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A characterization of solutions of the discrete-time algebraic Riccati equation based on quadratic difference forms

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Abstract

This paper is concerned with a characterization of all symmetric solutions to the discrete-time algebraic Riccati equation (DARE). Dissipation theory and quadratic difference forms from the behavioral approach play a central role in this paper. Along the line of the continuous-time results due to Trentelman and Rapisarda [H.L. Trentelman, P. Rapisarda, Pick matrix conditions for sign-definite solutions of the algebraic Riccati equation, SIAM J. Contr. Optim. 40 (3) (2001) 969–991], we show that the solvability of the DARE is equivalent to a certain dissipativity of the associated discrete-time state space system. As a main result, we characterize all unmixed solutions of the DARE using the Pick matrix obtained from the quadratic difference forms. This characterization leads to a necessary and sufficient condition for the existence of a non-negative definite solution. It should be noted that, when we study the DARE and the dissipativity of the discrete-time system, there exist two difficulties which are not seen in the continuous-time case. One is the existence of a storage function which is not a quadratic function of state. Another is the cancellation between the zero and infinite singularities of the dipolynomial spectral matrix associated with the DARE, due to the infinite

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generalized eigenvalues of the associated Hamiltonian pencil. One of the main contributions of this paper is to demonstrate how to resolve these difficulties.

Keywords: Discrete-time algebraic Riccati equation; Dissipative system; Quadratic difference forms; Behavioral approach; Spectral factorization; Storage function

1. Introduction

The algebraic Riccati equation (ARE) plays an important role in many control problems such as linear quadratic optimal control, H^∞ optimal control, optimal filtering, and so on. Since its introduction in control theory, the ARE has been studied extensively.

An important problem related to the ARE is to find a necessary and sufficient condition for the existence of a sign definite solution of the equation. For the continuous-time system, Willems [1] derived a necessary condition for the existence of a non-positive definite solution. But it turned out that this result was not a sufficient condition [2]. Molinari [3] derived a necessary and sufficient condition for the existence of a non-positive definite solution. However, it is impossible to numerically check this condition because it contains the non-negative definiteness of infinite number of matrices [4]. Since then, several attempts have been made to this open problem. From the viewpoint of the behavioral approach, Trentelman and Rapisarda [5] derived a characterization of all unmixed solutions of the ARE by using quadratic differential forms. Their characterization results in a necessary and sufficient condition for the existence of a sign definite solution in terms of a single finite dimensional matrix called the Pick matrix.

The purpose of this paper is to derive a characterization of all symmetric solution to the discrete-time algebraic Riccati equation (DARE) along the line of [5]. In the discrete-time system, a necessary and sufficient condition for the existence and uniqueness of an unmixed solution are obtained by Clements and Wimmer [6]. But, there has never been derived the characterization of the solutions of the DARE so far.

In order to obtain a characterization of solutions of the DARE, we have to overcome the following two difficulties which are not seen in the continuous-time case. One difficulty arises in the construction of a storage function. In the continuous system, since every storage function is a quadratic function of state [7], a solution of the ARE can be obtained from a weighting matrix of a storage function. In contrast, in the discrete-time case, a storage function is not necessarily expressed as a quadratic function of state [8]. Only sufficient conditions have been known so far [8]. Another difficulty is the cancellation between the zero and infinite singularities of the dipolynomial spectral matrix associated with the DARE. This cancellation is due to the well-known fact that the Hamiltonian pencil has zero and infinite generalized eigenvalues [9,10]. We will show how to resolve the above difficulties by developing a spectral factorization algorithm satisfying a certain biproperness condition.

This paper is organized as follows. In Section 2, we review the basic definitions and results from the behavioral system theory. In particular, quadratic difference forms are introduced to formulate the dissipativity of a linear discrete-time system. We give some results related to storage functions in terms of quadratic difference forms. In Section 3, we solve the discrete-time problems as described the above, and derive a necessary and sufficient condition for the existence of a symmetric solution of the DARE. In Section 4, we obtain a characterization of all unmixed solutions of the DARE using the Pick matrix as a main result of this paper. As a corollary of this, we obtain a necessary and sufficient condition for the existence of a non-negative definite solution of the DARE. In Section 5, a numerical example is given in order to demonstrate the procedure for the present characterization of all unmixed solutions. Several preliminary lemmas used in this paper are collected in Appendix A. The proofs of our results are given in Appendix B.

We give the notations used in this paper in the following:

$\mathbb{R}_s^{m\times m}$: the set of $m\times m$ real symmetric matrices
$\mathbb{R}[\xi]$: the set of polynomials with coefficients in $\mathbb{R}$
$\mathbb{R}^{m_1\times m_2}[\xi]$: the set of $m_1\times m_2$ polynomial matrices in the indeterminate ξ
$\mathbb{R}^{m_1\times m_2}(\xi)$: the set of $m_1\times m_2$ rational matrices in the indeterminate ξ
$\mathbb{R}^{m_1\times m_2}[\zeta,\eta]$: the set of $m_1\times m_2$ polynomial matrices in the indeterminates ζ and η
$\mathbb{R}_s^{m\times m}[\zeta,\eta]$: the set of $m\times m$ real symmetric polynomial matrices in the indeterminates ζ and η
$\mathbb{R}[\xi^{-1},\xi]$: the set of dipolynomials in the indeterminate ξ
$\mathbb{R}^{m\times m}[\xi^{-1},\xi]$: the set of $m\times m$ dipolynomial matrices in the indeterminate ξ
$\mathbb{W}^{\mathbb{T}}$: the set of maps from $\mathbb{T}$ to $\mathbb{W}$
$l_2^q := \left\{ w\in(\mathbb{R}^q)^{\mathbb{Z}} \,\middle|\, \sum_{t=-\infty}^{\infty}\|w(t)\|^2<\infty \right\}$
$R(\xi)^{\sim} := R(\xi^{-1})^{\top}$
$M^{(l)}(\xi)$: the lth derivative of the polynomial matrix $M(\xi)$
$\tilde{R} := \begin{bmatrix} R_0 & R_1 & \cdots & R_L \end{bmatrix}$: the coefficient matrix of the polynomial matrix $R(\xi) = R_0 + R_1\xi + \cdots + R_L\xi^L$
$\mathrm{col}(A_1, A_2, \ldots, A_n) = \begin{bmatrix} A_1^{\top} & A_2^{\top} & \cdots & A_n^{\top} \end{bmatrix}^{\top}$
$\mathrm{diag}(a_1, a_2, \ldots, a_m)$: $m\times m$ diagonal matrix with diagonal elements $\{a_1, a_2, \ldots, a_m\}$
rowdim (A): the row dimension of a matrix A
$\Lambda(E,A)$: the set of the generalized eigenvalues of a square matrix pencil $\xi E - A$. This set consists of the finite eigenvalues which are the roots of $\det(\xi E - A)$, and the infinite eigenvalues which are the reciprocals of the zero eigenvalues of $\eta A - E$ (see e.g. [9,10,11,12]).

2. Preliminaries

In this section, we will review the basic definitions and results from the behavioral system theory.

2.1. Linear discrete-time system [13,14,20]

In the behavioral system theory, a dynamical system is defined as a triple $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, where $\mathbb{T}$ is the time axis, and $\mathbb{W}$ is the signal space in which the trajectories take their values on. The behavior $\mathfrak{B} \subseteq \mathbb{W}^{\mathbb{T}}$ is the set of all possible trajectories. In this paper, we will consider a *linear* time-invariant *discrete-time system* whose time axis is $\mathbb{T} = \mathbb{Z}$ and signal space is $\mathbb{W} = \mathbb{R}^q$. Such a Σ is represented by a system of linear constant coefficient difference-algebraic equation as

$$R_0 w + R_1\sigma w + \cdots + R_L\sigma^L w = 0, \tag{1}$$

where $R_0, R_1, \ldots, R_L \in \mathbb{R}^{\bullet\times q}$ and $L \geqslant 0$. The variable $w\in(\mathbb{R}^q)^{\mathbb{Z}}$ is called the *manifest variable*. The operator σ is called the shift operator defined by $(\sigma w)(t) := w(t+1)$ and $(\sigma^{\mathrm{T}} w)(t) := w(t+T)$ for all $T\in\mathbb{Z}$. We call (1) a *kernel representation* of $\mathfrak{B}$. A short hand notation for (1) is

$$R(\sigma)w = 0,$$

where $R(\xi) := R_0 + R_1\xi + \cdots + R_L\xi^L \in \mathbb{R}^{\bullet\times q}[\xi]$. Hence, $\mathfrak{B}$ is given by

$$\mathfrak{B} = \left\{ w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid R(\sigma)w = 0 \right\}.$$

Whenever rank $R(\lambda)$ is constant for all $\lambda \in \mathbb{C}$, there exists a polynomial matrix $M \in \mathbb{R}^{q\times m}[\xi]$ satisfying $R(\xi)M(\xi) = 0$ with $m \geqslant \operatorname{rank} M = q - \operatorname{rank} R$ [13], where 'rank M' is viewed as the normal rank of a polynomial matrix $M(\xi)$. Then, for every $w \in \mathfrak{B}$, there always exists an $\ell \in (\mathbb{R}^m)^{\mathbb{Z}}$ such that

$$w = M(\sigma)\ell. \tag{2}$$

The above system representation is called an *image representation* of $\mathfrak{B}$, and ℓ is an auxiliary variable called a *latent variable* of $\mathfrak{B}$. In terms of the image representation, $\mathfrak{B}$ can be rewritten as

$$\mathfrak{B} = \{ w \in (\mathbb{R}^q)^{\mathbb{Z}} \mid \exists\, \ell \in (\mathbb{R}^m)^{\mathbb{Z}} \text{ s.t. } w = M(\sigma)\ell \}.$$

An image representation of $\mathfrak{B}$ is called *observable* if $M(\sigma)\ell = 0$ implies $\ell = 0$. This is the case if and only if $M(\lambda)$ is right prime, i.e. $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$ [13].

We introduce the notion of *state maps* [14]. $X \in \mathbb{R}^{n\times m}[\xi]$ is said to induce a *state map* for Σ and a latent variable $x = X(\sigma)\ell$ is called a *state variable* for Σ, if x satisfies the *axiom of state*

$$\left\{ \begin{bmatrix} w_1 \\ x_1 \end{bmatrix}, \begin{bmatrix} w_2 \\ x_2 \end{bmatrix} \in \mathfrak{B}_{\text{full}} \text{ and } x_1(0) = x_2(0) \right\} \Longrightarrow \begin{bmatrix} w_1 \\ x_1 \end{bmatrix} \wedge \begin{bmatrix} w_2 \\ x_2 \end{bmatrix} \in \mathfrak{B}_{\text{full}}, \tag{3}$$

where $\mathfrak{B}_{\text{full}}$ is a *full behavior* defined by

$$\mathfrak{B}_{\text{full}} := \{ \operatorname{col}(w, x) \in (\mathbb{R}^{q+n})^{\mathbb{Z}} \mid \exists\, \ell \in (\mathbb{R}^m)^{\mathbb{Z}} \text{ s.t. } w = M(\sigma)\ell,\ x = X(\sigma)\ell \}.$$

In (3), $(v_1 \wedge v_2)(t)$ denotes $(v_1 \wedge v_2)(t) = v_1(t)$ for $t < 0$ and $(v_1 \wedge v_2)(t) = v_2(t)$ for $t \geqslant 0$. It is easily seen that the state map $X(\sigma)$ is not unique. A state map $X(\sigma)$ is said to be *minimal*, if $\operatorname{rowdim}(X) \leqslant \operatorname{rowdim}(X')$ for any other $X' \in \mathbb{R}^{n'\times m}[\xi]$ which induces a state map for Σ [14].

If $w = M(\sigma)\ell$ is an observable image representation, there exists a partition $M(\xi) = \operatorname{col}(Y(\xi), U(\xi))$ satisfying $U \in \mathbb{R}^{m\times m}[\xi]$ is non-singular, and $Y(\xi)U(\xi)^{-1}$ is proper, possibly after permuting the components of w appropriately and, accordingly, the rows of $M(\xi)$ [8]. Such a partition is called a *proper input–output partition* of $M(\xi)$. We can regard $u = U(\sigma)\ell$ and $y = Y(\sigma)\ell$ as input and output, respectively.

Let $X \in \mathbb{R}^{n\times m}[\xi]$ induce a minimal state map for Σ, and let $x = X(\sigma)\ell$. Then, there exist matrices $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ satisfying $x(t+1) = Ax(t) + Bu(t)$ from Proposition IX.2 in [13]. Also, we have the next lemma.

Lemma 1 [7,15]. *Suppose that $X \in \mathbb{R}^{n\times m}[\xi]$ induces a minimal state map for Σ represented by the observable image representation $w = M(\sigma)\ell$. Let $M(\xi) = \operatorname{col}(Y(\xi), U(\xi))$ be a proper input–output partition. We introduce a new polynomial matrix $F \in \mathbb{R}^{p\times m}[\xi]$. Then, the following statements* (i)–(iii) *hold.*

(i) *There exists a matrix $C \in \mathbb{R}^{p\times n}$ satisfying $F(\xi) = CX(\xi)$ if and only if $F(\xi)U(\xi)^{-1}$ is strictly proper.*
(ii) *There exist $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$ satisfying $F(\xi) = CX(\xi) + DU(\xi)$ if and only if $F(\xi)U(\xi)^{-1}$ is proper.*
(iii) *In the case of $p = m$, there exist a matrix $C \in \mathbb{R}^{m\times n}$ and a non-singular matrix $D \in \mathbb{R}^{m\times m}$ satisfying $F(\xi) = CX(\xi) + DU(\xi)$ if and only if $F(\xi)U(\xi)^{-1}$ is biproper.*

2.2. Quadratic difference forms and dissipativity

Consider a two-variable polynomial matrix in $\mathbb{R}^{m_1\times m_2}[\zeta,\eta]$ described by

$$\Phi(\zeta,\eta)=\sum_{i=0}^{N}\sum_{j=0}^{N}\Phi_{ij}\zeta^i\eta^j, \tag{4}$$

where $\Phi_{ij}\in\mathbb{R}^{m_1\times m_2}$ and $N\geqslant 0$. This $\Phi(\zeta,\eta)$ induces a bilinear difference form

$$L_\Phi:(\mathbb{R}^{m_1})^{\mathbb{Z}}\times(\mathbb{R}^{m_2})^{\mathbb{Z}}\to\mathbb{R}^{\mathbb{Z}},\quad L_\Phi(\ell_1,\ell_2)(t):=\sum_{i=0}^{N}\sum_{j=0}^{N}\ell_1(t+i)^\top\Phi_{ij}\ell_2(t+j).$$

This means that ζ and η correspond to the shift operations on $\ell_1(t)$ and $\ell_2(t)$, respectively.

We call $\Phi(\zeta,\eta)$ symmetric when $m_1=m_2=:m$ and $\Phi(\zeta,\eta)^\top=\Phi(\eta,\zeta)$. In this case, $\Phi(\zeta,\eta)$ induces a *quadratic difference form* (QDF)

$$Q_\Phi:(\mathbb{R}^m)^{\mathbb{Z}}\to\mathbb{R}^{\mathbb{Z}},\quad Q_\Phi(\ell)(t):=L_\Phi(\ell,\ell)(t).$$

A QDF $Q_\Phi(\ell)$ is called the *rate of change* of $Q_\Psi(\ell)$ if $Q_\Psi(\ell)(t+1)-Q_\Psi(\ell)(t)=Q_\Phi(\ell)(t)$. In terms of two-variable polynomial matrices, this relationship is expressed equivalently as $(\zeta\eta-1)\Psi(\zeta,\eta)=\Phi(\zeta,\eta)$.

With every $\Phi\in\mathbb{R}_s^{m\times m}[\zeta,\eta]$ in (4), we define its *coefficient matrix* by

$$\tilde{\Phi}:=\begin{bmatrix}\Phi_{00} & \Phi_{01} & \cdots & \Phi_{0N}\\ \Phi_{10} & \Phi_{11} & \cdots & \Phi_{1N}\\ \vdots & \vdots & \ddots & \vdots\\ \Phi_{N0} & \Phi_{N1} & \cdots & \Phi_{NN}\end{bmatrix}\in\mathbb{R}_s^{(N+1)m\times(N+1)m}.$$

For $\Phi\in\mathbb{R}_s^{m\times m}[\zeta,\eta]$, a QDF $Q_\Phi(\ell)$ is called *non-negative* if $Q_\Phi(\ell)(t)\geqslant 0$ for all $\ell\in(\mathbb{R}^m)^{\mathbb{Z}}$ and $t\in\mathbb{Z}$. If $Q_\Phi(\ell)$ is non-negative, and if $Q_\Phi(\ell)=0$ implies $\ell=0$, then $Q_\Phi(\ell)$ is said to be *positive*. Clearly, $Q_\Phi(\ell)$ is non-negative if and only if $\tilde{\Phi}\geqslant 0$.

For $\Phi\in\mathbb{R}_s^{m\times m}[\zeta,\eta]$, its coefficient matrix can be factored as $\tilde{\Phi}=\tilde{M}^\top\Sigma_\Phi\tilde{M}$, where $\Sigma_\Phi\in\mathbb{R}_s^{\operatorname{rank}\tilde{\Phi}\times\operatorname{rank}\tilde{\Phi}}$, $\tilde{M}\in\mathbb{R}^{\operatorname{rank}\tilde{\Phi}\times(N+1)m}$ is of full row rank, and $\det\Sigma_\Phi\neq 0$, i.e. $\operatorname{rank}\Sigma_\Phi=\operatorname{rank}\tilde{\Phi}$. With such a factorization of $\tilde{\Phi}$, we obtain a *canonical factorization* of $\Phi(\zeta,\eta)$ as $\Phi(\zeta,\eta)=M(\zeta)^\top\Sigma_\Phi M(\eta)$, where $M(\xi):=\tilde{M}\operatorname{col}(I_m,\xi I_m,\ldots,\xi^N I_m)\in\mathbb{R}^{\operatorname{rank}\tilde{\Phi}\times m}[\xi]$.

The map ∂ associates a *dipolynomial matrix* with a two-variable polynomial matrix as follows. Given $\Phi\in\mathbb{R}_s^{m\times m}[\zeta,\eta]$, we define the map

$$\partial:\mathbb{R}_s^{m\times m}[\zeta,\eta]\to\mathbb{R}^{m\times m}[\xi^{-1},\xi],\quad \partial\Phi(\xi):=\Phi(\xi^{-1},\xi).$$

A QDF $Q_\Phi(\ell)$ is called *average non-negative*, if $\sum_{t=-\infty}^{\infty}Q_\Phi(\ell)(t)\geqslant 0$ for all $\ell\in l_2^m$. Then, from Proposition 3.1 in [16], $Q_\Phi(\ell)$ is average non-negative if and only if $\partial\Phi(e^{i\omega})\geqslant 0$ holds for all $\omega\in[0,2\pi)$.

Here, we introduce the notion of dissipativity.

Definition 1 [8]. Let $\Pi\in\mathbb{R}_s^{q\times q}[\zeta,\eta]$. A system $\Sigma=(\mathbb{Z},\mathbb{R}^q,\mathfrak{B})$ is called dissipative with respect to the supply rate $Q_\Pi(w)$ if $\sum_{t=-\infty}^{\infty}Q_\Pi(w)(t)\geqslant 0$ holds for all $w\in l_2^q\cap\mathfrak{B}$.

We can think of $Q_\Pi(w)$ as the power delivered to the system Σ. The dissipativity implies that the net flow of energy into the system is non-negative, i.e. the system dissipates energy. Hence, the rate of increase of the energy stored inside of the system does not exceed the power supplied to it.

In the remainder of this section, we assume that $\mathfrak{B}$ has an observable image representation $w = M(\sigma)\ell$, $M \in \mathbb{R}^{q\times m}[\xi]$. Then, Σ is dissipative with respect to the supply rate $Q_\Pi(w)$ if and only if the QDF $Q_\Phi(\ell)$ induced by $\Phi(\zeta,\eta) = M(\zeta)^\top \Pi(\zeta,\eta)M(\eta)$ is average non-negative. Hence, we can describe any claims on the dissipativity with a general QDF in terms of a latent variable.

Definition 2 [8,15]. Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$.

(i) The QDF $Q_\Psi(\ell)$ induced by $\Psi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$ is a storage function for $Q_\Phi(\ell)$ if

$$Q_\Psi(\ell)(t+1) - Q_\Psi(\ell)(t) \leqslant Q_\Phi(\ell)(t) \tag{5}$$

holds for all $t \in \mathbb{Z}$ and $\ell \in (\mathbb{R}^m)^\mathbb{Z}$. We call (5) the dissipation inequality.

(ii) The QDF $Q_\Delta(\ell)$ induced by $\Delta \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$ is a dissipation rate for $Q_\Phi(\ell)$ if

$$\sum_{t=-\infty}^{\infty} Q_\Phi(\ell)(t) = \sum_{t=-\infty}^{\infty} Q_\Delta(\ell)(t) \tag{6}$$

and $Q_\Delta(\ell)(t) \geqslant 0$ hold for all $t \in \mathbb{Z}$ and $\ell \in l_2^m$.

Moreover, there is a one-to-one relation between a storage function $Q_\Psi(\ell)$ and a dissipation rate $Q_\Delta(\ell)$ defined by

$$Q_\Psi(\ell)(t+1) - Q_\Psi(\ell)(t) = Q_\Phi(\ell)(t) - Q_\Delta(\ell)(t), \tag{7}$$

or equivalently,

$$(\zeta\eta - 1)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - \Delta(\zeta,\eta). \tag{8}$$

Eq. (7) is called the dissipation equality.

It follows from Lemma 3.1 in [16] that (6) is equivalent to $\partial\Phi(\lambda) = \partial\Delta(\lambda)$ for all non-zero $\lambda \in \mathbb{C}$.

The next theorem gives a characterization of average non-negativity of $Q_\Phi(\ell)$ in terms of a storage function and a dissipation rate.

Proposition 1 [8]. *Let* $\Phi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$. *The following statements* (i)–(iii) *are equivalent.*

(i) $Q_\Phi(\ell)$ *is average non-negative.*
(ii) $Q_\Phi(\ell)$ *admits a storage function.*
(iii) $Q_\Phi(\ell)$ *admits a dissipation rate.*

In the rest of this section, we restrict our attention to the case where a supply rate $Q_\Pi(w)$ for Σ is induced by a symmetric matrix $\Pi \in \mathbb{R}_s^{q\times q}$. Then, a QDF $Q_\Phi(\ell)$ is induced by a two variable polynomial matrix defined by

$$\Phi(\zeta,\eta) = M(\zeta)^\top \Pi M(\eta). \tag{9}$$

Let $X \in \mathbb{R}^{n\times m}[\xi]$ induce a minimal state map for Σ, and define $x := X(\sigma)\ell \in (\mathbb{R}^n)^\mathbb{Z}$. We factorize a dissipation rate $Q_\Delta(\ell)$ as $\Delta(\zeta,\eta) = F(\zeta)^\top F(\eta)$, $F \in \mathbb{R}^{p\times m}[\xi]$. Then, (8) reduces to

$$(\zeta\eta - 1)\Psi(\zeta,\eta) = M(\zeta)^\top \Pi M(\eta) - F(\zeta)^\top F(\eta). \tag{10}$$

Let $M(\xi) = \mathrm{col}(Y(\xi), U(\xi))$ be a proper input–output partition. Such a partition always exists by the observability assumption of the image representation $w = M(\sigma)\ell$. From Lemma 1 and (10), we obtain the following proposition.

Proposition 2. *Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ be defined by (9). Let $\Psi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ induce a storage function for $Q_\Phi(\ell)$ corresponding to the dissipation rate induced by $\Delta(\zeta, \eta) = F(\zeta)^\top F(\eta)$, $F \in \mathbb{R}^{p\times m}[\xi]$. Then, there exists a symmetric matrix $P \in \mathbb{R}_s^{n\times n}$ satisfying $\Psi(\zeta, \eta) = -X(\zeta)^\top P X(\eta)$, i.e. $Q_\Psi(\ell) = -x^\top P x$ if and only if $F(\xi)U(\xi)^{-1}$ is proper.*

Proof. See Appendix B. □

If a storage function $Q_\Psi(\ell)$ is expressed as $\Psi(\zeta, \eta) = -X(\zeta)^\top P X(\eta)$ for some $P \in \mathbb{R}_s^{n\times n}$, then $Q_\Psi(\ell)$ is said to be a *quadratic function of state*, or simply a *state function*.

Remark 1. In continuous-time systems,[1] since $F(\xi)U(\xi)^{-1}$ is always proper, every storage function is a state function [7]. On the contrary, in the discrete-time case, the same claim does not hold in general. Because there exists a dissipation rate induced by $\Delta(\zeta, \eta) = F(\zeta)^\top F(\eta)$ for which $F(\xi)U(\xi)^{-1}$ is not proper [8]. Only sufficient conditions have been known so far.

We give the following proposition about the smallest storage function under some biproperness restriction.

Proposition 3. *Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ be defined by (9). Assume that $\partial\Phi(\mathrm{e}^{\mathrm{i}\omega}) > 0$ holds for all $\omega \in [0, 2\pi)$. Let $\Psi \in \mathbb{R}_s^{q\times q}[\zeta, \eta]$ induces a storage function for $Q_\Phi(\ell)$ corresponding to the dissipation rate induced by $F(\zeta)^\top F(\eta)$ such that $F \in \mathbb{R}^{m\times m}[\xi]$, $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ and $F(\xi)U(\xi)^{-1}$ is biproper. Let $H \in \mathbb{R}^{m\times m}[\xi]$ be a Schur[2] polynomial matrix such that $\partial\Phi(\xi) = H(\xi)^\sim H(\xi)$ and $H(\xi)U(\xi)^{-1}$ is biproper. Then, the storage function induced by $\Psi^-(\zeta, \eta) = \frac{\Phi(\zeta,\eta) - H(\zeta)^\top H(\eta)}{\zeta\eta - 1}$ satisfies*

$$Q_{\Psi^-}(\ell) \leqslant Q_\Psi(\ell), \quad \forall t \in \mathbb{Z}, \quad \ell \in l_2^m \tag{11}$$

for any other $\Psi(\zeta, \eta)$ satisfying the above conditions.

Proof. See Appendix B. □

3. Solvability condition of the DARE

In this paper, we consider the DARE with the unknown matrix $P \in \mathbb{R}_s^{n\times n}$

$$\begin{aligned} &A^\top P A - P + Q - (A^\top P B + S^\top) V(P)^{-1} (B^\top P A + S) = 0, \\ &V(P) = B^\top P B + R, \end{aligned} \tag{12}$$

[1] In the continuous-time case, the dissipation equality of (8) is replaced by $(\zeta + \eta)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta)$. Also, $\partial\Phi(\xi)$ and $F(\xi)^\sim$ are defined by $\partial\Phi(\xi) := \Phi(-\xi, \xi)$ and $F(\xi)^\sim := F(-\xi)^\top$, respectively.

[2] For a polynomial matrix $F \in \mathbb{R}^{m\times m}[\xi]$, we call it *Schur* (respectively, *anti-Schur*) if $\det F(\lambda) \neq 0$ for all $\lambda \in \mathbb{C}$ such that $|\lambda| \geqslant 1$ (respectively, $|\lambda| \leqslant 1$).

where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $Q \in \mathbb{R}_s^{n\times n}$, $R \in \mathbb{R}_s^{m\times m}$, and $S \in \mathbb{R}^{n\times m}$. The DARE (12) is associated with the linear quadratic optimal control problem of minimizing the quadratic performance index

$$J = \sum_{t=0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\top \Pi \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad \Pi := \begin{bmatrix} Q & S^\top \\ S & R \end{bmatrix}$$

for the system described by the state space equation

$$x(t+1) = Ax(t) + Bu(t), \tag{13}$$

where $x(t) \in \mathbb{R}^n$ is a state variable and $u(t) \in \mathbb{R}^m$ is an input variable. We assume that (A, B) is reachable. Recall that (A, B) is reachable if and only if $\begin{bmatrix} A - \lambda I_n & B \end{bmatrix}$ has full row rank for all $\lambda \in \mathbb{C}$. Hence, the reachability of (A, B) is equivalent to the controllability in the behavioral system theory [13].

We define the manifest variable $w \in (\mathbb{R}^{n+m})^{\mathbb{Z}}$ by $w := \mathrm{col}(x, u)$. Then, the state space equation (13) is equivalent to the kernel representation $R(\sigma)w = 0$ with $R(\xi) := \begin{bmatrix} A - \xi I_n & B \end{bmatrix}$. Hence, this system is defined by $\Sigma := (\mathbb{Z}, \mathbb{R}^{n+m}, \mathfrak{B})$ with the behavior $\mathfrak{B} = \{w \in (\mathbb{R}^{n+m})^{\mathbb{Z}} \mid R(\sigma)w = 0\}$. Since $R(\lambda)$ is assumed to have full row rank for all $\lambda \in \mathbb{C}$, $(\xi I_n - A)^{-1}B$ has a right coprime factorization over the polynomial ring, namely

$$(\xi I_n - A)^{-1}B = X(\xi)U(\xi)^{-1}, \tag{14}$$

where $X \in \mathbb{R}^{n\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ are right coprime. Without loss of generality, we assume $\det U(\xi) = \det(\xi I_n - A)$. By using the coprime factors $X(\xi)$ and $U(\xi)$, the observable image representation of $\mathfrak{B}$ is obtained as

$$w(t) = \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} X(\sigma) \\ U(\sigma) \end{bmatrix} \ell(t), \tag{15}$$

where $\ell \in (\mathbb{R}^m)^{\mathbb{Z}}$ is a latent variable. Since we assumed that (A, B) is reachable, it can be shown that $x = X(\sigma)\ell$ is a minimal state variable for Σ.

Let the QDF

$$Q_\Pi(w) = w^\top \Pi w \tag{16}$$

be a supply rate for Σ. Define the symmetric two-variable polynomial matrix

$$\Phi(\zeta, \eta) := M(\zeta)^\top \Pi M(\eta) \in \mathbb{R}_s^{m\times m}[\zeta, \eta], \quad M(\xi) = \mathrm{col}(X(\xi), U(\xi)). \tag{17}$$

Since $Q_\Pi(w) = Q_\Phi(\ell)$ from (15)–(17), the dissipativity of Σ for the supply rate $Q_\Pi(w)$ is equivalent to the average non-negativity of $Q_\Phi(\ell)$ as explained in Section 2.2. Hence, from now on, we assume

Assumption 1

(i) $\partial\Phi(e^{i\omega}) \geqslant 0$, $\forall \omega \in [0, 2\pi)$. *This implies that the system Σ is dissipative with respect to the supply rate $Q_\Pi(w)$, or equivalently, $Q_\Phi(\ell)$ is average non-negative.*
(ii) $\det \partial\Phi(\xi) \neq 0$ *holds as an element of* $\mathbb{R}[\xi^{-1}, \xi]$.

For a given $P \in \mathbb{R}_s^{n\times n}$, we define

$$\Psi(\zeta, \eta) := -X(\zeta)^\top P X(\eta),$$

$$\Delta(\zeta, \eta) := M(\zeta)^\top L(P) M(\eta), \quad L(P) := \begin{bmatrix} A^\top PA - P + Q & A^\top PB + S^\top \\ B^\top PA + S & V(P) \end{bmatrix}.$$

We easily see from (14) that $\xi X(\xi) = AX(\xi) + BU(\xi)$. It thus follows that $\Psi(\zeta, \eta)$ and $\Delta(\zeta, \eta)$ satisfies

$$\Phi(\zeta, \eta) - \Delta(\zeta, \eta) = (1 - \zeta\eta)X(\zeta)^\top PX(\eta), \tag{18}$$

or equivalently,

$$Q_\Phi(\ell)(t) - Q_\Delta(\ell)(t) = Q_\Psi(\ell)(t+1) - Q_\Psi(\ell)(t) \tag{19}$$

holds for all $\ell \in (\mathbb{R}^m)^\mathbb{Z}$ and for all $t \in \mathbb{Z}$. We obtain the following proposition from Definition 2 and Proposition 1.

Lemma 2. *Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ be defined by* (17). *Then, for $P \in \mathbb{R}_s^{n\times n}$, the following statements* (i)–(iii) *are equivalent.*

(i) $L(P) \geqslant 0$.
(ii) *The QDF $Q_\Phi(\ell)$ is average non-negative, and the QDF $Q_\Psi(\ell) = -x^\top Px$ is a storage function for $Q_\Phi(\ell)$.*
(iii) *The QDF $Q_\Phi(\ell)$ is average non-negative, and the QDF $Q_\Delta(\ell) = w^\top L(P)w$ is a dissipation rate for $Q_\Phi(\ell)$.*

Proof. See Appendix B. □

We define the set of the solutions to the DARE (12) by

$$\mathscr{S} := \left\{P \in \mathbb{R}_s^{n\times n} | P \text{ satisfies the DARE (12) and } V(P) > 0\right\}.$$

Then, we have a necessary condition for $\mathscr{S} \neq \emptyset$ in the following lemma.

Lemma 3. *Let $X \in \mathbb{R}^{n\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ be a right coprime factorization of $(\xi I_n - A)^{-1}B$. Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ be defined by* (17). *Then, for any $P \in \mathscr{S}$, $\partial\Phi(\xi)$ is factorized as $\partial\Phi(\xi) = F_P(\xi)^\sim F_P(\xi)$, where $F_P(\xi) = V(P)^{\frac{1}{2}}(KX(\xi) + U(\xi))$ and $K := V(P)^{-1}(B^\top PA + S)$. Therefore, the following statements* (i)–(iii) *hold.*

(i) *The two-variable polynomial matrices defined by $\Psi(\zeta, \eta) = -X(\zeta)^\top PX(\eta)$ and $\Delta(\zeta, \eta) = F_P(\zeta)^\top F_P(\eta)$ satisfy the dissipation equality* (10). *Thus, they induce a storage function and a dissipation rate for $Q_\Phi(\ell)$, respectively.*
(ii) $\det F_P(\xi) = \sqrt{\det V(P)}\det(\xi I_n - A_P)$, $A_P := A - BV(P)^{-1}(B^\top PA + S)$.
(iii) $\deg\det F_P(\xi) = n$ *holds, and the rational matrix $F_P(\xi)U(\xi)^{-1} = V(P)^{\frac{1}{2}}K(\xi I_n - A)^{-1}B + V(P)^{\frac{1}{2}}$ is biproper.*

Proof. See Appendix B. □

We see from Lemma 3(iii) that a necessary condition for $\mathscr{S} \neq \emptyset$ is that there exists $F \in \mathbb{R}^{m\times m}[\xi]$ satisfying $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ and $F(\xi)U(\xi)^{-1}$ is biproper. The next proposition guarantees that this necessary condition is also sufficient for $\mathscr{S} \neq \emptyset$.

Proposition 4. *Let $X \in \mathbb{R}^{n\times m}[\xi]$ and $U \in \mathbb{R}^{m\times m}[\xi]$ be a right coprime factorization of $(\xi I_n - A)^{-1}B$. Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta, \eta]$ be defined by* (17). *Then, $\mathscr{S} \neq \emptyset$ holds if and only if there exists a polynomial matrix $F \in \mathbb{R}^{m\times m}[\xi]$ satisfying $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ and $F(\xi)U(\xi)^{-1}$ is biproper.*

Proof. See Appendix B. □

We define

$$\mathscr{F} := \left\{ f \in \mathbb{R}[\xi] \;\middle|\; \begin{array}{l} f(\xi) = f_0 + f_1\xi + \cdots + f_n\xi^n,\ f_n > 0, \\ \det \partial\Phi(\xi) = f(\xi)^\sim f(\xi) \end{array} \right\}.$$

In view of Lemma 3 and Proposition 4, the basic idea for solving the DARE (12) is as follows. If we can find a spectral factorization $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ such that $\det F(\xi) = f(\xi)$ and $F(\xi)U(\xi)^{-1}$ is biproper for $f \in \mathscr{F}$, then the solution P corresponding to $f(\xi)$ is obtained by solving the polynomial matrix equation

$$\Phi(\zeta,\eta) - F(\zeta)^\top F(\eta) = (1-\zeta\eta)X(\zeta)^\top P X(\eta). \tag{20}$$

Hence, in order to establish the solvability condition of the DARE (12), we need to show the existence of a factorization such that $F(\xi)U(\xi)^{-1}$ is biproper for any $f \in \mathscr{F}$. But, unlike the continuous-time case, it is not trivial to prove this for the following two reasons.

(i) There holds $\deg\det\partial\Phi(\xi) = 2r \leqslant 2n$ in discrete-time systems, while $\deg\det\partial\Phi(\xi) = 2n$ is always guaranteed in the continuous-time case.[3] Thus, $f \in \mathscr{F}$ can be described by $f(\xi) = \xi^{n-r} f_0(\xi)$, where $f_0 \in \mathbb{R}[\xi]$ is such that $\det\partial\Phi(\xi) = f_0(\xi)^\sim f_0(\xi)$, $\deg f_0(\xi) = r$, and $f_0(0) \neq 0$. Hence, the singularities of $\partial\Phi(\xi)$ are arranged as

$$\underbrace{0,\ldots,0}_{n-r},\ \underbrace{\lambda_1,\ldots,\lambda_r}_{r},\ \underbrace{\lambda_1^{-1},\ldots,\lambda_r^{-1}}_{r},\ \underbrace{\infty,\ldots,\infty}_{n-r},$$

where $\lambda_1,\ldots,\lambda_r$ and $\lambda_1^{-1},\ldots,\lambda_r^{-1}$ are the non-zero roots of $\det\partial\Phi(\xi) = 0$. Note that n finite singularities of $\partial\Phi(\xi)$ are the zeros of $f(\xi)$, while n other singularities including infinities are the zeros of $\xi^n f(\xi^{-1})$. There are cancellations between the zero and infinite singularities in $\det\partial\Phi(\xi)$. Moreover, the singularities of $\partial\Phi(\xi)$ coincide with the eigenvalues of A_P and their reciprocals from Lemma 3(ii). Actually, these are the generalized eigenvalues of the Hamiltonian pencil associated with the DARE (12) [9,10]. Although Popov [18] proved the existence of a factorization $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ such that $\det F(\xi) = f(\xi)$ for $f \in \mathscr{F}$, the biproperness of $F(\xi)U(\xi)^{-1}$ is not guaranteed yet.

(ii) As pointed out in Remark 1, from the existence of a dissipation rate such that $F(\xi)U(\xi)^{-1}$ is not proper, storage functions are not expressed as a state function for discrete-time systems in general. This implies that the necessary condition for $\mathscr{S} \neq \emptyset$ in Lemma 3(i) is not always satisfied.

Example. Consider the case where $X(\xi) = 1$, $U(\xi) = \xi$, $Q = 2$, $R = 1$, and $S = 0$. In this case, we have $n = 1$ and $\Phi(\zeta,\eta) = 2 + \zeta\eta$. It is clear that $\partial\Phi(\xi) = 3$, and hence $\deg\det\partial\Phi(\xi) = 0 < 2 = 2n$. Hence, the singularities of $\partial\Phi(\xi)$ are $\{0,\infty\}$. If we choose $F(\xi) = \sqrt{3}\xi^2$, then $F(\xi)U(\xi)^{-1} = \sqrt{3}\xi$ is not proper. Taking $\Delta(\zeta,\eta) = F(\zeta)^\top F(\eta) = 3\zeta^2\eta^2$ yields $\Psi(\zeta,\eta) = -2 - 3\zeta\eta$. Since $F(\xi)U(\xi)^{-1}$ is not proper, by Proposition 2, this $\Psi(\zeta,\eta)$ induces a storage function which cannot be expressed as a state function. Indeed, the induced storage function $Q_\Psi(\ell) = -2x^2 - 3u^2$ depends not only on the state but also on the input.

In the remainder of this section, we will discuss how to overcome the above difficulties peculiar to the discrete-time case. More specifically, we will present a method for constructing a spectral factor $F(\xi)$ that satisfies the biproperness condition of $F(\xi)U(\xi)^{-1}$.

[3] The degree of a dipolynomial $\phi(\xi) = \phi_L\xi^L + \cdots + \phi_l\xi^l$ ($\phi_L, \phi_l \neq 0$, $L \geqslant l$) is defined by $\deg\phi(\xi) = L - l$ [17].

For this purpose, we assume that $U(\xi)$ is column reduced without loss of generality. Otherwise, we can always obtain such a coprime factorization as follows. There always exists a unimodular matrix $V \in \mathbb{R}^{m\times m}[\xi]$ such that $U'(\xi) := U(\xi)V(\xi)$ is column reduced (see [19, p. 386]). Then, the image representation of (14) is equivalently rewritten as $w = M'(\sigma)\ell'$, where $M'(\xi) = M(\xi)V(\xi)$ and $\ell' = V(\sigma)^{-1}\ell$. Let n_j $(j = 1, 2, \ldots, m)$ be the degree of the jth column vector of $U(\xi)$. We define the diagonal polynomial matrix $U_\mathrm{d}(\xi) := \mathrm{diag}(\xi^{n_1}, \xi^{n_2}, \ldots, \xi^{n_m})$, $n_1 + n_2 + \cdots + n_m = n$. It is well-known that $U(\xi)$ is column reduced if and only if $U(\xi)U_\mathrm{d}(\xi)^{-1}$ is biproper [19]. Moreover, the identity $F(\xi)U(\xi)^{-1} = F(\xi)U_\mathrm{d}(\xi)^{-1} \cdot U_\mathrm{d}(\xi)U(\xi)^{-1}$ implies that $F(\xi)U(\xi)^{-1}$ is biproper if and only if $F(\xi)U_\mathrm{d}(\xi)^{-1}$ is biproper. Therefore, we have only to check the biproperness of $F(\xi)U_\mathrm{d}(\xi)^{-1}$. A spectral factor $F(\xi)$ satisfying the biproperness condition can be obtained by the following iterative algorithm.

Algorithm 1

Step 1: Let $f \in \mathscr{F}$ be given. Then, $f(\xi)$ is expressed as $f(\xi) = \xi^{n-r} f_0(\xi)$, where $f_0 \in \mathbb{R}[\xi]$ satisfies $\deg\det f_0(\xi) = r$ and $\det \partial\Phi(\xi) = f_0(\xi)^\sim f_0(\xi)$. It is clear from the definition that $f_0(0) \neq 0$. Find a factor $F_0 \in \mathbb{R}^{m\times m}[\xi]$ satisfying $\partial\Phi(\xi) = F_0(\xi)^\sim F_0(\xi)$ and $\det F_0(\xi) = f_0(\xi)$. The existence of such an $F_0(\xi)$ is guaranteed for any $f \in \mathscr{F}$ (see e.g. [18]). Note also that $F_0(\xi)U_\mathrm{d}(\xi)^{-1}$ is always proper.

Step 2: If $\deg\det F_k(\xi) = n$, then stop, and the desired factor $F(\xi)$ is obtained by $F(\xi) := F_k(\xi)$. Otherwise, go to Step 3.

Step 3: At the $(k+1)$th iteration, we define $H_k := \lim_{|\xi|\to\infty} F_k(\xi)U_\mathrm{d}(\xi)^{-1}$. Let ρ_k denote the rank deficiency of H_k, namely $\mathrm{rank}\, H_k = m - \rho_k$. There exists an orthogonal matrix $Z_k \in \mathbb{R}^{m\times m}$ such that $Z_k H_k = \begin{bmatrix} 0_{\rho_k\times m} \\ \hline \hat{H}_k \end{bmatrix}$, where $\hat{H}_k \in \mathbb{R}^{(m-\rho_k)\times m}$ is of full row rank. Then, $Z_k F_k(\xi)$ is expressed as

$$Z_k F_k(\xi) = \begin{bmatrix} f_{11}^{(k)}(\xi) & f_{12}^{(k)}(\xi) & \cdots & f_{1m}^{(k)}(\xi) \\ f_{21}^{(k)}(\xi) & f_{22}^{(k)}(\xi) & \cdots & f_{2m}^{(k)}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ f_{\rho_k 1}^{(k)}(\xi) & f_{\rho_k 2}^{(k)}(\xi) & \cdots & f_{\rho_k m}^{(k)}(\xi) \\ \hline & \hat{F}_k(\xi) & & \end{bmatrix}$$

where $\hat{F}_k \in \mathbb{R}^{(m-\rho_k)\times m}[\xi]$ and $f_{ij}^{(k)} \in \mathbb{R}[\xi]$ satisfy

$$\lim_{|\xi|\to\infty} \hat{F}_k(\xi)U_\mathrm{d}(\xi)^{-1} = \hat{H}_k \ : \ \text{full row rank},$$

$$\lim_{|\xi|\to\infty} f_{ij}^{(k)}(\xi)\xi^{-n_j} = 0 \quad (i = 1, 2, \ldots, \rho_k;\ j = 1, 2, \ldots, m).$$

Define $\mu_{ki} := \min_{j\in[1,m]}\{n_j - \deg f_{ij}^{(k)}(\xi)\}$ for $i = 1, 2, \ldots, \rho_k$, and form a unitary polynomial matrix $W_k(\xi) := \mathrm{diag}(\xi^{\mu_{k1}}, \xi^{\mu_{k2}}, \ldots, \xi^{\mu_{k\rho_k}}, 1, \ldots, 1) \in \mathbb{R}_\mathrm{s}^{m\times m}[\xi]$. Update the polynomial matrix $F_k(\xi)$ by $F_{k+1}(\xi) := W_k(\xi)Z_k F_k(\xi)$, and go back to Step 2.

We obtain the next lemma since it can be shown that $F(\xi)U_\mathrm{d}(\xi)^{-1}$ is biproper for the factor $F(\xi)$ obtained from Algorithm 1.

Lemma 4. *For every $f \in \mathscr{F}$, there exists a polynomial matrix $F \in \mathbb{R}^{m\times m}[\xi]$ satisfying $\partial\Phi(\xi) = F(\xi)^{\sim}F(\xi)$, $\det F(\xi) = f(\xi)$, and $F(\xi)U(\xi)^{-1}$ is biproper.*

Proof. See Appendix B. □

Summarizing Lemmas 3, 4 and Proposition 4, we conclude that the solvability condition of the DARE (12) is given by Theorem 1.

Theorem 1. *There exists a real symmetric solution of the DARE (12), i.e. $\mathscr{S} \neq \emptyset$ if and only if Assumption 1 is satisfied.*

Proof. See Appendix B. □

4. Characterization of all unmixed solutions

In this section, we derive a characterization of all unmixed solutions of the DARE (12). A solution $P \in \mathscr{S}$ is called *unmixed* if $\Lambda(I_n, A_P) \cap \Lambda(A_P, I_n) = \emptyset$ is satisfied,[4] where $A_P = A - BV(P)^{-1}(B^\top PA + S)$. We define the set of all unmixed solutions by $\mathscr{S}_{\rm unm}$. Also, we define

$$\mathscr{F}_{\rm cop} := \{f \in \mathscr{F} \mid f(\xi) \text{ and } \xi^n f(\xi^{-1}) \text{ are coprime}\}.$$

It is straightforward to verify under Assumption 1 that $\mathscr{F}_{\rm cop} \neq \emptyset$ if and only if $\partial\Phi(\mathrm{e}^{\mathrm{i}\omega}) > 0$ for all $\omega \in [0, 2\pi)$. Hence, we see that $\mathscr{S}_{\rm unm} \neq \emptyset$ if and only if $\mathscr{F}_{\rm cop} \neq \emptyset$ from Lemma 3(ii). In the following, we assume a more restrictive condition than Assumption 1.

Assumption 1′. $\partial\Phi(\mathrm{e}^{\mathrm{i}\omega}) > 0$ *holds for all* $\omega \in [0, 2\pi)$.

We define the map

$$\mathrm{Ric} : \mathscr{F}_{\rm cop} \to \mathscr{S}_{\rm unm}$$

as follows. For $f \in \mathscr{F}_{\rm cop}$, find a factorization $\partial\Phi(\xi) = F(\xi)^{\sim}F(\xi)$ such that $F(\xi)U(\xi)^{-1}$ is biproper and $\det F(\xi) = f(\xi)$. Then, $P_f = \mathrm{Ric}(f)$ is defined as a unique solution of the equation (20).

Proposition 5. *Under the Assumption 1′, the map* Ric *is well-defined and bijective.*

Proof. See Appendix B. □

We consider the relationship between the map Ric and the characterization of all unmixed solutions. For a given $f \in \mathscr{F}_{\rm cop}$, let $F \in \mathbb{R}^{m\times m}[\xi]$ be such that $\partial\Phi(\xi) = F(\xi)^{\sim}F(\xi)$ and $\det F(\xi) = f(\xi)$. Let $P_f = \mathrm{Ric}(f)$.

We now define the Pick matrix which plays a central role in our characterization of all unmixed solutions. For $f \in \mathscr{F}_{\rm cop}$, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{C}$ are the roots of $f(\xi) = 0$. Note that these roots are not necessarily distinct. Let λ_i have the partial multiplicity $d_i \geqslant 1$. Then, $d_1 +$

[4] Our definition of the unmixed solution is slightly different from the definition by [6] in that we do not allow A_P to have an eigenvalue on the unit circle.

$d_2 + \cdots + d_k = n$. Let $F^{(i)} \in \mathbb{R}^{m\times m}[\xi]$ be the ith derivative of $F(\xi)$. By the result of Theorem 3.2.16 in [20], there exist non-zero vectors $a_{ij} \in \mathbb{C}^m$ $(j = 0, 1, \ldots, d_i - 1)$ such that

$$\sum_{j=l}^{d_i-1} \binom{j}{l} F^{(j-l)}(\lambda_i)a_{ij} = 0 \quad (l = 0, 1, \ldots, d_i - 1). \tag{21}$$

Using these vectors, we form the matrix $V_i \in \mathbb{C}^{d_i m\times d_i}$ as

$$V_i := \begin{bmatrix} \binom{0}{0} a_{i0} & \binom{1}{1} a_{i1} & \cdots & \binom{d_i-2}{d_i-2} a_{id_i-2} & \binom{d_i-1}{d_i-1} a_{id_i-1} \\ \binom{1}{0} a_{i1} & \binom{2}{1} a_{i2} & \cdots & \binom{d_i-1}{d_i-2} a_{id_i-1} & 0 \\ \vdots & \vdots & & 0 & 0 \\ \binom{d_i-2}{0} a_{id_i-2} & \binom{d_i-1}{1} a_{id_i-1} & & \vdots & \vdots \\ \binom{d_i-1}{0} a_{id_i-1} & 0 & \cdots & 0 & 0 \end{bmatrix}. \tag{22}$$

For $i, j = 1, 2, \ldots, k$, we construct the matrix $\Lambda_{ij} \in \mathbb{C}^{d_j\times d_j}$ by

$$\Lambda_{ij} := \sum_{s=0}^{d_j-1} \left(\frac{\bar{\lambda}_i}{1-\bar{\lambda}_i\lambda_j}\right)^s L_j^s, \quad L_i := \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \ddots & 0 & 0 \\ 0 & 2 & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & d_i-1 & 0 \end{bmatrix}. \tag{23}$$

Finally, we define the *Pick matrix* associated with $f(\xi)$ by the matrix $T_f \in \mathbb{C}^{n\times n}$ whose (i, j)th block $T_{ij} \in \mathbb{C}^{d_i m\times d_j m}(i, j = 1, 2, \ldots, k)$ is given by

$$T_{ij} := \sum_{l=0}^{\min\{d_i,d_j\}-1} \frac{1}{(1-\bar{\lambda}_i\lambda_j)^{2l+1}} \left(\Lambda_{ji}^* L_i^\top\right)^l \Lambda_{ji}^* V_i^* \Theta_{ij} V_j \Lambda_{ij} \left(L_j\Lambda_{ij}\right)^l. \tag{24}$$

In (24), $\Theta_{ij} \in \mathbb{C}^{d_i m\times d_j m}$ is the matrix whose (r, s)th block is given by $\Phi^{(r,s)}(\zeta, \eta)$ $(r = 0, 1, \ldots, d_i - 1, s = 0, 1, \ldots, d_j - 1)$ and $\Phi^{(r,s)}(\zeta, \eta)$ denotes the (r, s)th derivative with respect to ζ and η of $\Phi(\zeta, \eta)$.

We derive the relationship between the solution P_f and the Pick matrix T_f in the following. Recall that P_f is uniquely determined by

$$\Phi(\zeta, \eta) - F(\zeta)^\top F(\eta) = (1 - \zeta\eta)X(\zeta)^\top P_f X(\eta). \tag{25}$$

The (r, s)th partial derivative of (25) is given by

$$\begin{aligned} &\Phi^{(r,s)}(\zeta, \eta) - F^{(r)}(\zeta)^\top F^{(s)}(\eta) \\ &\quad = X^{(r)}(\zeta)^\top P_f X^{(s)}(\eta) - \frac{\mathrm{d}^r}{\mathrm{d}\zeta^r}\{\zeta X(\zeta)\}^\top P_f \frac{\mathrm{d}^s}{\mathrm{d}\eta^s}\{\eta X(\eta)\} \end{aligned} \tag{26}$$

for $r = 0, 1, \ldots, d_i - 1$, $s = 0, 1, \ldots, d_j - 1$. Since

$$\frac{\mathrm{d}^l}{\mathrm{d}\xi^l}\{\xi X(\xi)\} = \xi X^{(l)}(\xi) + l X^{(l-1)}(\xi), \tag{27}$$

the right hand side of (26) can be rewritten as

$$\begin{aligned}\Phi^{(r,s)}(\zeta,\eta) &- F^{(r)}(\zeta)^\top F^{(s)}(\eta) \\ &= (1-\zeta\eta)X^{(r)}(\zeta)^\top P_f X^{(s)}(\eta) - rsX^{(r-1)}(\zeta)^\top P_f X^{(s-1)}(\eta) \\ &\quad - r\eta X^{(r-1)}(\zeta)^\top P_f X^{(s)}(\eta) - s\zeta X^{(r)}(\zeta)^\top P_f X^{(s-1)}(\eta).\end{aligned} \tag{28}$$

Substituting $(\zeta,\eta) = (\bar{\lambda}_i, \lambda_j)$ into (28) yields

$$\begin{aligned}\Phi^{(r,s)}(\bar{\lambda}_i,\lambda_j) &- F^{(r)}(\bar{\lambda}_i)^\top F^{(s)}(\lambda_j) \\ &= (1-\bar{\lambda}_i\lambda_j)X^{(r)}(\bar{\lambda}_i)^\top P_f X^{(s)}(\lambda_j) - rsX^{(r-1)}(\bar{\lambda}_i)^\top P_f X^{(s-1)}(\lambda_j) \\ &\quad - r\lambda_j X^{(r-1)}(\bar{\lambda}_i)^\top P_f X^{(s)}(\lambda_j) - s\bar{\lambda}_i X^{(r)}(\bar{\lambda}_i)^\top P_f X^{(s-1)}(\lambda_j).\end{aligned} \tag{29}$$

From (29), the direct calculation of $V_i^*\Theta_{ij}V_j$ yields

$$V_i^*\Theta_{ij}V_j = (1-\bar{\lambda}_i\lambda_j)S_i^*P_fS_j - L_i^\top S_i^*P_fS_jL_j - \lambda_jL_i^\top S_i^*P_fS_j - \bar{\lambda}_iS_i^*P_fS_jL_j, \tag{30}$$

where

$$S_i := \begin{bmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \cdots & X^{(d_i-1)}(\lambda_i)\end{bmatrix} V_i. \tag{31}$$

Notice that the terms involving $F^{(r)}(\bar{\lambda}_i)^\top F^{(s)}(\lambda_j)$ vanish, because a straightforward calculation shows

$$N_iV_i = V_iL_i, \quad N_i := \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ 0 & 0 & 2I_m & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & (d_i-1)I_m \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \tag{32}$$

and (21) implies

$$\begin{bmatrix} F(\lambda_i) & F^{(1)}(\lambda_i) & \cdots & F^{(d_i-1)}(\lambda_i)\end{bmatrix} V_i = 0. \tag{33}$$

Pre- and post-multiplying (30) by Λ_{ji}^* and Λ_{ij}, respectively, we obtain

$$\Lambda_{ji}^*V_i^*\Theta_{ij}V_j\Lambda_{ij} = (1-\bar{\lambda}_i\lambda_j)S_i^*P_fS_j - \frac{1}{1-\bar{\lambda}_i\lambda_j}\Lambda_{ji}^*L_i^\top S_i^*P_fS_jL_j\Lambda_{ij}. \tag{34}$$

From $(L_j\Lambda_{ij})^{d_j} = 0$, constructing the matrix T_{ij} yields

$$T_{ij} = S_i^*P_fS_j. \tag{35}$$

Since (35) holds for all $i, j = 1, 2, \ldots, k$, we obtain

$$T_f = S_f^*P_fS_f. \tag{36}$$

In (36), $S_f \in \mathbb{C}^{n\times n}$ is called the *zero state matrix* associated with $f(\xi)$ defined by

$$S_f := \begin{bmatrix} S_1 & S_2 & \cdots & S_k\end{bmatrix}. \tag{37}$$

We can prove that S_f is non-singular under Assumption 1′. Hence, we obtain a characterization of all unmixed solutions as a main result of this paper.

Theorem 2. *Under Assumption 1′, all unmixed solutions to the DARE (12) are parametrized by*

$$\mathrm{Ric}(f) = (S_f^*)^{-1} T_f S_f^{-1}, \quad f \in \mathscr{F}_{\mathrm{cop}}.$$

Proof. See Appendix B. □

Using Propositions 3, 5, and Theorem 2, the largest solution of the DARE (12) is given by $P_h = (S_h^*)^{-1} T_h S_h^{-1}$, where $h \in \mathscr{F}_{\mathrm{cop}}$ is Schur. Hence, we obtain a necessary and sufficient condition for the existence of a non-negative definite solution of the DARE (12).

Corollary 1. *Under Assumptions 1′, let $h \in \mathscr{F}_{\mathrm{cop}}$ be Schur. Then, the DARE (12) has a non-negative definite solution if and only if T_h is non-negative definite.*

Proof. See Appendix B. □

5. Numerical example

Consider the DARE (12) with the coefficient matrices

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -\frac{3}{2} \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}, \quad R = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Note that $n = 2$ in this case. One of the right coprime factorizations of $(\xi I_2 - A)^{-1} B$ is given by

$$X(\xi) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad U(\xi) = \begin{bmatrix} \xi & -1 \\ -1 & \xi + \frac{3}{2} \end{bmatrix}.$$

The corresponding two-variable polynomial and dipolynomial matrices are

$$\Phi(\zeta, \eta) = \begin{bmatrix} 3 + 2\zeta\eta & -2\zeta - 1 \\ -2\eta - 1 & 4 + \zeta + \eta \end{bmatrix} \quad \text{and} \quad \partial\Phi(\xi) = \begin{bmatrix} 5 & -2\xi^{-1} - 1 \\ -2\xi - 1 & 4 + \xi + \xi^{-1} \end{bmatrix},$$

respectively. It is easy to verify by direct calculation that $\partial\Phi(\mathrm{e}^{\mathrm{i}\omega}) > 0$ for all $\omega \in [0, 2\pi)$. Since $\det \partial\Phi(\xi) = 3\xi + 15 + 3\xi^{-1}$, we have $\deg\det\partial\Phi(\xi) = 2 < 4 = 2n$, i.e. there is a cancellation between ξ and ξ^{-1}. $\partial\Phi(\xi)$ has four singularities

$$\left\{ \lambda_1 = \frac{-5 + \sqrt{21}}{2},\ \lambda_2 = \frac{-5 - \sqrt{21}}{2},\ \lambda_3 = 0,\ \lambda_4 = \infty \right\}.$$

Then, $\mathscr{F}_{\mathrm{cop}}$ consists of two elements $h(\xi) = h'(\xi - \lambda_1)(\xi - \lambda_3)$ and $a(\xi) = a'(\xi - \lambda_2)(\xi - \lambda_3)$, where $h', a' > 0$ satisfy $h'a' = 15$.

We first choose the Schur polynomial $h(\xi)$, and compute the corresponding solution, i.e. the largest solution of the DARE (12). A spectral factor satisfying the biproperness condition and $\det H(\xi) = h(\xi)$ is given by

$$H(\xi) = \begin{bmatrix} \frac{(4-\sqrt{21})\sqrt{7395+1530\sqrt{21}}}{85}\xi & -\frac{\sqrt{7395+1530\sqrt{21}}}{85}\xi \\ \frac{\sqrt{782+102\sqrt{21}}}{17}\xi & \frac{(-23+3\sqrt{21})\sqrt{782+102\sqrt{21}}}{340} - \frac{\sqrt{782+102\sqrt{21}}}{34}\xi \end{bmatrix}.$$

Solving $H(\lambda_i)V_i = 0$ $(i = 1, 3)$ yield $V_1 = \begin{bmatrix} 1 \\ 4 - \sqrt{21} \end{bmatrix}$ and $V_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then, the zero state matrix S_h and Pick matrix T_h are given by $S_h = \begin{bmatrix} 1 & 1 \\ 4 - \sqrt{21} & 0 \end{bmatrix}$ and $T_h = \begin{bmatrix} \frac{15-3\sqrt{21}}{2} & -1 + \sqrt{21} \\ -1 + \sqrt{21} & 3 \end{bmatrix}$, respectively.

Since the eigenvalues of T_h are $\frac{47+3\sqrt{21}+\sqrt{758-78\sqrt{21}}}{20}$ and $\frac{47+3\sqrt{21}-\sqrt{758-78\sqrt{21}}}{20}$, the corresponding solution P_h is positive definite. Indeed, we obtain $P_h = \begin{bmatrix} 3 & -1 \\ -1 & \frac{17+3\sqrt{21}}{10} \end{bmatrix}$ which is positive definite. Moreover, the closed-loop zeros are 0 and $\frac{-5+\sqrt{21}}{2}$, which coincide with the roots of $h(\xi) = 0$. Thus, P_h is the stabilizing solution of the DARE (12).

Next, we choose $a(\xi)$. Similarly to the above case, the zero state matrix and Pick matrix are given by $S_a = \begin{bmatrix} 1 & 1 \\ 4+\sqrt{21} & 0 \end{bmatrix}$ and $T_a = \begin{bmatrix} \frac{15+3\sqrt{21}}{2} & -1-\sqrt{21} \\ -1-\sqrt{21} & 3 \end{bmatrix}$, respectively. We obtain the indefinite solution as $P_a = \begin{bmatrix} 3 & -1 \\ -1 & \frac{17-3\sqrt{21}}{10} \end{bmatrix}$, which is neither stabilizing nor anti-stabilizing solution of the DARE (12).

6. Conclusion

In this paper, we have derived the characterization of all unmixed solutions of the DARE (12) based on QDF. Moreover, we have obtained a necessary and sufficient condition for the existence of a non-negative definite solution.

Using the QDF and the dipolynomial matrix associated with the DARE, we have shown that the existence of a real symmetric solution of the DARE is equivalent to a certain biproperness condition of a spectral factorization of the dipolynomial matrix. It appeared that the discrete-time problem was that there does not always exist such a spectral factorization. We have solved this problem by developing a spectral factorization algorithm satisfying the above condition, and shown that the solvability of the DARE is equivalent to a certain dissipativity of the associated discrete-time state space system. Also, we have shown that the singularities of the dipolynomial matrix coincide with the generalized eigenvalues of the associated Hamiltonian pencil. Such a relationship has never been considered from a behavioral viewpoint so far.

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Appendix A. Preliminary lemmas

We here collect some preliminary lemmas that will be used in the proofs in Appendix B.

Lemma A 1. *For a given* $\Phi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$, *we assume that* $\partial\Phi(e^{i\omega}) \geqslant 0$ *holds for all* $\omega \in [0, 2\pi)$. *Let* $f_0 \in \mathbb{R}[\xi]$ *be an arbitrary polynomial satisfying* $\det \partial\Phi(\xi) = f_0(\xi)^{\sim} f_0(\xi)$ *and* $f_0(0) \neq 0$. *Then, there exists a polynomial matrix* $F_0 \in \mathbb{R}^{m\times m}[\xi]$ *satisfying* $\partial\Phi(\xi) = F_0(\xi)^{\sim} F_0(\xi)$, $\det F_0(\xi) = f_0(\xi)$ *and* $F_0(\xi)U(\xi)^{-1}$ *is proper.*

Proof. It is obvious from Theorem 1 in §37 of [18] that there exists a square polynomial matrix $F_0 \in \mathbb{R}^{m\times m}[\xi]$ satisfying $\partial\Phi(\xi) = F_0(\xi)^{\sim} F_0(\xi)$ and $\det F_0(\xi) = f_0(\xi)$. Then, $\Delta_0(\zeta,\eta) := F_0(\zeta)^{\top} F_0(\eta)$ induces a dissipation rate for $Q_\Phi(\ell)$. Let $F_0(\xi)$ be expressed by $F_0(\xi) = F_{0,0} + F_{0,1}\xi + \cdots + F_{0,r'}\xi^{r'}$, where $F_{0,0}, F_{0,1}, \ldots, F_{0,r'} \in \mathbb{R}^{m\times m}$ and $F_{0,r'} \neq 0$. Since $\det F_{0,0} = \det F_0(0) = f_0(0) \neq 0$, we obtain

$$\operatorname{rank} \Delta_0(0,0) = \operatorname{rank} F_{0,0}^\top F_{0,0} = \operatorname{rank} \tilde{F}_0^\top \tilde{F}_0 = \operatorname{rank} \tilde{\Delta}_0.$$

Hence, by Lemma A 2 below, the storage function corresponding to the dissipation rate $Q_{\Delta_0}(\ell)$ is given by $-X(\zeta)^\top P_0 X(\eta)$ for some $P_0 \in \mathbb{R}_s^{n\times n}$. This is equivalent to the properness of $F_0(\xi)U(\xi)^{-1}$ from Proposition 2. □

Lemma A 2 [8]. *Suppose that $X \in \mathbb{R}^{n\times m}[\xi]$ induces a minimal state map for Σ. Let $\Phi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$. Let $\Delta \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$ induce a dissipation rate for $Q_\Phi(\ell)$. Let $\Psi \in \mathbb{R}_s^{m\times m}[\zeta,\eta]$ induce a storage function for $Q_\Phi(\ell)$ corresponding to the dissipation rate induced by $\Delta(\zeta,\eta)$. Assume that* $\operatorname{rank} \tilde{\Delta} = \operatorname{rank} \Delta(0,0)$. *Then, there exists a symmetric matrix $P \in \mathbb{R}_s^{n\times n}$ satisfying $\Psi(\zeta,\eta) = -X(\zeta)^\top P X(\eta)$.*

Lemma A 3. *Let $M \in \mathbb{R}^{(n+m)\times m}[\xi]$ be defined by (17). Then, the mapping*

$$w_0 : (\mathbb{R}^m)^{\mathbb{Z}} \to \mathbb{R}^{n+m}, \quad w_0(\ell) := (M(\sigma)(\ell))\,(0)$$

is surjective. Therefore, the coefficient matrix $\tilde{M}$ has full row rank.

Proof. The proof is omitted because the lemma can be proved in the same way as the continuous-time case [5]. □

Appendix B. Proofs

Proof of Proposition 2. Let $\Psi(\zeta,\eta) = G(\zeta)^\top \Sigma_\Psi G(\eta)$, $G \in \mathbb{R}^{\operatorname{rank}\tilde{\Psi}\times m}[\xi]$ be a canonical factorization, where $\Sigma_\Psi \in \mathbb{R}_s^{\operatorname{rank}\tilde{\Psi}\times\operatorname{rank}\tilde{\Psi}}$ is non-singular. Substituting the above factorization and the proper input–output partition of $M(\xi)$ into (10) yield

$$(\zeta\eta - 1)G(\zeta)^\top \Sigma_\Psi G(\eta) = \begin{bmatrix} Y(\zeta) \\ U(\zeta) \end{bmatrix}^\top \Pi \begin{bmatrix} Y(\eta) \\ U(\eta) \end{bmatrix} - F(\zeta)^\top F(\eta). \tag{B.1}$$

Pre- and post-multiplying (B.1) by $U(\zeta)^{-\top}$ and $U(\eta)^{-1}$, we get

$$\begin{aligned}&(\zeta\eta - 1)U(\zeta)^{-\top} G(\zeta)^\top \Sigma_\Psi G(\eta) U(\eta)^{-1} \\ &\quad = \begin{bmatrix} Y(\zeta)U(\zeta)^{-1} \\ I_m \end{bmatrix}^\top \Pi \begin{bmatrix} Y(\eta)U(\eta)^{-1} \\ I_m \end{bmatrix} - U(\zeta)^{-\top} F(\zeta)^\top F(\eta) U(\eta)^{-1}.\end{aligned} \tag{B.2}$$

Since $Y(\xi)U(\xi)^{-1}$ is proper, we can see from (B.2) that $G(\xi)U(\xi)^{-1}$ is strictly proper if and only if $F(\xi)U(\xi)^{-1}$ is proper. From Lemma 1, the strict properness of $G(\xi)U(\xi)^{-1}$ is equivalent to the existence of $H \in \mathbb{R}^{\operatorname{rank}\tilde{\Psi}\times n}$ such that $G(\xi) = HX(\xi)$. In this case, $\Psi(\zeta,\eta)$ is expressed as $\Psi(\zeta,\eta) = -X(\zeta)^\top P X(\eta)$ with $P = -H^\top \Sigma_\Psi H$. This completes the proof. □

Proof of Proposition 3. The dissipation equalities associated with the dissipation rates $F(\zeta)^\top F(\eta)$ and $H(\zeta)^\top H(\eta)$ are given by

$$Q_\Psi(\ell)(t+1) - Q_\Psi(\ell)(t) = Q_\Phi(\ell)(t) - \|F(\sigma)\ell(t)\|^2, \tag{B.3}$$

$$Q_{\Psi^-}(\ell)(t+1) - Q_{\Psi^-}(\ell)(t) = Q_\Phi(\ell)(t) - \|H(\sigma)\ell(t)\|^2, \tag{B.4}$$

respectively. Subtracting (B.3) from (B.4) yields

$$Q_{(\Psi^- - \Psi)}(\ell)(t+1) - Q_{(\Psi^- - \Psi)}(\ell)(t) = \|F(\sigma)\ell(t)\|^2 - \|H(\sigma)\ell(t)\|^2. \tag{B.5}$$

Let $X \in \mathbb{R}^{n\times m}[\xi]$ induce a minimal state map for Σ, and define $x := X(\sigma)\ell$. Since both $F(\xi)U(\xi)^{-1}$ and $H(\xi)U(\xi)^{-1}$ is biproper, we see from Proposition 2 that $\Psi(\zeta,\eta)$ and $\Psi^-(\zeta,\eta)$ can be expressed as $\Psi(\zeta,\eta) = -X(\zeta)^\top PX(\eta)$ and $\Psi^-(\zeta,\eta) = -X(\zeta)^\top P_h X(\eta)$ for some $P \in \mathbb{R}_s^{n\times n}$ and $P_h \in \mathbb{R}_s^{n\times n}$, respectively. Thus, (B.5) can be rewritten as

$$\begin{aligned} &x(t+1)^\top(P_h - P)x(t+1) - x(t)^\top(P_h - P)x(t) \\ &\quad = \|F(\sigma)\ell(t)\|^2 - \|H(\sigma)\ell(t)\|^2. \end{aligned} \tag{B.6}$$

We now show that there exists a latent variable satisfying $H(\sigma)\ell(t) = 0$ $(t \geqslant 0)$ and $x(0) = (X(\sigma)\ell)(0) = x_0$ for any $x_0 \in \mathbb{R}^n$. Since $X(\xi)$ induces a minimal state map, there exist $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ satisfying

$$x(t+1) = Ax(t) + Bu(t), \tag{B.7}$$

where $u := U(\sigma)\ell$ serves as an input for Σ. By Lemma 1(iii), there exist a matrix $C_h \in \mathbb{R}^{m\times n}$ and a non-singular matrix $D_h \in \mathbb{R}^{m\times m}$ satisfying $H(\xi) = C_h X(\xi) + D_h U(\xi)$ since $H(\xi)U(\xi)^{-1}$ is biproper. This implies $H(\sigma)\ell(t) = C_h x(t) + D_h u(t)$. Then, $H(\sigma)\ell(t) = 0$ $(t \geqslant 0)$ is equivalent to

$$C_h x(t) + D_h u(t) = 0 \quad (t \geqslant 0). \tag{B.8}$$

It is obvious that the state space equation

$$x(t+1) = (A - BD_h^{-1}C_h)x(t), \quad x(0) = x_0$$

has a solution for any initial state $x_0 \in \mathbb{R}^n$. By taking $u(t) = -D_h^{-1}C_h x(t)$ $(t \geqslant 0)$ for such a solution, both (B.7) and (B.8) are fulfilled. This clearly implies that there always exists a latent variable satisfying the requirements described above.

To complete the proof, we choose the latent variable so that $H(\sigma)\ell(t) = 0$ $(t \geqslant 0)$. Recall that $\ell(t) \to 0$ $(t \to \infty)$ holds because $H(\xi)$ is Schur. Hence, we get $x(t) \to 0$ $(t \to \infty)$. Then, summing (B.6) up from $t = 0$ to $t = \infty$ along the above trajectory yields $x(0)^\top(P_h - P)x(0) = \sum_{t=0}^{\infty} \|F(\sigma)\ell(t)\|^2 \geqslant 0$. Since $x(0)$ is arbitrary, it follows that $P_h - P$ is non-negative definite. This is equivalent to $Q_{\Psi^-}(\ell)(t) \leqslant Q_\Psi(\ell)(t)$ for all $\ell \in (\mathbb{R}^m)^{\mathbb{Z}}$ and for all $t \in \mathbb{Z}$. $\square$

Proof of Lemma 2. (iii) ⇒ (i) We easily see from (iii) that $(M(\sigma)\ell)^\top L(P)M(\sigma)\ell \geqslant 0$ for all $\ell \in l_2^m$ and $t \in \mathbb{Z}$. This is the case if only if $\tilde{M}^\top L(P)\tilde{M} \geqslant 0$, which is equivalent to $L(P) \geqslant 0$ from Lemma A 3.

(i) ⇒ (iii) Summing up the dissipation equality (19) from $t = -\infty$ to $t = \infty$ yields $\sum_{t=-\infty}^{\infty} Q_\Phi(\ell)(t) - \sum_{t=-\infty}^{\infty} Q_\Delta(\ell)(t) = 0$. Since $Q_\Delta(\ell) = w^\top L(P)w$ and $L(P) \geqslant 0$, we obtain $\sum_{t=-\infty}^{\infty} Q_\Phi(\ell)(t) = \sum_{t=-\infty}^{\infty} Q_\Delta(\ell)(t) \geqslant 0$.

(ii) ⇔ (iii) The equivalence of (ii) and (iii) follows immediately from the fact that $\Psi(\zeta,\eta)$ and $\Delta(\zeta,\eta)$ satisfy the dissipation equality of (10). $\square$

Proof of Lemma 3. Let P be an element of $\mathscr{S}$. Since $V(P) > 0$ holds from the definition of $\mathscr{S}$, it follows from (12) and the definition of $L(P)$ that

$$L(P) = \begin{bmatrix} K & I_m \end{bmatrix}^\top V(P) \begin{bmatrix} K & I_m \end{bmatrix} \geqslant 0, \tag{B.9}$$

where $K = V(P)^{-1}(B^\top PA + S)$. Then, the QDF $Q_\Delta(\ell) = w^\top L(P)w$ induced by $\Delta(\zeta, \eta) = M(\zeta)^\top L(P)M(\eta)$ is a dissipation rate for $Q_\Phi(\ell)$ by Lemma 2. Moreover, pre- and post-multiplying (B.9) by $M(\zeta)^\top$ and $M(\eta)$ yield

$$\Delta(\zeta, \eta) = \{KX(\zeta) + U(\zeta)\}^\top V(P)\{KX(\eta) + U(\eta)\} = F_P(\zeta)^\top F_P(\eta).$$

This implies that $\partial\Phi(\xi) = F_P(\xi)^\sim F_P(\xi)$ holds, because $\partial\Phi(\xi) = \partial\Delta(\xi)$.

(i) It is clear that $\Psi(\zeta, \eta)$ and $\Delta(\zeta, \eta)$ satisfy the dissipation equality (10). It follows that they induce a storage function and a dissipation rate for $Q_\Phi(\ell)$, respectively.

(ii) By the identity

$$\begin{bmatrix} I_n & -X(\xi) \\ K & U(\xi) \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ K & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & KX(\xi) + U(\xi) \end{bmatrix} \begin{bmatrix} I_n & -X(\xi) \\ 0 & I_m \end{bmatrix},$$

we easily see that

$$\begin{aligned} &\det(\xi I_n - A)\det\{KX(\xi) + U(\xi)\} \\ &\quad = \det\begin{bmatrix} \xi I_n - A & B \\ 0 & I_m \end{bmatrix} \det\begin{bmatrix} I_n & -X(\xi) \\ K & U(\xi) \end{bmatrix} = \det\left\{\begin{bmatrix} \xi I_n - A & B \\ 0 & I_m \end{bmatrix}\begin{bmatrix} I_n & -X(\xi) \\ K & U(\xi) \end{bmatrix}\right\} \\ &\quad = \det\begin{bmatrix} \xi I_n - A + BK & 0 \\ K & U(\xi) \end{bmatrix} = \det U(\xi)\det(\xi I_n - A + BK). \end{aligned}$$

Since we have assumed $\det U(\xi) = \det(\xi I_n - A)$, we get

$$\det F_P(\xi) = \det\{V(P)^{\frac{1}{2}}\}\det\{KX(\xi) + U(\xi)\} = \sqrt{\det V(P)}\det(\xi I_n - A_P).$$

(iii) Since $\deg\det F_P(\xi) = n$ from (ii), we can get $F_P(\xi)U(\xi)^{-1} = V(P)^{\frac{1}{2}}\{K(\xi I_n - A)^{-1}B + V(P)^{\frac{1}{2}}\}$ by direct calculation using (14). Moreover, the inverse of $F_P(\xi)U(\xi)^{-1}$ is given by a proper rational matrix $\{F_P(\xi)U(\xi)^{-1}\}^{-1} = \{I - K(\xi I_n - A_P)^{-1}B\}V(P)^{-\frac{1}{2}}$. This implies that $F_P(\xi)U(\xi)^{-1}$ is biproper. □

Proof of Proposition 4. We have only to prove the sufficiency, since the necessity immediately follows from Lemma 3.

Let $F \in \mathbb{R}^{m\times m}[\xi]$ be a polynomial matrix such that $\partial\Phi(\xi) = F(\xi)^\sim F(\xi)$ and $F(\xi)U(\xi)^{-1}$ is biproper. From Lemma 1, there exist a matrix $C \in \mathbb{R}^{m\times n}$ and a non-singular matrix $D \in \mathbb{R}^{m\times m}$ satisfying $F(\xi) = CX(\xi) + DU(\xi)$. Let a dissipation rate $Q_\Delta(\ell)$ be induced by $\Delta(\zeta, \eta) = F(\zeta)^\top F(\eta)$. Then, by Proposition 2, the corresponding storage function is expressed as $\Psi(\zeta, \eta) = -X(\zeta)^\top PX(\eta)$ for some $P \in \mathbb{R}_s^{n\times n}$. Furthermore, it follows from Lemma 2 that $Q_\Delta(\ell)$ can be expressed as $Q_\Delta(\ell) = w^\top L(P)w$, or equivalently, $\Delta(\zeta, \eta) = M(\zeta)^\top L(P)M(\eta)$. Hence, we obtain

$$M(\zeta)^\top \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix} M(\eta) = M(\zeta)^\top L(P)M(\eta). \tag{B.10}$$

In terms of coefficient matrices, (B.10) is equivalent to $\tilde{M}^\top \begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix} \tilde{M} = \tilde{M}^\top L(P)\tilde{M}$. Since $\tilde{M}$ has full row rank from Lemma A 3, it reduces to

$$\begin{bmatrix} C & D \end{bmatrix}^\top \begin{bmatrix} C & D \end{bmatrix} = L(P). \tag{B.11}$$

Since D is nonsingular, we get rank $L(P) = m$. We also see that $V(P) = D^\top D > 0$ from the (2, 2) block of (B.11), which implies that rank $L(P) = \operatorname{rank} V(P) = m$. This is the case if and only if the Schur complement of $V(P)$ in $L(P)$ is equal to zero, namely

$$A^\top PA - P + Q - (A^\top PB + S^\top)V(P)^{-1}(B^\top PA + S) = 0.$$

It is clear from this equation that P satisfies the DARE (12). This completes the proof of sufficiency. □

Proof of Lemma 4. The proof is completed by showing that $F(\xi)U_\mathrm{d}(\xi)^{-1}$ is biproper for the polynomial matrix $F \in \mathbb{R}^{m\times m}[\xi]$ obtained from Algorithm 1. Notice that $F(\xi)U_\mathrm{d}(\xi)^{-1}$ is proper because, at each iteration, $W_k(\xi)$ and Z_k are chosen so that $H_{k+1} = \lim_{|\xi|\to\infty} F_{k+1}(\xi)U_\mathrm{d}(\xi)^{-1}$ be finite.

Suppose that Algorithm 1 stopped at $k = l$ with $F(\xi) = F_l(\xi)$ and $\deg\det F(\xi) = n$. To prove the biproperness, we assume on the contrary that $F(\xi)U_\mathrm{d}(\xi)^{-1}$ is proper but not biproper. In this case, we carry out Step 2 for $k = l$ once more to obtain $F_{l+1}(\xi) = W_l(\xi)Z_lF(\xi)$. It follows from the definitions of $W_k(\xi)$ and Z_k that $F_{l+1}(\xi)U_\mathrm{d}(\xi)^{-1}$ is proper, and hence $\lim_{|\xi|\to\infty} F_{l+1}(\xi)U_\mathrm{d}(\xi)^{-1}$ is finite. On the other hand, it is not difficult to see that

$$\deg\det F_{l+1}(\xi) = \sum_{i=1}^{\rho_l} \mu_{li} + n > n = \deg\det U_\mathrm{d}(\xi).$$

The strict inequality in the above equation immediately follows from the assumption that $\lim_{|\xi|\to\infty} F_{l+1}(\xi)U_\mathrm{d}(\xi)^{-1}$ is singular. Hence, we obtain

$$\lim_{|\xi|\to\infty} \det\{F_{l+1}(\xi)U_\mathrm{d}(\xi)^{-1}\} = \lim_{|\xi|\to\infty} \frac{\det F_{l+1}(\xi)}{\det U_\mathrm{d}(\xi)} = \infty.$$

This contradicts the properness of $F_{l+1}(\xi)U_\mathrm{d}(\xi)^{-1}$. Therefore, the proof of this lemma is completed. □

Proof of Theorem 1. The necessity is clear from the result of Lemma 3. The sufficiency can be shown by Proposition 4 and Lemma 4. □

Proof of Proposition 5

Proof of well-definedness: To prove the well-definedness of the map Ric, we introduce two polynomial matrices $F_i \in \mathbb{R}^{m\times m}[\xi]$ $(i = 1, 2)$ such that $\partial\Phi(\xi) = F_i(\xi)^\sim F_i(\xi)$, $\det F_i(\xi) = f(\xi)$, and $F_i(\xi)U(\xi)^{-1}$ is biproper for a given $f \in \mathscr{F}_\mathrm{cop}$.

We first show that $L(\xi) := F_2(\xi)F_1(\xi)^{-1}$ is a constant orthogonal matrix. Assume that $L(\xi)$ is a rational matrix. Then, it is biproper because $F_i(\xi)U(\xi)^{-1}$ $(i = 1, 2)$ is biproper and $L(\xi) = \{F_2(\xi)U(\xi)^{-1}\}\{F_1(\xi)U(\xi)^{-1}\}^{-1}$. Moreover, it follows from $F_1(\xi)^\sim F_1(\xi) = F_2(\xi)^\sim F_2(\xi) = \partial\Phi(\xi)$ that $L(\xi)$ is unitary, i.e. $L(\xi)^\sim L(\xi) = I_m$. Recall that, if $\lambda \in \mathbb{C}$ is a pole of a unitary rational matrix $L(\xi)$, then λ^{-1} is a zero of $L(\xi)$. If the pole λ is equal to zero, then $L(\xi)$ will have a zero at infinity. This contradicts the biproperness of $L(\xi)$, and hence $L(\xi)$ does not have any poles at $\xi = 0$.

Let $L(\xi) = L_2(\xi)L_1(\xi)^{-1}$, $L_1, L_2 \in \mathbb{R}^{m\times m}[\xi]$ be a right coprime factorization of $L(\xi)$. Then, by the unitarity of $L(\xi)$, we get

$$L_2(\xi)^\sim L_2(\xi) = L_1(\xi)^\sim L_1(\xi). \tag{B.12}$$

From the above discussion, the zeros of $L_i(\xi)$ $(i = 1, 2)$ are identical to the non-zero roots of $f(\xi) = \det F_i(\xi) = 0$. Suppose that $\lambda \neq 0$ is a root of $\det L_1(\xi) = 0$. Then, there exists a non-zero vector $v \in \mathbb{C}^m$ such that $L_1(\lambda)v = 0$. Substituting $\xi = \lambda$ into (B.12) and post-multiplying by v yield $L_2(\lambda^{-1})^\top L_2(\lambda)v = L_1(\lambda^{-1})^\top L_1(\lambda)v = 0$. Since $f(\xi)$ belongs to $\mathscr{F}_\mathrm{cop}$, $\det L_1(\lambda) = 0$

implies $\det L_1(\lambda^{-1}) = \det L_2(\lambda^{-1}) \neq 0$. It thus follows that $L_2(\lambda)v = 0$. This contradicts the right coprimeness of $L_1(\xi)$ and $L_2(\xi)$. Since λ is an arbitrary zero of $L_1(\xi)$, we conclude that $L_1(\xi)$ and $L_2(\xi)$ are non-singular constant matrices, namely they do not have any zeros. This implies that $L = L_2 L_1^{-1}$ is an orthogonal constant matrix.

We now complete the proof of the well-definedness of Ric. As was discussed in Section 3, the Riccati solution $P_i = \mathrm{Ric}(f)$ corresponding to $F_i(\xi)$ $(i = 1, 2)$ is obtained by solving the polynomial equation

$$\Phi(\zeta, \eta) - F_i(\zeta)^\top F_i(\eta) = (1 - \zeta\eta)X(\zeta)^\top P_i X(\eta). \tag{B.13}$$

We will prove the well-definedness by showing $P_1 = P_2$. Since $L = F_2(\xi)F_1(\xi)^{-1}$ is an orthogonal matrix, we have $F_1(\zeta)^\top F_1(\eta) = F_2(\zeta)^\top F_2(\eta)$. It follows from (B.13) that $X(\zeta)^\top P_1 X(\eta) = X(\zeta)^\top P_2 X(\eta)$. Since $X(\sigma)$ is a minimal state map for Σ, the map $\ell \mapsto (X(\sigma)\ell)(0)$ is surjective. Hence, we have $x_0^\top P_1 x_0 = x_0^\top P_2 x_0$ for all $x_0 \in \mathbb{R}^n$. Clearly, this implies $P_1 = P_2$.

Proof of bijectiveness: Let P be an element of $\mathscr{S}$. We assume that there exist two polynomials $f_1, f_2 \in \mathscr{F}_{\mathrm{cop}}$ satisfying $\mathrm{Ric}(f_1) = \mathrm{Ric}(f_2) = P$. Let $F_1, F_2 \in \mathbb{R}^{m\times m}[\xi]$ be polynomial matrices such that $\partial\Phi(\xi) = F_i(\xi)^\sim F_i(\xi)$, $\det F_i(\xi) = f_i(\xi)$ and $F_i(\xi)U(\xi)^{-1}$ is biproper $(i = 1, 2)$. Then, we obtain $F_1(\zeta)^\top F_1(\eta) = F_2(\zeta)^\top F_2(\eta)$, since

$$\Phi(\zeta, \eta) - F_i(\zeta)^\top F_i(\eta) = (1 - \zeta\eta)X(\zeta)^\top P X(\eta)$$

holds for $i = 1, 2$. This implies that $\det F_1(\zeta)\det F_1(\eta) = \det F_2(\zeta)\det F_2(\eta)$, i.e. $f_1(\zeta)f_1(\eta) = f_2(\zeta)f_2(\eta)$. Given that the highest degree coefficients of $f_1(\xi)$ and $f_2(\xi)$ are positive, it follows $f_1(\xi) = f_2(\xi)$. This concludes the proof of the injectiveness of Ric.

The surjectiveness is easily proved by taking $f(\xi) = \det F_P(\xi)$, where $F_P(\xi)$ is defined in Lemma 3. □

Proof of Theorem 2. We have only to show that S_f is non-singular under Assumption 1′.

There holds $\xi X(\xi) = AX(\xi) + BU(\xi)$ from (14). Then, it follows from (27) that $\xi X^{(l)}(\xi) + lX^{(l-1)}(\xi) = AX^{(l)}(\xi) + BU^{(l)}(\xi)$ for $l = 0, 1, \ldots, d_i - 1$. Hence, we obtain

$$\begin{aligned}&\begin{bmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \cdots & X^{(d_i-1)}(\lambda_i)\end{bmatrix}(\lambda_i I_{d_i} + N_i)\\ &\quad = \begin{bmatrix} A & B\end{bmatrix}\begin{bmatrix} M(\lambda_i) & M^{(1)}(\lambda_i) & \cdots & M^{(d_i-1)}(\lambda_i)\end{bmatrix}.\end{aligned} \tag{B.14}$$

Post-multiplying (B.14) by V_i, it follows from (31) and (32) that

$$\lambda_i S_i + S_i L_i = AS_i + B\begin{bmatrix} U(\lambda_i) & U^{(1)}(\lambda_i) & \cdots & U^{(d_i-1)}(\lambda_i)\end{bmatrix} V_i. \tag{B.15}$$

Since $F(\xi)U(\xi)^{-1}$ is biproper, there exist a matrix $C \in \mathbb{R}^{m\times n}$ and a non-singular matrix $D \in \mathbb{R}^{m\times m}$ satisfying $F(\xi) = CX(\xi) + DU(\xi)$ from Lemma 1. Thus, we get

$$\begin{aligned}&\begin{bmatrix} F(\lambda_i) & F^{(1)}(\lambda_i) & \cdots & F^{(d_i-1)}(\lambda_i)\end{bmatrix}\\ &\quad = C\begin{bmatrix} X(\lambda_i) & X^{(1)}(\lambda_i) & \cdots & X^{(d_i-1)}(\lambda_i)\end{bmatrix}\\ &\qquad + D\begin{bmatrix} U(\lambda_i) & U^{(1)}(\lambda_i) & \cdots & U^{(d_i-1)}(\lambda_i)\end{bmatrix}.\end{aligned}$$

Post-multiplying this by V_i, there holds from (31) and (33) that

$$\begin{bmatrix} U(\lambda_i) & U^{(1)}(\lambda_i) & \cdots & U^{(d_i-1)}(\lambda_i)\end{bmatrix} V_i = -D^{-1}CS_i. \tag{B.16}$$

Substituting (B.16) into (B.15) yields

$$(\lambda_i I_n - A + BD^{-1}C)S_i = -S_i L_i \quad (i = 1, 2, \ldots, k). \tag{B.17}$$

We now prove that S_i has full column rank. Partition S_i as $S_i = \begin{bmatrix} S_{i,0} & S_{i,1} & \cdots & S_{i,d_i-1} \end{bmatrix}$, $S_{i,l} \in \mathbb{C}^n$ $(l = 0, 1, \ldots, d_i - 1)$. (B.17) implies that $S_{i,0}, S_{i,1}, \ldots, S_{i,d_i-1}$ satisfy

$$(\lambda_i I_n - A + BD^{-1}C)S_{i,l} = -(l+1)S_{i,l+1} \quad (l = 0, 1, \ldots, d_i - 2), \tag{B.18}$$

$$(\lambda_i I_n - A + BD^{-1}C)S_{i,d_i-1} = 0. \tag{B.19}$$

It is sufficient to prove that $S'_{i,l} := \begin{bmatrix} S_{i,l} & S_{i,l+1} & \cdots & S_{i,d_i-1} \end{bmatrix}$ has full column rank for $l = d_i - 1, \ldots, 1, 0$ using a induction. For $l = d_i - 1$, (31) and (B.16) imply that there holds $M(\lambda_i)a_{i,d_i-1} = \mathrm{col}(I_n, -D^{-1}C)S'_{i,d_i-1}$. Since $a_{i,d_i-1} \neq 0$ and $M(\lambda_i)$ has full column rank from the observability, $S'_{i,d_i-1} \neq 0$, i.e. S'_{i,d_i-1} is of full column rank. Assume that $S'_{i,l}$ is of full column rank for $l = d_i - 1, d_i - 2, \ldots, k$. To deduce a contradiction, we also assume that $S'_{i,k-1} = \begin{bmatrix} S_{i,k-1} & S'_{i,k} \end{bmatrix}$ does not have full column rank. Then, there exists a non-zero vector $\nu_{i,k} \in \mathbb{C}^{d_i-k}$ satisfying $S_{i,k-1} = S'_{i,k}\nu_{i,k}$. Since $(\lambda_i I_n - A + BD^{-1}C)S'_{i,k} = -S'_{i,k}L_{i,k}$ from (B.18) and (B.19), we see $(\lambda_i I_n - A + BD^{-1}C)S_{i,k-1} = -S'_{i,k}L_{i,k}\nu_{i,k}$, where $L_{i,k} \in \mathbb{R}^{(d_i-k)\times(d_i-k)}$ is the lower subdiagonal matrix with elements $\{k, k+1, \ldots, d_i - 1\}$. Since $(\lambda_i I_n - A + BD^{-1}C)S_{i,k-1} = -S'_{i,k} \cdot \mathrm{col}(k, 0, \ldots, 0)$ from (B.18), we obtain $S'_{i,k}(L_{i,k}\nu_{i,k} - \mathrm{col}(k, 0, \ldots, 0)) = 0$. From the assumption that $S'_{i,k}$ is of full column rank, we have $L_{i,k}\nu_{i,k} - \mathrm{col}(k, 0, \ldots, 0) = 0$. This is a contradiction, because this is not the case for any choice of $\nu_{i,k}$. Hence, S_i is of full column rank.

From (B.17) and the fact S_i has full column rank, the column vectors of S_i are the eigenvectors of $A - BD^{-1}C$ for the eigenvalue λ_i with the partial multiplicity d_i. Thus, if $i \neq j$, the column vectors of S_i and S_j are linearly independent. Since $i, j = 1, 2, \ldots, k$ are arbitrary, S_f is non-singular. □

Proof of Corollary 1. Let $f \in \mathscr{F}_{\mathrm{cop}}$ be arbitrary. Under Assumption 1′, the map Ric is bijective from Proposition 5. Thus, the polynomial matrix which induces the corresponding storage function is given by $\Psi(\zeta, \eta) = -X(\zeta)^\top P_f X(\eta)$, where $P_f = \mathrm{Ric}(f)$. Similarly, from Proposition 3, the polynomial matrix which induces the smallest storage function is given by $\Psi^-(\zeta, \eta) = -X(\zeta)^\top P_h X(\eta)$, where $P_h = \mathrm{Ric}(h)$. This implies $-x^\top P_f x \geqslant -x^\top P_h x$ for all $\ell \in (\mathbb{R}^m)^{\mathbb{Z}}$ and $t \in \mathbb{Z}$. Since $X(\sigma)$ is a minimal state map for Σ, the map $\ell \mapsto (X(\sigma)\ell)(0)$ is surjective. Hence, we have $-x_0^\top P_f x_0 \geqslant -x_0^\top P_h x_0$ for all $x_0 \in \mathbb{R}^n$. This implies $P_f \leqslant P_h$ for all $f \in \mathscr{F}_{\mathrm{cop}}$. From Theorem 2, we see that $P_h = (S_h^*)^{-1}T_h S_h^{-1}$ gives the largest solution of the DARE (12). Since the existence of non-negative definite solution of the DARE (12) is equivalent to $P_h \geqslant 0$, this is the case if and only if $T_h \geqslant 0$, which completes the proof. □

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