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Lagrangian submanifolds in affine symplectic geometry

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Abstract

We uncover the lowest order differential invariants of Lagrangian submanifolds under affine symplectic maps, and find out what happens when they are constant.

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1. Introduction

In an affine space with translation invariant symplectic form, I can try to get two Lagrangian submanifolds into high order of contact, using maps that preserve the affine and symplectic structure. The lowest order local invariants that prevent this are described below, equated with the space of real cubic hypersurfaces in projective space, modulo projective automorphism. Setting these invariants to constants, we will see that there is still remarkable flexibility in the Lagrangian submanifold.

2. Generating functions

Definition 1. An *affine symplectic space* is a real finite-dimensional affine space with translation invariant symplectic form.

Example 1. $\mathbb{R}^{2n}$ with linear coordinates q^i, p_i and symplectic form $dp_i \wedge dq^i$ is an affine symplectic space.

I will work entirely in the smooth category, except as stated explicitly when I use the Cartan–Kähler Theorem. I record a result from symplectic geometry:

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Theorem 1. *Suppose that L is a Lagrangian submanifold of an affine symplectic space, and we choose a point $x \in L$. Then after an affine symplectomorphism, we can identify our affine symplectic space with $V \oplus V^*$ (where V is a vector space), with symplectic form*

$$(v, \xi), (v', \xi') \mapsto \langle \xi', v\rangle - \langle \xi, v'\rangle$$

so that our point becomes $(0,0)$ and the tangent space $T_x L$ becomes $V \oplus 0$. In such an identification, there is a function $S : V \to \mathbb{R}$ whose germ at the origin is uniquely determined, so that near $(0,0)$ the Lagrangian submanifold is identified with

$$\{(q, p):\ p = S'(q)\}$$

and so that $S(0) = 0$, $S'(0) = 0$, $S''(0) = 0$. Every such function S gives rise to a Lagrangian submanifold in this way.

Proof. Fixing a point of the affine symplectic space turns it into a symplectic vector space; for the rest see [1, pp. 161, 402]. □

If a Lagrangian submanifold is given as

$$\{p = S'(q)\} \subset V \otimes V^*$$

then the function S is called a *generating function* of the Lagrangian submanifold. The generating function in the theorem above is determined once the indicated identification is made; hence it is determined up to choices of such identification: the group of linear symplectic transformations of $V \oplus V^*$ which preserve the subspace $V \oplus 0$. Mere linear algebra shows that this group is

$$P = \left\{ \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \right\},$$

where $A \in GL(V)$ and $B : V^* \to V$ is symmetric. Another point of view: we can take V to be the tangent space $T_x L$ and then the choice of S is determined by choice of a Lagrangian subspace complementary to $T_x L$.

Lemma 1. *The subgroup*

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

of P preserves the lowest order term of the Taylor expansion of the generating function S.

Proof. If we change coordinates as

$$\begin{pmatrix} Q \\ P \end{pmatrix} = \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix},$$

then in the new coordinates, there will be some new generating function, $T(Q)$. At all points (q, p) on our Lagrangian submanifold, we have $p = S'(q)$ and $P = T'(Q)$, and

$$Q = q + Bp,$$
$$P = p,$$

so $T'(Q) = S'(q) = S'(Q - BP) = S'(Q - BT'(Q))$ or

$$T'(Q) = S'\big(Q - BT'(Q)\big).$$

Since T and S both vanish to second order, the above equation forces them to have the same lowest order terms in their Taylor expansions. □

The abelian group

$$\begin{pmatrix} I & B \\ 0 & I \end{pmatrix}$$

thus acts on Taylor expansions as a subgroup of the nilpotent group

$$\begin{pmatrix} I & 0 & 0 & \dots \\ * & I & 0 & \dots \\ * & * & I & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where each row represents an order of Taylor coefficients.

Lemma 2. *The group*

$$\begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

acts on generating functions by the contragredient representation of $GL(V)$:

$$S(q) \mapsto S(A^{-1}q).$$

Proof.

$$\frac{\partial}{\partial q} S(A^{-1}q) = (A^{-1})^t S'(A^{-1}q). \qquad \square$$

Theorem 2. *The space of* 2*nd order contact classes of Lagrangian submanifolds in an affine symplectic space of dimension* $2n$ *is canonically identified with the real cubic hypersurfaces in projective space* $\mathbb{C}P^{n-1}$ *modulo real projective automorphisms.*

Proof. We map each pointed Lagrangian submanifold to the third order terms in the Taylor expansion of its generating function. From the last two lemmas, this is well defined. $\square$

Example 2. The Clifford torus:

$$T^n = \{(q,p) \in \mathbb{R}^{2n}\colon q_i^2 + p_i^2 = r_i^2\}$$

where the r_i are some constants. The reader may check that the generating function given near $q = 0$, $p = r$ is naturally identified, by integrating $p = S'(q)$ with

$$S(q) = \sum \frac{r_i^2}{2}\left(\arcsin\frac{q_i}{r_i} + \frac{q_i}{r_i}\sqrt{1 - \frac{q_i^2}{r_i^2}}\right)$$

and that the Taylor expansion is

$$S(q) = \sum r_i q_i - \sum \frac{r_i}{6} q_i^3 + \cdots$$

ignoring 5th and higher order terms. But the subtraction of the linear term simply translates the Lagrangian submanifold, so we may ignore it. Hence, up to affine symplectic transformation, the Clifford torus has generating function

$$S(q) = \frac{1}{6}\sum q_i^3 + \cdots.$$

Example 3. Given any cubic form $C \in S^3V^*$ we can take the graph of its differential

$$L = \{(q, C'(q)) \mid q \in V\}.$$

It is easy to see that not only does this have C as its cubic form, but that the 2nd order contact class of L at *every point* is given by C. Therefore for every cubic form, there is at least one Lagrangian submanifold which has this cubic form representing its second order contact class at all points.

3. Singular behaviour

Recall that a homogeneous polynomial on a vector space is called *singular* if there exists a (possibly complex) line of critical points of the polynomial. Generically among the homogeneous polynomials of fixed degree, this does not happen (where "generic" here indicates the Zariski topology). If a critical line exists for a homogeneous cubic polynomial then a real critical line exists by Bezout's theorem. Generic Lagrangian submanifolds at general points will have generating functions with nonsingular cubic terms (where "generic" here indicates the smooth topology). Such a critical line for the cubic term of the generating function, if real, corresponds to a line with second order contact with the Lagrangian submanifold. So generally a Lagrangian submanifold has no such lines, i.e. it turns away from its own tangent space, like a curve in Euclidean space. By contrast, a surface in Euclidean space at a point of negative Gauss curvature, under "generic circumstances", cuts its own tangent space on a curve with a double point, hence has two such lines of 2nd order contact: the two tangent lines to this curve at the double point.

Example 4. (1) Consider the function

$$S(q) = \frac{1}{3}\left(R^2 - |q|^2\right)^{3/2}\left(|q|^2 - r^2\right)^{3/2},$$

where $r \leqslant |q| \leqslant R,\ q \in \mathbb{R}^n$. The reader may check that $S^1 \times S^{n-1}$ is immersed into $\mathbb{R}^{2n}$ as the Lagrangian submanifold

$$\left\{\left(q, \pm S'(q)\right):\ r \leqslant |q| \leqslant R\right\},$$

which is invariant under the linear symplectic maps induced by rotation of the q variable.

(2) There is another way to get $S^1 \times S^{n-1}$ to immerse as a Lagrangian submanifold: take any immersion of S^1 into the plane, and any Lagrangian immersion of S^{n-1} into $\mathbb{R}^{2(n-1)}$, for instance as

$$\left\{\left(q, \pm f'(q)\right):\ |q| \leqslant 1\right\}$$

where

$$f(q) = \frac{1}{3}\left(1 - |q|^2\right).$$

In either case, by computing the Taylor expansions of the functions f and S the reader may calculate that these Lagrangian submanifolds have everywhere singular cubic forms, but with somewhat different singularities. (In case (1) there is always a linear factor to the cubic term of any generating function about any point, for example.) Moreover these Lagrangian immersions are distinct under symplectomorphism, if in case (2) we take the map of the circle to the plane to be a Jordan curve, for instance, since in case (1) $\int p\,dq$ over the S^1 cycle is zero, which in case (2), $\int p\,dq$ is the area contained inside the Jordan curve.

4. Cubic hypersurfaces in projective space

We reach an insurmountable obstacle: cubic hypersurfaces in projective space cannot be classified up to projective automorphism, except in low dimensions. Homogeneous polynomials of degree d in n variables form a vector space of $\binom{n+d-1}{d}$ dimensions [7]. In the case of cubic polynomials, there are:

$$\begin{array}{ll} \dfrac{n(n-1)(n-2)}{6} & \text{ways to choose distinct indices } i, j, k, \\ n(n-1) & \text{ways to choose } i = j \neq k, \\ n & \text{ways to choose } i = j = k \end{array}$$

so $\binom{n+2}{3} = \frac{n^3+3n^2+2n}{6}$ dimensions. Since we have a group of n^2 dimensions, acting in a faithful representation, any sort of reasonable quotient space must have

$$\binom{n+2}{3} - n^2 = \binom{n}{3}$$

dimensions. For example:

n	$\binom{n}{3}$
3	1
4	4
5	10
6	20
7	35

To convince oneself that the group action is faithful, consider the equivariant map

$$V^* \to S^3 V^*, \quad \lambda \mapsto \lambda^3.$$

There are three types of cubic forms which we wish to single out. First, there are the singular ones.

Definition 2. Take V a finite-dimensional real vector space. We say that a form $F \in S^3 V^*$ is *singular* if its zero locus in $\mathbb{CP}V$ has a singular point, or equivalently if F has a critical point away from the origin in $V \otimes \mathbb{C}$.

Definition 3. For forms of any fixed degree and number of variables, there is an irreducible homogeneous polynomial in the coefficients of these forms, called the *discriminant*, which vanishes precisely on the singular forms. It is unique up to scaling.

For cubic forms in n variables, the discriminant has degree $n2^{n-1}$ [5, p. 79]. Thus the discriminant changes only by a positive factor under linear coordinate changes.

Secondly, there are the semistable cubic forms (see [8, p. 102], [9, p. 194]).

Definition 4. A form $F \in S^3 V^*$ is called *semistable* if there is a polynomial on $S^3 V^*$ (i.e. a polynomial in the coefficients of cubic forms), invariant up to scaling under the action of $GL(V)$ on $S^3 V^*$, which does not vanish on F. Semistable forms constitute a Zariski open set. Forms which are not semistable are called *null*.

Thirdly, there are the stable cubic forms.

Definition 5. A form $F \in S^3 V^*$ is *stable* if (1) only finitely many linear transformations of V leave F invariant, and (2) there is some polynomial p on $S^3 V^*$ invariant up to scaling, not vanishing on F, so that the orbits of $GL(V)$ on which $p \neq 0$ are algebraically closed. Stable forms also constitute a Zariski open set.

We quote some theorems of geometric invariant theory:

Theorem 3. *Nonsingular $\subset$ stable $\subset$ semistable, null $=$ singular.*

Proof. See [9, p. 79]. □

Theorem 4. *The standard representation of the general linear group $GL(V)$ on the space of cubic forms $S^3 V^*$ has a canonical choice of moduli space, $\mathfrak{M}_{3,n}$ ($n = \dim V$), given as the projective variety associated to the algebra of covariants. There is a well defined regular map of the semistable elements*

$$(S^3 V^*)^{ss} \to \mathfrak{M}_{3,n}$$

and the fiber of this map through any stable element is the orbit of that element.

I have a conjecture to make concerning homogeneous polynomials in general:

Definition 6. Say that a homogeneous polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree d is in *nice* form if it has the form

$$p(x) = \sum_i (-1)^{\epsilon_i} x_i^d + q(x),$$

where $q(x)$ has no terms of the form x_i^d or $x_i^{d-1} x_j$.

Alternatively, p is in nice form if it has 1 at each vertex of its Newton polyhedron, and 0 at each node of its Newton polyhedron which is next to a vertex.

Definition 7. A homogeneous polynomial is *nice* if it can be brought to nice form by a linear coordinate change.

Conjecture 1. *Nonsingular implies nice. For degree greater than* 2*, there is precisely one choice of coordinates in which a nice polynomial reaches nice form, up to permutation of coordinates, and* (*if the degree is even*) *switching signs of coordinates.*

The result is obvious for degree $d = 2$, with any n, and is known for $d = 3$, $n \leqslant 3$.

Proposition 1. *There is an open set* (*in the Euclidean topology*) *in* $S^d\mathbb{R}^n$ *of polynomials which are nice.*

Proof. Polynomials in nice form are in codimension n, and they stay in nice form only linear changes of variable given by matrices whose diagonal terms vanish. The result follows by dimension count. □

Corollary 1. *Nice polynomials form a Zariski constructible set, containing a Euclidean open set.*

Over the complex numbers, this proves that generic nonsingular homogeneous polynomials are nice.

Corollary 2. *Nice polynomials of odd degree form a Zariski open set.*

Proof. Use induction on the number of variables, by setting each of them to 0, and use the fact that reduction to nice form is generically possible over complex numbers, to show that the complex linear transformation reducing a real odd degree polynomial to nice form must actually be a real linear transformation. □

The hyperbolic geometry of the moduli space of cubic surfaces (see [2]) might be useful in studying Lagrangian submanifolds.

5. Three points on a projective line

A cubic form in two variables either vanishes everywhere or on at most 3 lines through the origin. From this observation it is easy to see that:

Lemma 3. *Every real cubic polynomial in two variables can be brought to precisely one of the* 5 *normal forms in Table* 1 *by linear change of variables. The second column provides the isotropy group of the normal form. The third describes the condition on a cubic form under which it has the given normal form. where* Σ_3*, the group of permutations of* 3 *letters, is represented by*

$$\Sigma_3 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \right\}.$$

Table 1
Cubic forms in 2 variables

Normal form	Isotropy group	Type
0	$GL(2,\mathbb{R})$	0
$\frac{1}{6}x^3$	$\left\{ \begin{pmatrix} 1 & 0 \\ c & d \end{pmatrix} : d \neq 0 \right\}$	linear^3
$\frac{1}{2}x^2y$	$\left\{ \begin{pmatrix} a & 0 \\ 0 & \frac{1}{a^2} \end{pmatrix} : a \neq 0 \right\}$	$\text{linear}_1 \cdot (\text{linear}_2)^2$
$\frac{1}{6}x^3 + \frac{1}{6}y^3$	$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\}$	(linear)(irred quadratic)
$\frac{1}{2}x^2y - \frac{1}{2}xy^2$	Σ_3	$\text{linear}_1 \cdot \text{linear}_2 \cdot \text{linear}_3$

Proof. Projectivizing, the points $[x:y]$ in $\mathbb{CP}^1$ on which the cubic form vanishes must be a triple of complex points, unless the cubic form vanishes. Because the cubic form is real, the triple of points must be invariant under complex conjugation. There could be three distinct real points, or a real double point and a distinct real point, or a triple real point, or a real point and a pair of complex conjugate points. For instance, if there is a real point and a pair of conjugate complex points, we can arrange that the real point move to $[x:y]=[1:-1]$, because the group of linear transformations acts triply transitively on $\mathbb{RP}^1$. Lets look at the two complex points now. The subset $\mathbb{RP}^1 \subset \mathbb{CP}^1$ splits $\mathbb{CP}^1$ into two hemispheres. Picking one of our complex points, we can check with a little calculation that the real linear fractional transformations which fix it and fix $[1:-1]$ are a finite group. Therefore the real linear fractional transformations fixing $[1:-1]$ have a 2-dimensional orbit on the nonreal points of $\mathbb{CP}^1$, so we can put our complex point wherever we like, as long as we keep it in the same hemisphere. So we put it at $[1/2+i\sqrt{3}/2:1]$. □

Of course the dimension of the manifold of cubic forms having a given normal form is the codimension of the isotropy group. Thus the generic cubic form has normal form among the last two on our list. Indeed these are precisely the nonsingular cubic forms, which we see immediately from the discriminant (see [7, p. 56]), which is

$$\delta(ax^3+3bx^2y+3cxy^2+dy^3)=a^2d^2-3b^2c^2+4b^3d+4ac^3-6abcd$$

and is positive on $\frac{1}{6}x^3+\frac{1}{6}y^3$ and negative on $\frac{1}{2}x^2y-\frac{1}{2}xy^2$.

Proposition 2. *The stable cubic forms in two variables are those on which the discriminant does not vanish. Everything else is unstable. The moduli space $\mathfrak{M}_{3,2}$ is a pair of points.*

6. Cubic forms in 3 variables

Lemma 4. *Every real cubic form in 3 variables can be brought to precisely one of the 15 normal forms described in Table 2 by linear change of variables. The second column provides the isotropy group, the third the condition on a cubic form that it have the given normal form. Here Σ_3 means the same group of 2×2 matrices that occurred in the previous lemma, P is the group of permutation matrices, and $O(2)_x$ is the orthogonal group fixing the x axis. In the case of a split linear times a Lorentzian quadratic, $a^2-b^2=1$. The parameter σ is an invariant of any nonsingular cubic form, and the indicated isotropy group is for each fixed value of σ.*

Proof. For the last row, see [4]. The rest is easy to calculate. For example, the symmetry group of xyz must act on projectivized points $[x:y:z]\in\mathbb{RP}^2$ permuting the lines $(x=0)$, $(y=0)$, $(z=0)$. Therefore, after action of the permutation group P, we have a linear transformation for which all three coordinate axes are eigenspaces, so given by a diagonal matrix. Plugging in a diagonal matrix, one immediately finds the symmetry group as stated for xyz. □

Proposition 3. *On the list in the previous lemma, the last entry (nonsingular) is stable, and the two next to the last entry (nodal) are semistable. The others are unstable. Stability corresponds to nonsingularity, while semistability corresponds to the presence of a single real double point in the associated cubic curve. The moduli space $\mathfrak{M}_{3,3}$ is topologically a figure eight. The forms corresponding to smooth cubic curves with one circuit lie on one loop of the figure eight, while those with two circuits lie on the other. The point in the middle corresponds to the cubic curves with double point.*

Proof. See [9]. □

7. Consequences for Lagrangian surfaces and 3 folds

Theorem 5. *A Lagrangian surface which has nonsingular cubic form invariant at a point, has the same cubic form invariant at all nearby points. If its cubic form invariant is never singular, it has a canonical choice of framing, up to switching the legs when the cubic form is*

$$\frac{x^3}{6}+\frac{y^3}{6}$$

or up to the previously indicated Σ_3 action when the cubic form is

$$\frac{x^2y}{2} - \frac{xy^2}{2}.$$

In the first (*second*) *case, if the fundamental group has no subgroup of index* 2 (*index* 2, 3 *or* 6), *then the surface bears two* [5] *canonical trivializations.*

Proof. The stability of the nonsingular cubic forms arises because they are defined by the condition that the discriminant is nonzero, an open condition. The canonical framing is the choice of linear symplectic coordinates in which the cubic form achieves normal form. □

The lowest order invariant is the choice of cubic form, i.e. third order terms of the generating function, which we have seen belongs to one of five classes, after affine symplectic transformation.

Table 2
Cubic forms in 3 variables

Normal form	Isotropy group	Type
0	$GL(3,\mathbb{R})$	
$\frac{x^3}{6}$	$\begin{pmatrix} 1 & 0 & 0 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$	perfect cube
$\frac{x^2y}{2}$	$\begin{pmatrix} a & 0 & 0 \\ 0 & \frac{1}{a^2} & 0 \\ \cdot & \cdot & \cdot \end{pmatrix}$	square · ind linear
$\frac{x^2y}{2} - \frac{xy^2}{2}$	$\begin{pmatrix} \Sigma_3 & & 0 \\ & & 0 \\ \cdot & \cdot & \cdot \end{pmatrix}$	3 distinct lin dep factors
$\frac{x(x^2+y^2)}{2}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ \cdot & \cdot & \cdot \end{pmatrix}$	lin · dep semidef quad
xyz	$P \cdot \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & (ab)^{-1} \end{pmatrix}$	3 ind factors
$\frac{z(x^2+y^2)}{2}$	$\begin{pmatrix} a \cdot O(2) & & 0 \\ & & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}$	lin · ind semidef quad
$\frac{x(xz-y^2)}{2}$	$\begin{pmatrix} a^2 & 0 & 0 \\ b & a^{-1} & 0 \\ b^2a^{-2} & 2ba^{-3} & a^{-4} \end{pmatrix}$	null lin · lorentz quad
$\frac{z(x^2+y^2-z^2)}{2}$	$O(2)_z$	def lin · lorentz quad
$\frac{x(x^2+y^2-z^2)}{2}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \quad a^2 - b^2 = 1$	split lin · lorentz quad
$\frac{x(x^2+y^2+z^2)}{2}$	$O(2)_x$	lin · def quad
$\frac{x^3}{6} - \frac{y^2z}{2}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & \frac{1}{a^2} \end{pmatrix}$	cuspidal
$\frac{x^3}{6} + \frac{x^2z}{2} - \frac{y^2z}{2}$	$y \mapsto \pm y$	real nodal
$\frac{x^3}{6} - \frac{x^2z}{2} - \frac{y^2z}{2}$	$y \mapsto \pm y$	imag nodal
$\frac{x^3}{6} + \frac{y^3}{6} + \frac{z^3}{6} + \sigma xyz, \quad \sigma \neq \frac{-1}{2}$	P	nonsingular

Theorem 6. *Suppose that L is a Lagrangian surface in an affine symplectic space, with constant cubic form invariant. Then if that cubic form is nonsingular, we have the trivializations indicated in the last result. If the cubic form is* 0 *our Lagrangian surface consists of open subsets of Lagrangian planes. If the cubic form is* $\frac{x^3}{6}$ *then the surface bears a canonical choice of nonvanishing one form* (*given in each tangent space by the function x*). *If the cubic form is* $\frac{1}{2}x^2y$ *then there is a global choice of a pair of line fields* ($x = 0$, $y = 0$).

Proof. If the cubic form vanishes, then the locally defined generating function S satisfies (in any local affine symplectic coordinates q, p)

$$\frac{\partial^3 S}{\partial q^i q^j q^k} = 0,$$

so that S is a quadratic function, and therefore the graph of the gradient of S (i.e. the Lagrangian submanifold) is linear. If the cubic form is x^3, then the symmetry group preserving x^3 in each tangent space, and also preserves x, a linear function on the tangent space and therefore a 1-form. □

These theorems have the obvious topological consequences: a sphere is simply connected, so a smooth choice of finitely many bases of each tangent space is as good as one, i.e. a trivialization. But the sphere has nontrivial tangent bundle. Therefore no Lagrangian sphere can have constant nonsingular cubic form. Moreover, a sphere cannot have cubic form $\frac{x^3}{6}$, because that would provide a global nonvanishing 1-form (given in each tangent space by x), so a global nowhere vanishing 1-form, which the sphere does not have.

Theorem 7. *A Lagrangian submanifold in an affine symplectic space with nonsingular* (*resp. stable*) *cubic form invariant at a point has a nonsingular* (*resp. stable*) *cubic form invariant at all nearby points.*

Proof. The condition for nonsingularity (or stability) is an open condition on the 3-jet. □

Theorem 8. *A Lagrangian* 3 *manifold with everywhere nonsingular cubic form invariant has a canonical framing up to permutation. A Lagrangian* 3 *manifold L with everywhere not unstable cubic form has a canonical homomorphism*

$$\pi_1(L) \mapsto \mathbb{Z} * \mathbb{Z}.$$

A Lagrangian 3 *manifold with constant cubic form invariant equal to*

$$\frac{x^3}{6} + \frac{x^2 z}{2} - \frac{y^2 z}{2}$$

has a canonical choice of framing up to switching the sign of the second vector in the framing. More generally, a Lagrangian 3 *manifold with constant cubic form invariant has a connection with holonomy in the indicated isotropy group.*

Proof. This homomorphism $\pi_1(L) \mapsto \mathbb{Z} * \mathbb{Z}$ is merely the monodromy of the canonical framing, since the moduli space $\mathfrak{M}_{3,3}$ is a figure 8, so has fundamental group $\pi_1(\mathfrak{M}_{3,3}) = \mathbb{Z} * \mathbb{Z}$. □

8. Integrability of the affine symplectic group and Heisenberg's canonical quantization

Theorem 9 *(Grunewald–van Hove). The polynomials of degree less than or equal to* 2 *on an affine symplectic space form a maximal "quantizable" set in the algebra of polynomials.*

For a proof, and a definition of quantizable (in the sense of Heisenberg) see [6].

Theorem 10. *The Hamiltonian flow of a function on an affine symplectic space preserves the affine structure precisely if it is a polynomial of degree at most* 2.

Proof. The Hamiltonian vector field in affine Darboux coordinates,

$$\dot{q} = \frac{\partial H}{\partial p}, \qquad \dot{p} = -\frac{\partial H}{\partial q}$$

is an infinitesimal affine transformation precisely if it is of the form constant + linear. □

Suppose that our affine symplectic space is a vector space V and that the symplectic structure is Ω. We define an antisymmetric linear map $J : V^* \to V$ by the equation

$$\Omega(J\xi, w) = \xi(w).$$

We can describe the "Heisenberg Lie algebra" as the central $\mathbb{R}$ extension of the affine symplectic Lie algebra:

$$\begin{pmatrix} JA & J\xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \mapsto \begin{pmatrix} JA & J\xi \\ 0 & 0 \end{pmatrix}$$

where $A : V \to V^*$ is symmetric, and $\xi \in V^*$. Now we identify the matrix on the left-hand side with the Hamiltonian function

$$h(v) = \frac{1}{2}\langle Av, v\rangle + \langle \xi, v\rangle + c$$

and this is an isomorphism between the Lie bracket and the Poisson bracket, i.e. if we choose

$$f(v) = \frac{1}{2}\langle Av, v\rangle + \langle \xi, v\rangle + c,$$
$$g(v) = \frac{1}{2}\langle Bv, v\rangle + \langle \eta, v\rangle + d,$$

then

$$\{f, g\} = \frac{1}{2}\big\langle (AJB - BJa)v, v\big\rangle + \langle AJ\xi - BJ\eta, v\rangle + \langle \xi, J\eta\rangle.$$

Lemma 5. *For the generic choice of symmetric linear map $A : V \to V^*$ on a symplectic vector space V there exist linear Darboux coordinates $q_1, p_1, \ldots, q_n, p_n$ for V so that A is diagonalized, i.e.*

$$A = \begin{pmatrix} a_1 & & & & \cdots \\ & b_1 & & & \cdots \\ & & a_2 & & \cdots \\ & & & b_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can also write this

$$\big\langle A(q, p), (q, p)\big\rangle = \sum \big(a_j q_j^2 + b_j p_j^2\big).$$

For a proof of this lemma, see [10].

Corollary 3. *The symmetric linear maps A_j*

$$\big\langle A_j(q, p), (q, p)\big\rangle = a_j q_j^2 + b_j p_j^2$$

provide linearly independent Hamiltonian functions

$$h_j(q, p) = \frac{1}{2}\big(a_j q_j^2 + b_j p_j^2\big)$$

(*harmonic oscillators*) *which Poisson commute with one another, and with the Hamiltonian function*

$$h(q, p) = \frac{1}{2}\big\langle A(q, p), (q, p)\big\rangle = \frac{1}{2}\sum \big(a_j q_j^2 + b_j p_j^2\big).$$

Corollary 4. *Suppose that A is also invertible.* (*This holds generically.*) *Define A_j as above, and*

$$\xi_j = A_j A^{-1}\xi.$$

Then the Lie algebra spanned by

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} JA_1 & J\xi_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \dots, \quad \begin{pmatrix} JA_n & J\xi_n & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is commutative, and has dimension $n+1$, and contains

$$\begin{pmatrix} JA & J\xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Moreover it projects to a commutative n-dimensional subalgebra of the affine symplectic Lie algebra under

$$\begin{pmatrix} JB & J\eta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} JB & J\eta \\ 0 & 0 \end{pmatrix}.$$

Proof. The proof is linear algebra, using the obvious result that $JA_jJA_k = 0$ for $j \neq k$. □

Theorem 11. *Every one parameter subgroup of the affine symplectic group is contained in an n parameter abelian subgroup.*

Proof. By Lemma 5, this is true for generic one parameter subgroups (in the Zariski topology), hence it follows for any one parameter subgroup. □

Thus we can reinterpret our results very heuristically: there is an integrable system at work, and we are following its action on its Lagrangian submanifolds. Since these are supposed to "carry the physical data", we want to study the local and global invariants of Lagrangian submanifolds to see what sort of invariant data they are capable of carrying.

9. Extreme points

Definition 8. An (weak) *extreme point* of a submanifold of an affine space is a point at which the restriction of an affine function attains a (degenerate) nondegenerate critical point with positive (semidefinite) definite Hessian.

Proposition 4. *For a Lagrangian surface with cubic form invariant at a given point among 0, $\frac{1}{6}x^3$, $\frac{1}{2}x^2y$ this point is a weak extreme point. If it has cubic form $\frac{1}{6}x^3 + \frac{1}{6}y^3$ then this point is an extreme point. Otherwise it is neither extreme nor weak extreme.*

Proof. For a Lagrangian surface with cubic form invariant 0, the generating function vanishes up to fourth order, say $S(x, y, z) = ax^4 + \cdots$. Then the Lagrangian submanifold is the graph of

$$p_1 = \frac{\partial S}{\partial x}, \qquad p_2 = \frac{\partial S}{\partial y}, \qquad p_3 = \frac{\partial S}{\partial z},$$

all cubic expressions in x, y, z, up to higher order terms. The linear function p_1 attains a critical point when restricted to the Lagrangian submanifold. □

Corollary 5. *A Lagrangian surface with nonsingular cubic form at all points must be everywhere extreme, or nowhere.*

Corollary 6. *A compact Lagrangian surface* (*in an affine symplectic space*) *with nonsingular cubic form at all points must have cubic form*

$$\frac{1}{6}x^3 + \frac{1}{6}y^3$$

(*in some linear coordinates, in each tangent space*).

Proof of the proposition. Consider the problem in Darboux coordinates, with $p = S'(q)$ as our Lagrangian surface, and suppose $S(0) = 0$, $S'(0) = 0$, $S''(0) = 0$. Our affine function f can be assumed to take value 0 at 0. It will vanish on the tangent space, so it looks like

$$f(q, p) = \sum u_i p_i$$

for $(u_1, u_2) \neq 0$. For f to have a nondegenerate maximum on L (i.e. on $p = \partial S/\partial q$),

$$0 = \frac{\partial f}{\partial q_j} = S''(0)u$$

which is vacuously satisfied, and

$$\left(\frac{\partial^2 S}{\partial q_i \partial q_j}\right) < 0$$

which says

$$\left(u_k \frac{\partial^3 S}{\partial q_i \partial q_j \partial q_k}\right) < 0.$$

The rest is a simple calculation. For instance if $u_1, u_2 < 0$ then we get a nondegenerate maximum for f in the expected case $S = \frac{1}{6}x^3 + \frac{1}{6}y^3$. □

Corollary 7. *A Lagrangian submanifold of any dimension in an affine symplectic space, which has nonsingular cubic form at all points, must have everywhere weak extreme points.*

Proof. We can slice by symplectic 3 planes to get to the case of a Lagrangian 3 manifold. The resulting 3 manifold clearly must be nonsingular, since a singularity in its cubic form would provide one in the original cubic form. Now we use our normal form for nonsingular cubic forms in 3 variables to get

$$S = \frac{1}{6}x^3 + \frac{1}{6}y^3 + \frac{1}{6}z^3 + \sigma xyz + \cdots$$

to 3rd order. Now set z and its conjugate momentum variable to 0. □

Example 5. The Clifford torus has extreme points at all points, since it sits in a sphere.

10. Differential equations for prescribing cubic form invariants

Let me fix affine Darboux coordinates on our affine symplectic space. Then the affine symplectic group can be identified with the bundle of symplectic framings:

$$g = \begin{pmatrix} u_1 & \dots & u_n & v^1 & \dots & v^n & x \\ 0 & \dots & 0 & 0 & \dots & 0 & 1 \end{pmatrix},$$

where

$$u_j = \begin{pmatrix} u_j^1 \\ \vdots \\ u_j^{2n} \end{pmatrix}, \qquad v^j = \begin{pmatrix} v_1^j \\ \vdots \\ v_{2n}^j \end{pmatrix}, \qquad x = \begin{pmatrix} q^1 \\ \vdots \\ q^{2n} \\ p_1 \\ \vdots \\ p_n \end{pmatrix}$$

are vectors in $\mathbb{R}^{2n}$ and u's and v's form a symplectic framing:

$$\Omega(u_i, u_j) = \Omega(v^i, v^j) = 0,$$

$$\Omega(u_i, v^j) = \delta_i^j,$$

where $\Omega = dq^i \wedge dp_i$ is our symplectic form. We write

$$g = \begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix}$$

for short. The equation

$$dg = \begin{pmatrix} du & dv & dx \\ 0 & 0 & 1 \end{pmatrix}$$

describes 1 forms on the affine symplectic group, while the left invariant one forms are given by, say

$$g^{-1}\,dg = \begin{pmatrix} \alpha & \beta & \omega \\ \gamma & -{}^t\alpha & \eta \\ 0 & 0 & 0 \end{pmatrix}$$

with β, γ symmetric (and I write ${}^tM$ for the transpose of a matrix M). By setting $dg = g \cdot g^{-1} \cdot dg$ we find the first structure equations:

$$du = u \cdot \alpha + v \cdot \gamma,$$
$$dv = u \cdot \beta - v \cdot {}^t\alpha,$$
$$dx = u \cdot \omega + v \cdot \eta.$$

The second structure equations, which we get from the equation

$$d(g^{-1}\,dg) = -(g^{-1}\,dg) \wedge (g^{-1}\,dg)$$

are

$$d\alpha = -\alpha \wedge \alpha - \beta \wedge \gamma,$$
$$d\beta = -\alpha \wedge \beta + \beta \wedge {}^t\alpha,$$
$$d\gamma = -\gamma \wedge \alpha + {}^t\alpha \wedge \gamma,$$
$$d\omega = -\alpha \wedge \omega - \beta \wedge \eta,$$
$$d\eta = -\gamma \wedge \omega + {}^t\alpha \wedge \eta.$$

Lemma 6. *Under the map*

$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto x$$

from the affine symplectic group to the affine space, the symplectic form from the affine space lifts to $\omega^i \wedge \eta_i$. (*N.B. the lifted form will not be symplectic.*)

Proof. The form $\omega^i \wedge \eta_i$ is left invariant, and the lift of the symplectic form will also be left invariant, since the group preserves the symplectic form. It suffices to check that these forms match at the identity element. □

To any Lagrangian submanifold L in the affine symplectic space, we assign the bundle $B = B_L$, which is the submanifold of the affine symplectic group given as those elements

$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix}$$

such that $x \in L$ and u a framing for T_xL. This is easily seen to be a principal P bundle, where P is a subgroup of the linear symplectic group:

$$P = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix} \begin{pmatrix} I & B \\ 0 & I \end{pmatrix} \right\}.$$

This is the stabilizer of the Lagrangian plane $p = 0$, which we have encountered before. We have the inclusion of $B \subset ASp$ where ASp is the affine symplectic group. Let me pull back the forms

$$\alpha,\ \beta,\ \gamma,\ \omega,\ \eta$$

to B.

Lemma 7. *$\eta_1, \dots, \eta_n$ vanish on B.*

Proof. By left invariance, it suffices to show that these forms vanish on some left translate of B, so I can assume that B contains the identity element of ASp, and show that each η_i vanishes on tangent vectors to B at this point. At the identity element, $\eta_i = dp_i$ and

$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix} = I$$

so $u_i = \frac{\partial}{\partial q^i}$. □

Lemma 8. *$\omega^1 \wedge \cdots \wedge \omega^n$ never vanishes on B.*

The proof is the same as that of the previous lemma.

On our bundle B the second structure equations simplify to

$$\begin{aligned}
d\alpha &= -\alpha \wedge \alpha - \beta \wedge \gamma, \\
d\beta &= -\alpha \wedge \beta + \beta \wedge {}^t\alpha, \\
d\gamma &= -\gamma \wedge \alpha + {}^t\alpha \wedge \gamma, \\
d\omega &= -\alpha \wedge \omega, \\
d\eta &= 0 = -\gamma \wedge \omega
\end{aligned}$$

with all the ω^j independent. The last equation, by Cartan's lemma, says that there exist unique functions S_{ijk}, symmetric in j, k so that $\gamma = S_{ijk}\omega^k$. Since $\gamma_{ij} = \gamma_{ji}$ the S functions are symmetric in all subscripts. Plugging $\gamma_{ij} = S\omega$ into the second structure equations gives

$$\left(dS_{ijk} - S_{ijl}\alpha^l_k - S_{ilk}\alpha^l_j - S_{ljk}\alpha^l_i\right) \wedge \omega^k = 0.$$

Again by Cartan's lemma, we then have well defined DS_{ijkl}, the covariant derivatives of the S, determined by

$$DS_{ijkl}\omega^l = dS_{ijk} - S_{ijl}\alpha^l_k - S_{ilk}\alpha^l_j - S_{ljk}\alpha^l_i.$$

The DS functions are also symmetric in all subscripts.

11. Interpreting the equations

We will need to split our matrix

$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix}$$

into more parts:

$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b & q \\ c & d & p \\ 0 & 0 & 1 \end{pmatrix}.$$

So

$$u_j = a^k_j \frac{\partial}{\partial q^k} + c^k_j \frac{\partial}{\partial p_k},$$

$$v^j = b_j^k \frac{\partial}{\partial q^k} + d_j^k \frac{\partial}{\partial p_k}.$$

Suppose that L is a Lagrangian submanifold of $\mathbb{R}^{2n}$ with the form

$$p = S'(q)$$

and that $S(0) = 0$, $S'(0) = 0$, $S''(0) = 0$.

Lemma 9. *The vector fields*

$$u_j = \frac{\partial}{\partial q^j} + \frac{\partial^2 S}{\partial q^j \partial q^k} \frac{\partial}{\partial p_k},$$
$$v^j = \frac{\partial}{\partial p_j}$$

form a symplectic framing, with the u vector fields tangent to the Lagrangian submanifold $p = S'(q)$.

This is a minor computation. We can map a point $(q, S'(q))$ of our Lagrangian submanifold to the affine symplectic group as

$$g(q) = \begin{pmatrix} I & 0 & q \\ S'' & I & S' \\ 0 & 0 & 1 \end{pmatrix}$$

so that the first columns give our vector fields u and v. Through this map we pull down all of our forms on the affine symplectic group to the Lagrangian submanifold.

$$dg = \begin{pmatrix} 0 & 0 & dq \\ dS'' & 0 & dS' \\ 0 & 0 & 0 \end{pmatrix},$$

so to find $g^{-1}\,dg$,

$$g^{-1} = \begin{pmatrix} I & 0 & -q \\ -S'' & I & -S' + S''q \\ 0 & 0 & 1 \end{pmatrix}$$

which gives

$$g^{-1}\,dg = \begin{pmatrix} 0 & 0 & dq \\ dS'' & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} \alpha & \beta & \omega \\ \gamma & -{}^t\alpha & \eta \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus $\gamma = dS''$ or

$$\gamma_{ij} = \frac{\partial^3 S}{\partial q^i \partial q^j \partial q^k} dq^k = \frac{\partial^3 S}{\partial q^i \partial q^j \partial q^k} \omega^k$$

which shows that the invariants S_{ijk} turn out to be the third derivatives of the generating function. Their covariant derivatives, immediate from the above expressions, turn out to be the fourth derivatives.

12. Setting our invariants to constants

If we have a Lagrangian submanifold on which the cubic form invariant is constant, we can choose (at least locally) a symplectic framing on which the S_{ijk} are constant, say C_{ijk}. This would be a submanifold of the affine symplectic group on which the forms

$$\gamma_{ij} - C_{ijk}\omega^k$$

all vanish. With the assumption that C is symmetric in its indices, a submanifold of the affine symplectic group on which these forms vanish and on which

$$\omega^1 \wedge \cdots \wedge \omega^n \neq 0$$

is precisely such a symplectic framing. We take these forms to generate a system of partial differential equations

$$\gamma_{ij} - C_{ijk}\omega^k = 0$$

whose solution manifolds we will study.

13. The one-dimensional case

All curves in the symplectic plane are Lagrangian. Our second structure equations are

$$\begin{aligned}
d\alpha &= -\beta \wedge \gamma, \\
d\beta &= -2\alpha \wedge \beta, \\
d\gamma &= -2\gamma \wedge \alpha, \\
d\omega &= -\alpha \wedge \omega - \beta \wedge \eta, \\
d\eta &= -\gamma \wedge \omega + \alpha \wedge \eta.
\end{aligned}$$

The tableau for Lagrangian curves is

$$d\eta = -\gamma \wedge \omega \mod \eta.$$

The solutions of course depend on 1 arbitrary function of 1 variable, since a Lagrangian curve can be seen as locally the graph of a function.

Setting $\gamma = C\omega$ for some constant C, and adding this to our ideal, we find the tableau:

$$d(\gamma - C\omega) = 3C\alpha \wedge \omega.$$

So on solutions, $\alpha = A\omega$ for some function A. Again we have 1 function of 1 variable for solutions, unless $C = 0$ in which can we obviously get only lines as solutions. Thus along any curve in the plane it is possible to choice a symplectic framing so that the function f determined by $\gamma = f\omega$ is constant.

Now suppose that we try to set A to a constant.

$$d(\alpha - A\omega) = -C\beta \wedge \omega.$$

So again 1 function of 1 variable, and we can arrange our framing to make A constant along with f. But in this framing $\beta = B\omega$ for some function B. We set this to a constant, and find

$$d(\beta - B\omega) = 0.$$

So our system is Frobenius, and there is a finite parameter family of solutions, given by

$$g^{-1}\,dg = \begin{pmatrix} A & B & 1 \\ C & -A & 0 \\ 0 & 0 & 0 \end{pmatrix} \omega$$

which determines our curve:

$$g(t) = \exp(tM)$$

(up to reparameterization) where

$$M = \begin{pmatrix} A & B & 1 \\ C & -A & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

By scaling, we can assume that $\det M = -1, 0$ or 1. If $\det M = 1$ then the curve through a point x is given by

$$g(t) = \begin{pmatrix} \cos t I + \sin t M & (\sin t I + (\cos t - 1) M^{-1}) x \\ 0 & I \end{pmatrix}.$$

The Lagrangian curve is an ellipse. If $\det M = -1$ then the curve similarly is a hyperbola, and if $\det M = 0$ the curve is a parabola. These are the homogeneous Lagrangian curves in the affine symplectic plane.

14. Lagrangian surfaces and 3-manifolds with fixed cubic form

Proposition 5. *The real analytic Lagrangian surfaces with constant cubic form invariant arise in the generality specified in Table 3.*

Proposition 6. *The real analytic Lagrangian 3 manifolds with constant cubic form invariant arise in the generality specified in Table 4.*

Proof. The precise meaning of the expression "k functions of n variables" comes from Cartan's work on systems of partial differential equations, and is explained in detail in [3]. The differential equations for constant nonsingular cubic invariant are in involution without any prolongation, except in the case of the Fermat cubic

$$x^3 + y^3 + z^3$$

for which a single prolongation reveals involution, with the same Cartan characters as any of the other nonsingular cases. Computational technique for handling Cartan's test is explained in detail in [3]. □

Table 3
Lagrangian surfaces with fixed cubic form

Normal form	Generality
0	all Lagrangian planes
perfect cube	2 functions of 1 variable
$\text{linear}_1 \cdot (\text{linear}_2)^2$	3 functions of 1 variable
linear · (irred quadratic)	1 function of 2 variables (generic)
$\text{linear}_1 \cdot \text{linear}_2 \cdot \text{linear}_3$	1 function of 2 variables (generic)

Table 4
Lagrangian 3 manifolds with fixed cubic form

Normal form	Generality
0	all Lagrangian affine planes
perfect cube	3 functions of 1 variable
square · indep linear	5 functions of 1 variable
3 distinct lin dep factors	1 function of 2 variables
lin · dep semidef quadratic	1 function of 2 variables
3 indep lin factors	1 constant
linear · ind semidef quad	1 constant
null linear · lorentz quad	1 function of 2 variables
def linear · lorentz quad	8 functions of 1 variable
split linear · lorentz quad	8 functions of 1 variable
linear · def quad	2 functions of 2 variables
cuspidal	2 functions of 2 variables
imaginary nodal	3 functions of 2 variables
real nodal	3 functions of 2 variables
nonsingular	3 functions of 2 variables

15. Ruled Lagrangian submanifolds

In the notation established in Section 10, the tangent space to a Lagrangian submanifold in an adapted frame is

$$u_1, \dots, u_n.$$

Moving along the adapted frame bundle, we find

$$\begin{aligned} du_i &= u_j \alpha_i^j + v^j \gamma_{ji} \\ &= u_j \alpha_i^j + v^j S_{ijk} \omega^k \end{aligned}$$

and

$$d(u_1 \wedge \dots \wedge u_n) = \sum_i (-1)^{i+1} \alpha_i^i u_1 \wedge \dots \wedge u_n + \sum_i (-1)^{i+1} u_1 \wedge \dots \wedge u_{i-1} \wedge S_{ijk} \omega^k v^j \wedge u_{i+1} \wedge \dots \wedge u_n.$$

We see that the tangent space is carried along in parallel by a vector tangent to our Lagrangian submanifold precisely if that vector is in the plane field determined by $\gamma = 0$, i.e. by $S_{ijk}\omega^k = 0$. These are precisely the tangent directions in which our cubic form on the given tangent space is invariant under translation, i.e. the "unused variables" of the cubic form. For instance in $\mathbb{R}^n$, the form $C(x_1, \dots, x_k)$ has among its "unused variables" any linear combination of

$$\frac{\partial}{\partial x^{k+1}}, \dots, \frac{\partial}{\partial x^n}.$$

This plane field $\gamma = 0$ is holonomic, since our structure equations give

$$d\gamma = -\gamma \wedge \alpha + {}^t\alpha \wedge \gamma.$$

Moreover, it is a field of Cauchy characteristics (see [3]), so that we can reduce the differential system.

Let $AL_d(\mathbb{R}^{2n})$ be the space of affine sublagrangian d planes in $\mathbb{R}^{2n}$. If we let $L_d(\mathbb{R}^{2n})$ be the linear sublagrangian d planes, then

$$AL_d(\mathbb{R}^{2n}) \to L_d(\mathbb{R}^{2n})$$

is a vector bundle of rank $2n - d$. The dimensions are

$$\begin{aligned} \dim L_d(\mathbb{R}^{2n}) &= k\frac{4n - 3k + 1}{2}, \\ \dim AL_d(\mathbb{R}^{2n}) &= k\frac{4n - 3k + 1}{2} + 2n - k. \end{aligned}$$

We can include $L_d(\mathbb{R}^{2n}) \subset Gr_d(\mathbb{R}^{2n})$, and identify

$$T_P Gr_d(\mathbb{R}^{2n}) \cong \operatorname{Lin}(P, \mathbb{R}^{2n}/P)$$

invariantly under the $GL(2n, \mathbb{R})$ action. The subspace of tangent vectors to $L_d(\mathbb{R}^{2n})$ is identified as follows: given a tangent vector $A \in \operatorname{Lin}(P, \mathbb{R}^{2n}/P)$ we can define the bilinear form

$$v, w \in P \mapsto \Omega(Av, w),$$

where Ω is our symplectic form on $\mathbb{R}^{2n}$.

$$T_P L_d(\mathbb{R}^{2n}) \cong \{A \in \operatorname{Lin}(P, \mathbb{R}^{2n}/P):\ \Omega(A\,\cdot, \cdot) \text{ antisymmetric}\}.$$

This gives a well defined section σ of the bundle $T^* L_d(\mathbb{R}^{2n}) \otimes \Lambda^2 U^*$ where U is the universal bundle on $L_d(\mathbb{R}^{2n})$ by

$$\sigma(A, v, w) = \Omega(Av, w).$$

I will need to refine my notation a little to explain how the Pfaffian system on $AL_d(\mathbb{R}^{2n})$ is determined. I will write my left invariant one forms on the affine symplectic group splitting them into subscripts running $1, \dots, d$ and

$d+1,\dots,n$, as, for example

$$\alpha = \begin{pmatrix} (\alpha)_{11} & (\alpha)_{12} \\ (\alpha)_{21} & (\alpha)_{22} \end{pmatrix},$$
$$\omega = \begin{pmatrix} (\omega)_1 \\ (\omega)_2 \end{pmatrix}.$$

This unfortunate notation allows me to distinguish between α_2^2 and $(\alpha)_{22}$, which hopefully the reader can distinguish as well.

The tangent spaces to the fibers of the projection

$$\pi : ASp(\mathbb{R}^{2n}) \to AL_d(\mathbb{R}^{2n})$$
$$\begin{pmatrix} u & v & x \\ 0 & 0 & 1 \end{pmatrix} \mapsto d \text{ plane through } x \text{ containing } u_1,\dots,u_d$$

are given by the equations

$$(\alpha)_{21} = (\gamma)_{11} = (\gamma)_{12} = \eta = (\omega)_2 = 0.$$

These equations cut out the Lie algebra of the isotropy group of $\mathbb{R}^d \subset \mathbb{R}^{2n}$ among the left invariant vector fields.

Now if we pick a cubic form $C \in S^3\mathbb{R}^n$ with its d unused variables being $x^1,\dots,x^d$, then we can consider the Pfaffian system $\gamma = C\cdot\omega$, $\eta = 0$ on $ASp(\mathbb{R}^{2n})$ and try to "push" it down to $AL_d(\mathbb{R}^{2n})$ along the projection. On each fiber, $ASp(\mathbb{R}^{2n})_P$ of $\pi : ASp(\mathbb{R}^{2n}) \to AL_d(\mathbb{R}^{2n})$ we have a foliation given by the holonomic plane field $\gamma_{ij} = C_{ijk}\omega^k$. The problem of solving the Pfaffian system we started with on $ASp(\mathbb{R}^{2n})$ should resolve in to two problems: solving the reduced system on $AL_d(\mathbb{R}^{2n})$, and then choosing the correct leaf of this foliation above each d plane.

16. The tableau product

We will make an algebraic description of the space of integral elements of our Pfaffian equations.

Definition 9. We define a product

$$A \otimes C \in \mathrm{Lin}(V\otimes V, V)\otimes S^3V^* \mapsto AC \in S^2V^*\otimes \Lambda^2 V^*$$

given by

$$\begin{aligned} AC(u_1,u_2,u_3,u_4) = {} & C\big(u_1,u_2,A(u_3,u_4)\big) + C\big(u_1,A(u_2,u_4),u_3\big) + C\big(A(u_1,u_4),u_2,u_3\big) \\ & - C\big(u_1,u_2,A(u_4,u_3)\big) - C\big(u_1,A(u_2,u_3),u_4\big) - C\big(A(u_1,u_3),u_2,u_4\big). \end{aligned}$$

We define another product

$$B\otimes C \in S^2V\otimes S^3V^* \mapsto BC \in \mathrm{Lin}(V\otimes V, V)$$

by

$$\big\langle \lambda, BC(u,v)\big\rangle := C(u,v,B\lambda)$$

where we take $u, v \in V$, $\lambda \in V^*$ and think of B as a symmetric linear map $B : V^* \to V$.

Proposition 7. *For any $C \in S^3V^*$, C is singular at $w \in V$ iff $AC = 0$ for some (and hence any) $A \in \mathrm{Lin}(V\otimes V, V)$ of the form*

$$A(u,v) = w\big\langle L(u), v\big\rangle$$

with $L : V \to V^*$ any nonzero linear map. Consequently, for C singular, the kernel of

$$A \mapsto AC$$

in $\mathrm{Lin}(V\otimes V, V)$ has dimension at least n^2. This dimension is n^3 precisely for $C = 0$.

Proposition 8. *If $A = BC$ then $AC = 0$. The map $B \mapsto BC$ is injective from S^2V for generic $C \in S^3V^*$, hence*

$$A \mapsto AC$$

has kernel in $\mathrm{Lin}(V \otimes V, V)$ *of dimension at least* $n(n+1)/2$.

Proposition 9. *For any $C \in S^3V^*$, the integral elements of the equations given above are parameterized by the vector space*

$$\{A \in \mathrm{Lin}(V \otimes V, V)\colon\ AC = 0\}.$$

Therefore to apply the Cartan–Kähler theorem to our differential equations, we have only to count the dimension of this vector space, and compare to the sum $\sum ks'_k$ given by the reduced Cartan characters s'_k.

Example 6. Consider the cubic form

$$C(x) = \frac{1}{6}\langle\lambda, x\rangle^3$$

for some nonzero $\lambda \in V^*$. We can choose linear coordinates $x^1, \ldots, x^n$ on our vector space in which $\lambda = x^1$. Then it is easy to compute that $AC = 0$ precisely if

$$A^1_{ij} = 0, \quad i, j > 1 \qquad A^1_{j1} = 3A^1_{1j}.$$

Therefore the kernel of $A \mapsto AC$ has dimension $n^2(n-1) + n$.

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