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Reflective inductive inference of recursive functions

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Abstract

In this paper, we investigate reflective inductive inference of recursive functions. A reflective IIM is a learning machine that is additionally able to assess its own competence.

First, we formalize reflective learning from arbitrary, and from canonical, example sequences. Here, we arrive at four different types of reflection: reflection in the limit, optimistic, pessimistic and exact reflection.

Then, we compare the learning power of reflective IIMs with each other as well as with the one of standard IIMs for learning in the limit, for consistent learning of three different types, and for finite learning.

Keywords: Learning theory; Inductive inference; Learning in the limit; Reflection; Self-assessment

1. Motivation and introduction

Learning theory aims to model and investigate the scenario that a learner has to identify a target object from indirect information, i.e. examples, about it. Consequently, since there are potentially infinitely many examples about an object, sequences of hypotheses have to be considered. Gold [1] introduced the model of learning in the limit where the learner (here called IIM) is required to converge in the sense that there exists a point in time at which the learner has identified the object and, furthermore, the learner's guesses do not change any more.

However, the point in time at which the stabilization happens is unknown and in general even undecidable. Therefore, since the early days of learning theory, alternative models has been developed to support the user of a learning device. One of the most strongest forms here is finite learning [1,2] where the user gets told the point when the learning process is finished.

However, from a user's point of view, even this model has a drawback. Usually, learning is considered with respect to a set $\mathcal{C}$ of target objects. Correct behaviour of an IIM is only required when presenting information about an object of $\mathcal{C}$. Outside of this scope the IIM may behave arbitrarily. In the worst case, it may happen that the IIM pretends to learn correctly while in reality its outputs are rubbish. However, a user relying on such an IIM cannot recognize this fact.

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1.1. Machine self-assessment

Therefore, models to prevent this behaviour have been developed. Reliable inference machines [3,4] only converge if they really learn an object. Kinber and Zeugmann [5–7] generalized this model by letting an IIM explicitly output some kind of error messages. Still, the messages output by such machines are limiting recursive in character.

Mukouchi and Arikawa [8] developed the model of refutable inference. Here, the learner is required either to identify the target object or to recognize that it is not contained in its hypothesis space and therefore to refute this hypothesis space. This allows the user to recognize a failure of the IIM and to react in some way. This basic idea has been refined in many different directions. For example, in [9], the cases in which a refutation has to take place, have been tailored towards more realistic needs. Jain *et al.* [10] considered weaker variants of refutation. Also, a lot of other investigations of the model of refutation, as well as applications in other scenarios, have taken place (cf., e.g., [11–14]).

In parallel to that evolution, Jantke [15] changed the focus from refuting hypothesis spaces to assessing the competence of an IIM. Whereas in the model of refutable learning, the IIM has to signal if the object to be learnt as a whole does not belong to the target class, the competence of an IIM is defined with respect to the information available. Here, a given chunk of information may be rejected when recognizing that this information does not describe any of the target objects. Such information pieces are called inacceptable in this paper.

We consequently extended Jantke's ideas and arrived at four so called reflection types: exact, pessimistic, optimistic and limit-reflection. An IIM which is exactly reflecting has to reject an inacceptable information (i.e. a set of examples which definitely don't describe one of the target objects) immediately. On the other hand, if some information piece is acceptable (which means, that it is consistent with at least one of the target objects), such an IIM must not reject this information, i.e. it must agree with it. Pessimistic and optimistic IIMs are allowed to make one-sided errors. A pessimistic IIM may underestimate its competence, i.e. it may erroneously reject acceptable information. In turn, an optimistic IIM may overestimate its competence, i.e. it may sometimes accept inacceptable information. Limit-reflecting machines, finally, are allowed both types of error. In any case, however, all types of reflecting machines have to converge in the limit to the correct assessment.

In this paper, we investigate the influence of the four reflection types on the learning of recursive functions. Here, we consider learning in the limit, different variants of consistent learning, and finite learning.

1.2. Results and messages

The main results are as follows. (A complete overview over the relations of the identification types investigated is displayed in Fig. 1.)

For learning in the limit (denoted as *LIM*, cf. Definition 1), the actual choice of the reflection type doesn't influence the learning power (cf. Corollary 1). More precisely, a class of functions is reflectively learnable in the limit iff it is embedded in an initially complete learnable class. This has the following consequences. First, there are learnable classes for which there exist no reflective IIMs. Second, reflective learning in the limit is only possible in cases when reflection is not necessary at all, since every information piece is acceptable and hence the IIM never has to reject something.

For finite learning (called *FIN*, cf. Definition 2), the picture changes completely. Here, depending on which reflection requirement we demand the IIM to fulfill, we arrive at different learning power (cf. Corollary 9). Also, there are classes which are finitely learnable but any IIM for it can not correctly assess its own competence.

Finally, we consider consistent identification. Here, three different formalizations of consistency have been published in the literature which turned out to be of different expressiveness. Surprisingly, these small differences heavily influence the reflecting abilities of consistent IIMs.

An IIM is said to work total-consistently (called *T-CONS*, cf. Definition 2), if it is defined for every input and, furthermore, each hypothesis is consistent with the information it was constructed from. Every such IIM can easily be enriched by reflecting abilities (cf. Corollary 2). However, this result is of the same quality as the one for learning in the limit since a class of functions is reflectively total-consistently learnable iff it is embedded in an initially complete learnable class.

Whereas for total-consistent identification consistency is required for any hypothesis, in the other two models of consistent learning this is weakened. Here, consistency is demanded for hypotheses built from acceptable information only. Whereas in the model *R-CONS* (see Definition 2) the IIM still needs to be defined on any input, a CONS-IIM (cf. Definition 2) may be undefined on inacceptable information. For both models, enlarging the IIMs by reflecting abilities has nearly the same effects (cf. Corollaries 6 and 7). First, exact and pessimistic reflection are of the same power. Second, these two are less powerful than optimistic reflection which itself is weaker than limit-reflection. Also here, there are IIMs which cannot be made reflective.

To sum up, there is a subtle interplay between learning and reflection requirements. The demand that an IIM additionally has to reflect about its own competence is orthogonal to the classical identification requirements. In some learning models, there is no difference between the reflection types. In other models, the differences are huge. Of special interest here is the relationship between optimistic and pessimistic behaviour. There is no unique trend, which of both is stronger than the other one. However, there seems to appear some tendency that exact and pessimistic reflection as well as optimistic and limit-reflection are similar in spirit. This nicely coincides with the well-known observation that proving something is much harder than disproving it. In our model, pessimistic and exact reflections have to prove that some information is acceptable whereas optimistic or limit-reflective ones have to disprove this fact.

Moreover, we learnt that in most cases a system cannot correctly assess its competence. If this is possible, then such a system is very constrained. There is no universal reflection mechanism.

The remaining paper is structured as follows. In Section 2, we provide basic knowledge from inductive inference. The scenario of reflection is introduced in Section 3. In Section 4, we investigate the learning power of reflective IIMs. This is done in that, for each of the three basic paradigms learning in the limit (Section 4.1), consistent learning (Section 4.2), and finite learning (Section 4.3) we compare different types of reflective identification types with each other as well as with standard identification types. Moreover, we investigate whether reflective IIMs can draw benefits from the knowledge that they know in advance the order in which the information is presented. Furthermore, we show some closedness properties.

2. Inductive inference

Most notation is adopted from [16]. $\mathbb{N}$ denotes the set of natural numbers. We consider computable functions on $\mathbb{N}$. The set of all unary computable and all unary total-computable (also called recursive) functions is denoted by $\mathcal{P}$ and $\mathcal{R}$, respectively. By $\wp(S)$ we denote the power set of a set S.

For the rest of the paper, we fix a Gödel numbering φ of all computable functions as well as a corresponding Blum complexity measure ϕ (cf. [16,17]). Let $i \in \mathbb{N}$ and $x \in \mathbb{N}$. By φ_i we denote the ith function in φ, i is called a program. If the computation of φ_i terminates on input x, we write $\varphi_i(x)\downarrow$, if not we write $\varphi_i(x)\uparrow$.

Let $f \in \mathcal{R}$. An example for f is a pair $(x, f(x))$. Let $\sigma = ((x_n, f(x_n)))_{n\in\mathbb{N}}$ be an infinite sequence of examples for f. σ is called a representation for f if $\{x_n \mid n \in \mathbb{N}\} = \mathbb{N}$. By σ^n we denote the initial segment $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$. σ is in canonical order if $x_n = n$, for all n. $Repr(f)$ denotes the set of all representations of f. Furthermore, by $[f]$ we denote the set of all initial segments of representations from $Repr(f)$. $Repr(\cdot)$ and $[\cdot]$ are canonically extended to function classes $\mathcal{C} \subseteq \mathcal{R}$ as well as to sets $\mathcal{IT} \subseteq \wp(\mathcal{R})$ of function classes. By $Repr^c(\cdot)$ and $[\cdot]^c$ we denote the set of all representations as well as the set of all initial segments of canonical representations of $\cdot$. Sometimes, we abuse notation and identify recursive functions with their canonical representations. Hence, we use f^n to denote the initial segment $(0, f(0)), \ldots, (n, f(n))$ of the canonical representation of f.

A class $C \subseteq \mathcal{R}$ is called initially complete if $[C] = [\mathcal{R}]$.

An inductive inference machine (IIM) is a computable function that receives finite segments of representations of recursive functions as input and outputs natural numbers. These numbers are interpreted as programs in our fixed Gödel numbering φ.

By $\langle\cdot,\cdot\rangle$ we denote Cantors pairing function which is canonically extended to an arbitrary number of arguments. For technical reasons, we prefer to let an IIM map from $\mathbb{N}$ to $\mathbb{N}$. For this, we implicitly assume that any finite segment $\tau = (x_0, y_0), \ldots, (x_n, y_n) \in [\mathcal{R}]$ is uniquely encoded into a natural number $\langle n, \langle x_0, y_0\rangle, \ldots, \langle x_n, y_n\rangle\rangle$ and identify a segment with its encoding. So, $\varphi_i(\tau)$ is used instead of $\varphi_i(\langle\tau\rangle)$, for example.

Let σ be an infinite and let τ and τ' be two finite sequences. By τ^+ we denote the set of all elements of τ. $\tau \diamond \tau'$ and $\tau \diamond \sigma$ denote the concatenations.

An infinite sequence of natural numbers $(x_n)_{n\in\mathbb{N}}$ is said to *converge in the limit* to x (denoted by $\lim_n x_n = x$) iff for all but finitely many n it holds $x_n = x$.

Now we are ready to define our learning models.

Definition 1 (*[1]*). Let $f \in \mathcal{R}$ and $M \in \mathcal{P}$ be an IIM. M is said to LIM^{arb}-identify f (denoted by $f \in LIM^{arb}(M)$) iff for every $\sigma \in Repr(f)$ holds:

(1) $M(\sigma^n)\downarrow$, for all $n \in \mathbb{N}$.
(2) There exists an h such that $\lim_n M(\sigma^n) = h$ with $\varphi_h = f$.

M LIM^{arb}-identifies a class $\mathcal{C} \subseteq \mathcal{R}$ iff M LIM^{arb}-identifies every $f \in \mathcal{C}$.
The identification type LIM^{arb} denotes the set of all classes that are LIM^{arb}-identifiable by any IIM.

In the above definition, *LIM* stands for *learning in the limit* and the superscript *a* denotes learning from arbitrary representations.

A vast amount of variations from Golds standard model have been investigated. We focus our attention to consistent and finite identification, which are defined as follows.

Definition 2 (*[1–3,18,19]*). Let $\mathcal{C} \subseteq \mathcal{R}$ and $\mathcal{IT}$ be one of the following identification types: $CONS^{arb}$, $R\text{-}CONS^{arb}$, $T\text{-}CONS^{arb}$, FIN^{arb}.

$\mathcal{C} \in \mathcal{IT}$ iff there is an IIM $M \in \mathcal{P}$ with $\mathcal{C} \subseteq LIM^{arb}(M)$ such that the following corresponding condition is fulfilled:

$CONS^{arb}$: For all $\tau \in [\mathcal{C}]$ and for all $(x, y) \in \tau^+$ it holds $\varphi_{M(\tau)}(x) = y$.
$R\text{-}CONS^{arb}$: $M \in \mathcal{R}$ and $\mathcal{C} \subseteq CONS^{arb}(M)$.
$T\text{-}CONS^{arb}$: $M \in \mathcal{R}$ and, for all $\tau \in [\mathcal{R}]$ and all $(x, y) \in \tau^+$, it holds $\varphi_{M(\tau)}(x) = y$.
FIN^{arb}: There exists a decision predicate $d \in \mathcal{R}$ such that for all $f \in \mathcal{C}$ and all $\tau \in [f]$ it holds $d(\tau) = 1$ iff $\varphi_{M(\tau)} = f$. (We call any hypothesis $M(\tau)$ with $d(\tau) = 1$ a final hypothesis.)

Finally, $\mathcal{C} \subseteq \mathcal{R}$ is *exactly enumerable* ($\mathcal{C} \in NUM!$), iff there exists a $g \in \mathcal{R}$ with $\mathcal{C} = \{\varphi_{g(n)} \mid n \in \mathbb{N}\}$. *NUM* is the smallest superset of *NUM*! that is closed under subsets.

Originally, learning of recursive functions has been considered from canonical representations, only. For any of the identification types $\mathcal{IT}^{arb}$ defined above, we let $\mathcal{IT}$ denote the resulting identification type for learning from canonical sequences where we require the conditions of the Definitions 1 and 2 for canonical representations, only.

Next, we summarize the relation between these identification types.

Theorem 1 (*[3,18–22]*). (1) $NUM! \subset NUM \subset T\text{-}CONS^{arb} \subset R\text{-}CONS^{arb} \subset CONS^{arb} \subset LIM^{arb}$.
(2) $FIN^{arb} \subset R\text{-}CONS^{arb}$.
(3) $FIN^{arb} \# T\text{-}CONS^{arb}$.
(4) $FIN^{arb} \# NUM!$.
(5) $LIM^{arb} = LIM$.
(6) $CONS^{arb} \subset CONS$.
(7) $R\text{-}CONS^{arb} \subset R\text{-}CONS$.
(8) $FIN^{arb} = FIN$.

For total-consistent learning, the relation was still open. The following proof is due to an anonymous referee who generalised the author's original result that $T\text{-}CONS^{arb} \subset T\text{-}CONS$ holds.

Theorem 2 (*[23]*). $T\text{-}CONS \setminus CONS^{arb} \neq \emptyset$.

Proof. We let $\mathcal{C}_{ffs}$ be the set of all functions of finite support, i.e. $\mathcal{C}_{ffs} = \{f \in \mathcal{R} \mid \exists n \in \mathbb{N}\, \forall x > n : f(x) = 0\}$. Additionally we let $\mathcal{C}_t$ be the set $\{f \in \mathcal{R} \mid \forall x \in \mathbb{N} : f(2x+1) = \phi_{f(0)}(x), f(2x+2) = \varphi_{f(0)}(x)\}$.

It is easy to see that $\mathcal{C}_{ffs} \cup \mathcal{C}_t$ belongs to *T-CONS*. An IIM M, when told the values $f(0), \ldots, f(n)$ simply checks that $f(1)$ coincides with $\phi_{f(0)}(0)$, $f(2)$ coincides with $\varphi_{f(0)}(0)$, and so on up to $f(n)$. Note that, since M is told the value for $\phi_{f(0)}(x)$ before actually computing $\varphi_{f(0)}(x)$, these tests can easily be done. In the case all these tests succeed, M outputs a program for $t_{f(0)}$, where $t_i(0) = i$, $t_i(2x+1) = \phi_i(x)$, and $t_i(2x+2) = \varphi_i(x)$, for all $x \in \mathbb{N}$. Otherwise, M outputs a program for the corresponding function of finite support.

On the other hand, $\mathcal{C}_{ffs} \cup \mathcal{C}_t \notin CONS^{arb}$, which can be verified as follows. We assume the contrary, i.e. let $\mathcal{C}_{ffs} \cup \mathcal{C}_t \subseteq CONS^{arb}(M)$ for some IIM M.

Let $(\alpha_n)_{n\in\mathbb{N}}$ be any fixed enumeration of $[\mathcal{R}]$. For any $i \in \mathbb{N}$, a function g_i is defined as follows, where α_i^+ denotes the set of all elements contained in α_i:

$$g_i(x) = \begin{cases} b : (2x+2, b) \in \alpha_i^+ \\ 0 : (2x+2, b) \notin \alpha_i^+ \text{ for all } b \text{ and } M(\alpha_i \diamond (2x+2, 0)) = M(\alpha_i) \\ 1 : \text{otherwise} \end{cases}$$

Since M learns $\mathcal{C}_{ffs}$, M is total and hence each g_i is, too.

Now, take any function $\varphi_i \in \mathcal{R}$ which is $\{0, 1\}$-valued and consider the function t_i. Clearly, $t_i \in \mathcal{C}_t$ holds. Hence, there exists a locking sequence (cf. [24]) τ for M on t_i, i.e. $M(\tau) = M(\tau \diamond \tau')$ for any $\tau' \in [t_i]$ and $\varphi_{M(\tau)} = t_i$ hold. Now, let j be such that $\alpha_j = \tau$ and consider g_j.

Let $x \in \mathbb{N}$. We show that $\varphi_i(x) = g_j(x)$. Remember that $t_i(2x+2) = \varphi_i(x)$. First, by definition, if $(2x+2, b) \in \alpha_j$ for some b, then $g_j(x) = t_i(2x+2) = b$ holds. Next, let $t_i(2x+2) = 0$. Since τ is a locking sequence and $(2x+2, 0)$ is consistent with t_i, $M(\alpha_j \diamond (2x+2, 0)) = M(\alpha_j)$ and therefore $g_j(x) = 0$. Finally, let $\varphi_i(x) = 1$, hence $t_i(2x+2) = 1$. This implies $\varphi_{M(\alpha_j)}(2x+2) = 1$. Since M works consistently and has to learn all functions of finite support, $M(\alpha_j \diamond (2x+2, 0)) \neq M(\alpha_j)$ and we arrive at $g_j(x) = 1$.

This implies that the set of all $\{0, 1\}$-valued computable functions is contained in $\{g_i \mid i \in \mathbb{N}\}$ which in turn is an enumerable set of total functions. Hence, the set of all $\{0, 1\}$-valued computable functions would belong to *NUM*, a contradiction. □

3. Reflective inductive inference

We want to enrich IIMs by the ability of assessing its own competence. So, first, we have to define what competence of an IIM means. In contrast to the approach of refutable learning (cf. [8]), where competence is defined with respect to complete representations, we define it based on initial segments. We extend the approach of Jantke [15], his reflecting and immediate reflecting IIMs correspond to our optimistic and exact reflection, respectively.

Let M be an IIM and $\mathcal{IT}^{arb}$ be any identification type. We only present the definitions for arbitrary example sequences, for canonical they are adapted analogously.

Let $\tau \in [\mathcal{R}]$ and $\sigma \in Repr(\mathcal{R})$. τ is said to be *acceptable* if $\tau \in [\mathcal{IT}^{arb}(M)]$, *inacceptable* otherwise. σ is *acceptable* iff $\sigma \in Repr(\mathcal{IT}^{arb}(M))$. σ is *inacceptable* if it has an initial segment which is inacceptable. The reader should note that by this definition some sequences may neither be acceptable nor inacceptable.

A reflection R for M with respect to $\mathcal{IT}^{arb}$ is a total-computable function $R : Repr(\mathcal{R}) \to \{0, 1\}$ that satisfies the following constraints:

(1) On every acceptable sequence σ, R converges to 1, i.e. $\lim_n R(\sigma^n) = 1$.
(2) On every inacceptable sequence σ, R converges to 0, i.e. $\lim_n R(\sigma^n) = 0$.

R works *optimistically*, if, for any $\tau \in [\mathcal{R}]$, $R(\tau) = 0$ implies that τ is inacceptable. R works *pessimistically*, if, for any $\tau \in [\mathcal{R}]$, $R(\tau) = 1$ implies that τ is acceptable. Finally, R is *exact*, if it both works optimistically and pessimistically.

If a reflection outputs 0 for some input τ, we say that R rejects τ. Analogously, R agrees with τ if it outputs 1.

Reflective IIMs are pairs of an ordinary IIM and a corresponding reflection.

Definition 3. Let $\mathcal{C} \subseteq \mathcal{R}$, $M \in \mathcal{P}$ be an IIM, $R \in \mathcal{R}$ be a reflection, and $\mathcal{IT}^{arb}$ be an identification type.

$\mathcal{C} \subseteq \mathcal{IT}^{arb}$-*Refl*$(M, R)$ iff $\mathcal{C} \subseteq \mathcal{IT}^{arb}(M)$ and R is a reflection for M with respect to $\mathcal{IT}^{arb}$.

$\mathcal{C} \subseteq \mathcal{IT}^{arb}$-*oRefl*$(M, R)$, $\mathcal{C} \subseteq \mathcal{IT}^{arb}$-*pRefl*$(M, R)$, $\mathcal{C} \subseteq \mathcal{IT}^{arb}$-*eRefl*$(M, R)$ iff $\mathcal{C} \subseteq \mathcal{IT}^{arb}(M)$ and R is an optimistic, pessimistic, and exact reflection for M with respect to $\mathcal{IT}^{arb}$, resp.

As usual, $\mathcal{IT}^{arb}$-*Refl* is the set of all classes $\mathcal{C}$ for which there are an IIM M and a corresponding reflection R with $\mathcal{C} \subseteq \mathcal{IT}^{arb}$-*Refl*$(M, R)$. For any $\lambda \in \{o, p, e\}$, $\mathcal{IT}^{arb}$-λ*Refl* is defined analogously.

The basic relations between these four reflective identification types directly implied by its definitions are as follows:

$$\begin{array}{ccc} & \mathcal{IT}^{arb}\text{-}Refl & \\ \rotatebox{45}{$\subseteq$} & & \rotatebox{-45}{$\supseteq$} \\ \mathcal{IT}^{arb}\text{-}oRefl & & \mathcal{IT}^{arb}\text{-}pRefl \\ \rotatebox{-45}{$\supseteq$} & & \rotatebox{45}{$\subseteq$} \\ & \mathcal{IT}^{arb}\text{-}eRefl & \end{array}$$

Moreover, for any identification type $\mathcal{IT}^{arb}$ we know $\mathcal{IT}^{arb}$-*Refl* $\subseteq \mathcal{IT}^{arb}$.

Depending on the underlying identification type, each of the above inclusions may be proper or not. So, as first question, we are interested in the actual instances of these relations for the learning types introduced.

In general, we are interested in the hierarchy of reflective identification types as well as their relation to the standard models.

4. Results

Now, we are ready to investigate reflective learning from arbitrary input sequences.

The results achieved are summarized in Fig. 1. Here, a line (or a path) between two identification types depicts that the lower one is a proper[1] subset of the upper one. If there is no connection, they are incomparable.[1]

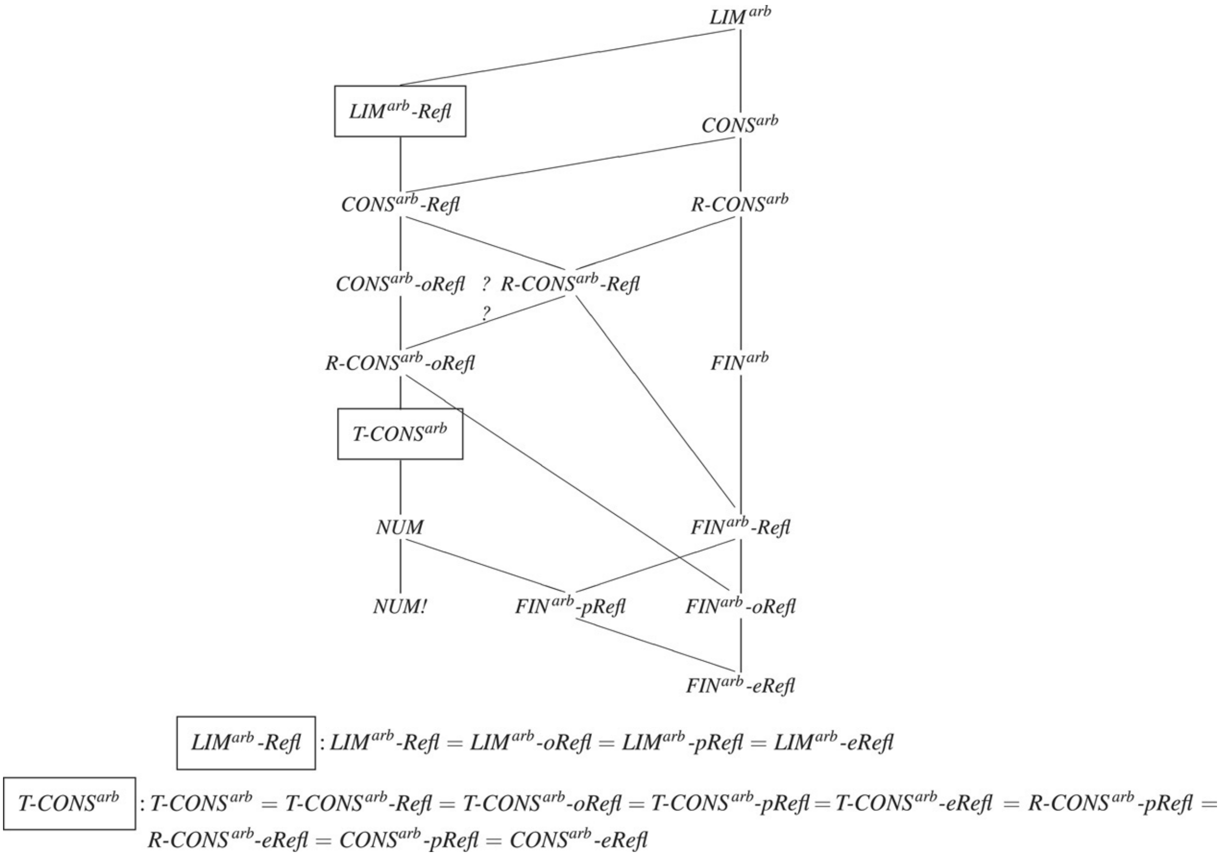

Fig. 1. The relations of the reflective and standard identification types for learning from arbitrary sequences. The relations for learning from canonical sequences are analogously.

Before going into the investigations of the single identification types, it is useful to have the following obvious lemma.

Lemma 1. *Let* $\mathcal{IT}_1, \mathcal{IT}_2 \in \{LIM^{arb}, CONS^{arb}, R\text{-}CONS^{arb}, T\text{-}CONS^{arb}, FIN^{arb}\}$ *with* $\mathcal{IT}_1 \subseteq \mathcal{IT}_2$. *Then, it holds* $\mathcal{IT}_1$-λ*Refl* $\subseteq \mathcal{IT}_2$-λ*Refl for all* $\lambda \in \{e, p, o, \varepsilon\}$.

The analogous statement holds for learning from canonical representations.

[1] The exact relationship between *R-CONS*^{arb}-*oRefl* and *R-CONS*^{arb}-*Refl* as well as *CONS*^{arb}-*oRefl* and *R-CONS*^{arb}-*Refl* is still open, we just know *R-CONS*^{arb}-*oRefl* $\subseteq$ *R-CONS*^{arb}-*Refl* and *CONS*^{arb}-*oRefl* $\setminus$ *R-CONS*^{arb}-*Refl* $\neq \emptyset$.

4.1. Learning in the limit

Jantke [15] already proved $LIM^{arb}\text{-}oRefl = LIM^{arb}\text{-}eRefl = LIM\text{-}oRefl = LIM\text{-}eRefl$. In fact, he showed $LIM^{arb}\text{-}oRefl = IC(LIM^{arb})$, where we call $IC(\mathcal{IT})$ the initially complete core of an identification type $\mathcal{IT}$. For any $\mathcal{IT} \subseteq \wp(\mathcal{R})$, $IC(\mathcal{IT})$ is the largest subset of $\mathcal{IT}$ such that every class contained is initially complete. Obviously, $IC(\mathcal{IT}) \subseteq \mathcal{IT}\text{-}eRefl$ holds, for any identification type $\mathcal{IT}$. The technique used in [15] can easily be adapted to show $LIM^{arb}\text{-}Refl \subseteq IC(LIM^{arb})$. Hence, we arrive at $LIM^{arb}\text{-}\lambda Refl = IC(LIM^{arb})$ as well as $LIM\text{-}\lambda Refl = IC(LIM)$, for all $\lambda \in \{e, p, o, \varepsilon\}$.

Furthermore, it is well-known that any class, which is initially complete and which contains the class $\mathcal{C}_s = \{f \in \mathcal{R} \mid \varphi_{f(0)} = f\}$ of self-referencing functions can not be identified in the limit. On the other hand, $\mathcal{C}_s$ can clearly be identified finitely, hence we have:

Theorem 3. *$FIN^{arb} \setminus LIM\text{-}Refl \neq \emptyset$.*

Since on the other hand there exist initially complete classes, which can not be learned consistently (cf. [22,25]), we also know:

Corollary 1.

$$\begin{array}{ccccccccc}
 & LIM^{arb} & & & = & & & LIM & \\
 & \cup & & & & & & \cup & \\
 & LIM^{arb}\text{-}Refl & & & & & & LIM\text{-}Refl & \\
\diagup & & \diagdown & & & & \diagup & & \diagdown \\
LIM^{arb}\text{-}oRefl & = & LIM^{arb}\text{-}pRefl & & = & & LIM\text{-}oRefl & = & LIM\text{-}pRefl \\
\diagdown & & \diagup & & & & \diagdown & & \diagup \\
 & LIM^{arb}\text{-}eRefl & & & & & & LIM\text{-}eRefl &
\end{array}$$

Reflective learning in the limit is orthogonal to all the standard identification types below LIM^{arb}.

Theorem 4. *For all $\lambda \in \{e, p, o, \varepsilon\}$,*

1. $LIM^{arb}\text{-}\lambda Refl \# CONS^{arb}$. $\quad LIM^{arb}\text{-}\lambda Refl \# CONS$.
2. $LIM^{arb}\text{-}\lambda Refl \# R\text{-}CONS^{arb}$. $\quad LIM^{arb}\text{-}\lambda Refl \# R\text{-}CONS$.
3. $LIM^{arb}\text{-}\lambda Refl \# FIN^{arb}$.
4. $CONS^{arb}\text{-}\lambda Refl \subset LIM^{arb}\text{-}\lambda Refl$. $\quad CONS\text{-}\lambda Refl \subset LIM\text{-}\lambda Refl$.

Proof. By Theorem 3, we know $FIN^{arb} \setminus LIM\text{-}Refl \neq \emptyset$.

To verify $LIM^{arb}\text{-}Refl \setminus CONS \neq \emptyset$, consider the following class introduced in [22]: $\mathcal{C}_{sl} = \{f \in \mathcal{R} \mid \exists n \in \mathbb{N} : f(n) > 1,\ \varphi_{f(n)} = f,\ \forall m > n : f(m) \leq 1\}$. It is well-known that $\mathcal{C}_{sl} \in LIM^{arb} \setminus CONS$ holds. Since $\mathcal{C}_{sl}$ is initially complete, $\mathcal{C}_{sl} \in LIM^{arb}\text{-}\lambda Refl$ holds as well. Together with Theorem 1 and Lemma 1 this yields the theorem. □

The class $\mathcal{C}_{sl}$ from the last proof as well as the class $\mathcal{C}_{ffs}$ of all recursive functions of finite support are $LIM^{arb}\text{-}Refl$ learnable, its union is not. This can easily be verified by standard techniques, hence we have:

Theorem 5. *$LIM^{arb}\text{-}\lambda Refl$ and $LIM\text{-}\lambda Refl$ are not closed under finite union for all $\lambda \in \{e, p, o, \varepsilon\}$.*

4.2. Consistent learning

Next, we investigate the initially complete cores of the different types of consistent learning: If constrained to initially complete classes, all types coincide.

Theorem 6. *$IC(T\text{-}CONS^{arb}) = IC(R\text{-}CONS^{arb}) = IC(CONS^{arb}) = T\text{-}CONS^{arb}$.*
$IC(T\text{-}CONS) = IC(R\text{-}CONS) = IC(CONS) = T\text{-}CONS$.

Proof. Clearly, an IIM that consistently learns an initially complete class is consistent on any input segment. Hence, $IC(CONS^{arb}) \subseteq T\text{-}CONS^{arb}$ holds.

On the other hand, $T\text{-}CONS^{arb}$ is closed under finite union (cf. [3,25]). Since the class $\mathcal{C}_{ffs}$ of all functions of finite support is clearly $T\text{-}CONS^{arb}$-identifiable, for every class $\mathcal{C} \in T\text{-}CONS^{arb}$ also $\mathcal{C} \cup \mathcal{C}_{ffs} \in T\text{-}CONS^{arb}$ holds. Hence, $T\text{-}CONS^{arb} \subseteq IC(T\text{-}CONS^{arb})$, and, since the same arguments also hold for learning from canonical sequences, the theorem follows. □

This directly yields that all types of reflective T-$CONS$-learning coincide.

Corollary 2.

$$\begin{array}{ccccccc}
 & T\text{-}CONS^{arb} & & \subset & & T\text{-}CONS & \\
 & \| & & & & \| & \\
 & T\text{-}CONS^{arb}\text{-}Refl & & & & T\text{-}CONS\text{-}Refl & \\
 /\!\!/ & & \backslash\!\!\backslash & & /\!\!/ & & \backslash\!\!\backslash \\
T\text{-}CONS^{arb}\text{-}oRefl & = & T\text{-}CONS^{arb}\text{-}pRefl & \subset & T\text{-}CONS\text{-}oRefl & = & T\text{-}CONS\text{-}pRefl \\
 \backslash\!\!\backslash & & /\!\!/ & & \backslash\!\!\backslash & & /\!\!/ \\
 & T\text{-}CONS^{arb}\text{-}eRefl & & & & T\text{-}CONS\text{-}eRefl &
\end{array}.$$

Since T-$CONS$- and FIN-learning is incomparable, Theorem 3 yields the following insights.

Corollary 3. *For all* $\lambda \in \{e, p, o, \varepsilon\}$,

1. $CONS^{arb}$-$\lambda Refl \# FIN^{arb}$. $\quad$ $CONS$-$\lambda Refl \# FIN^{arb}$.
2. R-$CONS^{arb}$-$\lambda Refl \# FIN^{arb}$. $\quad$ R-$CONS$-$\lambda Refl \# FIN^{arb}$.
3. T-$CONS^{arb}$-$\lambda Refl \# FIN^{arb}$. $\quad$ T-$CONS$-$\lambda Refl \# FIN^{arb}$.
4. FIN^{arb}-$\lambda Refl \subset T$-$CONS^{arb}$-$\lambda Refl$. $\quad$ FIN-$\lambda Refl \subset T$-$CONS$-$\lambda Refl$.

For R-$CONS$ and T-$CONS$-learning, the initially complete core exactly describes the exact and pessimistic reflective learnable classes.

Theorem 7. 1. $CONS^{arb}$-$pRefl = CONS^{arb}$-$eRefl = T$-$CONS^{arb}$.
2. $CONS$-$pRefl = CONS$-$eRefl = T$-$CONS$.
3. R-$CONS^{arb}$-$pRefl = R$-$CONS^{arb}$-$eRefl = T$-$CONS^{arb}$.
4. R-$CONS$-$pRefl = R$-$CONS$-$eRefl = T$-$CONS$.

Proof. The argumentation for learning from canonical representations is analogous to the one for learning from arbitrary sequences, therefore we just discuss the latter one.

Assertion 3 is a direct consequence of 1, therefore it suffices to verify 1 which, by Theorem 6, reduces to prove $CONS^{arb}$-$pRefl \subseteq IC(CONS^{arb})$. Let $\mathcal{C} \subseteq CONS^{arb}$-$pRefl(M, R)$. Then, an IIM M' that consistently identifies an initially complete superclass of $\mathcal{C}$ can work as follows.

On input τ, determine $R(\tau)$. If $R(\tau) = 1$, determine M's hypothesis h on input τ and output h. Otherwise, i.e. if $R(\tau) = 0$, output a fixed program for the function f_τ. Here, f_τ denotes the 'smallest' function of finite support consistent with τ, i.e. $f_\tau(x) = y$ for any $(x, y) \in \tau^+$ and $f_\tau(x) = 0$, otherwise.

It is not hard to see that M' works as required, i.e. $\mathcal{C} \subseteq CONS^{arb}(M')$ and $CONS^{arb}(M') \in IC(CONS^{arb})$ hold. □

We next investigate optimistic reflection.

Theorem 8. 1. R-$CONS^{arb}$-$oRefl \setminus T$-$CONS \neq \emptyset$.
2. $CONS^{arb}$-$oRefl \setminus R$-$CONS \neq \emptyset$.

Proof. The proof idea is based on the one usually used for diagonalization of consistent IIMs (cf. [25], e.g.). First, we define two function classes $\mathcal{C}_g$ and $\mathcal{C}_h$ as follows. Let $i, j \in \mathbb{N}$. Start with stage 0.

Stage 0: Set $\tau = (0, i)(1, j)$ and $x = 2$. Define $g_{i,j}(0) = h_{i,j}(0) = i$ and $g_{i,j}(1) = h_{i,j}(1) = j$. Goto stage 1.

Stage $n > 0$: Determine $a = \varphi_i(\tau)$, $b = \varphi_i(\tau \diamond (x, 1))$, and $c = \varphi_i(\tau \diamond (x, 2))$. Set $k = \langle \phi_i(\tau), \phi_i(\tau \diamond (x, 1)), \phi_i(\tau \diamond (x, 2))\rangle$. Set y_g and y_h according to the following cases:

(1) If $a \neq b$, set $y_g = 1$ and $y_h = 1$.
(2) If $a = b$ and $a \neq c$, set $y_g = 2$ and $y_h = 2$.
(3) If $a = b = c$, set $y_g = 1$ and $y_h = 2$.

Define $g_{i,j}(x) = y_g$, $h_{i,j}(x) = y_h$, and $g_{i,j}(x + 1) = h_{i,j}(x + 1) = k$. Set $\tau = \tau \diamond (x, y_g)(x + 1, k)$, $x = x + 2$, and goto stage $n + 1$.

We now let $\mathcal{C}_g$ and $\mathcal{C}_h$ be the sets of all functions $g_{i,j}$ and $h_{i,j}$, respectively, which are defined everywhere, i.e. $\mathcal{C}_g = \{g_{i,j} \mid i, j \in \mathbb{N},\ g_{i,j} \in \mathcal{R}\}$ and $\mathcal{C}_h = \{h_{i,j} \mid i, j \in \mathbb{N},\ h_{i,j} \in \mathcal{R}\}$. Obviously, $\mathcal{C}_g$ as well as $\mathcal{C}_h$ is not empty.

By construction, it holds $\mathcal{C}_g \notin T\text{-}CONS$ as well as $\mathcal{C}_g \cup \mathcal{C}_h \notin R\text{-}CONS$, which can be verified as follows.

Let M be a recursive IIM R-$CONS$-identifying $\mathcal{C}_g \cup \mathcal{C}_h$ and let i be one of its φ-programs. Since M is total, $g_{i,0} \in \mathcal{C}_g$ and $h_{i,0} \in \mathcal{C}_h$ holds. We show that M cannot identify both $g_{i,0}$ and $h_{i,0}$ and distinguish two cases.

Case 1: $g_{i,0} \neq h_{i,0}$.

Let n be the smallest number with $g_{i,0}(n) \neq h_{i,0}(n)$, hence $g_{i,0}^{n-1} = h_{i,0}^{n-1}$. By construction, n is even. By definition of $g_{i,0}$ and $h_{i,0}$ it holds $M(g_{i,0}^{n-1}) = M(g_{i,0}^{n-1} \diamond (n, 1)) = M(g_{i,0}^{n-1} \diamond (n, 2))$ as well as $g_{i,0}^{n-1} \diamond (n, 1) = g_{i,0}^{n}$ and $g_{i,0}^{n-1} \diamond (n, 2) = h_{i,0}^{n}$. Hence, M's hypothesis is inconsistent for at least one of these segments which violates our assumption.

Case 2: $g_{i,0} = h_{i,0}$.

Hence, in each stage $n > 0$, $g_{i,0}$ and $h_{i,0}$ are defined by statement 1 or 2, only. This implies that there are infinitely many n with $M(g_{i,0}^{n}) \neq M(g_{i,0}^{n+1})$ and therefore M does not converge on the canonical representation of $g_{i,0}$, which also violates our assumption.

This argument also shows that M cannot T-$CONS$-identify $\mathcal{C}_g$, hence $\mathcal{C}_g \notin T\text{-}CONS$ as well as $\mathcal{C}_g \cup \mathcal{C}_h \notin R\text{-}CONS$ hold.

On the other hand, both classes can be optimistically reflected. This is due to the fact that $g_{i,j}$'s as well as $h_{i,j}$'s value for every odd input $x > 2$ contains complexity information which, together with the knowledge about the results for the inputs $0, \ldots, x-2$, is sufficient to verify the output for $x - 1$. The reflection simply agrees with an input segment τ as long as consistency of τ with the corresponding $g_{i,j}$ resp. $h_{i,j}$ has not been disproved.

CONS^{arb}-identification of $\mathcal{C}_g \cup \mathcal{C}_h$ as well as R-*CONS*^{arb}-identification of $\mathcal{C}_g$ is trivial. The corresponding IIMs M_1 and M_2 may work as follows.

Let τ be any representation segment. Until i and j can be determined from τ, both machines output temporary, consistent hypotheses. When knowing i and j, M_1 tests consistency of $g_{i,j}$ with τ. If this consistency test does not terminate, also M_1 does not. If $g_{i,j}$ is consistent with τ, M_1 outputs a program for $g_{i,j}$, otherwise a program for $h_{i,j}$.

M_2 always hypothesizes $g_{i,j}$.

Clearly, $\mathcal{C}_g \cup \mathcal{C}_h = CONS^{arb}(M_1)$ and $\mathcal{C}_g = R\text{-}CONS^{arb}(M_2)$. Since $\mathcal{C}_g \cup \mathcal{C}_h$ as well as $\mathcal{C}_g$ can be optimistically reflected, we have $\mathcal{C}_g \in R\text{-}CONS^{arb}\text{-}oRefl \setminus T\text{-}CONS$ as well as $\mathcal{C}_g \cup \mathcal{C}_h \in CONS^{arb}\text{-}oRefl \setminus R\text{-}CONS$. □

This allows for the following conclusions.

Corollary 4. 1. *R-CONS-oRefl* $\setminus$ *T-CONS* $\neq \emptyset$.
2. *CONS-oRefl* $\setminus$ *R-CONS* $\neq \emptyset$.

Corollary 5.

1.	$R\text{-}CONS^{arb}\text{-}oRefl \subset CONS^{arb}\text{-}oRefl$.	$R\text{-}CONS\text{-}oRefl \subset CONS\text{-}oRefl$.
2.	$R\text{-}CONS^{arb}\text{-}Refl \subset CONS^{arb}\text{-}Refl$.	$R\text{-}CONS\text{-}Refl \subset CONS\text{-}Refl$.
3.	$CONS^{arb}\text{-}oRefl$ # $R\text{-}CONS^{arb}$.	$CONS^{arb}\text{-}oRefl$ # $R\text{-}CONS$.
	$CONS\text{-}oRefl$ # $R\text{-}CONS^{arb}$.	$CONS\text{-}oRefl$ # $R\text{-}CONS$.
4.	$CONS^{arb}\text{-}Refl$ # $R\text{-}CONS^{arb}$.	$CONS^{arb}\text{-}Refl$ # $R\text{-}CONS$.
	$CONS\text{-}Refl$ # $R\text{-}CONS^{arb}$.	$CONS\text{-}Refl$ # $R\text{-}CONS$.

We can also prove a separation between optimistic *CONS*^{arb} reflection and reflection in the limit.

Theorem 9. $CONS^{arb}\text{-}Refl \setminus CONS\text{-}oRefl \neq \emptyset$.

Proof. We show this by diagonalizing against all potential optimistic computable reflections. We assume any fixed enumeration $(\alpha_n)_{n\in\mathbb{N}}$ of $[\mathcal{R}]^c$. For any $i, j \in \mathbb{N}$, a function $r_{i,j}$ is defined as follows. We set $\mathcal{C}_r = \{r_{i,j} \mid i, j \in \mathbb{N},\ r_{i,j} \in \mathcal{R}\}$.

Search for the least n such that $\varphi_j(\alpha_n) = 0$ and $\alpha_n \in [\mathcal{R}_{i,j}]^c$, where $\mathcal{R}_{i,j} = \{f \in \mathcal{R} \mid f(0) = i, f(1) = j\}$. If this computation does not terminate, $r_{i,j}$ is undefined everywhere. If such an n has been found, we define $r_{i,j}(x) = y$ for all $(x, y) \in \alpha_n^+$ and $r_{i,j}(x) = 0$ otherwise.

An IIM that $CONS^{arb}$-identifies exactly $\mathcal{C}_g \cup \mathcal{C}_r$ can be defined analogously to M_1 in the previous proof.

Clearly, $\mathcal{C}_r$ can be pessimistically reflected by simulating the search for α_n. Since the class $\mathcal{C}_g \cup \mathcal{C}_r$ is discrete, $\mathcal{C}_g \cup \mathcal{C}_r$ can be reflected in the limit. Hence, we have $\mathcal{C}_g \cup \mathcal{C}_r \in CONS^{arb}\text{-}Refl$.

By construction, $\mathcal{C}_g \cup \mathcal{C}_r \notin CONS\text{-}oRefl$. This can be seen by assuming the contrary, let $\mathcal{C}_g \cup \mathcal{C}_r \subseteq CONS\text{-}oRefl(M, R)$. Let i and j be such that $\varphi_i = M$ and $\varphi_j = R$. Now, we distinguish two cases. First, if $r_{i,j} \in \mathcal{R}$ holds, one directly sees that R does not work optimistically since it rejects an acceptable segment, namely α_n which is used to define $r_{i,j}$.

Therefore, we now assume that $r_{i,j}$ is undefined for some input. Since $R \in \mathcal{R}$, this implies $R(\tau) = 1$, for any $\tau \in [\mathcal{R}_{i,j}]^c$. However, since in this case no $\tau \in [\mathcal{R}_{i,j}]^c$ is inacceptable, M is defined on every $\tau \in [\mathcal{R}_{i,j}]^c$, which in turn implies $g_{i,j} \in \mathcal{R}$ and $h_{i,j} \in \mathcal{R}$. However, by construction of $g_{i,j}$ and $h_{i,j}$, M cannot consistently identify both functions. Thus, at least one of its canonical initial segments is not acceptable, which violates our assumption on R.

Hence, $\mathcal{C}_g \cup \mathcal{C}_r$ can not be optimistically reflected wrt. a consistent IIM. $\square$

By putting all these insights together, the structure of reflective $CONS^{arb}$- and $R\text{-}CONS^{arb}$-identification is as follows, where the properness of $R\text{-}CONS^{arb}\text{-}oRefl \subseteq R\text{-}CONS^{arb}\text{-}Refl$ is still open.

Corollary 6.

$$
\begin{array}{ccccccc}
 & CONS^{arb} & & \subset & & CONS & \\
 & \cup & & & & \cup & \\
 & CONS^{arb}\text{-}Refl & & & & CONS\text{-}Refl & \\
\rotatebox{45}{$\subset$} & & \rotatebox{135}{$\subset$} & & \rotatebox{45}{$\subset$} & & \rotatebox{135}{$\subset$} \\
CONS^{arb}\text{-}oRefl & \supset & CONS^{arb}\text{-}pRefl & \subset & CONS\text{-}pRefl & \subset & CONS\text{-}oRefl \\
\rotatebox{135}{$\subset$} & & /\!/ & & \backslash\!\backslash & & \rotatebox{45}{$\subset$} \\
 & CONS^{arb}\text{-}eRefl & & & & CONS\text{-}eRefl &
\end{array}
$$

Corollary 7.

$$
\begin{array}{ccccccc}
 & R\text{-}CONS^{arb} & & \subset & & R\text{-}CONS & \\
 & \cup & & & & \cup & \\
 & R\text{-}CONS^{arb}\text{-}Refl & & & & R\text{-}CONS\text{-}Refl & \\
\rotatebox{45}{$\subseteq$} & & \rotatebox{135}{$\subset$} & & \rotatebox{45}{$\subset$} & & \rotatebox{135}{$\subseteq$} \\
R\text{-}CONS^{arb}\text{-}oRefl & \supset & R\text{-}CONS^{arb}\text{-}pRefl & \subset & R\text{-}CONS\text{-}pRefl & \subset & R\text{-}CONS\text{-}oRefl \\
\rotatebox{135}{$\subset$} & & /\!/ & & \backslash\!\backslash & & \rotatebox{45}{$\subset$} \\
 & R\text{-}CONS^{arb}\text{-}eRefl & & & & R\text{-}CONS\text{-}eRefl &
\end{array}
$$

Finally, we discuss closedness under union.

Theorem 10. (1) *$T\text{-}CONS^{arb}\text{-}\lambda Refl$ and $T\text{-}CONS\text{-}\lambda Refl$ are closed under recursively enumerable union for all $\lambda \in \{e, p, o, \varepsilon\}$.*

(2) *$CONS^{arb}\text{-}\lambda Refl$, $R\text{-}CONS^{arb}\text{-}\lambda Refl$, $CONS\text{-}\lambda Refl$, and $R\text{-}CONS\text{-}\lambda Refl$ are closed under recursively enumerable union for $\lambda \in \{e, p\}$.*

(3) *$CONS^{arb}\text{-}\lambda Refl$, $R\text{-}CONS^{arb}\text{-}\lambda Refl$, $CONS^{arb}\text{-}\lambda Refl$, and $R\text{-}CONS^{arb}\text{-}\lambda Refl$ are not closed under finite union for $\lambda \in \{o, \varepsilon\}$.*

Proof. $T\text{-}CONS^{arb}$ is closed under union [3,25], the same holds for $T\text{-}CONS$. Since $T\text{-}CONS^{arb}\text{-}\lambda Refl = T\text{-}CONS^{arb}$ as well as $T\text{-}CONS\text{-}\lambda Refl = T\text{-}CONS$, Assertion (1) holds. Furthermore, Assertion (2) follows from Theorem 7.

Let $\mathcal{C}_g$ be as in the proof of Theorem 8, i.e. $\mathcal{C}_g \in R\text{-}CONS^{arb}\text{-}oRefl$ and $\mathcal{C}_g \notin T\text{-}CONS^{arb}$. The latter implies $\mathcal{C} = \mathcal{C}_g \cup \mathcal{C}_{ffs} \notin T\text{-}CONS^{arb}$, which, since $\mathcal{C}$ is initially complete, implies $\mathcal{C} \notin CONS^{arb}$. This yields Assertion (3). $\square$

4.3. Finite learning

It is well-known that there does not exist any finitely learnable initially complete class. So, $IC(FIN^{arb}) = \emptyset$. But clearly, there exist function classes in FIN^{arb}-*eRefl*, hence $IC(FIN^{arb}) \subset FIN^{arb}$-*eRefl*. Nevertheless, the requirement that a class can both be finitely identified and pessimistically reflected is very strong.

Theorem 11. *FIN-pRefl* $\subseteq$ *NUM*.

Proof. Let $\mathcal{C} \in$ *FIN-pRefl*(M, R) and let $(\alpha_n)_{n\in\mathbb{N}}$ be a computable enumeration of $[\mathcal{R}]^c$. Furthermore, let d be the decision predicate for M as in Definition 2. A mechanism g enumerating $\mathcal{C}$ can be defined as follows. Let $n \in \mathbb{N}$.

On input n, first, compute $R(\alpha_n)$. If $R(\alpha_n) = 1$, then compute $M(\alpha_n)$ and $d(\alpha_n)$ (which has to be defined if $R(\alpha_n) = 1$). If $d(\alpha_n) = 1$, then output $M(\alpha_n)$. In all other cases, output some default value. □

For optimistic reflection, the learnable classes become much more powerful.

Theorem 12. FIN^{arb}-*oRefl* $\setminus$ *T-CONS* $\neq \emptyset$.

Proof. In fact, the IIM M_2 defined in the proof of Theorem 8 finitely identifies $\mathcal{C}_g$. Since $\mathcal{C}_g \notin$ *T-CONS*, we are done. □

For all the identification types $\mathcal{IT}$ considered until now, always $\mathcal{IT}$-*pRefl* $\subseteq$ $\mathcal{IT}$-*oRefl* holds. For finite learning, this is no longer the case.

Theorem 13. FIN^{arb}-*pRefl* $\setminus$ *FIN-oRefl* $\neq \emptyset$.

Proof. We prove the theorem by reducing the halting problem. We define $\mathcal{C}_c = \{f \in \mathcal{R} \mid i = f(0),\ \varphi_i(i)\downarrow,\ card\{x \in \mathbb{N} \mid f(x) > 0\} = \phi_i(i)\}$. $\mathcal{C}_c \in FIN^{arb}$-*pRefl* obviously holds: Our IIM M outputs a non-final hypothesis unless it can extract $i = f(0)$ and $\phi_i(i) = card(\{x \in \mathbb{N} \mid f(x) > 0\})$ holds. In this case, it constructs the corresponding function of finite support as final hypothesis. The corresponding pessimistic reflection outputs the value 1 if and only if M computes a final hypothesis and this hypothesis is consistent with the current data.

Now, assume to the contrary that $\mathcal{C}_c \in$ *FIN-oRefl*(M, R). Then, R can be used to decide the halting problem as follows: Let $i \in \mathbb{N}$. Search for a $\tau \in [\mathcal{R}_i]^c$ with $R(\tau) = 0$, where $\mathcal{R}_i$ denotes all recursive functions f with $f(0) = i$. (Since R is an optimistic reflection of M and since M finitely identifies $\mathcal{C}_c$, such a τ must exist.) Now determine $k = card\{(x, y) \in \tau^+ \mid y > 0\}$. If $\phi_i(i) \leq k$, set $\chi_K(i) = 1$, otherwise set $\chi_K(i) = 0$.

Obviously, χ_K is recursive. If $\chi_K(i) = 1$, $\varphi_i(i)$ has been proven to terminate. Now, assume $\chi_K(i) = 0$, but $\varphi_i(i)\downarrow$. Let τ be the segment found in the computation of χ_K. Since R works optimistically, τ is inacceptable. This implies $card\{(x, y) \in \tau^+ \mid y > 0\} > \phi_i(i)$ which contradicts the definition of χ_K. □

By the proof of Theorem 11, one is tempted to guess *FIN-pRefl* $\subseteq$ *NUM*!. However, this is not the case since *NUM*! is not closed under subsets, whereas this is the case for *FIN-pRefl*. An obvious example is the subset $\{f \in \mathcal{R} \mid \exists i \in \mathbb{N} : \varphi_i(i)\uparrow \text{ and } \forall x \in \mathbb{N} : f(x) = i\}$ of all constant functions which is obviously not exactly enumerable (i.e. $\notin$ *NUM*!), but clearly belongs to FIN^{arb}-*eRefl*.

Hence we have:

Corollary 8. *For all* $\lambda \in \{e, p, o, \varepsilon\}$, $\lambda_1 \in \{e, p\}$, *and* $\lambda_2 \in \{o, \varepsilon\}$,

1.	FIN^{arb}-λ*Refl* # *NUM*!.	*FIN*-λ*Refl* # *NUM*!.
2.	FIN^{arb}-λ_1*Refl* $\subset$ *NUM*.	*FIN*-λ_1*Refl* $\subset$ *NUM*.
3.	FIN^{arb}-λ_2*Refl* # *NUM*.	*FIN*-λ_2*Refl* # *NUM*.
4.	FIN^{arb}-λ_2*Refl* # *T-CONS*^{arb}.	*FIN*-λ_2*Refl* # *T-CONS*^{arb}.
	FIN^{arb}-λ_2*Refl* # *T-CONS*.	*FIN*-λ_2*Refl* # *T-CONS*.

For learning in the limit, and consistent learning, the relations between reflective learning from arbitrary and canonical input sequences could be derived from its characterizations by standard identification types. Though for finite learning this is not possible, we have the following equalities.

Theorem 14. 1. FIN^{arb}*-Refl* $=$ *FIN-Refl.*
2. FIN^{arb}*-oRefl* $=$ *FIN-oRefl.*
3. FIN^{arb}*-pRefl* $=$ *FIN-pRefl.*

Proof. First, it is easy to see that any IIM M finitely learning from canonical representations can be transformed into an IIM M' which identifies exactly the same class, i.e. $FIN(M) = FIN^{arb}(M')$ holds. Hence, it suffices to show that any reflection of some type λ working on canonical input sequences can be transformed into a λ-reflection for arbitrary input sequences.

In the following, we need some notation. Let $\tau \in [\mathcal{R}]$. By τ^c we denote the longest initial segment of a canonical representation which is contained in τ.

First, we consider optimistic reflection. So, let M be a finite IIM with decision predicate d and let R be an optimistic reflection for M. Furthermore, let M' (and its decision predicate d') be such that $FIN(M) = FIN^{arb}(M')$. Then, we define a reflection R' as follows. On input τ, determine τ^c and $R(\tau^c)$. Output $R(\tau^c)$.

It is obvious that R' is an optimistic reflection which works on arbitrary input sequences, hence *FIN-oRefl*$(M, R) =$ FIN^{arb}*-oRefl*(M', R') holds.

Analogously, FIN^{arb}*-Refl* $=$ *FIN-Refl* can be verified.

For proving FIN^{arb}*-pRefl* $=$ *FIN-pRefl*, let M be a finite IIM with decision predicate d and let R be a pessimistic reflection for M. Let M' be as above. Then, the following reflection R' obviously pessimistically reflects $FIN^{arb}(M')$ from arbitrary input sequences: On input τ, determine τ^c and $R(\tau^c)$. If $R(\tau^c) = 0$ output 0. Otherwise, i.e. if $R(\tau^c) = 1$, determine $M(\tau^c)$ and $d(\tau^c)$. If $d(\tau^c) = 0$, output 0. Otherwise, test whether or not $M(\tau^c)$ is consistent with τ. In the case $M(\tau^c)$ is consistent with τ, output 1, otherwise output 0. □

Hence, the picture regarding reflective finite learning is as follows.

Corollary 9.

$$\begin{array}{ccccccc}
 & FIN^{arb} & & = & & FIN & \\
 & \cup & & & & \cup & \\
 & FIN^{arb}\text{-}Refl & & = & & FIN\text{-}Refl & \\
\cup & & \cup & & \cup & & \cup \\
FIN^{arb}\text{-}oRefl & \# & FIN^{arb}\text{-}pRefl & = & FIN\text{-}pRefl & \# & FIN\text{-}oRefl \\
\cup & & \cup & & \cup & & \cup \\
 & FIN^{arb}\text{-}eRefl & & & & FIN\text{-}eRefl &
\end{array}$$

Finally, we discuss closedness under union. Consider the following two classes $\{f \in \mathcal{R} \mid \exists n \in \mathbb{N} \forall x \in \mathbb{N} : f(x) > 0 \iff x < n\}$ and $\{f \in \mathcal{R} \mid \forall x \in \mathbb{N} : f(x) = f(0)\}$. Both are FIN^{arb}*-eRefl* learnable, its union however is not discrete and hence $\notin$ *FIN*.

Theorem 15. *FIN^{arb}-λRefl and FIN-λRefl are not closed under finite union for all* $\lambda \in \{e, p, o, \varepsilon\}$.

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References

[1] E.M. Gold, Language identification in the limit, Inform. Control 10 (5) (1967) 447–474.
[2] B. Trakhtenbrot, J. Bārzdinš, Finite Automata – Behaviour and Synthesis, North-Holland, Amsterdam, 1973.
[3] L. Blum, M. Blum, Toward a mathematical theory of inductive inference, Inform. Control 28 (2) (1975) 125–155.
[4] E. Minicozzi, Some natural properties of strong identification in inductive inference, Theoret. Comput. Sci. 2 (3) (1976) 345–360.
[5] E. B. Kinber, T. Zeugmann, Inductive inference of almost everywhere correct programs by reliably working strategies, Elektronische Informationsverarbeitung und Kybernetik 21 (3) (1985) 91–100.
[6] E. Kinber, T. Zeugmann, One-sided error probabilistic inductive inference and reliable frequency identification, Inform. Comput. 92 (2) (1991) 253–284.
[7] T. Zeugmann, Algorithmisches Lernen von Funktionen und Sprachen, Habilitation Thesis, Technical University Darmstadt, Nov. 1993.

[8] Y. Mukouchi, S. Arikawa, Towards a mathematical theory of machine discovery from facts, Theoret. Comput. Sci. 137 (1) (1995) 53–84.
[9] S. Lange, P. Watson, Machine discovery in the presence of incomplete or ambiguous data, in: Algorithmic Learning Theory, 4th International Workshop on Analogical and Inductive Inference, AII '94, 5th International Workshop on Algorithmic Learning Theory, ALT '94, Reinhardsbrunn Castle, Germany, October 1994, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 872, Springer-Verlag, 1994, pp. 438–452.
[10] S. Jain, E. Kinber, R. Wiehagen, T. Zeugmann, On learning of functions refutably, Theoret. Comput. Sci. 298 (1) (2003) 111–143.
[11] S. Jain, Learning with refutation, J. Comput. Syst. Sci. 57 (3) (1998) 356–365.
[12] S. Matsumoto, A. Shinohara, Refutably probably aproximately correct learning, in: S. Arikawa, K. P. Jantke (Eds.), Algorithmic Learning Theory, 4th International Workshop on Analogical and Inductive Inference, AII '94, 5th International Workshop on Algorithmic Learning Theory, ALT '94, Reinhardsbrunn Castle, Germany, October 1994, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 872, Springer, 1994, pp. 469–483.
[13] W. Merkle, F. Stephan, Refuting learning revisited, Theoret. Comput. Sci. 298 (1) (2003) 145–177.
[14] Y. Mukouchi, M. Sato, Refutable language learning with a neighbor system, Theoret. Comput. Sci. 298 (1) (2003) 89–110.
[15] K.P. Jantke, Reflecting and self-confident inductive inference machines, in: Algorithmic Learning Theory, 6th International Workshop, ALT '95, Fukuoka, Japan, October 18-20, 1995, Proceedings, in: Lecture Notes in Artificial Intelligence, vol. 997, Springer, 1995, pp. 282–297.
[16] H. Rogers Jr., Theory of Recursive Functions and Effective Computability, McGraw-Hill, 1967.
[17] M. Blum, A machine independent theory of complexity of recursive functions, J. ACM 14 (2) (1967) 322–336.
[18] K. P. Jantke, H.-R. Beick, Combining postulates of naturalness in inductive inference, Elektronische Informationsverarbeitung und Kybernetik 17 (8–9) (1981) 465–484.
[19] R. Wiehagen, W. Liepe, Charakteristische Eigenschaften von erkennbaren Klassen rekursiver Funktionen, Elektronische Informationsverarbeitung und Kybernetik 12 (8–9) (1976) 421–438.
[20] J. Bārzdinš, Inductive inference of automata, functions and programs, in: Proc. International Congress of Math., Vancouver, 1974, pp. 455–460.
[21] R. Lindner, Algorithmische Erkennung, Habilitation Thesis, University Jena, 1972.
[22] R. Wiehagen, Limes-Erkennung rekursiver Funktionen durch spezielle Strategien, Elektronische Informationsverarbeitung und Kybernetik 12 (1/2) (1976) 93–99.
[23] Unknown Referee, Private communication.
[24] S. Jain, D. Osherson, J. S. Royer, A. Sharma, Systems that Learn. An Introduction to Learning Theory, 2nd edition, MIT Press, Cambridge, MA, London, England, 1999.
[25] R. Wiehagen, T. Zeugmann, Learning and consistency, in: K. P. Jantke, S. Lange (Eds.), Algorithmic Learning for Knowledge-Based Systems, GOSLER Final Report, in: Lecture Notes in Artificial Intelligence, vol. 961, Springer, 1995, pp. 1–24.