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The spectral radius of submatrices of Laplacian matrices for trees and its comparison to the fiedler vector ☆

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Abstract

We consider the effects on the spectral radius of submatrices of the Laplacian matrix for graphs by deleting the row and column corresponding to various vertices of the graph. We focus most of our attention on trees and determine which vertices v will yield the maximum and minimum spectral radius of the Laplacian when row v and column v are deleted. At this point, comparisons are made between these results and results concerning the Fiedler vector of the tree.

Keywords: Graph; Tree; Laplacian matrix; Spectral radius

1. Introduction and preliminaries

Consider an undirected weighted graph $\mathcal{G}$ on n vertices without loops or multiple edges, with vertices labelled $1, \ldots, n$. The *Laplacian matrix* associated with $\mathcal{G}$ is the $n \times n$ matrix $L = (\ell_{i,j})$ whose entries are given by

$$\ell_{i,j} = \begin{cases} -w, & \text{if } i \neq j \text{ and } i \text{ and } j \text{ are joined by and edge of weight } w > 0, \\ 0, & \text{if } i \neq j \text{ and } i \text{ is not adjacent to } j, \\ \sum_{j \neq i} |\ell_{ij}|, & \text{if } i = j. \end{cases}$$

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One of the basic results for Laplacian matrices is that for any graph $\mathscr{G}$, the corresponding Laplacian matrix is a positive semidefinite M-matrix. Hence the eigenvalues for L can be ordered as

$$0 = \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_{n-1} \leqslant \lambda_n. \tag{1.1}$$

Observe $\lambda_1 = 0$. It is known that $\lambda_2 = 0$ if and only if $\mathscr{G}$ is disconnected. Thus Merris [2] named λ_2 as the *algebraic connectivity of* $\mathscr{G}$. A good deal of attention has been devoted to the algebraic connectivity of graphs and to other sprectral properties of Laplacian matrices. For some initial results see Fiedler [2], and for an extensive survey on the Laplacian matrix see [10].

Given a tree $\mathscr{T}$ on n vertices and its Laplacian matrix L, let $L(v)$ be the $(n-1) \times (n-1)$ submatrix of L obtained by deleting the row and column of L corresponding to the vertex $v \in \mathscr{T}$. In this paper, we will investigate the properties of the spectral radius, $r(v)$, of these matrices for various vertices v. In Section 2, we investigate upper and lower bounds for $r(v)$ on trees and determine which vertices yield the maximum and minimum values for $r(v)$. We then observe how the value of $r(v)$ changes as we traverse various paths in $\mathscr{T}$. We prove a theorem (Theorem 2.10) which is surprisingly similar to the following well-known theorem from Fiedler [4]:

Theorem 1.1. *Let $\mathscr{T}$ be a weighted tree on n vertices, labelled* $1, \ldots, n$, *with Laplacian matrix L and algebraic connectivity μ. Let y be an eigenvector of L associated with μ. Then exactly one of the following cases occurs*:

(a) *Some entry of y is* 0. *In this case the subgraph of $\mathscr{T}$ induced by the set of vertices corresponding to the zero entries in y is connected. Moreover, there is a unique vertex k such that $y_k = 0$ and k is adjacent to a vertex m with $y_m \neq 0$. The entries of y are either increasing, decreasing, or identically* 0 *along any path in $\mathscr{T}$ which starts at k.*

(b) *No entry of y is* 0. *In this case there is a unique pair of vertices i and j such that i and j are adjacent in $\mathscr{T}$ with $y_i > 0$ and $y_j < 0$. Furthermore, the entries of y are increasing along any path in $\mathscr{T}$ which starts at i and does not contain j, while the entries of y are decreasing along any path in $\mathscr{T}$ which starts at j and does not contain i.*

An eigenvector y associated with μ is known as a *Fiedler vector*. Note that Theorem 1.1 discusses how the corresponding entries in a Fiedler vector change as we traverse various paths in $\mathscr{T}$. We will find similar results for the function $r(v)$.

Observe that Theorem 1.1 can be used to classify trees into two disjoint categories. If condition (a) of Theorem 1.1 holds, the tree $\mathscr{T}$ is of *Type I* and k is the *characteristic vertex* of $\mathscr{T}$. Likewise, if condition (b) holds, $\mathscr{T}$ is of *Type II* and i and j are the *characteristic vertices* of $\mathscr{T}$. (This terminology was suggested by Merris [9].) We will continue Section 2 by using the results concerning the function $r(v)$ to classify trees as Type A or Type B similarly to the way Merris classified trees as Type I or Type II. Since $L(v)$ is permutationally similar to a block diagonal matrix, this will be done by looking more closely at each vertex $v \in \mathscr{T}$ and determining which block(s) of $L(v)$ yields the value for $r(v)$.

Looking at the blocks of $L(v)$ will naturally motivate us to compare our results concerning the function $r(v)$ with results concerning *bottleneck matrices*. In a weighted tree $\mathscr{T}$, the bottleneck matrix at vertex $v \in \mathscr{T}$ is the $(n-1) \times (n-1)$ matrix $M := (L(v))^{-1}$. According to [8]

$$m_{i,j} = \sum_{e \in E} \frac{1}{w(e)}, \tag{1.2}$$

where E is the set of all edges e that lie simultaneously on the path from i to v and on the path from j to v, and $w(e)$ is the weight of edge e. The branch(es) at v corresponding to the block(s) of M with the largest spectral radius is known as the *Perron branch(es)* at v. By using the blocks of $L(v)$ for various vertices v to aid us in classifying trees according to $r(v)$, we will prove a theorem (Theorem 2.11) that is strikingly similar to the following theorem from [8]:

Theorem 1.2. *Let $\mathscr{T}$ be a weighted tree. Then*

(a) *$\mathscr{T}$ is a tree of Type I if and only if there is a unique vertex (namely the characteristic vertex) at which there are two or more Perron branches.*

(b) *$\mathscr{T}$ is a tree of Type II if and only if $\mathscr{T}$ has a unique pair of adjacent vertices i and j (namely the characteristic vertices) such that the branch at vertex i containing vertex j is the unique Perron branch at i, while the branch at vertex j containing vertex i is the unique Perron branch at j.*

(c) *If m is not a characteristic vertex of $\mathscr{T}$, then the unique Perron branch at m is the branch which contains the characteristic vertex (or vertices) of T.*

Finally, we end Section 2 by recalling Theorem 6 from [8] which states that the entries in a Fiedler vector corresponding to the vertices in an unweighted tree increase concave downward (in absolute value) as one travels away from the characteristic vertices of a tree. We answer the natural question of whether a similar result holds for the function $r(v)$.

In Section 3 of this paper, we apply the results from Section 2 to certain trees and compare their characteristic vertices with what we call their spectral vertices. We also compare our classification of trees with that of Merris and summarize these results here. Finally, in Section 4, we revisit Theorem 1.1 by studying graphs which are not trees.

2. Upper bounds and lower bounds for $r(v)$

Let L be the Laplacian matrix for a weighted graph on n vertices. If the eigenvalues of L are ordered as in (1.1) then it is known by an interlacing theorem of eigenvalues (see [6]) that

$$\lambda_{n-1} \leqslant r(v) \leqslant \lambda_n \tag{2.1}$$

for each vertex $v \in \mathscr{G}$. We begin by further investigating (2.1) for trees. Observe that this inequality has two components: $\lambda_{n-1} \leqslant r(v)$ and $r(v) \leqslant \lambda_n$. We begin our investigation by focusing on the latter component. To do this, we need the following matrix theoretic result which is based on Proposition 1 in [3]:

Theorem 2.1. *Let L be the Laplacian matrix for a weighted tree $\mathscr{T}$ on n vertices. If all entries of an eigenvector of L are different from zero, then the corresponding eigenvalue λ is simple. Moreover, all submatrices of order $n-1$ of the matrix $L-\lambda I$ are nonsingular.*

Therefore, we see that $r(v)$ can equal λ_n only if an eigenvector corresponding to λ_n has a zero entry. To see that this can never happen in the Laplacian matrix for a tree, we recall that all nontrivial trees are bipartite and we begin with the following definition:

Definition 2.2. Let $\mathscr{G}$ be a bipartite graph and L be its the Laplacian matrix. Let B be the matrix created from L by taking the absolute value of each entry. Then B is the *bipartite complement of L*.

The following lemma which is based on Proposition 2.2 in [5] gives a important relation between L and its bipartite complement B when $\mathscr{G}$ is a bipartite graph. If A is an $n \times n$ matrix and $S \subset \{1, \ldots, n\}$, then $A(S)$ is the matrix created from A by deleting the rows and columns corresponding to the elements in S.

Lemma 2.3. *Let $\mathscr{G}$ be a connected bipartite graph on n vertices. Let L be its Laplacian matrix and B be its bipartite complement. If S is a subset of $\{1, 2, \ldots, n\}$, then $L(S)$ and $B(S)$ are unitarily similar.*

Proof. Since $\mathscr{G}$ is bipartite, we can partition the vertices of $\mathscr{G}$ into two subsets V_1 and V_2 so that no two vertices in V_i are adjacent for $i = 1, 2$. Let U be the diagonal matrix in which the ith diagonal entry, u_{ii}, is as follows:

$$u_{ii} = \begin{cases} 1, & \text{if } v_i \in V_1, \\ -1, & \text{if } v_i \in V_2. \end{cases}$$

Observe that $U(S)L(S)(U(S))^{-1} = B(S)$. Thus $L(S)$ and $B(S)$ are unitarily similar. □

Since all trees are bipartite, it follows that for any subset S of $\{1, 2, \ldots, n\}$ and for any tree $\mathscr{T}$ that $L(S)$ and $B(S)$ are unitarily similar. Note that since we are allowing for the possibility of S to be empty, we should note the possibility of $L(S)$ and $B(S)$ equalling L and B, respectively. Hence Lemma 2.3 shows that L and B are unitarily similar. Recall that similar matrices have the same eigenvalues. This fact leads us to the following corollary:

Corollary 2.4. *Let S be a fixed subset of $\{1, 2, \ldots, n\}$ and let x be an eigenvector for $L(S)$ corresponding to the eigenvalue λ. Then $U(S)x$ is an eigenvector for $B(S)$ corresponding to λ. Moreover, for each entry x_i in x, $x_i = 0$ if and only if $(U(S)x)_i = 0$.*

Proof. Since $L(S)$ and $B(S)$ are unitarily similar, we can write

$$U(S)L(S) = B(S)U(S).$$

Hence

$$U(S)L(S)x = B(S)U(S)x.$$

Since x is an eigenvector of $L(S)$ corresponding to λ, it follows that

$$\lambda U(S)x = B(S)U(S)x.$$

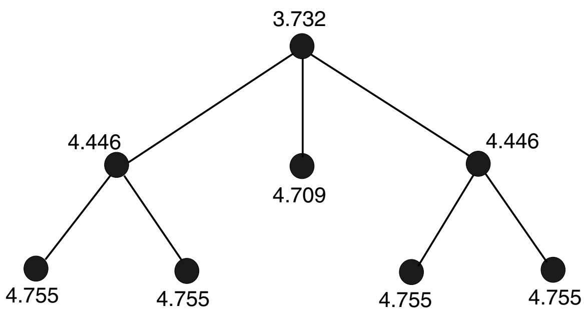
Thus $U(S)x$ is an eigenvector for $B(S)$ corresponding to λ. By the construction of $U(S)$, it is clear that $x_i = 0$ if and only if $(U(S)x)_i = 0$. □

Corollary 2.4 leads us to our main result concerning the relationship between $r(v)$ and λ_n, showing that the inequality is actually strict:

Theorem 2.5. *Let $\mathscr{T}$ be a weighted tree and L be it's Laplacian matrix. Then $r(v) < \lambda_n$ for all vertices $v \in \mathscr{T}$.*

Proof. Let x be an eigenvector of L corresponing to λ_n. Taking S to be empty, we see that since λ_n is the largest eigenvalue of L, then by Lemma 2.3 it must also be the largest eigenvalue for the bipartite complement B with Ux as its corresponding eigenvector. Observe that B is a nonnegative, irreducible matrix. So by Perron–Frobenius Theory (see [1]), it follows that Ux does not have a zero entry. Hence, by Corollary 2.4, x does not have a zero entry. The conclusion now follows from Theorem 2.1. □

We now focus our attention on the vertices v of $\mathscr{T}$ which yield a minimum value for $r(v)$. Recall from (2.1) that $r(v) \geqslant \lambda_{n-1}$. Unlike λ_n, we will see that there do exist trees in which this inequality is sharp. The following unweighted tree, labelling each vertex v with its value $r(v)$, is such an example:



We see that $\lambda_{n-1} = 3.732$ and there is a vertex in this tree with $r(v) = 3.732$. This leads us to the following definition:

Definition 2.6. A weighted tree is of *Type A* if there exists a vertex v such that $r(v) = \lambda_{n-1}$. A weighted tree is of *Type B* if no such vertex exists.

This definition is strikingly similar to Merris' classification of trees [9] into Type I and Type II which is based on the entries in the Fiedler vector. As with the theorems concerning Merris' classification of trees and bottleneck matrices, we will make use of the block diagonal structure of the submatrices of L (and B). Clearly, $L(v)$ is permutationally similar to a block diagonal matrix with each block corresponding to a branch at v. Thus for a vertex v, $r(v)$ attains its value from the block(s) in $L(v)$ with the maximum spectral radius. We will refer to this block(s) as the *spectral block(s) at* v and the corresponding branch(es) at the *spectral branch(es) at* v.

At this point, we investigate Type A trees and a vertex v such that $r(v) = \lambda_{n-1}$. We define this vertex as a *spectral vertex*. The following proposition gives useful properties of a spectral vertex:

Proposition 2.7. *Let v be a vertex in a weighted tree $\mathcal{T}$. Then v has more than one spectral branch if and only if $r(v) = \lambda_{n-1}$. Moreover, if such a vertex v exists, then it is unique.*

Proof. First suppose that $r(v) = \lambda_{n-1}$ and that, to the contrary, v has just one spectral branch. Without loss of generality, let $v = 1$. Per Lemma 2.3, we will prove this proposition using the bipartite complement, B. Permute B such that $v = 1$ and recall $B(1)$ is (permutationally similar to) a block diagonal with each block B_i corresponding to the branch b_i at $v = 1$. By Theorem 2.1, there exists an eigenvector x of B corresponding to λ_{n-1} such that the first entry is zero. Partition x in accordance with the blocks of B as follows:

$$x = \left[0 \mid y_1 \mid y_2 \mid \cdots \mid y_k\right],$$

where for each $i = 1, \ldots, k$, the vector y_i is a Perron vector of B_i. Suppose that $v = 1$ has only one spectral branch, say b_1. Since each block B_i is nonnnegative and irreducible and since $B_i y_i = \lambda_{n-1} y_i$ for each i, it follows from Perron–Frobenius Theory that y_1 has strictly positive entries while $y_i = 0$ for $2 \leqslant i \leqslant k$. Hence

$$x = \left[0 \mid y_1 \mid 0 \mid \cdots \mid 0\right].$$

Suppose y_1 has, say, p entries. Since the (1, 1) entry of B is positive and since there exists exactly one entry in the first $p + 1$ entries of the first row of B other than the (1, 1) entry that is positive, it follows by matrix multiplication that $(Bx)_1 > 0$ which contradicts the eigenvalue–eigenvector equation $Bx = \lambda_{n-1} x$. Hence v must have at least two spectral branches.

Now suppose v has $m \geqslant 2$ spectral branches for some $2 \leqslant m \leqslant k$, and show that $r(v) = \lambda_{n-1}$. Without loss of generality, let $v = 1$. Using the bipartite complement B, let $B_1, \ldots, B_m$ be the spectral blocks of B at $v = 1$. Consider the vector

$$x = \left[0 \mid y_1 \mid y_2 \mid \cdots \mid y_m \mid 0 \mid \cdots \mid 0\right],$$

where 0 is the first entry, y_i are the Perron vectors for the spectral blocks of B, and the remaining entries of x are all zero. Observe $B_i y_i = r(1) y_i$ for each i. Since $m \geqslant 2$, and since every entry in each subvector y_i is nonzero, it follows by matrix–vector multiplication that we can normalize $y_1, \ldots, y_m$ to create a vector $\hat{x}$ so that $B\hat{x} = r(1)\hat{x}$. Thus $r(1)$ is an eigenvalue of B. So by (2.1), Lemma 2.3, and Theorem 2.5, it follows that $r(1) = \lambda_{n-1}$.

To show uniqueness, let $v, w \in T$ be such that $v \neq w$ and $r(v) = r(w) = \lambda_{n-1}$. Then v and w must each have at least two spectral branches. Thus there exists a spectral branch at w that does not contain v. Let C be the block of $B(w)$ that corresponds to such a branch. If we let C_w be the block in $B(v)$ that contains the row/column corresponding to vertex w, then upon observing the fact that C is a proper submatrix of C_w and both matrices are nonnegative and irreducible, we obtain

$$r(v) \geqslant \rho(C_w) > \rho(C) = r(w) = \lambda_{n-1},$$

which contradicts the fact that $r(v) = \lambda_{n-1}$. Therefore, if there exists vertex such that $r(v) = \lambda_{n-1}$, then such a vertex is unique. $\square$

Thus, if a tree is of Type A, then the spectral vertex is the unique vertex that has more than one spectral branch. Observe that this definition is very similar in flavor to Merris' definiton of Type I trees (see Theorem 1.2). Therefore, it is natural to make an analogous definition for the spectral vertices for trees of Type B. In the following proposition, we show that there is such an analogous definition.

Proposition 2.8. *Let $\mathscr{T}$ be a weighted tree. Then there exists two adjacent vertices v and w such that a spectral branch at v contains w and a spectral branch at w contains v. Moreover, if $\mathscr{T}$ is of Type B, then this pair of vertices is unique.*

Proof. Consider the graph $\mathscr{G}$ with vertices $1, 2, \ldots, n$ created as follows: For each vertex i, place an edge $\{i, i'\}$ where i' is a neighbor of i that lies on a spectral branch at i. Since $\mathscr{T}$ has $n - 1$ edges and $\mathscr{G}$ has n edges, it follows that at least one of these edges is duplicated. The endpoints of a duplicated edge are such vertices v and w, hence showing existence.

To show uniqueness in the Type B case, let x and y be another pair of adjacent vertices such that the spectral branch at each of these vertices contains the other vertex. Without loss of generality, let the path from w to x contain no other spectral vertices. Let B be the bipartite complement of L. For distinct vertices i and j, let B_i^j be the block of $B(i)$ that contains the row/column corresponding to vertex j. Thus by the definition of spectral vertices and by the fact that B_i^j is irreducible for all $i, j \in \mathscr{T}$, we see that

$$\rho(B_w^v) > \rho(B_w^x) > \rho(B_x^y) > \rho(B_x^w) > \rho(B_w^v), \tag{2.2}$$

where the second and fourth inequalities follow from Perron-Frobenious theory. But (2.2) reduces to $\rho(B_w^v) > \rho(B_w^v)$, a contradiction. Hence in a Type B tree, such vertices v and w are unique. □

Therefore, in a Type B tree, we define the *spectral vertices* to be the unique pair of adjacent vertices v and w such that the spectral branch at v contains w and the spectral branch at w contains v. The inequalities in (2.2) of Proposition 2.8 imply the following corollary which concerns vertices that are not spectral vertices:

Corollary 2.9. *Let $\mathscr{T}$ be a weighted tree and suppose that m is not a spectral vertex of $\mathscr{T}$, then the unique spectral branch at m is the branch which contains the spectral vertex (or vertices) of $\mathscr{T}$.*

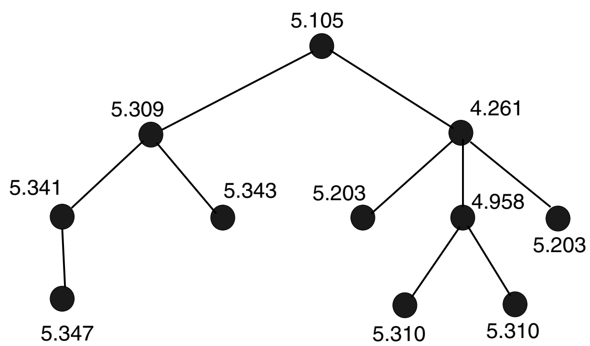
It should be noted that Corollary 2.9 applies to both trees of Type A and trees of Type B. At this point, we can summarize our results that parallel Theorem 1.1:

Theorem 2.10. *Let $\mathscr{T}$ be a weighted tree on n vertices labelled $1, \ldots, n$ with Laplacian matrix L. Then exactly one of the following occurs:*

(a) *The function $r(v) = \lambda_{n-1}$ for a unique vertex $k \in \mathscr{T}$. In addition, the value of the function $r(v)$ increases along any path in $\mathscr{T}$ which starts at k.*

(b) *The function $r(v)$ never attains λ_{n-1} for any vertex $v \in \mathscr{T}$. In this case, there exist a unique pair of adjacent vertices i and j such that the value of $r(v)$ increases along any path in $\mathscr{T}$ which starts at i and does not contain j, while the value of $r(v)$ increases along any path which starts at j and does not contain i.*

Theorem 2.10 implies that in any tree $\mathscr{T}$, the function $r(v)$ is minimized at a spectral vertex (and maximized at a pendant vertex). However, it is important to note that the two spectral vertices of a Type B tree are not necessarily the two vertices in which the function $r(v)$ obtains its two smallest values. Observe the following tree:



The vertices v with values for $r(v)$ of 4.261 and 4.958 are the two vertices with the minimum value for $r(v)$, yet the vertices with values 4.261 and 5.105 are the spectral vertices.

In Theorem 2.10, observe that if condition (a) holds, then $\mathscr{T}$ is a Type A tree with k as its spectral vertex, while if condition (b) holds then $\mathscr{T}$ is a Type B tree with i and j as its spectral vertices. At this point we summarize the results for Type A and Type B trees that parallel Theorem 1.2:

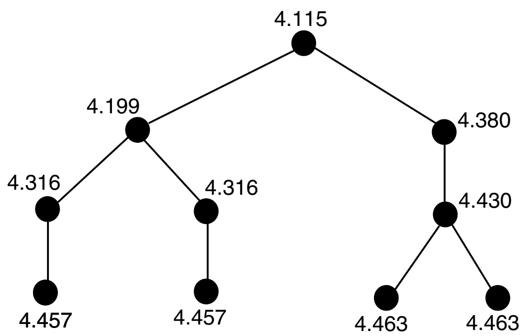
Theorem 2.11. *Let $\mathscr{T}$ be a weighted tree with L as its Laplacian matrix. Then*

(a) *If k is the unique spectral vertex of a Type A tree, then k is the unique vertex that has at least two spectral branches.*

(b) *If i and j are the spectral vertices of a Type B tree, then the unique spectral branch at i contains j while the unique spectral branch at j contains i.*

(c) *If m is not a spectral vertex in $\mathscr{T}$, the the unique spectral branch at m contains all of the spectral vertices of $\mathscr{T}$.*

We end this section by recalling that Kirkland, Neumann, and Shader proved in [8] that as one travels away from the characteristic vertex (vertices) of an unweighted tree, the corresponding entries in the Fiedler vector increase concave downward (in absolute value). So it is natural to end this section by asking if the same idea holds true for the function $r(v)$. While it is often the case that the same idea holds true, it is not always the case as seen below:



Observe that this is a Type A tree where the vertex with $r(v) = 4.115$ is the unique spectral vertex. However, the values of $r(v)$ for the vertices do not increase in a concave downward fashion as one travels away from the spectral vertex.

3. Comparing the characteristic and spectral vertices in certain trees

The definitions and theorems for Type A trees and Type I trees have a similar style in terms of special vertices, either characteristic or spectral. The same holds true for Type B and Type II trees. Therefore, it is natural to ask if there is a relationship between the spectral vertices and characteristic vertices of a tree. In this section, we give several answers to this question. We consider some common genre of trees:

We commence with the least difficult case—trees which are symmetric about a vertex or symmetric about an edge. We summarize these results in the following theorem:

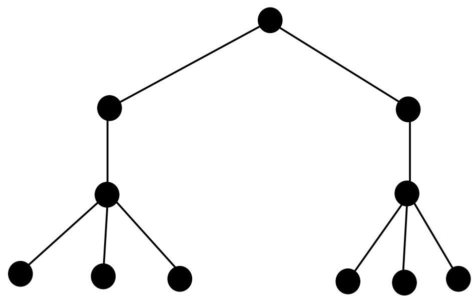
Theorem 3.1. *If $\mathscr{T}$ is a weighted tree that is symmetric about a vertex v, then $\mathscr{T}$ is both a Type I tree and a Type A tree with v as both the characteristic vertex and the spectral vertex. Moreover, if $\mathscr{T}$ is a weighted tree that is symmetric about an edge e which is incident to vertices v and w, then $\mathscr{T}$ is both a Type II tree and a Type B tree with v and w as both the characteristic vertices and spectral vertices.*

Proof. Let $\mathscr{T}$ be a tree that is symmetric about a vertex v and let L be its Laplacian. Consider the matrix $L(v)$. Observe that $L(v)$ is permutationally similar to a 2×2 block diagonal matrix in which the blocks are permutationally similar to each other. The first result follows immediately from Theorem 1.2 and Thereom 2.11. Similarly, if $\mathscr{T}$ is symmetric about the edge e which is incident to vertices v and w. Then the block of the matrix $L(v)$ which contains the row and column corresponding to w is permutationally similar to the block of the matrix $L(w)$ which contains the row and column corresponding to v. The second result now follows similarly. □

It is possible to create unweighted trees from a tree described in Theorem 3.1 by adding branches to v such that the $L(v)$ is a block diagonal matrix where the blocks associated with these additional branches at v are such that their spectral radii are less than or equal to the spectral radii of the original blocks, and the spectral radii of the inverses of these blocks are less than that of the spectral radii of the inverses of the orginal blocks. The tree that is provided preceeding Definition 2.6 in Section 2 is an example of such a tree this is both of Type I and Type A.

Continuing with our discussion of trees which are both Type I and Type A, observe that the example of the unweighted tree given at the end of Section 2 is of Type I and Type A, but that their spectral/Perron branches are not isomorphic. Therefore, at this juncture, it is natural to consider such trees. We now turn our attention to a class of Type I trees that was considered by Kirkland [7]. In [7], given positive integers $k_1, \dots, k_m$, the unweighted rooted tree $T(k_1, \dots, k_m)$ with root vertex r is defined inductively as follows: Let $T(k_1)$ be the star of $k_1 + 1$ vertices. To obtain $T(k_1, \dots, k_{j+1})$ from $T(k_1, \dots, k_j)$ when $j \geqslant 1$, take each pendant vertex

p of $T(k_1, \ldots, k_j)$ and add in k_{j+1} new pendant vertices, each adjacent to p. For example, the tree $T(2, 1, 3)$ is as follows:



Suppose we create a tree $\mathscr{T}$ by taking a vertex x and making it adjacent to the root vertices of $T(k_1, \ldots, k_m)$ and $T(k_m, \ldots, k_1)$. Then according to Corollary 2.1 of [7], T is a Type I tree with characteristic vertex x. We wish to show that T is also a Type A tree with spectral vertex x. We do this in the following theorem:

Theorem 3.2. *Suppose we create an unweighted tree $\mathscr{T}$ by taking a vertex x and making it adjacent to the root vertices of $T(k_1, \ldots, k_m)$ and $T(k_m, \ldots, k_1)$. Then $\mathscr{T}$ is a Type A tree with x as its spectral vertex.*

Proof. Let L be the Laplacian matrix for $\mathscr{T}$ and let B be its bipartite complement. We will prove this theorem using B as per Lemma 2.3. Observe that the matrix $B(x)$ is permutationally similar to a 2×2 block diagonal matrix with one block, say B_1, corresponding to $T(k_1, \ldots, k_m)$ and the other block, say B_2, corresponding to $T(k_m, \ldots, k_1)$. By Theorem 2.11, it suffices to show that $\rho(B_1) = \rho(B_2)$.

Observe $T(k_1, \ldots, k_m)$ has y vertices where

$$y = 1 + \sum_{i=1}^{m} \prod_{j=1}^{i} k_j.$$

Let $d(v)$ denote the distance from vertex v to vertex x. Label the vertices of $T(k_1, \ldots, k_m)$ with the positive integers $1, \ldots, y$ in such a way that $d(a) > d(b)$ implies that a gets a larger label than b. (Note that this forces the root vertex to be labelled 1, its descendents to be labelled $2, \ldots, k_1 + 1$, etc.) Permute the matrix B_1 according to this labelling. Label $T(k_m, \ldots, k_1)$ in a similar manner with the integers $1, \ldots, z$ (where z is the number of vertices in $T(k_m, \ldots, k_1)$) and permute the matrix B_2 accordingly. By doing such a labelling of the vertices, it becomes easier to observe that $\rho(B_1) = \rho(Q_1)$ and $\rho(B_2) = \rho(Q_2)$ where

$$Q_1 = \begin{bmatrix} k_1+1 & k_1 & & & & & \\ 1 & k_2+1 & k_2 & & & \mathbf{0} & \\ & 1 & k_3+1 & k_3 & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \\ & \mathbf{0} & & & 1 & k_m+1 & k_m \\ & & & & & 1 & 1 \end{bmatrix}$$

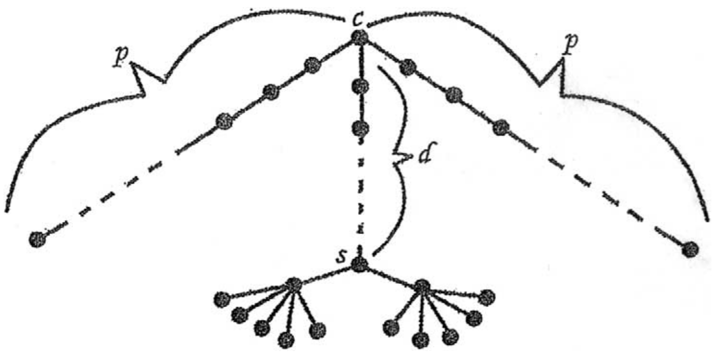
and

$$Q_2 = \begin{bmatrix} k_m+1 & k_m & & & & & \\ 1 & k_{m-1}+1 & k_{m-1} & & & \mathbf{0} & \\ & 1 & k_{m-2}+1 & k_{m-2} & & & \\ & & \cdots & \cdots & \cdots & & \\ & & & \cdots & \cdots & \cdots & \\ & \mathbf{0} & & & 1 & k_1+1 & k_1 \\ & & & & & 1 & 1 \end{bmatrix}.$$

However, Q_1 is similar to Q_2. Therefore

$$\rho(B_1) = \rho(Q_1) = \rho(Q_2) = \rho(B_2). \qquad \square$$

In the previous two theorems, we see frequently occurring trees that are both Type I and Type A. Observe that in both of these theorems that the characteristic vertex is the same as the spectral vertex. Therefore, it is tempting to ask: If a tree is both of Type I and Type A, then is the characteristic vertex necessarily the same as the spectral vertex? The answer to this question is "no." In fact, we can create a tree that is of both Type I and Type A in which the distance from the characteristic vertex to the spectral vertex is as large as one likes. We do so by considering the following unweighted tree:

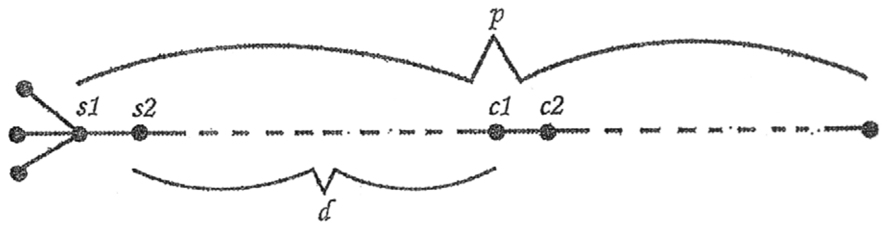


Observe that vertex s has two branches each consisting of one vertex adjacent to five vertices (other than s). Letting L be the Laplacian matrix for this tree, note that the spectral radius of the submatrix of L corresponding to each of these branches is 6.854. Observe that in the remaining branch at s, the spectral radius of the submatrix of L corresponding to this branch cannot be greater than 6 according to the Geršgorin Disc Theorem (see [6], p. 344). Hence s has two spectral branches and thus is the unique spectral vertex in accordance with Theorem 2.11. However, by the construction of bottleneck matrices (see (1.2)), as p increases, the spectral radius of each bottleneck matrix at vertex c containing each branch not containing s increases without bound. Hence, for each fixed d, we can make p large enough such that each of these branches are the characteristic branches at c. By Theorem 1.2, c would be the characteristic vertex. Thus we have created a tree that is of Type I and Type A where the distance between the spectral vertex and characteristic vertex is d.

Below, we see the minimum value that p must be in order for the distance between the characteristic vertex and spectral vertex to be d.

d	1	2	3	4	5
p	7	9	11	12	14

Similarly, if a tree is of Type II and Type B, we can create a tree in which the minimum distance between a characteristic vertex and a spectral vertex is as large as we would like. Consider the following unweighted tree below:



Observe that the spectral radius of the submatrix of the Laplacian matrix corresponding to the branch at s_1 containing the three pendant vertices is 1 while the spectral radius of the submatrix corresponding to the other branch is clearly larger than 1. Similarly, the spectral radius of the submatrix corresponding to the branch at s_2 containing the three pendant vertices is 4.792 while the submatrix corresponding to the other branch at s_2 is less than 4 by the Geršgorin Disc Theorem. Thus the unique spectral branch at s_1 contains s_2, and vice versa. Therefore, s_1 and s_2 are the spectral vertices of the tree in accordance with Proposition 2.8. Letting c_1 and c_2 be the characteristic vertices, it is clear by the construction of bottleneck matrices in accordance with (1.2), that as p increases without bound, the distance d between c_1 and s_2 increases without bound. Below, we see appropriate values of p for each given distance d between s_2 and c_1:

d	1	2	3	4	5
p	7	9	11	13	15

Finally, we close this section by showing that a Type I tree need not be a Type A tree. Similarly, a Type II tree need not be a Type B tree. Below we have examples to the contrary.

In Fig. 1, vertex 1 is the unique characteristic vertex, yet vertices 1 and 2 are the spectral vertices. Thus this tree is of Type I and Type B. In Fig. 2, vertices 1 and 2 are the characteristic vertices yet vertex 1 is the unique spectral vertex. Hence this tree is of Type II and of Type A. As with the Type I-A trees and the Type II-B trees, we can create trees in which the distance between the characteristic vertices and spectral vertices is as large as we desire. Since the process is similar to the above processes, we omit the details.

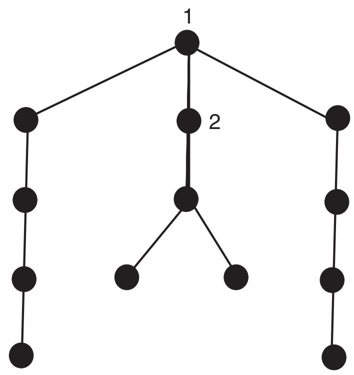


Fig. 1. A Type I and Type B tree.

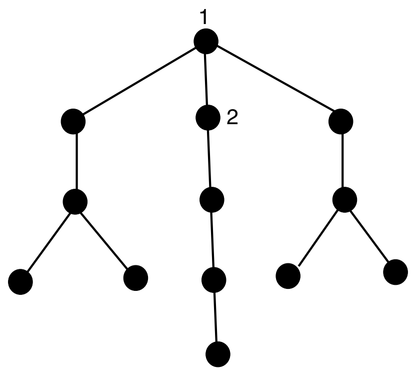


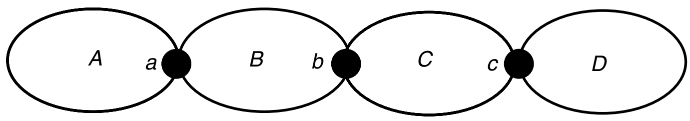
Fig. 2. A Type II and Type A tree.

4. Properties of $r(v)$ in graphs which are not trees

In this section, we consider the values for the function $r(v)$ at various vertices of graphs which are not trees. We do so in the context of comparing these results with the results in [4] that are a generalization of Theorem 1.1. Since not all graphs are bipartite, we are unable to use the bipartite complement of the Laplacian matrix as freely and as often as we did in Sections 2 and 3. Thus the proof of the main theorem of this section will be quite different:

Theorem 4.1. *Let $\mathscr{G}$ be a connected weighted graph with Laplacian matrix L. Let P be a path that begins at a cut point a in $\mathscr{G}$ in which the next cut point b on P is such that $r(b) > r(a)$. Then P has the property that the values of $r(v)$ at the cut points contained in P form an increasing sequence.*

Proof. Consider the representation of a graph $\mathscr{G}$ below:



and consider its corresponding Laplacian matrix:

$$L = \left[\begin{array}{c|c|c|c|c|c|c} A & x & 0 & 0 & 0 & 0 & 0 \\ \hline x^{\mathrm{T}} & a & y^{\mathrm{T}} & \alpha & 0 & 0 & 0 \\ \hline 0 & y & B & z & 0 & 0 & 0 \\ \hline 0 & \alpha & z^{\mathrm{T}} & b & w^{\mathrm{T}} & \beta & 0 \\ \hline 0 & 0 & 0 & w & C & v & 0 \\ \hline 0 & 0 & 0 & \beta & v^{\mathrm{T}} & c & u^{\mathrm{T}} \\ \hline 0 & 0 & 0 & 0 & 0 & u & D \end{array}\right].$$

Suppose that $r(b) > r(a)$. Then it suffices to show that $r(c) > r(b)$. For visual clarity, observe the matrices $L(a)$, $L(b)$, and $L(c)$ as follows:

$$L(a) = \left[\begin{array}{c|c|c|c|c|c} A & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & B & z & 0 & 0 & 0 \\ \hline 0 & z^{\mathrm{T}} & b & w^{\mathrm{T}} & \beta & 0 \\ \hline 0 & 0 & w & C & v & 0 \\ \hline 0 & 0 & \beta & v^{\mathrm{T}} & c & u^{\mathrm{T}} \\ \hline 0 & 0 & 0 & 0 & u & D \end{array}\right],$$

$$L(b) = \left[\begin{array}{c|c|c|c|c|c} A & x & 0 & 0 & 0 & 0 \\ \hline x^{\mathrm{T}} & a & y^{\mathrm{T}} & 0 & 0 & 0 \\ \hline 0 & y & B & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & C & v & 0 \\ \hline 0 & 0 & 0 & v^{\mathrm{T}} & c & u^{\mathrm{T}} \\ \hline 0 & 0 & 0 & 0 & u & D \end{array}\right],$$

$$L(c) = \left[\begin{array}{c|c|c|c|c|c} A & x & 0 & 0 & 0 & 0 \\ \hline x^{\mathrm{T}} & a & y^{\mathrm{T}} & \alpha & 0 & 0 \\ \hline 0 & y & B & z & 0 & 0 \\ \hline 0 & \alpha & z^{\mathrm{T}} & b & w^{\mathrm{T}} & 0 \\ \hline 0 & 0 & 0 & w & C & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & D \end{array}\right].$$

Each of these matrices can be partitioned into a 2×2 block matrix in which the off-diagonal blocks are zero. Let a_1 and a_2 be the $(1, 1)$ and $(2, 2)$ blocks of $L(a)$. Define b_1, b_2, c_1, and c_2 in a similar fashion for $L(b)$ and $L(c)$, respectively. Observe that $r(b) > r(a)$ implies that

$$r(b) = \max\{\rho(b_1), \rho(b_2)\} > \max\{\rho(a_1), \rho(a_2)\} = r(a).$$

At this point, we will show that $\rho(a_2) > \rho(b_2)$ which necessarily implies $r(b) = \rho(b_1)$. Let $\widehat{A}$ be the matrix created from a_2 by taking all entries which are also entries in b_2 and replacing them with zero. Let $\widehat{B}$ be the matrix that is the same size as A that created from b_2 by bordering the matrix with zeros that correspond to the non-b_2 entries of $\widehat{A}$. In other words:

$$\widehat{A} = \left[\begin{array}{c|c|c|c|c} B & z & 0 & 0 & 0 \\ \hline z^{\mathrm{T}} & b & w^{\mathrm{T}} & \beta & 0 \\ \hline 0 & w & 0 & 0 & 0 \\ \hline 0 & \beta & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and

$$\widehat{B} = \left[\begin{array}{c|c|c|c|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & C & v & 0 \\ \hline 0 & 0 & v^{\mathrm{T}} & c & u^{\mathrm{T}} \\ \hline 0 & 0 & 0 & u & D \end{array}\right].$$

Observe that $\lambda_1(\widehat{A}) = 0$ since the set of vectors e_i, where i is an entry corresponding to the rows/columns which correspond to the matrix D of $\widehat{B}$, forms a basis for the eigenspace of $\widehat{A}$ corresponding to the eigenvalue zero. (Recall that e_i is the vector with 1 in the i^{th} position and zeros in all other positions.) Using the Courant–Fischer Theorem (see [6]), we see that

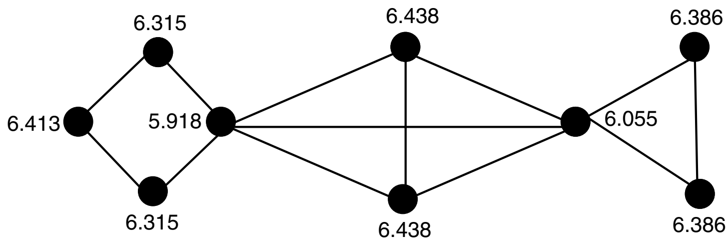
$$\begin{aligned}\rho(a_2) = \rho(\widehat{A} + \widehat{B}) &= \max_{x^{\mathrm{T}}x=1} [x^{\mathrm{T}}(\widehat{A} + \widehat{B})x] \\ &= \max_{x^{\mathrm{T}}x=1} [x^{\mathrm{T}}\widehat{A}x + x^{\mathrm{T}}\widehat{B}x] \\ &\geqslant \max_{x^{\mathrm{T}}x=1} [\lambda_1(\widehat{A}) + x^{\mathrm{T}}\widehat{B}x] \\ &= \max_{x^{\mathrm{T}}x=1} [x^{\mathrm{T}}\widehat{B}x] \\ &= \rho(\widehat{B}) \\ &= \rho(b_2).\end{aligned}$$

However, observe that the inequality in this case will be strict since no unit vector which is in the eigenspace of $\widehat{A}$ corresponding to $\lambda_1(\widehat{A}) = 0$ is an eigenvector of $\widehat{B}$. Thus $\rho(\widehat{A} + \widehat{B}) > \rho(\widehat{B})$ and hence $\rho(a_2) > \rho(b_2)$. Therefore $r(b) = \rho(b_1)$. By using a similar argument as above, $r(c) = \rho(c_1)$. Since b_1 is a proper submatrix of c_1, it follows that

$$r(c) = \rho(c_1) > \rho(b_1) = r(b). \qquad \square$$

We see from Theorem 4.1 that if we chose a cut vertex u such that the value of $r(u)$ is minimized over all cut vertices, then any path P that begins at u that contains cut vertices other than u will have the property that the values of $r(v)$ at the cutpoints in $v \in P$ will be an increasing sequence as P is traversed. This parallels the results in [4] that are a generalization of those listed in Theorem 1.1.

It is also known from [4] that for each block $B \in \mathscr{G}$, the entries of the Fiedler vector corresponding to the vertices in B that are not cutpoints in $\mathscr{G}$ are bounded above and below by the entries of the Fiedler vector corresponding to the of vertices in B that are cutpoints. So it is natural to ask if this idea holds true for the function $r(v)$. Not only does this idea turn out *not* to be true, we also see by the example given below that the vertex in which $r(v)$ is maximized or minimized need not be a cut point. The numbers next to each vertex v represent the value of $r(v)$.



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