Using Recurrence Relations to Count Certain Elements in Symmetric Groups

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We use the fact that certain cosets of the stabilizer of points are pairwise conjugate in a symmetric group S_n in order to construct recurrence relations for enumerating certain subsets of S_n. Occasionally one can find 'closed form' solutions to such recurrence relations. For example, the probability that a random element of S_n has no cycle of length divisible by q is $\prod_{d=1}^{\lfloor n/q \rfloor} (1 - \frac{1}{dq})$.

1. INTRODUCTION

Let S_n denote the symmetric group of degree n. If $\Sigma \subseteq S_n$, then let $\mathcal{OM}_q(\Sigma)$, $\mathcal{OD}_q(\Sigma)$, $\text{Œ}_q(\Sigma)$ denote the number of elements in Σ having order: a multiple of q, dividing q, and equal to q, respectively. Similarly, let $\mathcal{CM}_q(\Sigma)$, $\mathcal{CD}_q(\Sigma)$, $\mathcal{CE}_q(\Sigma)$ denote the number of elements in Σ having a cycle (in its disjoint cycle decomposition) of length: a multiple of q, dividing q, and equal to q, respectively. It is not hard to write down recurrence relations satisfied by $\mathcal{OM}_q(C_{n,k}), \ldots, \mathcal{CE}_q(C_{n,k})$ where $C_{n,k}$ is a certain coset of a stabilizer of $k-1$ points. Given a function N, denote by $\overline{N}$ the function defined by $\overline{N}(\Sigma) = |\Sigma| - N(\Sigma)$. We shall give a 'closed form' solution to the recurrence relation for the number, $\overline{\mathcal{CM}_q}(C_{n,k})$, of elements in $C_{n,k}$ having n cycles of length divisible by q.

Asymptotic properties of the order and cycle decomposition, of a random element of the symmetric group, S_n, were studied by Erdös and Turán in a series of seven papers entitled 'On some problems in a statistical group theory' published between 1956 and 1972. It is shown in [2] that the distribution X_n of $\log |\tau|$, where τ is a uniformly random element of S_n, approaches (as $n \to \infty$) the normal distribution $N(\mu, \sigma^2)$ where $\mu = \frac{1}{2} \log^2 n$ and $\sigma^2 = \frac{1}{3} \log^3 n$. The expected order, E_n, of a uniformly random element of S_n was shown by Goh and Schmutz [3] to satisfy $\log E_n \sim O(\sqrt{n/\log n})$ as $n \to \infty$. This is substantially smaller than the maximal order, M_n, of an element of S_n as $\log M_n \sim \sqrt{n \log n}$ as $n \to \infty$ (see [5, p. 222]).

The number of $x \in S_n$ satisfying $x^q = 1$ is $\mathcal{OD}_q(S_n)$. Wilf [9] showed for fixed q that $\mathcal{OD}_q(S_n)/n! \sim g_q(n)$ where $g_q(n)$ is a given function of q and n. In a similar vein, Pavlov [8] showed that certain random variables associated with cycle structure are asymptotically normal when $n \to \infty$.

One can show that the probability that an element of S_n has no cycles of lengths $a_1, \ldots, a_m$ is at most $(\sum_{k=1}^{m} a_k^{-1})^{-1}$ (see [1, Theorem VI]). To estimate $\overline{\mathcal{CM}_q}(S_n)$, take $a_k = qk$ and $m = \lfloor n/q \rfloor$. Comparisons with integrals show that $q(1 + \log m)^{-1} \le (\sum_{k=1}^{m} (qk)^{-1})^{-1} \le q(\log(m+1))^{-1}$. This is unhelpful if $1 + \log m \le q$ (as probabilities are always ≤ 1).

The motivation for this work arose from the following problem in probabilistic group theory. Given $\varepsilon > 0$ and a group G isomorphic to precisely one of the groups $G_1, G_2, \ldots$, then (when possible) determine with probability $\ge 1 - \varepsilon$ whether G is isomorphic, or is not isomorphic, to G_k after testing the order of $N(\varepsilon, G)$ randomly chosen elements of G. This problem seems most likely to be successful if the sequence $G_1, G_2, \ldots$, comprises groups that are finite and simple (or with few composition factors). For such groups the set of orders of elements of G frequently characterizes G (see, for example, [6]). The task is clearly impossible if different groups G_k and G_ℓ have the same proportions of elements of each order. It follows from the

'law of large numbers' [7] and the above result of Erdös and Turán that this task is possible if G is a symmetric group and $G_k = S_k$ for all k. By using additional information we can give a smaller value of $N(\varepsilon, G)$. Thus it is important to be able to quickly calculate *actual* values of $\mathcal{OM}_q(S_n)$, $\mathcal{OD}_q(S_n)$, etc and not merely *asymptotic* approximations as $n \to \infty$.

If q is a prime power, then [1, Lemma I] can be interpreted as giving a formula for the number $\overline{\mathcal{OM}}_q(S_n)$ of elements of S_n whose order is not a multiple of q. Note that $\overline{\mathcal{OM}}_q(\Sigma) \leq \overline{\mathcal{CM}}_q(\Sigma)$, and equality holds if q is a prime power. We shall give a more general formula in the next section for $\overline{\mathcal{CM}}_q(C_{n,k})$ which specializes when $k = 1$ to $\overline{\mathcal{CM}}_q(S_n) = \prod_{j=1}^{n} (j - [q \mid j])$ where $[q \mid j]$ equals 1 if q divides j, and 0 otherwise. If P is a logical proposition, then $[P]$ denotes 1 if P is true, and 0 otherwise. This notation, attributed to Iverson [4, p. 24], is useful for reducing a collection of formulas involving different cases, to one formula.

2. Recurrence Relations

The symmetric group S_n acts naturally on the set $\{1, \ldots, n\}$. If $k \in \{0, \ldots, n\}$, let $G_{n,k}$ denote the subgroup of S_n that fixes each of $1, 2, \ldots, k$. If $k \in \{1, \ldots, n\}$, let $C_{n,k}$ denote the coset $G_{n,k-1}(1, 2, \ldots, k)$. Note that $G_{n,k}$ is permutationally isomorphic to S_{n-k}. Furthermore $C_{n,1} = S_n$ and $C_{n,n} = \{(1, 2, \ldots, n)\}$.

The six recurrence relations below use the ordering: $(n', k') < (n, k)$ if and only if $n' < n$, or $n' = n$ and $k' > k$.

LEMMA 1. *Let q, n, k be positive integers where $k \leq n$. Let $q = q_1, \ldots, q_r$ where $q_1, \ldots, q_r$ are powers of distinct primes. Let $\Delta(q, k) = \prod_{j=1}^{r} q_j^{[q_j \nmid k]}$ and $\triangledown(q, k) = \prod_{j=1}^{r} q_j^{[q_j \mid k]}$. If $k < n$, then*

$$\overline{\mathcal{OM}}_q(C_{n,k}) = \overline{\mathcal{OM}}_{\Delta(q,k)}(C_{n-k,1}) + (n-k)\overline{\mathcal{OM}}_q(C_{n,k+1}), \tag{1}$$

$$\mathcal{OD}_q(C_{n,k}) = [k \mid q]\mathcal{OD}_q(C_{n-k,1}) + (n-k)\mathcal{OD}_q(C_{n,k+1}), \tag{2}$$

$$\overline{\mathcal{OE}}_q(C_{n,k}) = \sum_{d \,\mid\, \triangledown(q,k)} \overline{\text{Œ}}_{d\Delta(q,k)}(C_{n-k,1}) + (n-k)\overline{\mathcal{OE}}_q(C_{n,k+1}), \tag{3}$$

$$\overline{\mathcal{CM}}_q(C_{n,k}) = [q \nmid k]\overline{\mathcal{CM}}_q(C_{n-k,1}) + (n-k)\overline{\mathcal{CM}}_q(C_{n,k+1}), \tag{4}$$

$$\overline{\mathcal{CD}}_q(C_{n,k}) = [k \nmid q]\overline{\mathcal{CD}}_q(C_{n-k,1}) + (n-k)\overline{\mathcal{CD}}_q(C_{n,k+1}), \tag{5}$$

$$\overline{\mathcal{CE}}_q(C_{n,k}) = [q \neq k]\overline{\mathcal{CE}}_q(C_{n-k,1}) + (n-k)\overline{\mathcal{CE}}_q(C_{n,k+1}), \tag{6}$$

where the respective initial conditions are

$$\overline{\mathcal{OM}}_q(C_{n,n}) = [q \nmid n], \qquad \mathcal{OD}_q(C_{n,n}) = [n \mid q], \qquad \overline{\mathcal{OE}}_q(C_{n,n}) = [q \neq n],$$

$$\overline{\mathcal{CM}}_q(C_{n,n}) = [q \nmid n], \qquad \overline{\mathcal{CD}}_q(C_{n,n}) = [n \nmid q], \qquad \overline{\mathcal{CE}}_q(C_{n,n}) = [q \neq n].$$

PROOF. The initial conditions are easily verified. Suppose now that $k < n$, and consider the coset decomposition

$$G_{n,k-1} = G_{n,k} \cup \bigcup_{\ell > k} G_{n,k}(k, \ell).$$

Post-multiplying by $(1, \ldots, k)$ gives

$$C_{n,k} = G_{n,k}(1, 2, \ldots, k) \cup \bigcup_{\ell > k} G_{n,k}(1, 2, \ldots, k, \ell). \tag{7}$$

Note that the set of elements moved by $a \in G_{n,k}$ (i.e., the support of a) is disjoint from the support of $b = (1, \ldots, k)$. Also, if $\ell > k$, then $G_{n,k}(1, 2, \ldots, k, \ell)$ is the conjugate of $C_{n,k+1}$

by $(k + 1, \ell)$. Now ab has no cycle of length a multiple of q if and only if k is not a multiple of q, and a has no cycle of length a multiple of q. That is,

$$\overline{\mathcal{CM}_q}(G_{n,k}(1, 2, \ldots, k)) = [q \nmid k]\overline{\mathcal{CM}_q}(G_{n,k}) = [q \nmid k]\overline{\mathcal{CM}_q}(C_{n-k,1}).$$

It follows from Eqn. (7) that

$$\overline{\mathcal{CM}_q}(C_{n,k}) = [q \nmid k]\overline{\mathcal{CM}_q}(C_{n-k,1}) + (n - k)\overline{\mathcal{CM}_q}(C_{n,k+1}).$$

The recurrence relations (5) and (6) are derived similarly, and the recurrence relations (1)–(3) can be easily derived from the facts below. Note that the order of ab satisfies $|ab| =$ lcm$(|a|, |b|)$. Hence (1) $q \nmid |ab|$ if and only if $\Delta(q, k) \nmid |a|$; (2) $|ab| \mid q$ if and only if $|a| \mid q$ and $k \mid q$; and (3) $|ab| = q$ if and only if $|a| = d\Delta(q, k)$ where $d \mid \nabla(q, k)$. $\square$

The recurrence relations for the complementary numbers $\mathcal{OM}_q(C_{n,k})$, $\overline{\mathcal{OD}_q}(C_{n,k})$ etc, can be determined from determined from those above using the fact that $\overline{N}(\Sigma) = |\Sigma| - N(\Sigma)$. We shall give a surprising 'closed form' solution to the recurrence relation for $\overline{\mathcal{CM}_q}(C_{n,k})$. Let $n \bmod q$ be the unique integer r satisfying $n \equiv r \pmod q$ and $0 \le r < n$. The 'mod' function is notorious for not preserving order, so the formula for $\overline{\mathcal{CM}_q}(C_{n,k})$ below is curious as it involves both '$\le$' and 'mod'.

THEOREM 2. *If q, n, k are positive integers and $1 \le k \le n$, then*

$$\overline{\mathcal{CM}_q}(C_{n,k}) = f_q(n - k + 1) - [(-k) \bmod q \le s]f_q(n - k) \tag{8}$$

where $f_q(n) = \prod_{j=1}^{n}(j - [q \mid j])$ and $s = q - 2 - (n \bmod q)$. In particular, $\overline{\mathcal{CM}_q}(S_n) = f_q(n)$.

PROOF. The result is trivially true when $q = 1$. Assume henceforth that $q > 1$. We use induction on (n, k) ordered via $(n', k') < (n, k)$ when $n' < n$, or $n' = n$ and $k' > k$. Consider formula (8) when $k = n$. By Lemma 1, $\overline{\mathcal{CM}_q}(C_{n,n}) = [q \nmid n]$. Since

$$(-n) \bmod q = [q \nmid n]q - (n \bmod q),$$

it follows that

$$[(-n) \bmod q \le q - 2 - (n \bmod q)] = [[q \nmid n]q \le q - 2] = [q \mid n].$$

The right-hand side of (8) is $f_q(1) - [q \mid n]f_q(0) = 1 - [q \mid n] = [q \nmid n]$, and so (8) is true when $k = n$. Assume now that (8) is true for $(n', k') < (n, k)$ where $1 \le k < n$. Thus

$$\overline{\mathcal{CM}_q}(S_{n-k}) = \overline{\mathcal{CM}_q}(C_{n-k,1}) = f_q(n - k)$$

as $[q - 1 \le q - 2 - (n \bmod q)] = 0$.

Observe that $[(-k) \bmod q \le s] = \sum_{j=0}^{s}[q \mid k + j]$ where $[q \mid (k + j)]$ is abbreviated $[q \mid k + j]$. When $n \bmod q = q - 1$, then $s = -1$ and both sides are zero. (A sum $\sum_{j=0}^{-1} a_j$ is zero by convention.) Suppose that $n \bmod q < q - 1$. Then at most one summand $[q \mid k + j]$ is non-zero, and the equation $[q \mid k + j] = 1$ is equivalent to the equation $(-k) \bmod q = j$. Hence $\sum_{j=0}^{s}[q \mid k + j] = [(-k) \bmod q \le s]$, as required.

We shall now prove that

$$\overline{\mathcal{CM}_q}(C_{n,k}) = f_q(n - k + 1) - f_q(n - k)\sum_{j=0}^{s}[q \mid k + j].$$

We shorten this equation to $\overline{\mathcal{CM}_q}(C_{n,k}) = F_{k-1} - F_k \sum_{j=0}^{s}[q \mid k + j]$. The first equality below is justified by Eqn. (4), and the second follows from the inductive hypothesis

$$\overline{\mathcal{CM}_q}(C_{n,k}) = [q \nmid k]\overline{\mathcal{CM}_q}(C_{n-k,1}) + (n - k)\overline{\mathcal{CM}_q}(C_{n,k+1})$$

$$= [q \nmid k]F_k + (n - k)\left\{ F_k - F_{k+1} \sum_{j=0}^{s} [q \mid k + 1 + j] \right\}$$

$$= (n - k + 1)F_k - [q \mid k]F_k - (n - k)F_{k+1} \sum_{j=0}^{s} [q \mid k + 1 + j]$$

$$= (n - k + 1)F_k - [q \mid k]F_k - (n - k)F_{k+1} \sum_{j=1}^{s+1} [q \mid k + j].$$

The last step involved a change in summation variable.

The equation $[q \mid n - k] \sum_{j=1}^{s+1}[q \mid k + j] = 0$ is helpful. This is clearly true when $[q \mid n - k] = 0$. If $[q \mid n - k] = 1$, then $k \equiv n \pmod{q}$ and so $[q \mid k + j] = [j = s + 2]$. Thus in either case the expression is zero. Using the equation $F_k = (n - k - [q \mid n - k])F_{k+1}$, therefore gives

$$\overline{\mathcal{CM}_q}(C_{n,k}) = (n - k + 1)F_k - [q \mid k]F_k - F_k \sum_{j=1}^{s+1} [q \mid k + j]$$

$$= (n - k + 1)F_k - F_k \sum_{j=0}^{s+1} [q \mid k + j]$$

$$= {}^*(n - k + 1 - [q \mid n - k + 1])F_k - F_k \sum_{j=0}^{s} [q \mid k + j]$$

$$= F_{k-1} - F_k \sum_{j=0}^{s} [q \mid k + j]$$

*where the second to last equality uses $[q \mid n - k + 1] = [q \mid k + s + 1]$ since $s \equiv -n - 2 \pmod{q}$. This completes the inductive proof. $\qquad\square$

3. ESTIMATIONS AND APPLICATIONS

The recurrence relations of Lemma 1 give algorithms which are quadratic in n for computing these numbers. As the conjugacy classes of S_n correspond bijectively to partitions of n, these numbers can be computed by summing over certain partitions. This gives rise to slower algorithms for computing these numbers. In practice, however, we need not compute all the significant digits of these numbers, usually the first four suffice. Good lower bounds may be found quickly by considering some of the large relevant conjugacy classes.

It is a simple (and somewhat surprising) consequence of Theorem 2 that the proportion, $p_{q,n}$, of elements of S_n having no cycles of length divisible by q is the same for $n = mq$, $mq + 1, \ldots, mq + q - 1$. Estimates for $p_{q,n} = \overline{\mathcal{CM}_q}(S_n)/n!$ are obtained below.

If $q = 1$, then $p_{q,n} = 0$. Assume henceforth that $q \geq 2$. Useful upper and lower bounds for

$p_{q,mq} = \prod_{k=1}^{m}(1 - (qk)^{-1})$ may be deduced from

$$\left| \sum_{k=1}^{m} \log(1 - (qk)^{-1}) + \sum_{k=1}^{m} (qk)^{-1} \right| \leq \sum_{k=1}^{m} \left| \log(1 - (qk)^{-1}) + (qk)^{-1} \right|$$

$$\leq \sum_{k=1}^{m} \sum_{i=2}^{\infty} \frac{(qk)^{-i}}{2} \leq \sum_{k=1}^{m} (qk)^{-2} < C$$

where $C = \sum_{k=1}^{\infty}(qk)^{-2} = q^{-2}\pi^2/6$. It follows from

$$-\frac{1}{q}(1 + \log m) - C \leq \sum_{k=1}^{m} \log(1 - (qk)^{-1}) \leq -\frac{1}{q}\log m + C$$

that $c_q^{-1}(em)^{-1/q} \leq \prod_{k=1}^{m}(1 - (qk)^{-1}) \leq c_q m^{-1/q}$ where $c_q = e^{q^{-2}\pi^2/6}$.

Recall that one motivation for computing the numbers $\overline{\mathcal{OM}}_q(S_n)$ etc, arose from probabilistic computational group theory. Suppose we are given a 'black box' group G which is known to be isomorphic to S_n for some n. How do we find n? The relative frequency of finding an element of G of odd order should be close to the probability $\overline{\mathcal{OM}}_2(S_k)/k!$ for precisely two values of k, say m and $m + 1$. If p is the smallest prime divisor of m or $m + 1$, then by determining the relative frequency of elements of G of order co-prime to p, one can determine, with quantifiable probability, whether n equals m of $m + 1$.

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