\mathcal{G})$. (Of course for a C^*-algebra $\mathbb{K}(\mathcal{G}) \cong \mathcal{G}$.)*

Proof. Clearly $\mathcal{G}$ is an essential Hilbert C^*-extension of itself. ∎

In particular, the last corollary holds for any injective operator space $\mathcal{G}$.

The following corollary seems to be due to Hamana, Ruan, and perhaps others.

COROLLARY A.5. *Suppose that $T: Z \to W$ is a surjective complete isometry between Hilbert C^*-modules. Then T is an imprimitivity bimodule isomorphism.*

Proof. Clearly (Z, T^{-1}) is an essential Hilbert C^*-extension of W. The proof of A.2 (iii) produces a Hilbert W-isomorphism $R: Z \to \mathcal{T}(W) = W$. Since $R \circ T^{-1} = Id_W$, we have $R = T$. ∎

COROLLARY A.6. *Suppose that $R: (W_1, i_1) \to (W_2, i_2)$ is an X-isomorphism between Hilbert C^*-extensions of X. Then R is a Hilbert X-isomorphism. Consequently R is an imprimitivity bimodule isomorphism.*

COROLLARY A.7. *Suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ are C^*-algebras. Suppose that for $k = 1, 2$, X is a right Hilbert C^*-module over $\mathcal{D}_k$. Thus X has two different right module actions and two corresponding inner products. Suppose that the canonical operator space structure on X induced by each inner* *product coincides. If* $R: X \to X$, *then* $R \in B_{\mathscr{D}_1}(X)$ (*resp.* $R \in \mathbb{B}_{\mathscr{D}_1}(X)$, $R \in \mathbb{K}_{\mathscr{D}_1}(X)$, R *is as a rank* 1 *with respect to the first inner product*) *if and only if* $R \in B_{\mathscr{D}_2}(X)$ (*resp.* $R \in \mathbb{B}_{\mathscr{D}_1}(X)$, $R \in \mathbb{K}_{\mathscr{D}_2}(X)$, R *is a rank* 1 *with respect to the second inner product*). *Thus if* X *is full with respect to both actions, then* $\mathscr{D}_1$ *and* $\mathscr{D}_2$ *give the same subset of* $\mathscr{M}_r(X) \subset CB(X)$.

Proof. By Cohen's factorization theorem, if necessary, we may suppose w.l.o.g. that X is a full C^*-module over each $\mathscr{D}_k$. By the previous corollary, I_X may be supplemented by two *-isomorphisms $\mathbb{K}_{\mathscr{D}_1}(X) \to \mathbb{K}_{\mathscr{D}_2}(X)$ and $\theta: \mathscr{D}_1 \to \mathscr{D}_2$, becoming an imprimitivity bimodule isomorphism. Hence it is eacy to check from the definition of the latter term that rank one operators on X with respect to the one inner product are rank one with respect to the other inner product. Hence $\mathbb{K}_{\mathscr{D}_1}(X) = \mathbb{K}_{\mathscr{D}_2}(X)$. The rest is fairly clear. For example, if one takes the definition of adjointability with respect to the $\mathscr{D}_1$-action and apply θ to this equation, then one sees the $\mathscr{D}_2$-adjointability. Finally note that $R \in B_{\mathscr{D}_1}(X)$ iff $R \in LOB(X)$. ∎

We remind the reader of Rieffel's theory of quotient Morita contexts, from [54, Section 3]. By this theory there is, for any $\mathscr{C}$–$\mathscr{D}$-imprimitivity bimodule Z, a lattice isomorphism between the lattices of closed $\mathscr{C}$–$\mathscr{D}$-submodules of Z, and ideals in $\mathscr{C}$ or ideals in $\mathscr{D}$. Although it is not spelled out there, it is not hard to see that these are also in 1–1 correspondence with ideals in $\mathscr{L}(Z)$ and that the correspondence is the following one: If K is a $\mathscr{C}$–$\mathscr{D}$-submodule of a $\mathscr{C}$–$\mathscr{D}$-imprimitivity bimodule, and if $\mathscr{I}(K)$ is the associated ideal in $\mathscr{L}(Z)$, then $\mathscr{I}(K) \cong \mathscr{L}(K)$. Also Z/K is an imprimitivity bimodule, and $\mathscr{L}(Z/K) \cong \mathscr{L}(Z)/\mathscr{L}(K)$ as C*-algebras. This is no doubt well known.

Its also important to notice that Rieffel's quotient mechanism works perfectly with regard to the natural operator space structures on the quotients he works with. See also [45, Section 6], where some operator space generalizations of Rieffel's quotients are worked out.

Definition A.8. Suppose that (Z, i) is a Hilbert C^*-extension of an operator space X, and suppose that Z is a $\mathscr{C}$–$\mathscr{D}$-imprimitivity bimodule. A *boundary submodule* for X in Z is a closed $\mathscr{C}$–$\mathscr{D}$-submodule K of Z such that the quotient map $q: Z \to Z/K$ is completely isometric on X. The *Shilov boundary submodule* $\mathscr{N}(X)$ for X in Z is the largest boundary submodule for X in Z, if such exists.

Lemma A.9. *If* K *is a boundary submodule for* X *in* Z, *then* $(Z/K, q \circ i)$ *is a Hilbert* C^*-*extension of* X.

Proof. By [54], Z/K is a Hilbert C^*-module. By the above correspondences, $q(i(X))$ generates $\mathscr{L}(Z/K)$. ∎

We will not really use the following, but imagine it may be useful elsewhere:

PROPOSITION A.10. *K is a boundary submodule for X in (Z, i) if and only if $\mathscr{I}(K)$ is a boundary ideal for $\mathscr{S}(X)$ in $\mathscr{L}^1(Z)$, if and only if $\mathscr{I}(K)$ as a boundary ideal for $c(X)$ in $\mathscr{L}(Z)$.*

Proof. $I = \mathscr{I}(K)$ is an ideal in $\mathscr{L}(Z)$ and therefore also in $\mathscr{L}^1(Z)$. We view $\mathscr{S}(X) \subset \mathscr{L}^1(Z)$, so that there is a canonical completely contractive map $\mathscr{S}(X) \to \mathscr{L}^1(Z)/I$. By Paulsen's lemma, this map is a complete order isomorphism if and only if the restriction of this map to the 1–2-corner X is a complete isometry. However, this restriction maps into $\mathscr{L}(Z)/I$. Since $\mathscr{L}(Z)/I \cong \mathscr{L}(Z/K)$, we see that it is necessary and sufficient that the map $X \to Z/K$ be a complete isometry. ∎

THEOREM A.11 (Hamana). *If Z is a Hilbert C^*-extension of X, then the Shilov boundary submodule $\mathscr{N}(X)$ for X in Z exists. Moreover, $Z/\mathscr{N}(X) \cong \mathscr{T}(X)$ Hilbert X-isomorphically.*

Proof. This follows as in [37], from the Lemma, and so there is no advantage in rewriting it here. Alternatively, one can use the last proposition and the corresponding result in [34]. ∎

By analogy to the remark on [34, p. 782] this all implies that:

COROLLARY A.12 (Hamana). *The Hilbert C^*-envelope of X may be taken to be any Hilbert C^*-extension (Z, i) of X such that the Shilov boundary submodule of $i(X)$ in Z is (0). Moreover, given two Hilbert C^*-extensions (Z_1, i_1) and (Z_2, i_2) of X, there exists a unique surjective Hilbert X-isomorphism R: $Z_1/N_1 \to Z_2/N_2$, where N_1 and N_2 are the Shilov boundary submodules of X in Z_1 and Z_2, respectively.*

The last corollary is useful in calculating $\mathscr{T}(\cdot)$. As a typical application, we list the following result, which was used in Section 4:

THEOREM A.13. *For operator spaces X and Y, the following (completely isometric) imprimitivity bimodule isomorphisms are valid:*

(i) $\mathscr{T}(X \oplus_\infty Y) \cong \mathscr{T}(X) \oplus_\infty \mathscr{T}(Y)$.

(ii) $\mathscr{T}(C_I(X)) \cong C_I(\mathscr{T}(X))$ *and more generally* $\mathscr{T}(\mathbb{K}_{I,J}(X)) \cong \mathbb{K}_{I,J}(\mathscr{T}(X))$, *for any cardinals I, J.*

Also $\mathscr{T}(X \otimes_{spatial} Y)$ is not isomorphic to the spatial (or exterior tensor product of $\mathscr{T}(X)$ and $\mathscr{T}(Y)$, in general. Similarly, if X is a right A-module *and Y is a left A-module, then $\mathcal{T}(X \otimes_{hA} Y)$ is not isomorphic to any kind of tensor product of $\mathcal{T}(X)$ and $\mathcal{T}(Y)$, in general.*

Proof. (ii) It is enough to prove the first assertion; for then by symmetry there is a matching assertion for $R_J(X)$, and then one can use the relation $\mathbb{K}_{I,J}(X) = C_I(R_J(X))$. We will use Corollary A.12. Let ε_i: $\mathcal{T}(X) \to C_I(\mathcal{T}(X))$ be the ith inclusion map, which is an isometric $\mathcal{F}(X)$-module map. Suppose that W is a boundary submodule for $C_I(X)$ in $C_I(\mathcal{T}(X))$. Thus W is a $\mathbb{K}_I(\mathcal{E})$–$\mathcal{F}$-submodule of $C_I(\mathcal{T}(X))$, and the canonical map Q: $C_I(X) \to C_I(\mathcal{T}(X))/W$ is a complete isometry. Clearly W is also a $\mathbb{K}_I(M(\mathcal{E}))$-submodule of $C_I(\mathcal{T}(X))$. Letting $W_i = E_{ii} W = \varepsilon_i(\mathcal{T}(X)) \cap W$, we see that this may be identified as an $\mathcal{E}$–$\mathcal{F}$-submodule W_i' of $\mathcal{T}(X)$. If we can show that the canonical map $X \to \mathcal{T}(X)/W_i'$ is a complete isometry then we would be done, since $\mathcal{T}(X)$ has no nontrivial boundary submodules.

To this end notice first that $\mathcal{T}(X)/W_i' \cong \varepsilon_i(\mathcal{T}(X))/W$ completely isometrically, and second notice that the map $Q \circ (\varepsilon_{i|_X})$ is a complete isometry from $X \to C_I(\mathcal{T}(X))/W$, which maps into $\varepsilon_i(\mathcal{T}(X))/W$.

A similar but easier argument proves (i). Namely, any boundary submodule for $X \oplus_\infty Y$ in $\mathcal{T}(X) \oplus_\infty \mathcal{T}(Y)$ may be written as $K_1 \oplus_\infty K_2$, where K_i are submodules of $\mathcal{T}(X)$ and $\mathcal{T}(Y)$, respectively. It is clear that K_1 is a boundary submodule for X in $\mathcal{T}(X)$ and is consequently trivial, and so on. Another way to prove this result is first to prove that $I(X \oplus_\infty Y) = I(X) \oplus_\infty I(Y)$, which is quite easy to see.

One way to see the failure of the second tensor product relation is to set $X = R_n$, $Y = C_n$, $A = \mathbb{C}$. Then $\mathcal{T}(X \otimes_{hA} Y) = \mathcal{T}(M_n^*)$, which has dimension at least n^2, whereas $\mathcal{T}(X) \otimes \mathcal{T}(Y)$ has dimension n^2. If there was some quotient of the latter isomorphic to the former, then $\mathcal{T}(M_n^*)$ would have dimension n^2. This implies that $\mathcal{T}(M_n^*) = M_n^*$ and that M_n^* is a finite dimensional C^*-module, which implies that $MAX(\ell_2^1)$ is injective, since it is a complemented summand of M_n^* and every finite dimensional C^*-module is clearly injective. However, there are only three distinct two dimensional injective operator spaces, as is clear from a result of Smith's from 1989 (written up recently in [59]), and none of these is ℓ_2^1.

To see the failure of the first tensor product relation, take $X = Y = MIN(\ell_2^2)$. In this case $X \otimes_{spatial} Y \cong MIN(M_2)$, and therefore one can identify $\mathcal{E}(X)$ and $\mathcal{E}(X \otimes_{spatial} Y)$ from [66] (see also Corollary 4.24 above). ∎

From these statements it is easy to calculate the multiplier C^*-algebras of $X \oplus^\infty Y$ and $\mathbb{K}_{I,J}(X)$. For example, $M_l(X \oplus^\infty Y) = \mathcal{M}_l(X) \oplus^\infty \mathcal{M}_l(Y)$ and $\mathcal{M}_l(C_1(X)) = \mathbb{K}_I(\mathcal{M}_l(X))$.

One can write down a more general version of (i) of the last theorem for an arbitrary number of summands.

In general, if $X \subset Y$, there does not seem to be any nice relationship between $\mathcal{T}(X)$ and $\mathcal{T}(Y)$. Of course, if A is a unital C^*-subalgebra of a C^*-algebra B, then $\mathcal{T}(A) = A \subset \mathcal{T}(B) = B$. However, the disk algebra $A(\mathbb{D}) \subset C(\bar{\mathbb{D}})$, but $\mathcal{T}(A(\mathbb{D})) = C(\mathbb{T})$, whereas $\mathcal{T}(C(\bar{\mathbb{D}})) = C(\bar{\mathbb{D}})$. Even if, in addition, there is a completely contractive projection P: $Y \to X$ there does not seem to be much one can say in general. Indeed, by looking at the case when X, Y are minimal operator spaces it seems that we need some M-structure [1, 4] to be able to say anything. It would be interesting to pursue such a noncommutative M-theory further.[3]

Here is a sample application of the last theorem. If A is an operator algebra with c.a.i., then from 4.17 and relation (ii) above, we have $\mathcal{T}(C_I(A)) = C_I(C_e^*(A))$. Thus, since any c.a.i. for A is also one for $C_e^*(A)$, we have:

$$\mathbb{K}_l(C_i(A)) = \{[b_{ij}] \in \mathbb{K}_I(C_e^*(A)) : b_{ij}A \subset A\} = \mathbb{K}_I(A).$$

More generally, $\mathbb{K}_l(Y) = \mathbb{K}(Y)$ if Y is a strong Morita equivalence bimodule in the sense of [15]. One can show also that A.5 generalizes to this case; indeed one can recover all the "equivalence data" from the linear operator space structure of Y alone, up to equivalence bimodule isomorphism. This is all false if Y is merely a rigged module in the sense of [8].

APPENDIX B: SOME CONNECTIONS WITH THE INJECTIVE ENVELOPE; AND MORE RECENT RESULTS

In joint work with Paulsen [17], we found that for any operator space X, the 1–1 corner I_{11} of $I(\mathcal{S}(X))$ may be identified with $\mathcal{M}_l(I(X))$, and the latter also equals $\mathcal{A}_l(I(X))$ in this case. We also showed that $I(X)$ is a self-dual C^*-module. We defined another multiplier algebra $IM_l(X) = \{S \in I_{11} : SX \subset X\}$. There is a canonical sequence of completely contractive 1–1 homomorphisms $IM_l(X) \to \mathcal{M}_l(X) \to CB(X)$ given by restriction of domain. We then showed that the first of these homomorphisms is a complete isometry, thus $IM_l(X) \cong \mathcal{M}_l(X)$ as operator algebras. This gives a fourth description of the multiplier algebra.

We deduced from this that the A–B-action on an operator A–B-bimodule X extends to an A–B-action on the injective envelope $I(X)$.

If X is e.n.v. then the above points are easy to see from results in earlier sections. For example, Lemma 4.1 implies that if X is e.n.v. then $1_{\mathcal{E}(X)} = 1_{I_{11}}$. Thus one sees that $I(X)$ is an algebraically finitely generated C^*-module, and so $\mathbb{K}_l(I(X))$ is unital (see Section 15.4 in [63]), so that

[3] We are currently pursuing such a theory with Effros and Zarikian [69].

$I(X)$ is self-dual and also $I_{11} = \mathscr{A}_l(I(X))$. If $S \in IM_l(X)$, then $SX \subset X$, so that $S\mathscr{E}(X) \subset \mathscr{E}(X)$ by the first term principle. Thus $S = S1_{\mathscr{E}(X)} \in \mathscr{E}(X)$.

Consequently $IM_l(X) = \mathscr{M}_l(X)$ as subsets of I_{11}. Similar arguments work for $\mathscr{M}_r$. Thus $I(X)$ is algebraically finitely generated as a C^*-module on both sides and is an operator $\mathscr{M}_l(X)$–$\mathscr{M}_r(X)$-bimodule. It then follows from 5.3 that if X is also an operator A–B-bimodule, then the bimodule action extends to an operator A–B-bimodule action on $I(X)$.

We now apply some of these points to prove a Banach–Stone-type theorem for non-self-adjoint operator algebras with c.a.i. If A is such an algebra, then we saw in 4.17 that $I(A) = I(C_e^*(A)) = I(A^1)$, which is a unital C^*-algebra. By Theorem 4.18, the 1–1-corner I_{11} of $I(\mathscr{S}(A))$ is $I(A)$. Hence, by 4.17 again and the above mentioned result from [17], $LM(A) = \mathscr{M}_l(A) \cong IM_l(A) \subset I(A)$. That is, $LM(A) = \{T \in I(A) : TA \subset A\}$. Similarly $M(A) = \{T \in I(A) : TA \subset A, AT \subset A\}$. These last facts are related to results from [33, 17].

Corollary B.1. Let A and B be operator algebras with c.a.i., and let $T: A \to B$ be a completely isometric linear surjection. Then there exists a completely isometric surjective homomorphism $\pi: A \to B$ and a unitary u with $u, u^{-1} \in M(A)$ such that $T(a) = u^{-1}\pi(a)$ for all $a \in A$.

Proof. In the case that A and B are unital C^*-algebras this is probably well known (cf. [39]). We will sketch a quick proof of this case for the readers interest, using Theorem A.5. In this case, T is an imprimitivity bimodule isomorphism or triple isomorphism. If $T(1) = u$, and $T(v) = 1$, then

$$u^*u = T(1)^*\, T(1) = T(v)\, T(1)^*\, T(1) = T(v.1.1) = 1.$$

Similarly $u^*u = 1$. So u is unitary. Then $\theta(\cdot) = u^{-1}T(\cdot)$ is a unital isometric homomorphism, since $u^{-1}T(x)\, u^{-1}T(y) = u^{-1}T(x.1.y)$, for $x, y \in A$. It is therefore a *-homomorphism (or this may be proved directly from the definition of triple morphism: $\theta(x^*) = u^{-1}T(1 \cdot x^* \cdot 1) = u^{-1}uT(x)^*\, u = (\theta(x))^*$).

In the more general case of operator algebras with c.a.i., extend T to a completely isometric linear surjection $I(A) \to I(B)$ (which is clearly possible by injectivity and rigidity). Then by the first part, there exists a faithful *-isomorphism $\pi: I(A) \to I(B)$ and a unitary $u \in I(B)$ such that $T(a) = u\pi(a)$ for all $a \in A$. Inside $I(B)$, we have that $u^{-1}B = \pi(A)$ is an operator algebra with c.a.i. We have $u^{-1}Bu^{-1}B = u^{-1}B$ by Cohen's factorization theorem. Thus $Bu^{-1}B = B$. Also, $u^{-1}BB = u^{-1}B$, by Cohen's theorem again. Thus by [15, Theorem 4.15], $u^{-1}B = B$. Thus $uB = B$ and $\pi(A) = B$. Hence $u, u^{-1} \in LM(B)$ by the characterization of this space above the statement of B.1. A similar argument applied to Bu^{-1} shows that $u, u^{-1} \in RM(B)$. So $u, u^{-1} \in M(B)$. ∎

The last result may be given a slightly more elementary proof by using A^{**} instead of $I(A)$ (modify the first paragraph of the proof so that it works for unital operator algebras, and modify the second paragraph by replacing $I(\cdot)$ with the second dual. Thus one avoids results from [17]). In any case, the result is new as far as I am aware in this generality (although such theorems have been explored since [2] as a nice application of the noncommutative Shilov theory).

We now mention some recent observations which the reader may find interesting. First, concerning left adjointable maps on operator spaces, it is interesting to know which of the classical results for adjointable maps on C^*-modules go through for operator spaces. Strikingly, of the four sections of Chapter 15 of [63] devoted to the basic theory of C^*-modules, almost all of the results in the first three sections concerning adjointable maps go through with the same proofs! In particular, Section 15.3 concerning the polar decomposition of adjointable maps is valid. Here is a sample result:

THEOREM B.2. *Suppose that T is a left adjointable map on an operator space X, with $T(X)$ closed in X. Then T has a polar decomposition $T = V\,|T|$ for a left adjointable partial isometry V satisfying* $\ker(V) = \ker(T)$, $\ker(V^*) = \ker(T^*)$, $V(X) = T(X)$, *and* $V^*(X) = T^*(X)$.

As corollaries of this we can show, for example, that if $T \in \mathscr{A}_l(X)$ then T is invertible in $\mathscr{A}_l(X)$ iff T is 1–1 and surjective and iff T and T^* are bounded away from 0. In this case $(T^{-1})^* = (T^*)^{-1}$. Also, a linear surjective adjointable isometry $T: X \to X$ is a unitary in $\mathscr{A}_l(X)$.

Here are some sample observations relating to the question of the closure of the classes of adjointable maps and left multipliers, with respect to some canonical constructions. For example these classes are closed w.r.t. the spatial tensor product of maps or the $\oplus^\infty$ sum of maps. Also, T is a left multiplier iff T^{**} is. The same is not quite true for adjointable maps (one direction fails without an extra hypothesis). All of these assertions follow fairly easily from 4.11 and 4.10.

Finally, the reader is directed to a forthcoming work of the author with Effros and Zarikian [69] where we lay the foundations of a 1-sided M-structure theory. This work has very strong connections to topics studied here; in particular we find there a deeper characterization of multipliers and adjointable maps. For example, $T \in \mathrm{Ball}(\mathscr{M}_l(X))$ if and only if $T \oplus Id$ is completely contractive $C_2(X) \to C_2(X)$. (Paulsen has subsequently found a simpler proof.) Note that this result immediately gives our oplication theorem 5.2, and BRS, and many other consequences. For example, using this, the author was able to find [68] weak*-versions of most of the results in Section 5, including a characterization of the σ-weakly closed unital subalgebras of $B(H)$ as exactly the operator algebras (with c.a.i.) having an operator space predual.

These newer ways from [17, 69] of looking at the multiplier algebras give cleaner approaches to some results in the present paper.

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