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Solvable Jordan algebras of compact operators☆

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ABSTRACT

It is proved that a Jordan algebra of compact operators which is closed is either an Engel Jordan algebra, or contains a nonzero finite rank operator; Moreover, it is showed that any solvable Jordan algebra of compact operators on an infinite dimensional Banach space is triangularizable.

1. Introduction

The Invariant Subspace Problem that the existence of a nontrivial closed subspace invariant under an operator or a set of operators is one of the famous open questions in the operator theory. One of the classical results on the subject was due to Lomonosov [3] in 1973. He proved that every irreducible algebra of compact operators on a Banach space $\mathcal{X}$ is dense in the algebra $\mathcal{B}(\mathcal{X})$ of all bounded operators on $\mathcal{X}$, with respect to the weak operator topology. Wojtyński [10] extends the results of Lomonosov to Lie algebras of compact operators, who established the following result that a Lie algebra of compact operators which is closed is either an Engel Lie algebra, or contains a nonzero finite rank operator. Some

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more general results, such as semigroups of compact operators and semigroups of Volterra operators, were given in [4,9], respectively.

It could be said that one of the aims of Invariant Subspace Theory is to establish triangularizability results. The classical examples are the famous Engel and Lie theorems which state that nilpotent and, respectively, solvable Lie algebras of operators on finite-dimensional spaces are triangularizable. They were extended to compact operators in [7] for Engle Lie algebra and in [6] for solvable Lie algebras. Historically, the theoretical development of Jordan algebras has closely corresponded to the theoretical development of Lie algebras, a fact which has been utilized to verify many results in both areas. Recently, Kennedy et al. [2] showed an analog of Engel's theorem for Jordan algebras of operators which says that a Jordan algebra of Volterra operators is triangularizable.

It is natural and very interesting to ask whether Wojtyński's result extends to a Jordan algebra of compact operators on a Banach space, and whether an analogous result of Lie's theorem for Jordan algebras of compact operators on a Banach space is valid. We will give affirmative answers to the above two questions in this paper. The main results of this paper are the following: (1) a Jordan algebra of compact operators which is closed is either an Engel Jordan algebra, or contains a nonzero finite rank operator; (2) any solvable Jordan algebra of compact operators is triangularizable.

2. Preliminaries

We now introduce some definitions and notations. Let $\mathcal{X}$ be a complex Banach space, $\mathcal{B}(\mathcal{X})$, $\mathcal{K}(\mathcal{X})$ and $\mathcal{F}(\mathcal{X})$ the sets of all bounded linear operators on $\mathcal{X}$, all compact operators, all finite rank operators correspondingly. For $A \in \mathcal{B}(\mathcal{X})$, by $\sigma(A)$ we denote the spectrum of A. A is said to be quasinilpotent if $\sigma(A) = \{0\}$ or equivalently if $lim_{n\to\infty}(\|A^n\|)^{\frac{1}{n}} = 0$. Note that an operator is called Volterra if it is compact and quasinilpotent; a set of operators is Volterra if its elements are Volterra.

A Lie algebra $\mathcal{L}$ of operators is a subspace of $\mathcal{B}(\mathcal{X})$ which is closed under the Lie product $[A, B] = AB - BA$, for $A, B \in \mathcal{L}$. An element $A \in \mathcal{L}$ induces a linear transformation adA on $\mathcal{B}(\mathcal{X})$ defined by $adA(B) = AB - BA$, for $B \in \mathcal{B}(\mathcal{X})$. Note that $\mathcal{L}$ is invariant under adA; by $ad_{\mathcal{L}}A$ we denote the restriction of adA to $\mathcal{L}$. The operator $ad_{\mathcal{L}}A$ is called the adjoint representation of A. Recall that if $\mathcal{L}$ is a normed Lie subalgebra, then $\mathcal{L}$ is an Engel Lie algebra if all operators $ad_{\mathcal{L}}A : B \longmapsto [A, B]$ on $\mathcal{L}$ are quasinilpotent for any $A \in \mathcal{L}$.

A Jordan algebra $\mathcal{J}$ of operators is a subspace of $\mathcal{B}(\mathcal{X})$ if it is closed under the Jordan product $A \circ B = AB + BA$, for $A, B \in \mathcal{J}$. An element $A \in \mathcal{J}$ induces the multiplication operator $\mathcal{T}_A$ on $\mathcal{B}(\mathcal{X})$ defined by $\mathcal{T}_A(B) = A \circ B$, for $B \in \mathcal{B}(\mathcal{X})$. Note that $\mathcal{J}$ is invariant under $\mathcal{T}_A$; by $\mathcal{T}_A|_{\mathcal{J}}$ we denote the restriction of $\mathcal{T}_A$ to $\mathcal{J}$. Let $\mathcal{J}$ is a normed Lie subalgebra, then $\mathcal{J}$ is an Engel Jordan algebra if all operators $\mathcal{T}_A|_{\mathcal{J}} : B \longmapsto A \circ B$ on $\mathcal{J}$ are quasinilpotent for any $A \in \mathcal{J}$.

The Jordan triple product is defined by

$$\{ABC\} = ABC + CBA, \tag{1}$$

for any $A, B, C \in \mathcal{B}(\mathcal{X})$. It is easy to check that the formula

$$2\{ABC\} = (A \circ B) \circ C + (B \circ C) \circ A - (A \circ C) \circ B \tag{2}$$

holds. Especially, we have that

$$\{ABA\} = A \circ (A \circ B) - A^2 \circ B. \tag{3}$$

Similarly, for any $A_1, \ldots, A_n \in \mathcal{B}(\mathcal{X})$, we will write

$$\{A_1 \cdots A_n\} = A_1 \cdots A_n + A_n \cdots A_1. \tag{4}$$

It is well known that, if $n \geqslant 4$, this cannot be expressed in terms of the Jordan product, even there exist Jordan subalgebras of $\mathcal{B}(\mathcal{X})$ which are not closed under the above multilinear product (4). Further, it is easy to see that

$$[A, [B, C]] = (A \circ B) \circ C - (A \circ C) \circ B. \tag{5}$$

As we all know, for $A, B, C, D \in \mathcal{J}$, one has the following identities:

$$\begin{aligned} & A \circ ((B \circ C) \circ D) + B \circ ((C \circ A) \circ D) + C \circ ((A \circ B) \circ D) \\ & \quad = (B \circ C) \circ (A \circ D) + (C \circ A) \circ (B \circ D) + (A \circ B) \circ (C \circ D), \quad (6) \\ & D \circ ((B \circ C) \circ A) + B \circ ((C \circ D) \circ A) + C \circ ((D \circ B) \circ A) \\ & \quad = (B \circ C) \circ (A \circ D) + (C \circ A) \circ (B \circ D) + (A \circ B) \circ (C \circ D), \quad (7) \\ & \mathcal{T}_A \mathcal{T}_{B \circ C} + \mathcal{T}_B \mathcal{T}_{C \circ A} + \mathcal{T}_C \mathcal{T}_{A \circ B} = \mathcal{T}_{A \circ (B \circ C)} + \mathcal{T}_B \mathcal{T}_A \mathcal{T}_C + \mathcal{T}_C \mathcal{T}_A \mathcal{T}_B, \quad (8) \\ & 2\mathcal{T}_A \mathcal{T}_B \mathcal{T}_C = \mathcal{T}_{C \circ A} \mathcal{T}_B + \mathcal{T}_{A \circ B} \mathcal{T}_C + \mathcal{T}_{B \circ C} \mathcal{T}_A - \mathcal{T}_{(C \circ A) \circ B} + [\mathcal{T}_B, \mathcal{T}_C]\mathcal{T}_A + \mathcal{T}_B[\mathcal{T}_A, \mathcal{T}_C] + [\mathcal{T}_A, \mathcal{T}_B]\mathcal{T}_C. \quad (9) \end{aligned}$$

Letting $\delta_{A,B} = [\mathcal{T}_A, \mathcal{T}_B]$ for $A, B \in \mathcal{J}$, it is well known that $\delta_{A,B}$ is a Jordan derivation of Jordan algebra $\mathcal{J}$, i.e.

$$\delta_{A,B}(C \circ D) = (\delta_{A,B}C) \circ D + C \circ (\delta_{A,B}D) \quad (10)$$

for any $C, D \in \mathcal{J}$. Recall that a Jordan algebra $\mathcal{J}$ is called solvable if there exists a natural number n such that $\mathcal{J}^{(n)} = \{0\}$, where the chains $\mathcal{J}^{(n)}$ is defined by the rules

$$\mathcal{J}^{(0)} = \mathcal{J}, \mathcal{J}^{(k+1)} = \mathcal{J}^{(k)} \circ \mathcal{J}^{(k)}$$

for $k = 0, 1, \cdots$. The smallest n such that $\mathcal{J}^{(n)} = \{0\}$ is called the solvability length of $\mathcal{J}$. Clearly, the descending chains

$$\mathcal{J}^{(0)} \supseteq \mathcal{J}^{(1)} \supseteq \mathcal{J}^{(2)} \supseteq \cdots \supseteq \mathcal{J}^{(n)} \supseteq \cdots.$$

Note that for algebras in which its square is its an ideal, and for Lie algebras, each solvable power $\mathcal{J}^{(n)}$ is again its an ideal. In the case of Jordan algebras, this is not necessarily so, i.e. it is not necessarily true that $\mathcal{J}^{(n)} \circ \mathcal{J} \nsubseteq \mathcal{J}^{(n)}$ for any $n \in \mathcal{N}$.

For $A \in \mathcal{B}(\mathcal{X})$ and $\lambda \in \mathbb{C}$, where $\mathbb{C}$ denotes the complex field, let

$$\varepsilon_\lambda(A) = \{x \in \mathcal{X} : lim_{n \to \infty} \|(A - \lambda)^n x\| = 0\}.$$

The set $\varepsilon_\lambda(A)$ is called an elementary spectral manifold. It is known [10] that $\varepsilon_\lambda(A)$ is closed and nonzero if λ is an isolated point in the spectrum of A, and that A is quasinilpotent if $\varepsilon_0(A) = \mathcal{X}$. It is clear that $\varepsilon_\lambda(A) \neq \{0\}$ implies that $\lambda \in \sigma(A)$ (Note that $\lambda \in \sigma(A)$ does not imply that $\varepsilon_\lambda(A) \neq \{0\}$.)

A closed subspace $\mathcal{Y} \subseteq \mathcal{X}$ is invariant under a operator $T \in \mathcal{B}(\mathcal{X})$ if $T\mathcal{Y} \subseteq \mathcal{Y}$. A set $\mathcal{M}$ of bounded linear operators on a complex Banach space $\mathcal{X}$ is said to be reducible if there is a non-trivial subspace invariant under all the operators in the set. The set $\mathcal{M}$ is said to be triangularizable if there exists a maximal subspace chain consisting of closed subspaces which are invariant under all the operators in the set $\mathcal{M}$.

For Lie algebras $\mathcal{L}$ and $\mathcal{M}$, the designation $\mathcal{M} \lhd \mathcal{J}$ means that $\mathcal{M}$ is an ideal of $\mathcal{L}$, i.e. $[\mathcal{L}, \mathcal{M}] \subseteq \mathcal{M}$. Recall that if $\mathcal{J}$ is a Jordan algebra, then a Jordan ideal $\mathcal{K}$ is a subspace of $\mathcal{J}$ such that $\mathcal{J} \circ \mathcal{K} \subseteq \mathcal{K}$. For a Jordan algebra of operators $\mathcal{J}$, $\mathcal{A}$ denotes the unital enveloping associative algebra generated by $\mathcal{J}$, and if a set $\mathcal{M} \subseteq \mathcal{J}$, then $\mathcal{A}(\mathcal{M})$ denotes the ideal of $\mathcal{A}$ generated by $\mathcal{M}$, $\mathcal{J}(\mathcal{M})$ the Jordan ideal of $\mathcal{J}$ generated by $\mathcal{M}$.

3. The main results

We are now ready for the main results of this section. Throughout, we use the notations introduced above. To prove the main results of this section, we will need the following technical lemma.

Lemma 3.1. *For any $n \in \mathcal{N}$ and $A, B \in \mathcal{B}(\mathcal{X})$, we have that*

(1) $\mathcal{T}_A(BA) = (\mathcal{T}_A B)A$.
(2) $A \circ (A^n B) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(B))A^k$.

Proof. (1) $\mathcal{T}_A(BA) = A \circ (BA) = (A \circ B)A = (\mathcal{T}_A B)A$.

(2) We proceed by induction on n. Obviously,

$$A \circ (AB) = A^2B + ABA = A(A \circ B) = A \circ (A \circ B) - (A \circ B)A.$$

If $n > 2$, for $k \leqslant n$, by (1), applying the equations $AB = \mathcal{T}_A B - BA$ and

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k},$$

we have that

$$\begin{aligned}
A \circ (A^{n+1}B) = {} & A \circ (A^n(AB)) = \sum_{k=0}^{n}(-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(AB))A^k \\
= {} & \sum_{k=0}^{n}(-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(\mathcal{T}_A B - BA))A^k \\
= {} & (\mathcal{T}_A)^{n+1}(\mathcal{T}_A B - BA) + (-1)^n((\mathcal{T}_A)(\mathcal{T}_A B - BA))A^n \\
& + \sum_{k=1}^{n-1}(-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(\mathcal{T}_A B - BA))A^k \\
= {} & (\mathcal{T}_A)^{n+1}(\mathcal{T}_A B - BA) + (-1)^n((\mathcal{T}_A)(\mathcal{T}_A B - BA))A^n \\
= {} & \sum_{k=1}^{n-1}(-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(\mathcal{T}_A B))A^k + \sum_{k=1}^{n-1}(-1)^{k+1} \binom{n}{k} ((\mathcal{T}_A)^{n+1-k}(B))A^{k+1} \\
= {} & (\mathcal{T}_A)^{n+2}(B) + (-1)^{n+1}(\mathcal{T}_A(BA))A^n + \sum_{k=1}^{n}(-1)^k \binom{n}{k} ((\mathcal{T}_A)^{n+2-k}(B))A^k \\
& + \sum_{k=1}^{n}(-1)^k \binom{n}{k-1} ((\mathcal{T}_A)^{n+2-k}(B))A^k \\
= {} & \sum_{k=0}^{n+1}(-1)^k \binom{n+1}{k} ((\mathcal{T}_A)^{n+2-k}(B))A^k.
\end{aligned}$$

The conclusions follows. □

Lemma 3.2. *Let $\mathcal{J}$ be a closed Jordan algebra of compact operator. For any $A \in \mathcal{J}$, we have that*

$$\sigma(\mathcal{T}_A|_{\mathcal{J}}) \subseteq \sigma(A) + \sigma(A).$$

Moreover, $\sigma(\mathcal{T}_A|_{\mathcal{J}})$ is countable compact.

Proof. For any $A \in \mathcal{J}$, note that $\mathcal{T}_A : \mathcal{K}(\mathcal{X}) \to \mathcal{K}(\mathcal{X})$ defined by

$$\mathcal{T}_A(B) = AB + BA \text{ for any } B \in \mathcal{K}(\mathcal{X}).$$

By Rosenblum's theorem, we have that $\sigma(\mathcal{T}_A) \subseteq \sigma(A) + \sigma(A)$. Hence, $\sigma(\mathcal{T}_A)$ is countable. Recall that $\mathcal{T}_A|_{\mathcal{J}}$ is the restriction of $\mathcal{T}_A$ to its invariant subspace $\mathcal{J}$. So we have that $\sigma(\mathcal{T}_A|_{\mathcal{J}}) \subseteq \sigma(\mathcal{T}_A)$. The conclusion follows. □

Lemma 3.3. *Let $A \in \mathcal{K}(\mathcal{X})$ and $B \in \mathcal{B}(\mathcal{X})$. If there exists $0 \neq \lambda \in \sigma(\mathcal{T}_A)$ such that $\|(\mathcal{T}_A - \lambda)^n B\|^{\frac{1}{n}} \to 0$ for $n \to \infty$, then $B \in \mathcal{F}(\mathcal{X})$.*

Proof. Recall that 0 is the only accumulation point of $\sigma(A)$ and $\sigma(\mathcal{T}_A) \subseteq \sigma(A) + \sigma(A)$. So we can write $\lambda = \lambda_1 + \lambda_2$, where $\lambda_1, \lambda_2 \in \sigma(A)$. Let us set

$$\sigma_1 = \{\delta \in \sigma(A) : \delta \neq 0 \text{ and } \eta \in \sigma(A) \text{ such that } \lambda = \delta + \eta\}, \quad \text{and} \quad \sigma_2 = \sigma(A) \setminus \sigma_1.$$

Here, we claim below that σ_1 is a finite subset of $\sigma(A)$. Suppose, to the contrary, σ_1 is not a finite subset of $\sigma(A)$. So if $\{\delta_n\} \subseteq \sigma_1$ and $\lambda = \delta_n + \eta_n$, then there is $\{\delta_{n_k}\} \subseteq \{\delta_n\}$ and $\{\eta_{n_k}\} \subseteq \{\eta_n\}$ such that $\delta_{n_k} \to \delta_0$ and $\eta_{n_k} \to \eta_0$ for $k \to \infty$, which contradicts the hypothesis $\lambda \neq 0$. Hence we get that σ_1 and σ_2 are both closed subsets of $\sigma(A)$. By the Riesz decomposition theorem, then there exists an unique direct sum decomposition $\mathcal{X} = \mathcal{M}_1 \bigoplus \mathcal{M}_2$ such that, for $i = 1, 2$, $\mathcal{M}_i$ is invariant under A, and $\sigma(A|_{\mathcal{M}_1}) = \sigma_1$ and $\sigma(A|_{\mathcal{M}_2}) = \sigma_2$. Let P_i, $i = 1, 2$, be the projection onto $\mathcal{M}_i$. Obviously, we have that

$$\mathcal{B}(\mathcal{X}) = P_1\mathcal{B}(\mathcal{X})P_1 + P_1\mathcal{B}(\mathcal{X})P_2 + P_2\mathcal{B}(\mathcal{X})P_1 + P_2\mathcal{B}(\mathcal{X})P_2.$$

Since $\mathcal{M}_i$ are invariant under A for $i = 1, 2$, one easily see that

$$P_1AP_1 = A|_{\mathcal{M}_1}, P_1AP_2 = P_2AP_1 = 0, P_2AP_2 = A|_{\mathcal{M}_2},$$

and the spaces $P_i\mathcal{B}(\mathcal{X})P_j$ are invariant under $\mathcal{T}_A$ for $i, j = 1, 2$. In particular, $\mathcal{T}_A$ restricted to $P_2\mathcal{B}(\mathcal{X})P_2$ is equal to $\mathcal{T}_{A|_{\mathcal{M}_2}}$. Recall that $\sigma(\mathcal{T}_{A|_{\mathcal{M}_2}}) \subseteq \sigma(A|_{\mathcal{M}_2}) + \sigma(A|_{\mathcal{M}_2}) = \sigma_2 + \sigma_2$ and $\lambda \in \sigma_1$, which implies that $\lambda \notin \sigma(\mathcal{T}_{A|_{\mathcal{M}_2}})$. Clearly, there is a constant $\mathcal{M}$ such that

$$\begin{aligned} \|(\mathcal{T}_{A|_{\mathcal{M}_2}} - \lambda)^n P_2BP_2\|^{\frac{1}{n}} &= \|P_2((\mathcal{T}_A - \lambda)^n B)P_2\|^{\frac{1}{n}} \\ &\leqslant \mathcal{M}\|(\mathcal{T}_A - \lambda)^n B\|^{\frac{1}{n}} \to 0 \end{aligned}$$

for $n \to \infty$. Since $\lambda \notin \sigma(A|_{\mathcal{M}_2}) = \sigma(P_2AP_2)$, then we get that $P_2BP_2 = 0$. So we have that

$$B = P_1BP_1 + P_1BP_2 + P_2BP_1.$$

Note that since σ_1 is a finite subset of $\sigma(A)$ not containing 0 and A is compact, the space $\mathcal{M}_1$ is finite dimensional and hence $P_1 \in \mathcal{F}(\mathcal{X})$. Thus B is a finite rank operator. $\square$

We now state some equalities concerning Jordan operator algebras. For any $A, B, C \in \mathcal{J}$, from the equation $2ABA = A \circ (A \circ B) - A^2 \circ B$, we get that $ABA, ABCBA, (A \circ B)C(A \circ B) \in \mathcal{J}$. Further, from $[A, B]^2 = (ABA) \circ B - AB^2A - BA^2B$ and $[A, B]C[A, B] = (A \circ B)C(A \circ B) - 2(ABCBA + BACAB)$, we see that

$$[A, B]^2, [A, B]C[A, B] \in \mathcal{J}. \tag{11}$$

Letting $A_1, A_2, A_3, A_4 \in \mathcal{J}$, it is easy to check that $A_1 \circ \{A_2A_3A_4\} = \{A_1A_2A_3A_4\} + \{A_2A_3A_4A_1\}$. Then obviously yields

$$\{A_1A_2A_3A_4\} = -\{A_2A_3A_4A_1\} + \mathcal{J} \circ \mathcal{J}. \tag{12}$$

Note that

$$(A_1 \circ A_2) \circ (A_3 \circ A_4) = \{A_1A_2A_3A_4\} + \{A_2A_1A_3A_4\} + \{A_3A_4A_2A_1\} + \{A_3A_4A_1A_2\}.$$

By applying the cyclic permutation twice, we get that $\{A_3A_4A_2A_1\} \in \{A_2A_1A_3A_4\} + \mathcal{J} \circ \mathcal{J}$ and $\{A_3A_4A_1A_2\} \in \{A_1A_2A_3A_4\} + \mathcal{J} \circ \mathcal{J}$. Hence we have that

$$\{A_1A_2A_3A_4\} = -\{A_2A_1A_3A_4\} + \mathcal{J} \circ \mathcal{J}. \tag{13}$$

From (12) and (13), it is easy to check that

$$\{A_1A_2A_3A_4\} \in (-1)^{\sigma}\{A_{\sigma(1)}A_{\sigma(2)}A_{\sigma(3)}A_{\sigma(4)}\} + \mathcal{J} \circ \mathcal{J}, \tag{14}$$

where σ is a permutation of $\{1, 2, 3, 4\}$ and $(-1)^{\sigma}$ its sign. In what follows, we shall frequently use these identities. Here, $[[\mathcal{J}, \mathcal{J}], \mathcal{J}^{(n)}]$ denotes the set $\{[[A, B], C] : A, B \in \mathcal{J}, C \in \mathcal{J}^n\}$, and $\{\mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)}\}$ denotes the set $\{\{ABC\} : A \in \mathcal{J}, B, C \in \mathcal{J}^n\}$.

Lemma 3.4. *For any $n \in \mathcal{N}$, we have that*

(a) $[[\mathcal{J}, \mathcal{J}], \mathcal{J}^{(n)}] \subseteq \mathcal{J}^{(n)}$.
(b) $\mathcal{J}^{(n+1)} \circ \mathcal{J} \subseteq \mathcal{J}^{(n)}$.

(c) $\{\mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)}\} \subseteq \mathcal{J}^{(n)}$.
(d) $((\mathcal{J}^{(n)} \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(n)} \subseteq \mathcal{J}^{(n)}$.
(e) $\mathcal{A}((\mathcal{J}^{(n)})^5) \subseteq \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)})$.

Proof. (a) We carry out the proof by induction on n. It is clear that $\delta_{A,B}C = [[A, B], C]$ for $A, B, C \in \mathcal{J}$. Suppose that $\delta_{\mathcal{J},\mathcal{J}}(\mathcal{J}^{n-1}) \subseteq \mathcal{J}^{n-1}$. By (10), we have that

$$\delta_{\mathcal{J},\mathcal{J}}(\mathcal{J}^{n}) = \delta_{\mathcal{J},\mathcal{J}}(\mathcal{J}^{n-1} \circ \mathcal{J}^{n-1}) = \delta_{\mathcal{J},\mathcal{J}}(\mathcal{J}^{n-1}) \circ \mathcal{J}^{n-1} + \mathcal{J}^{n-1} \circ \delta_{\mathcal{J},\mathcal{J}}(\mathcal{J}^{n-1}) \subseteq \mathcal{J}^{n}.$$

The conclusion follows.

(b) Let us assume that $\mathcal{J}^{(k)} \circ \mathcal{J} \subseteq \mathcal{J}^{(k-1)}$ for some natural number k. From (8) and the fact that $\mathcal{J}^{(k)} \subseteq \mathcal{J}^{(k-1)}$, we get that

$$\begin{aligned}\mathcal{J}^{(k+1)} \circ \mathcal{J} = {} & (\mathcal{J}^{(k)} \circ \mathcal{J}^{(k)}) \circ \mathcal{J} = (\mathcal{J}^{(k)} \circ (\mathcal{J}^{(k-1)} \circ \mathcal{J}^{(k-1)})) \circ \mathcal{J} \\ \subseteq {} & \mathcal{J}^{(k)} \circ ((\mathcal{J}^{(k-1)} \circ \mathcal{J}^{(k-1)}) \circ \mathcal{J}) + \mathcal{J}^{(k-1)} \circ ((\mathcal{J}^{(k-1)} \circ \mathcal{J}^{(k)}) \circ \mathcal{J}) \\ & + \mathcal{J}^{(k-1)} \circ (\mathcal{J}^{(k)} \circ (\mathcal{J}^{(k-1)} \circ \mathcal{J})) \\ \subseteq {} & \mathcal{J}^{(k-1)} \circ (\mathcal{J}^{(k)} \circ \mathcal{J}) \subseteq \mathcal{J}^{(k)}.\end{aligned}$$

The conclusion follows.

(c) From (a), we have that

$$(\mathcal{J} \circ \mathcal{J}^{(n)}) \circ \mathcal{J}^{(n)} \subseteq \mathcal{J}^{(n-1)} \circ \mathcal{J}^{(n)} \subseteq \mathcal{J}^{(n)}.$$

From (2), we get that

$$\{\mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)}\} \subseteq (\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)}) \circ \mathcal{J} + [\mathcal{J}, [\mathcal{J}^{(n)}, \mathcal{J}^{(n)}]].$$

By (a) and (b), the conclusion follows.

(d) Now suppose that $((\mathcal{J}^{(k)} \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(k)} \subseteq \mathcal{J}^{(k)}$ for some natural number k. Then by (6), we obtain that

$$\begin{aligned}((\mathcal{J}^{(k+1)} \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(k+1)} = {} & (((\mathcal{J}^{(k)} \circ \mathcal{J}^{(k)}) \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(k+1)} \\ \subseteq {} & (((\mathcal{J}^{(k)} \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(k)}) \circ \mathcal{J}^{(k+1)} \\ & + ((\mathcal{J}^{(k)} \circ \mathcal{J}^{(k)}) \circ (\mathcal{J} \circ \mathcal{J})) \circ \mathcal{J}^{(k+1)} \\ & + ((\mathcal{J}^{(k)} \circ \mathcal{J}) \circ (\mathcal{J}^{(k)} \circ \mathcal{J})) \circ \mathcal{J}^{(k+1)}\end{aligned}$$

from which and (a), we get that

$$((\mathcal{J}^{(k+1)} \circ \mathcal{J}) \circ \mathcal{J}) \circ \mathcal{J}^{(k+1)} \subseteq \mathcal{J}^{(k+1)}.$$

The conclusion follows.

(e) Let us assume that $A, B, C, D, E \in \mathcal{J}^{(n)}$. From the equation

$$ABCD + BCDA = (A \circ B)CD + BC(A \circ D) - B(A \circ C)D,$$

we get that

$$ABCD + BCDA,\ EABC + ABCE,\ BCDE + CDEB,\ ACDE + CDEA \in \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)}).$$

It can be easily seen that

$$\begin{aligned}4ABCDE - \{ABCD\} \circ E = {} & (ABCD + BCDA)E - \{BCD\}AE + ABC(D \circ E) - (EABC + ABCE)D \\ & + A(BCDE + CDEB) - (ACDE + CDEA)B + \{CDE\}AB - EDC(A \circ B),\end{aligned}$$

which implies that $ABCDE - \frac{1}{4}\{ABCD\} \circ E \in \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)})$. Since $\mathcal{J}^{(n)} = \mathcal{J}^{(n-1)} \circ \mathcal{J}^{(n-1)}$, for any $T \in \mathcal{J}^{(n)}$, we can write that $T = UV + VU$, where $U, V \in \mathcal{J}^{(n-1)}$. Note that

$$UV + VU = (U + V)^2 - U^2 - V^2.$$

It suffices to prove that $\{P^2Q^2R^2S^2\} \in \mathcal{J}^{(n)} + \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)})$ with $P, Q, R, S \in \mathcal{J}^{(n-1)}$. Here, we claim that

$$\{P^2Q^2R^2S^2\} - \{(\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q P^2)QRS\} \in \mathcal{J}^{(n)}$$

for any $P, Q, R, S \in \mathcal{J}^{(n-1)}$. Indeed, since $Q^2P^2 = Q(Q \circ P^2) - QP^2Q$, we have that

$$\{Q^2P^2R^2S^2\} = \{Q(\mathcal{T}_Q P^2)R^2S^2\} - \{(QP^2Q)R^2S^2\},$$

which implies that $\{Q^2P^2R^2S^2\} \in \{Q(\mathcal{T}_Q P^2)R^2S^2\} + \mathcal{J}^{(n)}$. By (14), one easily checks that

$$\begin{aligned}\{P^2Q^2R^2S^2\} \in & -\{Q^2P^2R^2S^2\} + \mathcal{J}^{(n)}\\ & \in \{(QP^2Q)R^2S^2\} + \{(R(Q \circ P^2)R)QS^2\} + \{(S(R \circ (Q \circ P^2))S)QR\}\\ & \quad + \{(\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q P^2)QRS\} + \mathcal{J}^{(n)}.\end{aligned}$$

This gives that $\{P^2Q^2R^2S^2\} - \{(\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q P^2)QRS\} \in \mathcal{J}^{(n)}$, as claimed.

Further, it follows from (9) that

$$\begin{aligned}2\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q = & \ \mathcal{T}_{Q\circ S}\mathcal{T}_R + \mathcal{T}_{S\circ R}\mathcal{T}_Q + \mathcal{T}_{R\circ Q}\mathcal{T}_S - \mathcal{T}_{(Q\circ S)\circ R}\\ & + [\mathcal{T}_R, \mathcal{T}_Q]\mathcal{T}_S + \mathcal{T}_R[\mathcal{T}_S, \mathcal{T}_Q] + [\mathcal{T}_S, \mathcal{T}_R]\mathcal{T}_Q.\end{aligned} \tag{15}$$

Note that $\mathcal{T}_{Q\circ S}\mathcal{T}_R P^2 = (Q \circ S) \circ (R \circ P^2) \in (\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)})$, which implies that $\{(\mathcal{T}_{Q\circ S}\mathcal{T}_R P^2)QRS\}$, $\{(\mathcal{T}_{S\circ R}\mathcal{T}_Q P^2)QRS\}$, $\{(\mathcal{T}_{R\circ Q}\mathcal{T}_S P^2)QRS\}$, $\{(\mathcal{T}_{R\circ(Q\circ S)} P^2)QRS\} \in \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)})$. From (15), we get that

$$\begin{aligned}\{(\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q P^2)QRS\} \in & \ \{([\mathcal{T}_R, \mathcal{T}_Q]\mathcal{T}_S P^2)QRS\} + \{(\mathcal{T}_R[\mathcal{T}_S, \mathcal{T}_Q]P^2)QRS\}\\ & + \{([\mathcal{T}_S, \mathcal{T}_R]\mathcal{T}_Q P^2)QRS\} + \mathcal{J}^{(n)} + \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)}).\end{aligned}$$

Since $QR = \frac{1}{2}(R \circ Q) - \frac{1}{2}[R, Q]$ and $[\mathcal{T}_R, \mathcal{T}_Q]\mathcal{T}_S P^2 = [[R, Q], S \circ P^2]$, by (14), then we have that

$$\begin{aligned}\{([\mathcal{T}_R, \mathcal{T}_Q]\mathcal{T}_S P^2)QRS\} = & \ \frac{1}{2}\{([[R, Q], S \circ P^2])(R \circ Q)S\} - \frac{1}{2}\{([[R, Q], S \circ P^2])[R, Q]S\}\\ = & \ \frac{1}{2}\{([[R, Q], S \circ P^2])(R \circ Q)S\} - \frac{1}{2}\{([R, Q](S \circ P^2)[R, Q])S\}\\ & + \frac{1}{2}\{(S \circ P^2)[R, Q]^2 S\}\\ = & \ \frac{1}{2}\{([[R, Q], S \circ P^2])(R \circ Q)S\} - \frac{1}{2}\{([R, Q](S \circ P^2)[R, Q])S\}\\ & - \frac{1}{2}\{[R, Q]^2(S \circ P^2)S\} + \mathcal{J}^{(n)}.\end{aligned}$$

By (5) and (11), we get that $\{([\mathcal{T}_R, \mathcal{T}_Q]\mathcal{T}_S P^2)QRS\} \in \mathcal{J}^{(n)}$. Similarly, we have that

$$\{([\mathcal{T}_S, \mathcal{T}_R]\mathcal{T}_Q P^2)QRS\}, \{([\mathcal{T}_S, \mathcal{T}_R]P^2)QRS\} \in \mathcal{J}^{(n)}.$$

On the other hand, by (14), we get that

$$\begin{aligned}\{(\mathcal{T}_R[\mathcal{T}_S, \mathcal{T}_Q]P^2)QRS\} = & \ \{(R \circ [[S, Q], P^2])QRS\}\\ = & \ \{R(R \circ [[S, Q], P^2])QS\} + \mathcal{J}^{(n)}\\ = & \ R \circ \{R[[S, Q], P^2]QS\} + \{(R[[S, Q], P^2]R)QS\} - \{R\{[[S, Q], P^2]QS\}R\} + \mathcal{J}^{(n)}\\ = & \ R \circ \{R([\mathcal{T}_S, \mathcal{T}_Q]P^2)QS\} + \{(R[[S, Q], P^2]R)QS\} - \{R\{[[S, Q], P^2]QS\}R\} + \mathcal{J}^{(n)}\end{aligned}$$

from which and (14) we obtain that $\{(\mathcal{T}_R[\mathcal{T}_S, \mathcal{T}_Q]P^2)QRS\} \in \mathcal{J}^{(n)}$. Therefore, we have obviously that

$$\{(\mathcal{T}_S\mathcal{T}_R\mathcal{T}_Q P^2)QRS\} \in \mathcal{A}(\mathcal{J}^{(n)} \circ \mathcal{J}^{(n)}) + \mathcal{J}^{(n)}.$$

The conclusion follows. □

Lemma 3.5. *For any $n, m \in \mathcal{N}$, we have that*

(i) $(\mathcal{J}(\mathcal{J}^{(n)}))^{2m} \subseteq (\mathcal{A}(\mathcal{J}^{(n)}))^{2m} \subseteq \mathcal{A}((\mathcal{J}^{(n)})^m)$.
(ii) $(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))^{10^n} \subseteq (\mathcal{A}(\mathcal{J} \circ \mathcal{J}))^{10^n} \subseteq \mathcal{A}(\mathcal{J}^{(n)})$.

Proof. (i) We proceed by induction on n. Suppose that $(\mathcal{A}(\mathcal{J}^{(n)}))^{2m} \subseteq \mathcal{A}((\mathcal{J}^{(n)})^m)$ for some natural number m. Recall that $\mathcal{A}(\mathcal{J}^{(n)}) = \mathcal{J}^{(n)} + \mathcal{A}\mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A} + \mathcal{A}\mathcal{J}^{(n)}\mathcal{A}$ and $\mathcal{A} = \mathcal{J} + \mathcal{J}\mathcal{A} = \mathcal{J} + \mathcal{A}\mathcal{J}$. Then we get that

$$(\mathcal{A}(\mathcal{J}^{(n)}))^{2(m+1)} \subseteq \mathcal{A}((\mathcal{J}^{(n)})^m)(\mathcal{J}^{(n)} + \mathcal{A}\mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A} + \mathcal{A}\mathcal{J}^{(n)}\mathcal{A})^2.$$

Note that $\mathcal{A}((\mathcal{J}^{(n)})^{m+1}) = (\mathcal{J}^{(n)})^{m+1} + \mathcal{A}(\mathcal{J}^{(n)})^{m+1} + (\mathcal{J}^{(n)})^{m+1}\mathcal{A} + \mathcal{A}(\mathcal{J}^{(n)})^{m+1}\mathcal{A}$. The statement will be proved if we show that $\mathcal{A}\mathcal{J}^{(n)}\mathcal{J}^{(n)}, \mathcal{A}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)} \subseteq \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}$. By Lemma 3.4 (c), we obtain that $\mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)} \subseteq \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}$.

For any $A, B \in \mathcal{J}$ and $C \in \mathcal{J}^{(n)}$, it is easy to check that

$$ABC = \frac{1}{2}((A \circ B)C + \frac{1}{2}[[A, B], C] + \frac{1}{2}C[A, B].$$

Hence we get that

$$\mathcal{J}\mathcal{J}\mathcal{J}^{(n)} \subseteq \mathcal{J}\mathcal{J}^{(n)} + [[\mathcal{J}, \mathcal{J}], \mathcal{J}^{(n)}] + \mathcal{J}^{(n)}\mathcal{J}\mathcal{J}.$$

By Lemma 3.4(a), we have that $[[\mathcal{J}, \mathcal{J}], \mathcal{J}^{(n)}] \subseteq \mathcal{J}^{(n)}$. By induction, we have that

$$\mathcal{A}\mathcal{J}^{(n)} \subseteq \mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}.$$

Then we get that

$$\begin{aligned}\mathcal{A}\mathcal{J}^{(n)}\mathcal{J}^{(n)} &\subseteq \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}\\ &\subseteq \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}(\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}) + \mathcal{J}^{(n)}\mathcal{A}\\ &\subseteq \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}.\end{aligned}$$

Similarly, we have that

$$\begin{aligned}&\mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}\mathcal{J}^{(n)}, \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)}\mathcal{A}, \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)}, \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)}\\ &\subseteq \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}.\end{aligned}$$

Further, it is easy to verify that

$$\begin{aligned}\mathcal{A}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)} \subseteq\ & \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}^{(n)}\mathcal{A}\\ &+ \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{A}\mathcal{J}\mathcal{J}^{(n)}\\ \subseteq\ & \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)} + \mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)}\mathcal{A} + \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{A}.\end{aligned}$$

On the other hand, note that $\mathcal{J}\mathcal{J}^{(n)} \subseteq \mathcal{J} \circ \mathcal{J}^{(n)} + \mathcal{J}^{(n)}\mathcal{J}$. Then we have that

$$\mathcal{J}\mathcal{J}^{(n)}\mathcal{J}\mathcal{J}^{(n)} \subseteq \{(\mathcal{J} \circ \mathcal{J}^{(n)})\mathcal{J}\mathcal{J}^{(n)}\} + \mathcal{J}^{(n)}\mathcal{A}.$$

By (2) and (5), it is easily seen that

$$\{(\mathcal{J} \circ \mathcal{J}^{(n)})\mathcal{J}\mathcal{J}^{(n)}\} = \{\mathcal{J}^{(n)}\mathcal{J}(\mathcal{J} \circ \mathcal{J}^{(n)})\} \subseteq ((\mathcal{J} \circ \mathcal{J}^{(n)}) \circ \mathcal{J}) \circ \mathcal{J}^{(n)} + [\mathcal{J}^{(n)}, [\mathcal{J}, \mathcal{J}^{(n)} \circ \mathcal{J}]].$$

By Lemma 3.4(d) and (a), the conclusion follows.

(ii) Let us suppose that $(\mathcal{A}(\mathcal{J}\circ\mathcal{J}))^{10^k}\subseteq\mathcal{A}(\mathcal{J}^{(k)})$. Using Lemma 3.4 (e) and (i), we have that

$$(\mathcal{A}(\mathcal{J}\circ\mathcal{J}))^{10^{k+1}}\subseteq(\mathcal{A}(\mathcal{J}^{(k)}))^{10}\subseteq\mathcal{A}((\mathcal{J}^{(k)})^5)\subseteq\mathcal{A}(\mathcal{J}^{(k)}\circ\mathcal{J}^{(k)})=\mathcal{A}(\mathcal{J}^{(k+1)}).$$

The conclusion follows. □

Proposition 3.1. *If $A\in\mathcal{K}(\mathcal{X})$, then for nonzero $\lambda\in\mathbb{C}$, the set $\varepsilon_\lambda(\mathcal{T}_A)$ consists of finite rank operators.*

Proof. This is a consequence of Lemma 3.3. □

Proposition 3.2. *Let $A,B\in\mathcal{B}(\mathcal{X})$ and $\lambda\in\mathbb{C}$. If $\|\mathcal{T}^n_{A-\lambda}(B)\|^{\frac{1}{n}}\to 0$ for $n\to\infty$, then $\varepsilon_\lambda(A)$ is invariant under B.*

Proof. Let y be in $\varepsilon_\lambda(A)$. By Lemma 3.1, we have that

$$(A-\lambda)\circ(A-\lambda)^nB=\sum_{k=0}^{n}(-1)^k\binom{n}{k}(\mathcal{T}^{n+1-k}_{A-\lambda}(B))(A-\lambda)^k.$$

By induction on n, one easily checks that

$$(A-\lambda)^{n+1}B=\sum_{i=0}^{n}(-1)^i((A-\lambda)\circ((A-\lambda)^{n-i}B))(A-\lambda)^i+(-1)^{n+1}B(A-\lambda)^{n+1}$$

from which we get that

$$(A-\lambda)^{n+1}B=\sum_{i=0}^{n}(-1)^i\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}(\mathcal{T}^{n+1-k-i}_{A-\lambda}(B))(A-\lambda)^{k+i}+(-1)^{n+1}B(A-\lambda)^{n+1}.$$

Let us write $a_j=2^j\|\mathcal{T}^j_{A-\lambda}(B)\|$, $b_j=2^j\|(A-\lambda)^jy\|$, $c_n=max_{j\leqslant n}a_{n-j+1}b_j$. Recall that $a_n^{\frac{1}{n}}\to 0$, $b_n^{\frac{1}{n}}\to 0$ for $n\to\infty$. It is easy to prove that $c_n^{\frac{1}{n}}\to 0(n\to\infty)$. Hence we have that

$$\begin{aligned}\|(A-\lambda)^{n+1}By\|^{\frac{1}{n}}&\leqslant\left(\sum_{i=0}^{n}\sum_{k=0}^{n-i}\binom{n-i}{k}\|(\mathcal{T}^{n+1-k-i}_{A-\lambda}(B))(A-\lambda)^{k+i}y\|\right)^{\frac{1}{n}}\\&\quad+\|B\|\|(A-\lambda)^{n+1}y\|^{\frac{1}{n}}\\&\leqslant\left(\sum_{i=0}^{n}(\frac{1}{2})^{i+1}\sum_{k=0}^{n-i}\binom{n-i}{k}a_{n+1-k-i}b_{k+i}\left(\frac{1}{2}\right)^{n-i-k}\left(\frac{1}{2}\right)^{k}\right)^{\frac{1}{n}}\\&\quad+\|B\|\|(A-\lambda)^{n+1}y\|^{\frac{1}{n}}\\&\leqslant c_n^{\frac{1}{n}}+\|B\|\|(A-\lambda)^{n+1}y\|^{\frac{1}{n}}.\end{aligned}$$

So we have that $\|(A-\lambda)^{n+1}By\|^{\frac{1}{n}}\to 0$ for $n\to\infty$, which means that $By\in\varepsilon_\lambda(A)$. □

Theorem 3.1. *Let $\mathcal{J}$ be a closed Jordan subalgebra of compact operator. Then $\mathcal{J}$ is either an Engel Jordan algebra, or contains a nonzero finite rank operator.*

Proof. If $\mathcal{J}$ is not an Engel Jordan algebra, then there is an $A\in\mathcal{J}$ such that $\sigma(\mathcal{T}_A|_{\mathcal{J}})\neq\{0\}$. Since $\sigma(\mathcal{T}_A|_{\mathcal{J}})$ is countable compact by Lemma 3.2, it contains a nonzero isolated point λ. Putting $\sigma_1=\{\lambda\}$ and $\sigma_2=\sigma(\mathcal{T}_A|_{\mathcal{J}})\setminus\{\lambda\}$, and repeating the proof of the above Lemma 3.3, we get that $\mathcal{J}=\mathcal{J}_1\bigoplus\mathcal{J}_2$ as the decomposition of $\mathcal{J}$, where $\mathcal{J}_i$ is invariant under $\mathcal{T}_A|_{\mathcal{J}}$ for $i=1,2$, $\sigma(\mathcal{T}_A|_{\mathcal{J}_1})=\sigma_1$ and

$\sigma(\mathcal{T}_A|_{\mathcal{J}_2}) = \sigma_2$. Since $\sigma_1 = \{\lambda\} = \sigma(\mathcal{T}_A|_{\mathcal{J}_1})$, it is easy to check that $\mathcal{J}_1 = \varepsilon_\lambda(\mathcal{T}_A|_{\mathcal{J}_1})$. Particularly, we have that $\varepsilon_\lambda(\mathcal{T}_A|_{\mathcal{J}_1}) \neq \{0\}$. So there exists a $0 \neq C \in \mathcal{J}$ such that $\|(\mathcal{T}_A - \lambda)^n C\|^{\frac{1}{n}} \to 0$ for $n \to \infty$. By Lemma 3.3, we have that C is a finite rank operator, which completes the proof. $\square$

Recall that a subspace $\mathcal{N}$ of $\mathcal{B}(\mathcal{X})$ is a Lie triple system if it is closed under the Lie triple product $[A, [B, C]]$ for all $A, B, C \in \mathcal{N}$. Given a subspace $\mathcal{N}$ of $\mathcal{B}(\mathcal{X})$, we denote by $\mathcal{L}(\mathcal{N})$ the Lie algebra generated by $\mathcal{N}$. If $\mathcal{N}$ is a Lie triple system then it is clear that $\mathcal{L}(\mathcal{N}) = \mathcal{N} + [\mathcal{N}, \mathcal{N}]$, where $[\mathcal{N}, \mathcal{N}] = span\{[A, B] : A, B \in \mathcal{N}\}$. The equality $[A, [B, C]] = (A \circ B) \circ C - (A \circ C) \circ B$ shows that every Jordan algebra is also a Lie triple system. For a Jordan algebra $\mathcal{J}$ of operators, the Lie algebra $\mathcal{L}(\mathcal{J})$ generated by $\mathcal{J}$ coincides with $\mathcal{J} + [\mathcal{J}, \mathcal{J}]$. Further, if $\mathcal{I}$ is a Jordan ideal of $\mathcal{J}$, putting $\mathcal{L}(\mathcal{J}, \mathcal{I}) = \mathcal{I} + [\mathcal{I}, \mathcal{J}]$, applying the Jacobi identity, it is easy to check that $\mathcal{L}(\mathcal{J}, \mathcal{I})$ is a Lie algebra, $\mathcal{L}(\mathcal{I})$ is a Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{I})$, and $\mathcal{L}(\mathcal{J}, \mathcal{I})$ is a Lie ideal of $\mathcal{L}(\mathcal{J})$, where $\mathcal{L}(\mathcal{I})$ and $\mathcal{L}(\mathcal{J})$ are Lie algebras generated by $\mathcal{I}$ and $\mathcal{J}$, respectively. Regarding the existence of invariant subspaces for Lie algebras of compact operators, a number of reducibility criteria are given (see [2,6–8]). We require the following known results.

Lemma 3.6 (See [2, Theorem 2.5]). *Let $\mathcal{L}$ be a Lie algebra of compact operators. The following conditions are equivalent.*

(i) *$\mathcal{L}$ is triangularizable.*
(ii) *$[\mathcal{L}, \mathcal{L}]$ is an Engel Lie algebra.*

Lemma 3.7 (See [2, Remark 9.6] and [2, Lemma 10.1]). *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{X})$ be a Lie triple system and $\mathcal{L}(\mathcal{M}) = \mathcal{M} + [\mathcal{M}, \mathcal{M}]$ a Lie algebra generated by $\mathcal{M}$. If $\mathcal{M}$ is Volterra, then $\mathcal{L}$ is triangularizable.*

Lemma 3.8 (See [8, Theorem 1.1]). *Let $\mathcal{L}$ be a Lie algebra of compact operators. If $\mathcal{L}$ has a nonzero Engel ideal, then $\mathcal{L}$ is reducible.*

Let $\mathcal{L} \subseteq \mathcal{B}(\mathcal{X})$ be an operator Lie algebra. The normalizer $Nor(\mathcal{L})$ of $\mathcal{L}$ in $\mathcal{B}(\mathcal{X})$ is defined as $Nor(\mathcal{L}) := \{S \in \mathcal{B}(\mathcal{X}) : [S, \mathcal{L}] \subseteq \mathcal{L}\}$, if $\mathcal{M}$ is a Lie ideal of $\mathcal{L}$, it is clear that $\mathcal{L} \subseteq Nor(\mathcal{M})$. A Lie ideal $\mathcal{M}$ of $\mathcal{L}$ is inner-characteristic if $\mathcal{M}$ is invariant for all adS with $S \in Nor(\mathcal{L})$, i.e. $Nor(\mathcal{L}) \subseteq Nor(\mathcal{M})$. The following lemma is useful.

Lemma 3.9 (See [8, Theorem 5.20]). *Let $\mathcal{L}$ be a Lie algebra of compact operators. Then $\mathcal{L}$ has the largest Engel ideal $\mathcal{E}(\mathcal{L})$ such that $\mathcal{E}(\mathcal{L})$ is closed in $\mathcal{L}$ and is inner-characteristic.*

Theorem 3.2. *Any solvable Jordan algebra $\mathcal{J}$ of compact operators is triangularizable.*

Proof. Let us assume that $\mathcal{J}^{(n)} = \{0\}$ and $\mathcal{J}^{(n-1)} \neq \{0\}$ for some positive integer n. So it follows immediately that $\mathcal{A}(\mathcal{J}^{(n)}) = \{0\}$. By Lemma 3.5, then we have obviously that

$$(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))^{10^n} \subseteq (\mathcal{A}(\mathcal{J} \circ \mathcal{J}))^{10^n} \subseteq \mathcal{A}(\mathcal{J}^{(n)}),$$

which implies that $\mathcal{J}(\mathcal{J} \circ \mathcal{J})$ is nilpotent. Now let us put that

$$\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})) = \mathcal{J}(\mathcal{J} \circ \mathcal{J}) + [\mathcal{J}(\mathcal{J} \circ \mathcal{J}), \mathcal{J}].$$

Further, it is easy to check that $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a Lie algebra, $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ and $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a Lie ideal of $\mathcal{L}(\mathcal{J})$, where $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ and $\mathcal{L}(\mathcal{J})$ are Lie algebras generated by $\mathcal{J}(\mathcal{J} \circ \mathcal{J})$ and $\mathcal{J}$, respectively. Moreover, by Lemma 3.9, let us suppose that $\mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))$ is the largest Engel ideal in $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$.

Recall that $\mathcal{J}(\mathcal{J} \circ \mathcal{J})$ is nilpotent and $\mathcal{J}(\mathcal{J} \circ \mathcal{J})$ generates $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$. By Lemmas 3.6 and 3.7, we have that $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is triangularizable and therefore $[\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J})), \mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))]$ is an Engel Lie algebra.

Note that $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$. By the Jacobi identity, it is easy to see that $[\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J})), \mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))]$ is also a Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$.

(i) If $[\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J})), \mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))] \neq \{0\}$, then it is a nonzero Engel Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$.

(ii) If $[\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J})), \mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))] = \{0\}$, then $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is commutative, and hence $\mathcal{L}(\mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a nonzero Engel Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$.

Therefore, $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ has a nonzero Engel Lie ideal. Recall that $\mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))$ is the largest Engel Lie ideal of $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$. That means that $\mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))$ is nonzero. Further, since $\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))$ is a Lie ideal of $\mathcal{L}(\mathcal{J})$, then we have that $\mathcal{L}(\mathcal{J}) \subseteq Nor(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))$. It follows directly from Lemma 3.9 that

$$ad\mathcal{L}(\mathcal{J})(\mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))) = [\mathcal{L}(\mathcal{J}), \mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))] \subseteq \mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J}))).$$

So we obtain that $\mathcal{L}(\mathcal{J})$ contains a nonzero Engel Lie ideal $\mathcal{E}(\mathcal{L}(\mathcal{J}, \mathcal{J}(\mathcal{J} \circ \mathcal{J})))$. By Lemma 3.8, we have that $\mathcal{L}(\mathcal{J})$ is reducible, which implies that $\mathcal{J}$ is reducible. In other words, any solvable Jordan algebra has a nontrivial invariant subspace. Since the property of being a solvable Jordan algebra of compact operators is inherited by quotients. That is, the Lie algebras induced on the gaps of a maximal subchain of invariant subspaces of $\mathcal{J}$ are clearly solvable. By the Triangularization Lemma (see [5, Lemma 1] or [1, Lemma 3.1]), Triangularizability is proved. □

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