Discrete Mathematics 313 (2013) 35–49

Contents lists available at SciVerse ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Perspective

Survey of two-dimensional acute triangulations

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A R T I C L E I N F O

Article history:
Received 11 January 2010
Received in revised form 7 September 2012
Accepted 12 September 2012
Available online 9 October 2012

Keywords:
Geodesic triangulation
Acute triangle
Double planar convex body
Polygon
Polyhedral surface

A B S T R A C T

We give a brief introduction to the topic of two-dimensional acute triangulations, mention results on related areas, survey existing achievements – with emphasis on recent activity – and list related open problems, both concrete and conceptual.

0. Introduction

A *triangulation* of a two-dimensional space means a collection of triangles (considered full, i.e. consisting of all points belonging to or surrounded by the three sides) covering the space such that the intersection of any two triangles is empty, or a vertex, or an edge (of both triangles). Triangulations are particular instances of *dissections* (for which the above intersection condition reduces to requiring only empty interior, and any kind of polygons can be considered instead of triangles).

A triangulation is *geodesic* if all of its triangles are geodesic, meaning that their edges are *segments*, i.e. shortest paths between the corresponding vertices. In this paper we shall always refer to geodesic triangulations. In [18], Colin de Verdière shows how to transform a triangulation of a compact surface of non-positive curvature into a geodesic triangulation. The corresponding planar problem was treated by Wagner [75]. See also [29,70].

A triangulation is *non-obtuse* (*acute*) if all angles within its triangles are at most (strictly less than) $\pi/2$. A *balanced* triangulation is an acute triangulation with all angles measuring more than $\pi/6$, and a *right* triangulation is defined as one where all triangles are right, a special case of a non-obtuse triangulation.

While we mainly focus on recent results about acute triangulations in the Euclidean plane, we also survey various findings on non-obtuse triangulations on surfaces, again concentrating on new developments.

The problem may also be treated in higher dimensions. There, one asks for triangulations with simplices (in three dimensions sometimes called *tetrahedrizations*) having acute or non-obtuse dihedral angles. For results therein, many of which are fascinating, read the survey [12] by Brandts et al., which also includes interesting research problems. In this direction, see also the paper [26] by Eppstein et al., which reveals the first known acute tetrahedrization of $\mathbb{R}^3$ and the 2009 breakthrough by VanderZee et al. [73], which gave an acute tetrahedrization of the cube. Kopczyński et al. [52] give a good summary of recent activity on higher-dimensional triangulations.

Concerning the further organization of this survey: the first part deals mainly with acute triangulations of polygons, covering existence, asymptotic upper bounds, mesh generation algorithms, concrete upper bounds, and a detour into dissections and acute triangulations of planar graphs. The second part surveys results on triangulating surfaces, reviewing

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doi:10.1016/j.disc.2012.09.016

what is known for Platonic surfaces, double planar convex bodies, Riemannian manifolds, and various other classical objects from geometry such as the sphere and the Klein bottle.

In order to give a taste of techniques typically used in the results featured in Sections 2.1, 2.3 and 3, we (i) give a short overview of the methods used by Saraf [65] to prove the deep Burago–Zalgaller Theorem [14] (see Section 2.1), where she uses a result of Yuan [76] which we discuss in detail. Yuan's Theorem provides a method to transform a non-obtuse triangulation of a polygon into an acute one. Furthermore, (ii) we reproduce from [38] a combinatorial proof of the fact that the size of the smallest acute triangulation of the boundary of the cube is 24 (Theorem 3.1.1) and (iii) exhibit further geometric aspects of these proofs by showing that every flat Möbius band admits an acute triangulation of size at most 9 (Theorem 3.4.3), where we followed [81]. Lastly, in Section 4, we give a list of eight open problems.

The concept of this survey rests on two pillars. On one hand we see the purely geometric aspect of treating geodesic triangulations, on the other the applications. In some applications, including mesh generation, only those triangulations of polyhedral surfaces are relevant which use all edges of the polyhedron as edges of the triangulations. However, in this survey we do not impose this restriction and treat the more general case in which we can find pieces of edges or even vertices of a polyhedral surface in interiors of triangles of a triangulation.

1. Motivation and history

Interest in non-obtuse and acute triangulations is widespread, and we shall indicate here only a fraction of the papers requiring "nicely" triangulated domains. What "nicely" means varies; it is often related to conditions concerning the angles of the triangulation. An important motivation is anchored in numerical analysis, where very flat and very sharp angles are undesirable—see the classical papers [5,32,68]. Let us now give samples from various fields of mathematics that necessitate (or yield stronger results on) domains that can be non-obtusely or acutely triangulated.

To our knowledge, one of the earliest instances of interest in non-obtuse triangulations was published in 1953, when MacNeal [55] studied the discretization of partial differential equations. In 1973, working on the proof of the discrete maximum principle, Ciarlet and Raviart [17], and later Strang and Fix [69] and also Santos [64], were interested in non-obtuse triangulations. Vavasis [74] studied an elliptic boundary value problem, where the error bound of the linear system depends on the angle of the triangulation employed; thus, a non-obtuse triangulation was desirable.

Acutely triangulating certain domains often yields stronger results, as can be seen in Elliott's and Stuart's 1993 paper [24], where the global dynamics of discrete semilinear parabolic equations were studied, and in Sethian's 1999 paper [66], which investigated the fast marching method (i.e. solving the eikonal equation). Recently, Erickson et al. showed interest in acutely triangulated domains, presenting in [27] an algorithm that constructs meshes suitable for spacetime discontinuous Galerkin finite-element methods. In an earlier paper [71] on the same subject, the triangulation was in fact required to be acute.

The theoretical (i.e. non-algorithmic) investigation of acute triangulations began circa 1960, with two dissimilar directions of attack: on one hand we have the deep Burago–Zalgaller Theorem [14], which proves the existence of acute triangulations on polyhedral surfaces—unfortunately, following their argument leads to a huge number of triangles and gives no practical information on how (non-huge) triangulations might be constructed in practice. On the other hand, we have the problem of finding the exact minimum size of the acute triangulation of a given convex n-gon, especially for small n. The latter started with two identical problems, one by Stover, reported by Gardner in his Mathematical Games section of the *Scientific American*, and the other proposed by Goldberg in the *American Mathematical Monthly*, both in 1960. More about this appears in Section 2.3.

Further developments and more detailed accounts of each line of research will be given in each section separately. Also, additional definitions that are essential for understanding certain results will be mentioned when needed.

2. Polygons

2.1. Existence and asymptotic upper bounds

In 1960, Burago and Zalgaller [14] – and 49 years later, with a simpler construction, Saraf [65] – proved the existence of acute triangulations for general polyhedral surfaces. It is notable that, in both papers, no specific upper bounds on the number of triangles needed are given. A corollary of Burago's and Zalgaller's work [14] on embedding an abstract polyhedron in $\mathbb{R}^3$ is as follows.

Theorem 2.1.1 (*Burago and Zalgaller [14]*). *Polygons (possibly with holes) and polyhedral surfaces can be acutely triangulated.*

We now give a short account of Saraf's proof of this fundamental result. The main challenge she tackled was the following. Consider a polygon that has been subdivided into polygonal regions. We now want to triangulate with non-obtuse triangles all of these regions so that their boundary vertices coincide (to get a triangulation, not a dissection, of the initial polygon). Saraf employed here the *divide-and-conquer* technique: an algorithm that recursively divides a given problem into subproblems of related type until the subdivided problems become simple enough to be solved directly. In this way she obtained a non-obtuse triangulation of a general polyhedral surface by subtriangulating each triangle separately while taking care that their boundary vertices coincide. Combining the aforementioned with results by Maehara [57] and

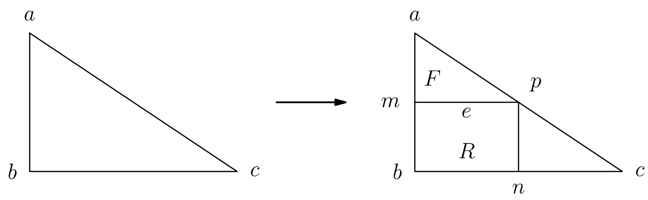


Fig. 1. The elementary subdivision of a right triangle.

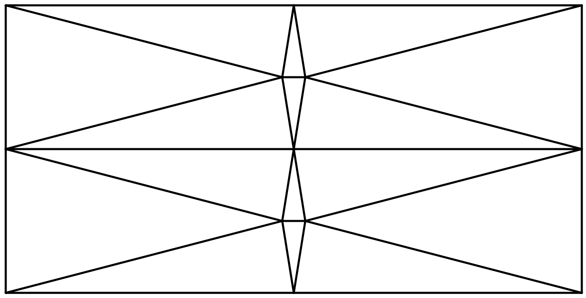


Fig. 2. The basic triangulation of a rectangle.

Yuan [76], which ascertained methods of transforming a non-obtuse triangulation of a given polygon into an acute one, she obtained an acute triangulation of an arbitrary polyhedral surface. We will give a sketch of Yuan's method (versus the one shown in [57]) for transforming a non-obtuse triangulation $\mathcal{T}$ of a polygon Γ into an acute one [76]. We include some detailed parts showing the main tools used by Yuan. Several arguments are akin to Saraf's divide-and-conquer approach.

For any triangle T, the segments joining the midpoints of the sides divide it into four congruent triangles similar to T. This is an *elementary subdivision* of T. The first step is to subdivide $\mathcal{T}$ into a tiling $\mathcal{T}_1$ according to the following two rules. Rule 1: Any acute triangle in $\mathcal{T}$ will be divided into four acute triangles by an elementary subdivision. Rule 2: Any right triangle in $\mathcal{T}$ will be divided into a rectangle and two right triangles as shown in Fig. 1, where the point on each side is its midpoint.

A face-to-face tiling $\mathcal{T}_1$ of Γ is obtained, consisting of a family $\mathcal{S}_1$ of acute triangles, a family $\mathcal{R}_1$ of rectangles and a family $\mathcal{F}_1$ of right triangles; $\mathcal{T}_1 = \mathcal{S}_1 \cup \mathcal{R}_1 \cup \mathcal{F}_1$. Each triangle $F \in \mathcal{F}_1$ is obtained according to Rule 2 (see Fig. 1) and has a rectangle R as a neighbor. This rectangle R is called the *basic rectangle* of F, and the common edge of F and R is called the *basic edge* of F. Thus, any $F \in \mathcal{F}_1$ has exactly one basic rectangle and one basic edge.

In the second step we subdivide $\mathcal{T}_1$ into $\mathcal{T}_2$, again following two rules. Rule 1: Apply an elementary subdivision to every element in $\mathcal{S}_1 \cup \mathcal{F}_1$. Rule 2: Triangulate every rectangle in $R \in \mathcal{R}_1$ as follows. Take the midpoints of two opposite sides and join them. This cuts R into two rectangular halves. Now apply the triangulation from Fig. 6 (which applies to any rectangle) to each half. The result is shown in Fig. 2. This is a *basic triangulation* of R.

We obtain a non-obtuse triangulation $\mathcal{T}_2$ of Γ. It remains to show that $\mathcal{T}_2$ can be converted into an acute triangulation of Γ. Let $\mathcal{S}_2$, $\mathcal{R}_2$ and $\mathcal{F}_2$ denote all triangles in $\mathcal{T}_2$ obtained from $\mathcal{S}_1$, $\mathcal{R}_1$, $\mathcal{F}_1$, respectively. Every triangle in $\mathcal{S}_2 \cup \mathcal{R}_2$ is acute. Hence, we need only consider the triangles in $\mathcal{F}_2$. The right triangles in $\mathcal{F}_2$ are obtained from $\mathcal{F}_1$ by elementary subdivision, so we can classify them according to their corresponding right triangles in $\mathcal{F}_1$. For any acute triangle (right triangle) $S \in \mathcal{S}_1$ $(F \in \mathcal{F}_1)$, let $\varphi(S)$ $(\varphi(F))$ denote the family of four acute triangles (four right triangles) obtained from the elementary subdivision of S (F). Similarly, for any rectangle $R \in \mathcal{R}_1$, let $\varphi(R)$ denote the family of sixteen acute triangles obtained from the elementary subdivision of the basic triangulation of R. We call the triangle in $\varphi(F)$ that has empty intersection with the basic edge e of F the *opposite triangle* with respect to e. If $\mathcal{S}_1 = \{S_1, S_2, \ldots, S_t\}$, $\mathcal{F}_1 = \{F_1, F_2, \ldots, F_n\}$, and $\mathcal{R}_1 = \{R_1, R_2, \ldots, R_m\}$, then

$$\mathcal{S}_2 = \bigcup_{i=1}^{t} \varphi(S_i), \qquad \mathcal{F}_2 = \bigcup_{i=1}^{n} \varphi(F_i), \quad \text{and} \quad \mathcal{R}_2 = \bigcup_{i=1}^{m} \varphi(R_i).$$

We prove that for any $F \in \mathcal{F}_1$ with basic rectangle $R \in \mathcal{R}_1$ and basic edge e, the triangulation can be perturbed so that all triangles in $\varphi(F) \cup \varphi(R)$, except the opposite triangle with respect to e (in $\varphi(F)$), become or remain acute. Let F have vertices a, b, c and basic edge bc, and let n be the midpoint of bc, as shown in Fig. 3.

We can slide n slightly in direction $\overrightarrow{nb}$ so that both mnp and pnc become acute triangles while all triangles in $\varphi(R)$ remain acute. Then we slide n slightly in direction $\overrightarrow{ba}$ so that mbn becomes acute while mnp, pnc and all triangles in $\varphi(R)$ remain acute.

Let e_i denote the basic edge of $F_i \in \mathcal{F}_1$ $(i \in \{1, \ldots, n\})$. The third step consists of converting the triangulation $\mathcal{T}_2$ into $\mathcal{T}_3$ by applying the sliding described above to each $\varphi(F_i)$. After Step 3, we know that in each $\varphi(F_i)$, only the opposite triangle with respect to e_i is not changed into an acute triangle. We denote by $\varphi_3(F_i)$ the set of four triangles in $\mathcal{T}_3$ resulting from $\varphi(F_i)$. There remain exactly n right triangles in $\mathcal{T}_3$, one in each of $\varphi_3(F_i)$. The case that $F \in \mathcal{F}_1$ has no adjacent right triangle

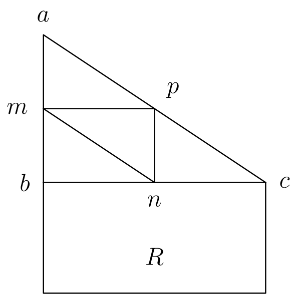
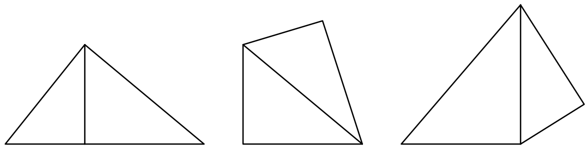


Fig. 3.



(a) Type I. (b) Type II. (c) Type III.

Fig. 4. Three kinds of adjacent triangles in $\mathcal{F}_1$.

is easily treated. (Two triangles of a triangulation are *adjacent* if they share a common edge.) Let F_i and F_j be two adjacent triangles in $\mathcal{F}_1$. If they have a common leg (hypotenuse), then we call them Type I (II); otherwise, we call them Type III, see Fig. 4.

Yuan then shows with a case-by-case analysis of the three possible Types I, II, III (which we do not reproduce in detail here), that for any two adjacent triangles F_i and F_j in $\mathcal{F}_1$, we can transform all the triangles in $\varphi_3(F_i) \cup \varphi_3(F_j)$ into acute triangles, leaving unchanged all other triangles in $\mathcal{T}_3 - (\varphi_3(F_i) \cup \varphi_3(F_j))$. This completes the proof.

In [57] Maehara had also proved the existence of an acute triangulation for any polygon on the basis of the existence of a non-obtuse triangulation. At the same time, he gave the upper bound $2 \cdot 6^5 N$ for the number of triangles in the acute triangulation, where N denotes the number of triangles in the existing non-obtuse triangulation. Refining Maehara's method, Yuan was able to considerably improve the factor from $2 \cdot 6^5$ to 24 in [76]. This is easy to show by using the above discussion.

Let $\mathcal{T}$ be a non-obtuse triangulation of a polygon, and let ν_1 (ν_2) denote the number of all acute (right) triangles in $\mathcal{T}$. Thus $N = \nu_1 + \nu_2$. Note that $|\mathcal{S}_1| = 4\nu_1$, $|\mathcal{R}_1| = \nu_2$, and $|\mathcal{F}_1| = 2\nu_2$ in $\mathcal{T}_1$. We have $|\mathcal{T}_2| = 4(|\mathcal{S}_1| + |\mathcal{F}_1|) + 16|\mathcal{R}_1| = 16\nu_1 + 24\nu_2 = 16(\nu_1 + \nu_2) + 8\nu_2 \leq 24N$. This implies $|\mathcal{T}_3| = |\mathcal{T}_2| \leq 24N$.

In her thesis [77], Yuan was able to slightly improve her own result (presented above).

Theorem 2.1.2 (*Yuan [77]*). *If a polygon can be triangulated into N non-obtuse triangles, then it can be triangulated into at most* $22N$ *acute triangles.*

Moreover, Yuan [76] established a concrete upper bound for the size of a non-obtuse triangulation of an n-gon, obtaining $N \leq 106n - 216$ on the basis of the circle packing method introduced in [11]. Combining Theorem 2.1.2 and the last sentence, one obtains the following result, which gives the best known asymptotic bound for acutely triangulating polygons.

Theorem 2.1.3 (*Yuan [77]*). *Every n-gon admits a triangulation into at most* $22 \cdot (106n - 216)$ *acute triangles.*

2.2. Mesh generation algorithms

Mesh generation refers to finding a triangulation for which the domain is specified, being typically a polygon (admitting holes) or a polyhedral surface. We already mentioned that mesh generation applications require that the angles not be too flat or very sharp; for motivation see [5] or [32]. The upper bound of 90° is of special interest in mesh generation applications, as it necessarily yields a Delaunay triangulation. (A planar triangulation $\mathcal{T}$ with vertices P is a *Delaunay triangulation* [20] if no vertex in P lies inside the circumcircle of any triangle in $\mathcal{T}$.) Additionally, non-obtuse meshes provide matrices with desirable numerical properties (for details, see [6]). Although its focus is not on acute triangulations, an excellent survey on algorithms for generating triangulations is [9]. We give here only a short overview.

In 1988, Baker et al. [6] published the first provably correct algorithm for constructing non-obtuse triangulations of polygons. However, this algorithm gives no size guarantee. Therefore, the output size may be arbitrarily large.

Four years later, Bern and Eppstein [8] provided an algorithm creating polynomial-size non-obtuse triangulations of n-gons (admitting holes), which runs in $O(n^2)$ time. Bounding all angles above by 90° is best possible for a polynomial-size triangulation, in the sense that any smaller fixed bound would require the number of triangles to depend not only on the size of the input, but also on the ratio between shortest height and longest side of the triangles allowed. See also [10], but note that this primarily addresses the triangulation of point sets.

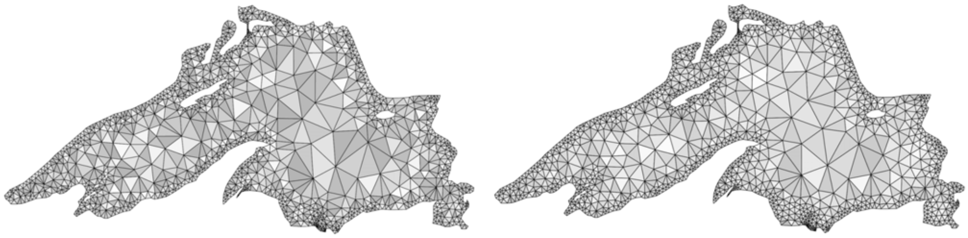


Fig. 5. Meshes of lake superior. The left figure shows the mesh constructed in [28], while the right figure depicts the improved mesh from [72], where the number of angles close to 90° has been dramatically reduced.
Source: Reproduced from Ref. [72].

In the same year, Melissaratos and Souvaine [61] presented an algorithm for constructing non-obtuse triangulations of polygons (admitting holes), using the results from [10]—note that earlier versions of [10] appeared since 1990. The special feature of this algorithm is that it can vary the level of refinement.

Also in 1992, Edelsbrunner et al. [23] gave an algorithm running in $O(n^2 \log n)$ time, which, given a constraint set of points and line-segments in the plane, yields a triangulation that minimizes the maximum angle (without adding points). If an acute triangulation exists, then this algorithm finds it. See also [7], where it is shown how to construct triangulations that minimize the maximum angle.

The "quadtree algorithm" is a useful tool in finding non-obtuse triangulations; see e.g. [31]. It involves a recursive partition of a region of the plane into axis-aligned squares. One square, the root, covers the entire region. A square can be divided into four child squares, by splitting it with horizontal and vertical line segments through its center. The collection of squares then forms a tree, with smaller squares at lower levels of the tree, the *quadtree*.

Based on the quadtree approach, Bern et al. published in [10] the following important algorithms. The first algorithm, given n points in the plane, yields a triangulation of the convex hull of the n points such that (i) the n points are vertices of the triangulation and (2) all angles are larger than some constant. Additionally, it can be modified (at a complexity cost) so that it also yields a non-obtuse triangulation. The second algorithm, given n points in the plane, provides a non-obtuse triangulation in linear time. To achieve acute triangulations, one may (i) modify the first algorithm so that all angles measure between 36° and 80°, or (ii) modify the second algorithm, which yields an acute triangulation. However, the angles then may be arbitrarily close to 90°.

In 1995, an interesting algorithm based on circle packings was published by Bern et al. [11], yielding non-obtuse planar triangulations in $O(n)$ time.

In 2002, Maehara [57] showed how to acutely triangulate a planar domain in $O(n)$ time and provided a method to modify a non-obtuse triangulation in order to obtain an acute one. He also gave a short proof that one can triangulate a polygon into right triangles.

Yuan [76] – based on [11] – published in 2005 her already mentioned, concrete upper bounds (improving results in [57]) for the number of triangles used in the triangulation of a polygon.

We also mention the recent results by Erten and Üngör [28], and by VanderZee et al. [72] (see Fig. 5). In the former, which appeared in 2007, the authors present a variant of a Delaunay refinement algorithm for acute triangulations. In the latter, published in 2010, an iterative algorithm is presented that transforms a given planar triangle mesh into an acute one by moving the interior vertices while keeping the connectivity fixed. The authors' approach is based on minimizing a certain energy, and their algorithm appears to be the first known strategy for acuteness that may generalize well to higher dimensions.

2.3. *Upper bounds for n-gons*

As mentioned in the introduction, concrete upper bounds for the size of acute triangulations have always attracted much interest, especially for basic geometrical objects such as polygons with few vertices. Since the Burago–Zalgaller Theorem from [14] mentioned in the introduction provides no further insight, other more constructive methods had to be developed. We will give in the following an account of the best known bounds (few are optimal) depending on the number of vertices of the polygon.

The problem of finding the exact minimum size of the acute triangulation of a given polygon had one of its origins in a problem of Stover reported in 1960 by Gardner in his Mathematical Games section of the *Scientific American* (see [33,34]). Stover asked whether a triangle with one obtuse angle can be cut into smaller triangles, all of them acute (a dissection!). By sheer coincidence, in the same year, Goldberg proposed the same problem in the *American Mathematical Monthly* (E1406, see [37]). Replying to the problem proposed by Goldberg, Manheimer [60] produced in 1960 a solution, which happens to be not only a dissection, but also a triangulation. Among the solvers we find Knuth and Federico (independently obtaining the

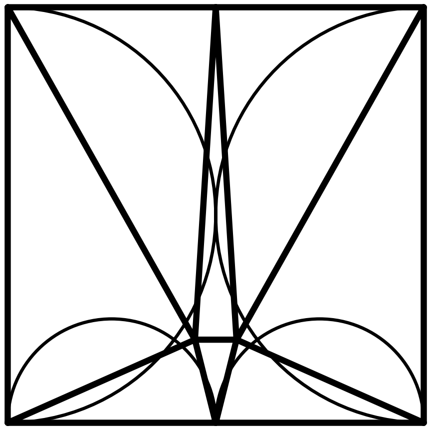


Fig. 6. The combinatorially unique triangulation of the square into 8 acute triangles (the purpose of the semicircles is to help check acuteness).

result). The latter gave an analysis of the problem based on Euler's formula. Also Burago and Zalgaller gave – unknowingly – in [14] the same answer in the same year, 1960: any obtuse triangle can be triangulated into 7 triangles, and this is best possible.

In the same busy year of 1960, Lindgren [54] described an acute triangulation of the square (see Fig. 6), proving that it can be done with 8 triangles and that this is optimal—Federico solved this independently in the same year. In 1966, Gardner also gave a construction, which he reports in one of his mathematical columns (reprinted in [35]), saying: "*For days I was convinced that nine was the answer; then suddenly I saw how to reduce it to eight*". We remark that, in fact, Gardner was trying to find dissections of the square, and it happened that his configuration on eight triangles is a triangulation. If he were looking for triangulations, this would have contradicted the following matter.

Cassidy and Lord [15] continued the investigations of acutely triangulating the square, publishing their results in 1980. They gave an alternative proof of the minimality (and combinatorial uniqueness) of the Federico–Lindgren construction on 8 triangles. They also showed that there is no triangulation consisting of exactly 9 triangles, and proved that there exist acute triangulations of the square with k triangles whenever $k \geq 10$.

Eppstein [25] discusses a slightly different problem, posed initially by Tromp in 1996. How do we minimize the maximal angle? For the solution using 8 triangles, he found a placement of the vertices in which the maximum angle is only about 85° and asked if more triangles would achieve an even better bound. Eppstein also produced a triangulation requiring 14 triangles, with all angles measuring 45°, 54°, 63°, or 72°. This was motivated by Tromp's question and a result of Gerver [36]. Gerver showed how to compute a dissection (not a triangulation!) of a polygon with no angles larger than 72°, assuming all interior angles of the input measure at least 36°.

A different approach appeared in 2000, when the subject was rediscovered and treated in a broader manner. Since then, many advancements have been made. Hangan et al. [38] proved several results and, among other things, generalized the Federico–Lindgren result:

Theorem 2.3.1 (*Hangan et al. [38]*)**.** *Every rectangle can be triangulated with* 8 *acute triangles, and this is the best possible estimate.*

The above result uses (combinatorially) the same construction as shown in Fig. 6, and already Gerver [36] pointed this out, albeit parenthetically. Yuan [77] studied triangulations of rectangles with all angles in the triangulation smaller than $\pi/2$ and larger than some $\varepsilon > 0$. Let $\delta_{\mathcal{T}}$ denote the smallest angle occurring in a given triangulation $\mathcal{T}$. Yuan [77] showed that the square admits an acute triangulation $\mathcal{T}$ with $|\mathcal{T}| = 14$ and $\delta_{\mathcal{T}} = \pi/4$, thus independently reproving the aforementioned result by Eppstein [25]. Jia et al. [48] proved that the square admits a balanced triangulation (i.e. $\delta_{\mathcal{T}} > \pi/6$) of size 11. Both bounds are shown to be best possible. For further results on rectangles we suggest consulting [77, pp. 69–74]; for results concerning balanced triangulations, see [48].

Theorem 2.3.2 (*Yuan [79]*)**.** *Every trapezoid that is not a rectangle admits an acute triangulation of size at most* 7*.*

By Theorems 2.3.1 and 2.3.2 one obtains a characterization of the rectangle via acute triangulations. In 2000, Maehara [56] investigated acute triangulations of (not necessarily convex) quadrilaterals and showed the following.

Theorem 2.3.3 (*Maehara [56]*)**.** *Every quadrilateral admits an acute triangulation of size at most* 10*, and this is best possible.*

Maehara also showed that any convex quadrilateral can be triangulated with at most 9 acute triangles; it was unknown whether this bound is optimal, as all known examples required at most 8 triangles (see Problem 1 in Section 4—this was answered recently by Cavicchioli, see the Note at the end of this survey). We continue by studying pentagons.

Theorem 2.3.4 (*Yuan [78]*). *Every pentagon can be triangulated into at most* 54 *acute triangles.*

It is interesting to notice the following difference between quadrilaterals and pentagons. While for quadrilaterals the maximum amount 10 is needed only in certain cases when the quadrilateral is not convex, all non-convex pentagons can be triangulated with at most 21 triangles (see Lemma 3.2 in [78]), and the above bound 54 emerges from the proof in one of the cases when the pentagon is convex! However, it seems unlikely that this bound is tight.

For convex n-gons with $6 \le n \le 56$, the best upper bounds are given by the following result.

Theorem 2.3.5 (*Zamfirescu [85]*). *If P is a convex n-gon with $n \ge 6$, then P admits an acute triangulation of size at most a_n, where*

$$a_n = \begin{cases} \frac{2}{3}n^3 + 2n^2 - \frac{65}{3}n + 20 & \text{for even } n \\ \frac{2}{3}n^3 + 2n^2 - \frac{89}{3}n + 68 & \text{for odd } n. \end{cases}$$

For $n \ge 57$ (in the non-convex case for all $n \ge 6$), the best known bound remains that given by Theorem 2.1.3.

2.4. Dissections

We now discuss dissections of polygons into triangles the interior angles of which satisfy certain conditions; this summary is not exhaustive, and gives only a glimpse into the topic. Evidently, any triangulation is a dissection (but not vice versa), whence all results we have seen in Sections 2.1–2.3 are fully applicable. Note that in the first paragraphs of Section 2.3 one clearly sees that this line of research treated dissections, which, in some cases, happened to be triangulations! Most importantly, we once more summon the Burago–Zalgaller Theorem [14], which provides the existence of dissections of polygons (admitting holes or not admitting holes) and polyhedral surfaces into acute triangles.

Hoggatt and Denman [40] answered Goldberg's problem E1406 [37] by dissecting an obtuse triangle into acute isosceles triangles, which is to our knowledge one of the first results on dissecting polygons into acute isosceles triangles.

Theorem 2.4.1 (*Hoggatt and Denman [40]*). *An obtuse triangle abc can be dissected into* 8 *acute isosceles triangles. If $\angle abc > 90^\circ$, $\angle abc - \angle cab < 90^\circ$, and $\angle abc - \angle bca < 90^\circ$, then only* 7 *are needed.*

We next consider dissections of polygons into acute triangles (or with even stronger angle conditions) in general and provide a short summary of the technically complex result of Gerver [36]. He states necessary conditions on the partitions of the vertices of a given polygon in order to guarantee an upper bound α on the largest occurring interior angle within all triangles in the dissection (where the discussion for $\pi/3 \le \alpha < 2\pi/5$ interests us). Gerver goes on to conjecture that these conditions are also sufficient. As in many results of Sections 2.3 and 3, Euler's formula plays a role (here for regular polygons). Gerver also gives results on irregular polygons, by extending his work on regular polygons using Riemann's mapping theorem.

In a different frame, Kaiser [49] proved the existence of dissections of polygons in Euclidean, elliptic, and hyperbolic space. For convex polygons (Euclidean and non-Euclidean) he established a function that yields the minimal number of acute triangles one requires, given the number of vertices of the polygon and non-acute angles. Kaiser also determined that the sphere (elliptic plane) admits a dissection into 20 [10] acute triangles and that both bounds are optimal.

2.5. Acute triangles in planar graphs

There have recently been several interesting results in geometric Graph Theory that deal with acute triangles. Now, finite topological triangulations of the plane are viewed as planar graphs. These may also be seen as *maximal planar graphs*, which are graphs to which no edge may be added without losing planarity. In this context, an *acute (non-obtuse) almost-triangulation* is a planar graph embedded in the plane such that all faces are acute (non-obtuse) triangles, with the exception of the unbounded face. We remark that the authors initiating this study used the term *pseudo-triangulation*, which in Computational Geometry has a completely different meaning. Since this survey concentrates on geometric aspects, we permit ourselves to change the term.

A few definitions are in order. In a planar graph, an *enclosing cycle* is a cycle in whose interior there is at least one vertex. A *separating cycle* of a graph G is a cycle whose deletion disconnects G.

Kaneko et al. [50] showed that an almost-triangulation consisting of m triangles can be transformed into a straight-line embedding in which at least $\lceil m/3 \rceil$ triangles are acute. It is well known that all planar graphs can be embedded into the plane such that every edge is a Euclidean segment [75]. Such an embedding is sometimes called a *planar straight-line graph*. Kaneko, Maehara, and Watanabe also showed that any acute almost-triangulation with no inner vertices admits a straight-line embedding.

A few years later, Maehara proved the following results.

Theorem 2.5.1 (*Maehara [58]*). *An almost-triangulation admits an acute straight-line embedding in the plane if and only if the triangulation has no enclosing cycle of length at most* 4.

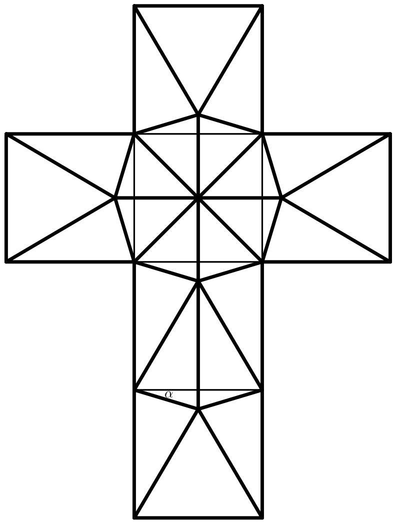


Fig. 7. One of several combinatorially distinct acute triangulations of the unfolded boundary of a cube with 24 triangles.

Theorem 2.5.2 (*Maehara [58]*). *An almost-triangulation with five or more vertices admits a non-obtuse straight-line embedding if and only if*

(i) *the set of vertices of degree* 4 *not belonging to the boundary of the unbounded face contains no adjacent pair, and*
(ii) *the graph obtained by deleting the vertices of degree* 4 *not belonging to the boundary of the unbounded face has no separating cycle of length at most* 4.

In 2005, two closely related results appeared on the subject. Kawarabayashi et al. [51] showed that every 4-connected triangulation with m bounded faces other than the octahedron can be drawn in the plane so that at least $(m+3)/2$ faces are acute triangles; furthermore, the bound is sharp. Koyama and Nakamoto [53] proved that the aforementioned statement holds under similar conditions, weakening 4-connectedness to minimum degree at least 4.

3. Surfaces

3.1. Platonic surfaces

A convex surface is the boundary of a compact convex set in $\mathbb{R}^3$ with non-empty interior. Ellipsoids, (bounded) cylinders, (bounded) cones, and polytopes are examples of convex surfaces. We shall call the boundaries of the five Platonic solids *Platonic surfaces*. They have been investigated extensively. Until recently, the sizes of their minimal triangulations were known except for the dodecahedron; now this case has also been solved [48]. There are two facets in every proof underlying each of these theorems. First, for the geometric construction, one provides a triangulation, acute or non-obtuse, with as few triangles as possible. In some cases, the construction is based on a prior result: one obtains for instance an acute triangulation of the boundary of the cube by using three acutely triangulated rectangles [38], where each rectangle can be acutely triangulated as in Theorem 2.3.1. Combinatorially, this coincides with the construction provided in Fig. 6. Proving correctness is done through a somewhat tedious case-by-case analysis, dealing with every angle in part. Second, the minimality of (or at least a good bound on) the number of triangles that are used in the construction is proved. This involves combinatorial techniques (like Euler's formula) and often further geometrical arguments. We will illustrate this in the case of the cube.

Let us remark that all results within this section can be improved to balanced triangulations (which, we recall, are acute triangulations having only triangles featuring angles greater than $\pi/6$) without requiring more triangles [48]. This is done by shifting vertices of the acute triangulation until the desired lower bound (i.e. $\pi/6$) is attained or by completely new constructions. We now list the results on acute triangulations of Platonic surfaces in their natural order, giving the simple cubic case special treatment—we do not mention the tetrahedral and octahedral case, as their edges directly provide a minimal triangulation.

Theorem 3.1.1 (*Hangan et al. [38]*). *The boundary of a cube admits an acute triangulation with* 24 *triangles, and this is best possible.*

An acute triangulation can be seen in Fig. 7. If we choose there $\alpha = \pi/6$, we obtain a balanced triangulation.

We now prove, following [38], that every acute triangulation of the (boundary of the) cube S requires at least 24 triangles. Since each vertex of the cube has curvature $\pi/2$, every triangle on S containing it in its interior has as sum of angles at least $3\pi/2$ and therefore cannot be acute. Since our triangulations must be geodesic, as defined in the introduction, and geodesics

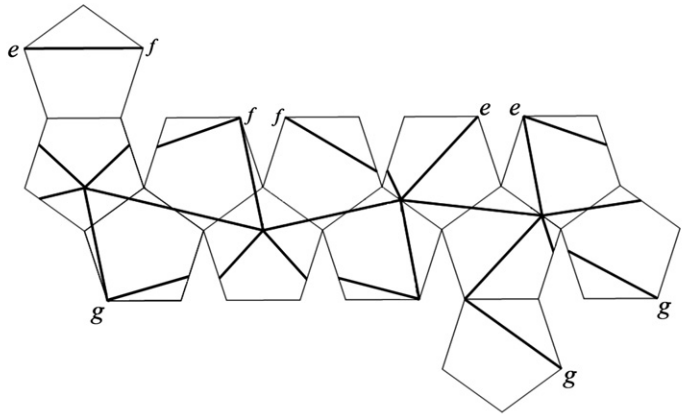


Fig. 8. The acute triangulation of the unfolded dodecahedral surface with 12 triangles constructed in [48], where the thin segments denote the edges of the dodecahedron, while the bold segments are the edges of the triangulation.

do not pass through vertices of S, every vertex of the cube must be a vertex of any acute triangulation. Suppose an acute triangulation $\mathcal{T}$ has at least 15 vertices. Each vertex in $\mathcal{T}$ that is also a vertex of S has degree at least 4, as the total angle around it is $3\pi/2$. Each of the other $p \geq 7$ vertices has degree at least 5, the total angle there being 2π. Thus, we have at least $32 + 35$ edges, counted twice. The number of edges, counted twice, equals $3t$, where t is the number of triangles in $\mathcal{T}$. Hence t is at least 24 (this number must be even).

Suppose now an acute triangulation $\mathcal{T}'$ has at most 14 vertices. Consider the four vertices $\{a, b, c, d\}$ of a face F of S. Clearly, some edge must start in a and go through the interior of F. If no vertex of $\mathcal{T}'$ is interior to F, then the edge ends outside the interior of F, and either from b or from d no edge can start and go through the interior of F. Hence, each face of S contains a vertex of $\mathcal{T}'$ in its interior, in fact precisely one vertex, as $\mathcal{T}'$ has at most six vertices different from those of the cube. This vertex must then be joined by an edge with each of the four vertices of the face. Indeed, suppose the vertex v interior to some face $abcd$ is not joined with a. Let ae be the edge of the triangulation starting at a and going through the interior of the face $abcd$. Then v must be separated by ae from b or d in $abcd$. Now a non-acute angle appears at b or d, and we obtain a contradiction. Let the faces F_1 and F_2 have the common edge ab. Let v_1 and v_2 be the vertices of $\mathcal{T}'$ interior to F_1 and F_2, respectively. Obviously, if ab is an edge of $\mathcal{T}'$, then v_1v_2 is not, and if v_1v_2 is an edge, then ab is not. However, one of the two edges must be present to prevent a quadrilateral in the triangulation. Hence, we may again count the edges, once each, by counting 4 for each vertex of the triangulation interior to some face, plus an edge for every edge of the cube. A priori there might be other edges as well. So, the number of edges is at least $24 + 12$. This means that $3t \geq 72$, whence $t \geq 24$. This completes the proof. It is easy to construct a non-obtuse triangulation of the boundary of the cube using only 4 triangles, and that this is smallest possible is easily seen, too.

We mention another interesting result concerning acute triangulations on the surface of the cube. A triangulation of a polyhedral surface P (i.e. the boundary of a polyhedron) is *proper* [59] (or *planar* [43]) if all edges of P appear in the triangulation, possibly in subdivided forms. Itoh and Maehara [43] proved that the surface of the cube admits a proper acute triangulation of size 56, that this is best possible, and that their construction is combinatorially unique. Going further in this direction, Maehara [59] proved (by using the main result from [65]) that every polyhedral surface admits a proper acute triangulation.

Two remarks are in order: (i) [59] contains a solution to an open question raised by Saraf [65]; she used Maehara's method from [57] to transform her proper non-obtuse triangulation into an acute one, but during this process the triangulation lost its attribute of being proper. (ii) Maehara's approach to construct proper triangulations is closer to applications in numerical analysis, as triangulations with triangles spanning multiple facets are rarely used. This is due to the fact that the linear approximation used in the application ought to accurately reflect the geometry of the underlying surface.

Theorem 3.1.2 (*Jia et al. [48]*). *The regular dodecahedral surface admits an acute triangulation with* 12 *triangles, and this is best possible.*

Fig. 8 illustrates the result from [48]. In fact, the theorem appearing in [48] is a stronger result than the one cited above, as the triangulation of size 12 obtained is not only acute but also balanced. For the non-obtuse case we refer to [47], where it was shown that there exists a non-obtuse triangulation of the regular dodecahedral surface with 10 triangles and that this is best possible.

Theorem 3.1.3 (*Itoh and Zamfirescu [46]*). *The regular icosahedral surface admits an acute, respectively non-obtuse, triangulation with* 12, *respectively* 8, *triangles, and these bounds are best possible.*

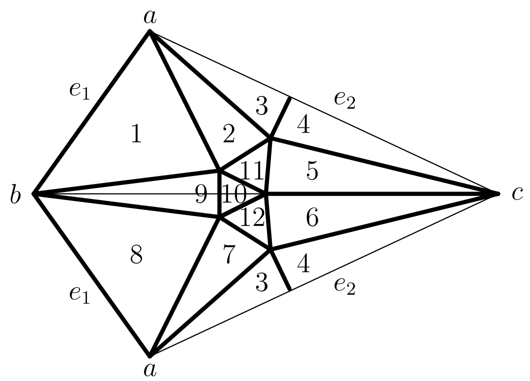


Fig. 9. Triangulating a double triangle T with 12 acute triangles, labeled 1 through 12. T is formed by gluing two triangles T_1 and T_2 along their boundaries. (The edges e_1 and e_2 are identified with their respective counterpart.) T_1 and T_2 are the sides of T. The triangles 3, 4, 9, and 10 have interior points on both sides of T, whereas the remaining triangles of the triangulation lie on precisely one of the two sides of T.

There also exists a balanced triangulation of the regular icosahedral surface of size 12 [48]. For non-regular polyhedra nothing is known (except for existence by the Burago–Zalgaller Theorem [14]). Even for the class of all tetrahedral surfaces no concrete upper bound was established so far; see Problem 5 in Section 4.

3.2. Double planar convex bodies

Let us start off with a simple observation. Two congruent planar convex bodies K and K' can be identified along their (entire) boundaries in accordance with the congruence, creating a surface $S = 2K$, the *double of* K. This is obviously homeomorphic to the sphere. K and K' are the *sides* of $2K$. Such a surface $2K$ is also called a *degenerate convex surface* because it is the limit of a sequence of convex surfaces. Even this case is not settled in regard to acute (or even non-obtuse, for that matter) triangulations, i.e. it is unknown whether acute triangulations of such surfaces exist. In fact, results are spread so thinly that we can give an exhaustive list, which follows shortly.

In order to give triangulations of double polygons, we note that in general one cannot simply triangulate – acutely for instance – a polygon and then apply this same triangulation to the copy, as a situation might occur where two triangles have two edges in common, which contradicts the definition of a triangulation! Even if it is possible to simply copy the triangulation, this is often not desirable, as better configurations (i.e. triangulations of smaller size) might exist.

We now give a short example from [84] that aims to clarify what an acute triangulation of a double triangle is. Consider the congruent copies T_1 and T_2 of an equilateral triangle as faces of our double triangle T. T is the (degenerate) convex surface we want to triangulate. Trisect the angles of T_1. The pairs of trisectors closer to each side of T_1 meet at the vertices of an equilateral triangle abc. The bisectors of T_2 meet at d. Then the trisectors of T_1, the bisectors of T_2, the sides of abc and the segments ad, bd, cd (which cross the boundary of T_1) determine an acute triangulation of T with 10 triangles. Many double triangles not too different from T can be acutely triangulated in the same combinatorial manner. However, the problem we want to solve is this: find the minimal integer N such that every double triangle can be triangulated with at most N acute triangles. We will see that N is not 10. The following is to our knowledge the first result on this topic (see Fig. 9).

Theorem 3.2.1 (*Zamfirescu [84]*)**.** *Every double triangle can be triangulated with at most* 12 *acute triangles. There exist triangles for which no smaller acute triangulation is possible.*

Theorem 3.2.2 (*Yuan and Zamfirescu [80]*)**.** *Every double convex quadrilateral admits an acute triangulation of size at most* 20.

Theorem 3.2.3 (*Yuan [78]*)**.** *Every double convex pentagon admits an acute triangulation of size at most* 76.

For double convex n-gons with $6 \le n \le 56$, the best bounds are given by $2a_n$, where a_n is defined in Theorem 2.3.5. For all larger n, the best known upper bound, $2 \cdot 22 \cdot (106n - 216)$, can be found in [77].

We now consider the non-polygonal case. Let $K \subset \mathbb{R}^2$ be a convex body with the x-axis and y-axis as orthogonal axes of symmetry, and C^2-boundary $\operatorname{bd} K$. Furthermore, let $(a, 0)$ and $(0, b)$ be points on $\operatorname{bd} K$ with $0 < b \le a$. Now let A be the arc $\{(x, y) \in \operatorname{bd} K : x, y \ge 0\}$; we notice that A determines K. We can now formulate the following result.

Theorem 3.2.4 (*Yuan and Zamfirescu [82]*)**.** *If the curvature of A is monotone (i.e. non-increasing or non-decreasing) or bounded above by* $2/b$*, then* $2K$ *admits an acute triangulation with at most* 72 *triangles.*

Thus, any double ellipse admits a triangulation with 72 acute triangles.

We now present an interesting class of surfaces which may be investigated regarding acute (or non-obtuse) triangulations. This requires a theorem by Alexandrov, which has an interesting history, as described by O'Rourke [62].

Alexandrov published his well-known Gluing Theorem in [3] and included it in his book from 1950 on convex polyhedra [1], unbeknownst to most Western mathematicians, as both publications appeared in Russian and in Russian only. The book was soon, in 1958, translated into German [2], but only in 2005 did an English translation of this great work appear [4].

A vertex of an abstract polyhedral surface has *non-negative curvature* if around it the angle sum is not greater than 2π. The Gluing Theorem has the following polyhedral version.

Theorem 3.2.5 (*Alexandrov [3]*)**.** *Every polyhedral surface with all vertices of non-negative curvature folds to a convex polytope.*

Next, we state a corollary of the Gluing Theorem.

Corollary 3.2.6. *Two planar convex bodies of equal perimeter can be identified along their boundaries, according to any isometry between the boundaries, creating a (possibly degenerate) convex surface.*

When the two bodies are congruent, and the boundary isometry respects the global isometry, we get the case of double planar convex bodies. Not only can we consider isometries between the boundaries not in accordance with the global isometry, we may even pick two different bodies (imagine gluing a square and a circle of the same perimeter). We point out that in the recent book [21] by Demaine and O'Rourke, the objects from 3.2.6 are called *D-forms*. They also study a simpler case, so-called *pita-forms*: these are objects obtained by taking a planar convex body and gluing one half of its perimeter to the other half. See [22] for a recent discussion on pita-forms. We may now ask for acute or non-obtuse triangulations on *D*-forms or pita-forms; here, nothing is known.

3.3. Riemannian manifolds

On Riemannian manifolds there are the following important existence results (which concern themselves with the stronger notion of almost equilateral triangulations), see [67,19], where the latter yields the following.

Theorem 3.3.1 (*Colin de Verdière and Marin [19]*)**.** *Let* $\varepsilon > 0$*.*

(i) *If X is the sphere S^2 (a torus) we can triangulate (X, g) for any metric g, such that all angles are between $3\pi/10 - \varepsilon$ and $2\pi/5 + \varepsilon$ ($\pi/3 - \varepsilon$ and $\pi/3 + \varepsilon$), and these bounds are best possible.*

(ii) *If (X, g) is an orientable compact surface of genus at least 2, we can triangulate (X, g), such that all angles are between $2\pi/7 - \varepsilon$ and $5\pi/14 + \varepsilon$, and these bounds are best possible.*

As far as we know there are no papers on a constructive approach, and no existent technique indicates how one might estimate the concrete number of triangles needed to acutely (or non-obtusely) triangulate a Riemannian surface in general.

We now turn from this very general frame to several classical geometric objects.

3.4. Sphere, flat Möbius strips, flat tori, and flat Klein bottles

Acute triangulations (even balanced ones, as is easily seen) of the sphere are readily available, and one cannot do better than 20 concerning the size of the triangulation: Brinkmann and McKay [13] enumerate the triangulations of the sphere such that all vertices have degree at least 5. There is (combinatorially) exactly one such triangulation into 20 triangles (the vertices of which can be placed such that it is acute), and none into 22. Nonetheless, investigations were made regarding what other acute triangulation sizes may occur, revealing the following.

Theorem 3.4.1 (*Itoh [41]*)**.** *The sphere admits acute triangulations of all even orders greater than or equal to* 20*, excluding* 22*.*

We note that in [41] the cases 28 and 34 remained open, but were resolved later by the same author, yielding the result above. Another result concerning acute triangulations on spheres is the following.

Theorem 3.4.2 (*Itoh and Zamfirescu [45]*)**.** *Every spherical geodesic triangle with angles smaller than π can be triangulated with at most* 10 *acute triangles, and this is the best possible.*

Flat Möbius strips have also been investigated. These are obtained from a flat rectangle R with sides of length 1 and $\alpha > 0$, by identifying pairs of points symmetric about the center of R and lying on the sides of length 1. Denote this Möbius strip by M_R. For an example, see Fig. 10. There, the sides of length 1 are a_1b_1 and b_2a_2 (these are being identified as explained above to form the Möbius strip) and the sides of length α are a_1b_2 and b_1a_2.

Theorem 3.4.3 (*Yuan and Zamfirescu [81]*)**.** *Every flat Möbius strip admits an acute triangulation of size at most* 9*, and this upper bound is best possible.*

The size of an optimal triangulation of M_R depends on α. For $\alpha \geq \sqrt{5/3} = 1.29099\ldots$ this size is 5 (see below for a proof); for $1 < \alpha < \sqrt{5/3}$ the (not proven as optimal) construction presented in [81] requires 8 triangles, and only for all $\alpha \leq 1$, 9 triangles are indeed needed.

To give an insight into the geometric aspect of the proofs of many results given in Sections 2.3 and 3, we now show that if $\alpha \geq \sqrt{5/3}$, then the Möbius strip M_R can be triangulated into 5 acute triangles, and no smaller acute triangulation is possible. We follow here [81]. The *relative interior* of a segment $xy \subset \mathbb{R}^2$, abbreviated with (xy), is the set $\{\lambda x + (1-\lambda)y : 0 < \lambda < 1\}$.

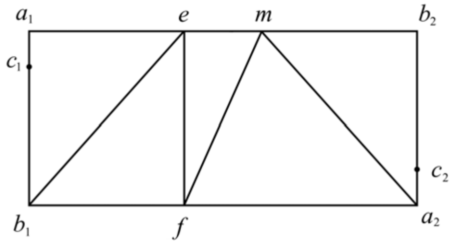


Fig. 10.

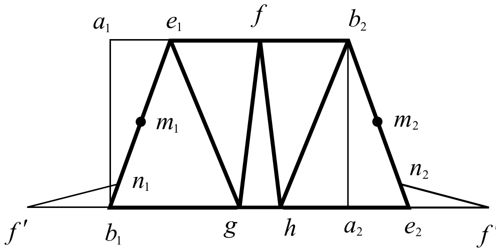


Fig. 11. Thick line-segments denote edges of the triangulation.

Case 1. $\alpha \geq \sqrt{3}$. Let $e, m \in a_1b_2$ in R, such that $\|b_1 - e\| = \|b_2 - e\|$ and $\|a_1 - m\| = \|a_2 - m\|$ (see Fig. 10).

Clearly, $e \in (a_1m)$ in R. Let f be the orthogonal projection of e on b_1a_2 in R; then $\angle mfa_2 < \pi/2$. By construction, it is easy to check that the line-segments a_1b_1, a_1e, b_1e, b_1f, em, ef, mb_2, ma_2 and fa_2 are all segments in M_R. Furthermore, since

$$\|m - f\| = \frac{\sqrt{\alpha^2+1}}{\alpha}, \qquad \|b_1 - f\| = \|m - b_2\| = \frac{\alpha^2-1}{2\alpha} \quad \text{and} \quad \alpha \geq \sqrt{3},$$

we have

$$\|m - f\| \leq \|m - b_2\| + \|b_1 - f\| \leq \|m - c_2\| + \|c_1 - f\|$$

for any point $c_i \in a_ib_i$ (c_1 and c_2 are identical), which implies that the line-segment mf is also a segment in M_R. Since ma_2 has length $\frac{\alpha^2+1}{2\alpha}$, the segment mf is not longer than ma_2, whence $\angle ma_2f \leq \pi/3$.

Thus M_R can be triangulated into 5 non-obtuse geodesic triangles, as shown in Fig. 10. Now we replace in this triangulation the vertices f and a_2 by two vertices on the side edge fa_2, close to f, respectively a_2. In this manner we obtain an acute triangulation of M_R of size 5.

Case 2. $\sqrt{5/3} \leq \alpha < \sqrt{3}$. Let $e_1 \in a_1b_2$ in R, such that $\|b_1 - e_1\| = \|b_2 - e_1\|$. The Möbius strip M_R is the isosceles trapezoid $e_1b_1e_2b_2 \subset \mathbb{R}^2$, see Fig. 11, where e_1 coincides with e_2 in M_R.

Let f be the midpoint of the side edge e_1b_2, and let $g, h \in b_1e_2$ in $\mathbb{R}^2$ be such that $\|e_1 - g\| = \|e_2 - g\| = \|b_1 - h\| = \|b_2 - h\|$. Since $\|b_1 - e_2\| - (\|b_1 - g\| + \|h - e_2\|) = \frac{3-\alpha^2}{2\alpha} > 0$, $g \in (b_1h)$. From our construction, it is easy to check that the line-segments b_1e_1, e_1f, e_1g, b_1g, gh, fb_2, b_2h, $he_2 \subset \mathbb{R}^2$ are indeed segments in M_R. Since

$$\|f - g\| = \frac{\sqrt{\alpha^4 + 10\alpha^2 + 9}}{4\alpha}, \qquad \|f - b_2\| + \|b_1 - g\| = \frac{5\alpha^2 - 3}{4\alpha} \quad \text{and} \quad \alpha \geq \sqrt{5/3},$$

we have

$$\|f - g\| \leq \|f - b_2\| + \|b_1 - g\| < \|f - e_1\| + \|e_2 - g\|.$$

Let m_i be the midpoint of $e_ib_i \subset \mathbb{R}^2$ ($i \in \{1, 2\}$). m_1 and m_2 coincide in M_R. If $n_1 \in b_1m_1$, then

$$\|f - b_2\| + \|b_1 - g\| = \|f' - b_1\| + \|b_1 - g\| \leq \|f' - n_1\| + \|n_1 - g\|$$

where $f' = f$ in M_R (see Fig. 11). If $n_1 \in e_1m_1$, then

$$\|f - e_1\| + \|e_2 - g\| = \|f'' - e_2\| + \|e_2 - g\| \leq \|f'' - n_2\| + \|n_2 - g\|,$$

where $f'' = f$ and $n_2 = n_1$ in M_R (see Fig. 11). Hence $fg \subset R$ is a segment in M_R. Analogously, $fh \subset R$ is also a segment in M_R. Furthermore, since $\sqrt{5/3} \leq \alpha < \sqrt{3}$, we have

$$\tan \frac{1}{2}\angle gfh = \frac{3-\alpha^2}{4\alpha} < 1 \quad \text{and} \quad \tan \frac{1}{2}\angle b_1e_1g = \frac{\alpha^2-1}{2\alpha} < 1.$$

Thus, all triangles in Fig. 11 are acute. We obtained an acute triangulation of M_R of size 5.

Now let $\mathcal{T}$ be an arbitrary acute triangulation of M_R with t triangles. If $\mathcal{T}$ has at least one interior vertex, then clearly $t \geq 5$. If $\mathcal{T}$ has no interior vertex, then we assume that it has s side vertices. Notice that every side vertex has degree at least 4, so $s \geq 5$. Now denote by e the number of edges of $\mathcal{T}$, and let $V(\mathcal{T})$ be the vertex set of $\mathcal{T}$. Since $3t + s = 2e = \sum_{x \in V(\mathcal{T})} \deg(x) \geq 4s$, we have $t \geq s \geq 5$, which completes the proof.

Let us turn our attention to flat tori.

Theorem 3.4.4 (*Itoh and Yuan [44]*). *Every flat torus can be triangulated into* 16 *acute triangles, and there are flat tori requiring* 14 *acute triangles.*

The natural question whether 16 or 14 is the optimal bound remains unanswered.

Theorem 3.4.5 (*Itoh [42]*). *Every flat Klein bottle admits an acute triangulation of size* 16.

We remark that, again, it is unknown whether this bound is optimal; see Problem 4.

4. Open problems

There are many interesting problems of varying difficulty regarding acute triangulations. We select eight such problems; we note that they stem from theoretical approaches and concern the two-dimensional case. For a better overview of recent trends and open problems on acute triangulations in applied mathematics, see e.g. [72]. For a series of intriguing conjectures in higher dimensions, [12] is a good source.

1. Does every convex quadrilateral admit an acute triangulation of size 8? It is known [56] that convex quadrilaterals may be acutely triangulated using (at most) 9 triangles and that rectangles need 8.
   This problem was recently solved by Cavicchioli. Please see the Note at the end of this survey.
2. Any flat torus can be acutely triangulated with 16 triangles, and in [44] a flat torus is constructed requiring 14 acute triangles, leaving open the question of the optimal upper bound.
3. A 72°-*triangulation* consists exclusively of triangles the interior angles of which measure at most 72°. Is 14 the minimum number of triangles in a 72°-triangulation of the square [25]?
4. Is 16 the optimal upper bound for the size of an acute triangulation of a flat Klein bottle [42]?
5. Find an upper bound N such that any tetrahedral surface admits an acute triangulation requiring at most N triangles [83].
6. Does there exist a number N such that every closed convex surface in $\mathbb{R}^3$ admits an acute triangulation with at most N triangles [38]? In view of Corollary 3.2.6, we propose the investigation of Problem 7, which reveals an interesting special case of Problem 6.
7. Consider a (possibly degenerate) closed convex surface obtained by gluing two planar convex isoperimetric bodies, i.e. identifying them along their boundaries. Does there exist a number N such that every surface obtained in this manner admits an acute triangulation with at most N triangles?
8. The following problem appeared in [83]. Let (X, ρ) be a metric space. For any three points $a, b, c \in X$, we say that the angle abc is *acute* if $\rho(a, c)^2 < \rho(a, b)^2 + \rho(b, c)^2$. A triple $\{a, b, c\}$ is a *triangle* if $\rho(a, c) < \rho(a, b) + \rho(b, c)$ and the other two analogous inequalities hold. A triangle is *acute* if all its three angles are acute. A *combinatorial triangulation* in X is a finite set of triangles combinatorially equivalent to some usual triangulation of a closed surface. Such a combinatorial triangulation is *acute* if all angles are acute. Let us now formulate two intertwined problems.
   (i) Given a metric space (X, ρ), which acute combinatorial triangulations exist in X?
   (ii) What is the smallest possible number of acute triangles?

Note. Cavicchioli succeeded in solving Problem 1, see [16]: she showed that every convex quadrilateral admits an acute triangulation of size 8, and that this is best possible. In another recent development, Feng and Yuan [30] showed that the boundary surface of the cuboctahedron can be triangulated into 12 acute triangles, and that this is best possible. Pambuccian [63] and Hociotă and Pambuccian [39] study acute triangulations from an axiomatic point of view.

Acknowledgments

I thank Iulia Mihai, Joseph O'Rourke, Douglas B. West, Tudor Zamfirescu, and the anonymous referees for their many helpful suggestions and constructive criticism.

References

[1] A.D. Alexandrov, Vypuklye Mnogogranniki, Gosydarstvennoe Izdatelstvo Tehno-Teoreticheskoi Literaturu, 1950 (in Russian).
[2] A.D. Alexandrov, Konvexe Polyeder, Akademie Verlag, Berlin, 1958, Math. Lehrbücher und Monographien, Translation of the 1950 Russian edition (in German).
[3] A.D. Alexandrov, Existence of a convex polyhedron and a convex surface with a given metric, in: Yu.G. Reshetnyak, S.S. Kutateladze (Eds.), A.D. Alexandrov: Selected Works: Part I, Gordon and Breach, Australia, 1996, pp. 169–173; Translation from Dokl. Akad. Nauk SSSR 30 (2) (1941) 103–106.
[4] A.D. Alexandrov, Convex Polyhedra, in: Monographs in Mathematics, Springer-Verlag, Berlin, 2005, Translation of the 1950 Russian edition by N.S. Dairbekov, S.S. Kutateladze, A.B. Sossinsky.
[5] I. Babuška, A.K. Aziz, On the angle condition in the finite element method, SIAM J. Numer. Anal. 13 (1976) 214–226.

[6] B.S. Baker, E. Grosse, C.S. Rafferty, Nonobtuse triangulations of polygons, Discrete Comput. Geom. 3 (1988) 147–168.
[7] M. Bern, H. Edelsbrunner, D. Eppstein, S. Mitchell, T.S. Tan, Edge insertion for optimal triangulations, Discrete Comput. Geom. 10 (1993) 47–65.
[8] M. Bern, D. Eppstein, Polynomial-size nonobtuse triangulation of polygons, Int. J. Comput. Appl. Math. 2 (3) (1992) 241–255.
[9] M. Bern, D. Eppstein, Mesh generation and optimal triangulation, in: Computing in Euclidean Geometry, World Scientific, 1995, pp. 47–123.
[10] M. Bern, D. Eppstein, J. Gilbert, Provably good mesh generation, J. Comput. System Sci. 48 (3) (1994) 384–409.
[11] M. Bern, S. Mitchell, J. Ruppert, Linear-size nonobtuse triangulations of polygons, Discrete Comput. Geom. 14 (1995) 411–428.
[12] J. Brandts, S. Korotov, M. Křížek, J. Šolc, On nonobtuse simplicial partitions, SIAM Rev. 51 (2) (2009) 317–335.
[13] G. Brinkmann, B.D. McKay, Fast generation of planar graphs, MATCH Commun. Math. Comput. Chem. 58 (2) (2007) 323–357.
[14] Y.D. Burago, V.A. Zalgaller, Polyhedral embedding of a net, Vestnik St. Petersburg Univ. Math. 15 (1960) 66–80 (in Russian).
[15] Ch. Cassidy, G. Lord, A square acutely triangulated, J. Recr. Math. 13 (1980–1981) 263–268.
[16] M. Cavicchioli, Acute triangulations of convex quadrilaterals, Discrete Appl. Math. 160 (7–8) (2012) 1253–1256.
[17] P.G. Ciarlet, P.A. Raviart, Maximum principle and uniform convergence for the finite element method, Comput. Methods Appl. Mech. Engrg. 2 (1973) 17–31.
[18] Y. Colin de Verdière, Comment rendre géodésique une triangulation d'une surface? Enseign. Math. (2) 37 (1991) 201–212 (in French).
[19] Y. Colin de Verdière, A. Marin, Triangulations presque équilatérales des surfaces, J. Differential Geom. 32 (1) (1990) 199–207 (in French).
[20] B. Delaunay, Sur la sphère vide, Bull. Acad. Sci. URSS Ser. VII (6) (1934) 793–800 (in French).
[21] E.D. Demaine, J. O'Rourke, Geometric Folding Algorithms: Linkages, Origami, Polyhedra, Cambridge Univ. Press, 2007.
[22] E.D. Demaine, G.N. Price, Generalized *D*-forms have no spurious creases, Discrete Comput. Geom. 43 (1) (2010) 179–186.
[23] H. Edelsbrunner, T.S. Tan, R. Waupotitsch, An $O(n^2 \log n)$ time algorithm for the minmax angle triangulation, SIAM J. Sci. Stat. Comput. 13 (4) (1992) 994–1008.
[24] C.M. Elliott, A.M. Stuart, The global dynamics of discrete semilinear parabolic equations, SIAM J. Numer. Anal. 30 (6) (1993) 1622–1663.
[25] D. Eppstein, Acute square triangulation. From the geometry junkyard, computational and recreational geometry. Accessible online at: http://www.ics.uci.edu/~eppstein/junkyard/acute-square/.
[26] D. Eppstein, J.M. Sullivan, A. Üngör, Tiling space and slabs with acute tetrahedra, Comput. Geom. 27 (3) (2004) 237–255.
[27] J. Erickson, D. Guoy, J.M. Sullivan, A. Üngör, Building spacetime meshes over arbitrary spatial domains, Eng. Comput. 20 (2005) 342–353.
[28] H. Erten, A. Üngör, Computing acute and non-obtuse triangulations, in: Proc. CCCG 2007, pp. 205–208.
[29] I. Fáry, On straight-line representation of planar graphs, Acta Sci. Math. (Szeged) 11 (1948) 229–233.
[30] X. Feng, L. Yuan, Acute triangulations of the cuboctahedral surface, Lecture Notes in Comput. Sci. (LNCS) 7033 (2011) 73–83.
[31] R.A. Finkel, J.L. Bentley, Quad trees: a data structure for retrieval on composite keys, Acta Inform. 4 (1974) 1–9.
[32] I. Fried, Condition of finite element matrices generated from nonuniform meshes, AIAA J. 10 (1972) 219–221.
[33] M. Gardner, Mathematical games. A fifth collection of "brain-teasers", Sci. Am. 202 (2) (1960) 150–154.
[34] M. Gardner, Mathematical games. The games and puzzles of Lewis Carroll, and the answers to February's problems, Sci. Am. 202 (3) (1960) 172–182.
[35] M. Gardner, New Mathematical Diversions from Scientific American, Simon and Schuster, 1966 (Chapter 3).
[36] J.L. Gerver, The dissection of a polygon into nearly equilateral triangles, Geom. Dedicata 16 (1984) 93–106.
[37] M. Goldberg, Problem E1406: dissecting an obtuse triangle into acute triangles, Amer. Math. Monthly 67 (1960) 923.
[38] Th. Hangan, J. Itoh, T. Zamfirescu, Acute triangulations, Bull. Math. Soc. Sci. Math. Roumanie 43 (91(3–4)) (2000) 279–285.
[39] I. Hociotă, V. Pambuccian, Acute triangulation of a triangle in a general setting revisited, J. Geom. 102 (1–2) (2011) 81–84.
[40] V.E. Hoggatt Jr., R. Denman, Acute isosceles dissection of an obtuse triangle, Amer. Math. Monthly 68 (1961) 912–913.
[41] J. Itoh, Acute triangulations of sphere and icosahedron, Josai Math. Monogr. 3 (2001) 53–62.
[42] J. Itoh, Acute triangulations on the surfaces, in: Proc. 1st Int. Conf. IMBIC on MSAST, 2007, pp. 9–10.
[43] J. Itoh, H. Maehara, On triangulations of the surface of a cube into planar acute triangles, Ryukyu Math. J. 21 (2008) 15–22.
[44] J. Itoh, L. Yuan, Acute triangulations of flat tori, European J. Combin. 30 (1) (2009) 1–4.
[45] J. Itoh, T. Zamfirescu, Acute triangulations of triangles on the sphere, Rend. Circ. Mat. Palermo (2) 70 (2002) 59–64.
[46] J. Itoh, T. Zamfirescu, Acute triangulations of the regular icosahedral surface, Discrete Comput. Geom. 31 (2004) 197–206.
[47] J. Itoh, T. Zamfirescu, Acute triangulations of the regular dodecahedral surface, European J. Combin. 28 (4) (2007) 1072–1086.
[48] L. Jia, L. Yuan, C.T. Zamfirescu, T.I. Zamfirescu, Balanced triangulations (in press).
[49] H. Kaiser, Zerlegungen euklidischer und nichteuklidischer Polygone in spitzwinklige Dreiecke, Rad Jugoslav. Akad. Znan. Umjet. 444 (1989) 75–86 (in German).
[50] A. Kaneko, H. Maehara, M. Watanabe, On the number of acute triangles in a straight-line embedding of a maximal planar graph, J. Combin. Theory Ser. B 75 (1) (1999) 110–115.
[51] K. Kawarabayashi, A. Nakamoto, Y. Oda, M. Watanabe, Acute triangles in 4-connected maximal plane graphs, Discrete Math. 292 (1–3) (2005) 95–106.
[52] E. Kopczyński, I. Pak, P. Przytycki, Acute triangulations of polyhedra and $\mathbb{R}^n$, Combinatorica 32 (1) (2012) 85–110.
[53] K. Koyama, A. Nakamoto, Acute triangles in triangulations on the plane with minimum degree at least 4, Discrete Math. 303 (1–3) (2005) 104–116.
[54] H. Lindgren, A quadrilateral dissection, Aust. Math. Teach. 16 (1960) 64–65.
[55] R.H. MacNeal, An asymmetrical finite difference network, Quart. Appl. Math. 11 (1953) 295–310.
[56] H. Maehara, On acute triangulations of quadrilaterals, Lecture Notes in Comput. Sci. (LNCS) 2098 (2001) 237–243.
[57] H. Maehara, Acute triangulations of polygons, European J. Combin. 23 (2002) 45–55.
[58] H. Maehara, Plane graphs with straight edges whose bounded faces are acute triangles, J. Combin. Theory Ser. B 88 (2) (2003) 237–245.
[59] H. Maehara, On a proper acute triangulation of a polyhedral surface, Discrete Math. 311 (17) (2011) 1903–1909.
[60] W. Manheimer, Solution to problem E1406: dissecting an obtuse triangle into acute triangles, Amer. Math. Monthly 67 (1960) 923.
[61] E.A. Melissaratos, D.L. Souvaine, Coping with inconsistencies: a new approach to produce quality triangulations of polygonal domains with holes, in: Proc. 8th Symp. Comput. Geom., 1992, pp. 202–211.
[62] J. O'Rourke, Folding polygons to convex polyhedra, in: Timothy V. Craine, Rheta Rubenstein (Eds.), Understanding Geometry for a Changing World: National Council of Teachers of Mathematics, 71st Yearbook, National Council of Teachers of Mathematics, Reston, VA, 2009, pp. 77–87.
[63] V. Pambuccian, Acute triangulation of a triangle in a general setting, Canad. Math. Bull. 53 (3) (2010) 534–541.
[64] V.R. Santos, On the strong maximum principle for some piecewise linear finite element approximate problems of non-positive type, J. Fac. Sci. univ. Tokyo Sect. 1A 29 (1982) 473–491.
[65] S. Saraf, Acute and nonobtuse triangulations of polyhedral surfaces, European J. Combin. 30 (2009) 833–840.
[66] J.A. Sethian, Fast marching methods, SIAM Rev. 41 (2) (1999) 199–235.
[67] G.B. Shabat, V.A. Voevodsky, Equilateral triangulations of Riemann surfaces, and curves over algebraic number fields, Sov. Math. Dokl. 39 (1) (1989) 38–41; Translation from Dokl. Akad. Nauk SSSR 304 (2) (1989) 265–268.
[68] J.R. Shewchuk, What is a good linear element? interpolation, conditioning, and quality measures, in: Proc. 11th International Meshing Roundtable, Ithaca, New York, Sandia National Laboratories, 2002, pp. 115–126.
[69] G. Strang, G.J. Fix, An Analysis of the Finite Element Method, Prentice-Hall, Englewood Cliffs, NJ, 1973.
[70] W. Tutte, How to draw a graph, Proc. Lond. Math. Soc. (3) 13 (1963) 743–768.
[71] A. Üngör, A. Sheffer, Pitching tents in space–time: mesh generation for discontinuous Galerkin method, Internat. J. Found Comput. Sci. 13 (2) (2002) 201–221.
[72] E. VanderZee, A.N. Hirani, D. Guoy, E.A. Ramos, Well-centered triangulation, SIAM J. Sci. Comput. 31 (6) (2010) 4497–4523.
[73] E. VanderZee, A.N. Hirani, V. Zharnitsky, D. Guoy, A dihedral acute triangulation of the cube, Comput. Geom. 43 (5) (2010) 445–452.
[74] S.A. Vavasis, Stable finite elements for problems with wild coefficients, SIAM J. Numer. Anal. 33 (3) (1996) 890–916.
[75] K. Wagner, Bemerkungen zum Vierfarbenproblem, Jber. DMV 46 (1936) 26–32 (in German).

[76] L. Yuan, Acute triangulations of polygons, Discrete Comput. Geom. 34 (2005) 697–706.
[77] L. Yuan, Acute triangulations, Doctoral Thesis Defended at TU Dortmund, Germany, 2006.
[78] L. Yuan, Acute triangulations of pentagons, Bull. Math. Soc. Sci. Math. Roumanie 53 (4) (2010) 393–410.
[79] L. Yuan, Acute triangulations of trapezoids, Discrete Appl. Math. 158 (10) (2010) 1121–1125.
[80] L. Yuan, C.T. Zamfirescu, Acute triangulations of doubly covered convex quadrilaterals, Boll. Unione Mat. Ital. (8) 10-B (2007) 933–938.
[81] L. Yuan, T. Zamfirescu, Acute triangulations of flat Möbius strips, Discrete Comput. Geom. 37 (4) (2007) 671–676.
[82] L. Yuan, T. Zamfirescu, Acute triangulations of double planar convex bodies, Publ. Math. Debrecen 81 (1–2) (2012) 121–126.
[83] T. Zamfirescu, Acute triangulations: a short survey, in: Proc. 6th RSMS, 2002, pp. 9–17.
[84] C.T. Zamfirescu, Acute triangulations of the double triangle, Bull. Math. Soc. Sci. Math. Roumanie 47 (3–4) (2004) 189–193.
[85] C.T. Zamfirescu, Improved bounds for acute triangulations of convex polygons, Util. Math. (in press).