JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **150**, 41–53 (1990)

Overtaking Optimal Control Problem of Age-Dependent Populations with Infinite Horizon*

W. L. CHAN AND GUO BAO ZHU[†]

Department of Mathematics, The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

Submitted by E. Stanley Lee

Received January 10, 1989

The optimal birth control of an age-dependent population model with unbounded time interval is considered. The minimum principle which must be satisfied by the overtaking optimal control is established. Large time behaviour and the turnpike property of the overtaking optimal trajectory are studied. Existence results are also presented. © 1990 Academic Press, Inc

1. INTRODUCTION

The problem of controlling and managing age-dependent biological populations has been studied in an optimal control setting by the authors in [1, 2] with a finite or infinite time horizon and various terminal conditions (see also [3, 4]). The aim of this paper is to study conditions under which the optimal birth control over an infinite time horizon of the McKendrick model has a stabilizing effect. As opposed to [1], here, we do not a priori assume that the cost functional, an improper integral, converges. This leads us to consider a weaker type of optimality, known as the overtaking optimality. Such a concept has a long history in the economic and operation research literature. It is hoped that our study will lead to a proper understanding of the open-endedness of the future in age-dependent population management.

Recently in [5], the overtaking optimal control of an infinite dimensional linear control system with unbounded time interval has been considered. However, the results there cannot be applied directly to our situation since the McKendrick model involves a bilinear (nonlinear) boundary birth control of a distributed system discribed by a first-order

* This work was carried out with the aid of a grant from the International Development Research Centre, Ottawa, Canada.

[†] Permanent address: Beijing Institute of Information and Control, Beijing, China.

0022-247X/90 $3.00

differential equation. We are, in fact, extending some of the results of [5] to a nonlinear case.

The paper is organized as follows. In Section 2 the optimal birth control problem is formulated. In Section 3 the minimum principle which must be satisfied by the overtaking optimal control is established via an associated finite horizon optimal control problem. Section 4 deals with the large time behaviour of the overtaking optimal trajectory, i.e., the turnpike property. Generally speaking, this property says that an optimal trajectory on any finite horizon will stay most of the time in the vicinity of an extremal steady state and will ultimately converge to it if the time interval becomes unbounded. Finally in Section 5, some existence results for overtaking optimal control are presented.

2. Problem Statement

We consider the population evolution system described by the following first-order partial differential equation with boundary control

$$\begin{aligned} &\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)\,p(r,t), && 0<r<r_m, \quad t>0, \\ &p(r,0)=p_0(r), && 0\leqslant r\leqslant r_m, \\ &p(0,t)=\beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr, && t\geqslant 0 \end{aligned} \tag{1}$$

in which $p(r, t)$ is the population density, r denotes age, t represents time; r_m is the maximum age; $\beta(t)$, the control variable, is the specific fertility rate of females at time t; $k(r)$ and $h(r)$ denote, respectively, the female ratio and the fertility pattern; $[r_1, r_2]$ is the fertility interval with $\int_{r_1}^{r_2} h(r)\,dr = 1$. The initial population density $p_0(r)$ is a nonnegative function and the mortality rate $\mu(r)$ satisfies

$$\int_0^r \mu(\rho)\,d\rho < +\infty, \qquad r<r_m,$$

$$\int_0^{r_m} \mu(\rho)\,d\rho = +\infty.$$

Assume that the population parameters in Eq. (1) are nonnegative and measurable functions. Furthermore, let β, h, and k be bounded functions whose values outside their domain of definitions are zero.

By the method of characteristics, the solution of Eq. (1) can be written (formally) as

$$p(r,t)=\begin{cases} p_0(r-t)\,e^{-\int_{r-t}^{r}\mu(\rho)\,d\rho}, & r\geqslant t,\\ \beta(t-r)\displaystyle\int_{r_1}^{r_2} k(s)\,h(s)\,p(s,t-r)\,ds\,e^{-\int_0^r\mu(\rho)\,d\rho}, & r<t. \end{cases} \tag{2}$$

The classical solution of (1) is a solution of (2). Under certain smoothness conditions on the population parameters, the two are equivalent. For a detailed discussion, see [6].

For an arbitrary $p_0(r)\in L^2(0,r_m)$, Eq. (2) in $L^2(0,r_m)$ has a unique solution $p(r,t)\in \mathbf{C}(0,\infty;L^2(0,r_m))$.

Because of the above reasons, we call the solution of Eq. (2) as a weak solution of Eq. (1). Unless otherwise stated, in what follows when we speak of solution of Eq. (1) we shall mean the weak solution.

Consider now the optimal control problem. The performance of the system on any interval $[0,t]$ is evaluated by the cost functional

$$J(\beta,p,t)=\int_0^t\int_0^{r_m} L(p(r,t),\beta(t))\,dr\,dt, \tag{3}$$

where $L\colon L^2(0,r_m)\times[0,\infty)\to L^2(0,r_m)$ is a continuously differentiable function. We call $\mathscr{A}(p_0)$ the set of pairs (β,p) which satisfy

(1) $\beta(\cdot)\in U_{ad}=\{\beta(t)\,|\,0\leqslant\beta_0\leqslant\beta(t)\leqslant\beta_1,\quad t\in[0,\infty)$ a.e., $\beta(t)$ is measurable on $[0,\infty)\}$.

(2) $p(\cdot,\cdot)$ is given by (2).

Then $\beta(\cdot)$ is called an admissible control at p_0, and $p(\cdot,\cdot)$ is the associated trajectory.

In this paper, we consider our problem on an infinite horizon, and we do not a priori assume the convergence of (3) as $t\to\infty$. Hence we need to consider the following weaker notions of optimality.

DEFINITION 1. $(\beta^*,p^*)\in\mathscr{A}(p_0)$ is overtaking optimal at p_0 if for any other pair $(\beta,p)\in\mathscr{A}(p_0)$

$$\varliminf_{t\to\infty}\,[J(\beta,p,t)-J(\beta^*,p^*,t)]\geqslant 0. \tag{4}$$

In other words, for every $(p,\beta)\in\mathscr{A}(p_0)$, any fixed $T>0$, and every $\varepsilon>0$, there exists t with $t\geqslant T$ such that

$$J(\beta^*,p^*,t)\leqslant J(\beta,p,t)+\varepsilon. \tag{5}$$

For any fixed T and an overtaking optimal control pair (β^*, p^*), define the finite horizon optimal control problem:

Minimize $J(\beta, p, T)$

subject to

$$\begin{aligned}
&\frac{\partial p(r,t)}{\partial t}+\frac{\partial p(r,t)}{\partial r}=-\mu(r)\,p(r,t), && 0<r_m,\ t>0,\\
&p(r,0)=p_0(r), && 0\leqslant r\leqslant r_m,\\
&p(0,t)=\beta(t)\int_{r_1}^{r_2} k(r)\,h(r)\,p(r,t)\,dr, && t\geqslant 0,\\
&p(r,T)=p^*(r,T),\ \beta(\cdot)\in U_{ad}.
\end{aligned} \tag{6}$$

For notational convenience, we denote the infinite horizon problem the IHP problem, and the associate finite horizon problem the FHP problem. First, we have the following apparent result:

PROPOSITION 1. *If (β^*, p^*) is optimal for* IHP, *then it is optimal for* FHP.

Proof. If (β^*, p^*) is not FHP optimal for IHP, then for some $(\hat{\beta}, \hat{p})$ satisfying (6), $\hat{\beta}(\cdot)\in U_{ad}$, and some $\varepsilon>0$ we have

$$\int_0^T\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t))\,dr\,dt<\int_0^T\int_0^{r_m} L(p^*(r,t),\beta^*(t))\,dr\,dt-\varepsilon.$$

Let (β, p) be defined by

$$\begin{aligned}
(\beta(t), p(r,t)) &= (\beta^*(t), p^*(r,t)) && \text{for all} \quad t\in(T,\infty),\\
&= (\hat{\beta}(t), \hat{p}(r,t)) && \text{for all} \quad t\in[0,T].
\end{aligned}$$

We then have $(\beta, p)\in\mathscr{A}(p_0)$ and

$$\int_0^t\int_0^{r_m} L(\hat{p}(r,t),\hat{\beta}(t))\,dr\,dt<\int_0^t\int_0^{r_m} L(p^*(r,t),\beta^*(t))\,dr\,dt-\varepsilon$$

for all $t\geqslant T$. This last statement contradicts the optimality of (β^*, p^*). This concludes the proof of the proposition.

We proved minimum principle for FHP problem in [1].

THEOREM 1. *Let (β^*, p^*) be the solution of* FHP; *then there exist $\lambda_{0T}\geqslant 0$, $\alpha_T(r)\in L^2(0, r_m)$, not both zero, such that the following minimum principle holds,*

$$\beta^*(t)\,H_\beta(\beta^*, p^*)=\max_{\beta_0\leqslant\beta\leqslant\beta_1}\beta H_\beta(\beta^*, p^*), \qquad \forall t\in[0,T]\ a.e., \tag{7a}$$

where

$$H(\beta, p) = q_T(t)\,\beta(t) \int_{r_1}^{r_2} k(r)\,h(r)\,p(r, t)\,dr - \lambda_{0T} L(p, \beta),$$

$$H_\beta(\beta^*, p^*) = \frac{\partial H(\beta^*, p^*)}{\partial \beta},$$

$q_T(t)$ *is the solution of the adjoint equation*

$$\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r)\,q(r, t) - \beta^*(t)\,k(r)\,h(r)\,q(t) + \lambda_{0T} \frac{\partial L(p^*, \beta^*)}{\partial p}, \tag{7}$$

$$q(r, T) = \alpha_T(r),$$

$$q(0, t) = q_T(t).$$

As with Eq. (1), we call solutions (weak solutions) of Eq. (7) to be the solutions of

$$\begin{aligned} q_T(t) &= e^{-\int_0^{T-t} \mu(\rho)\,d\rho} \alpha_T(T-t) \\ &\quad + \int_t^T e^{-\int_0^{s-t} \mu(\rho)\,d\rho} \beta^*(s)\,k(s-t)\,h(s-t)\,q_T(s)\,ds \\ &\quad - \lambda_{0T} \int_t^T e^{-\int_0^{s-t} \mu(\rho)\,d\rho} \left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(s-t, s)} ds, \\ q(r, t) &= e^{-\int_r^{r+T-t} \mu(\rho)\,d\rho} \alpha_T(r+T-t) \\ &\quad + \int_t^T e^{-\int_r^{r+s-t} \mu(\rho)\,d\rho} \beta^*(s)\,k(r+s-t)\,h(r+s-t)\,q_T(s)\,ds \\ &\quad - \lambda_{0T} \int_t^T e^{-\int_r^{r+s-t} \mu(\rho)\,d\rho} \left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r+s-t, s)} ds, \\ &\quad 0 \leqslant t \leqslant T, \quad 0 \leqslant r \leqslant r_m. \end{aligned} \tag{8}$$

Proposition 1 tells us that if (β^*, p^*) is optimal for IHP, then it must satisfy the minimum principle (7) on $[0, T]$. Equation (7a) is equivalent to

$$\left[q_T(t) \int_{r_1}^{r_2} k(r)\,h(r)\,p^*(r, t)\,dr - \int_0^{r_m} \left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r,t)} dr \right] \cdot [\beta - \beta^*(t)] \leqslant 0, \qquad \forall \beta \in [\beta_0, \beta_1],\ t \in [0, T] \text{ a.e.} \tag{9}$$

Since λ_{0T}, $\alpha_T(r)$ cannot vanish simultaneously, we may assume that $\|(\lambda_{0T}, p_{T_i}(r, 0)\|$, as $T_i \to \infty$ to be a monotone increasing series, such that

$\lambda_{0T_t} \to \lambda_\infty$ and $p_{T_t}(r) \to \alpha(r)$ (in the weak sense). By (8), it can be shown easily that

$$q_{T_t} \to q(t),$$

$$q(t) = \int_0^{t+r_m} e^{-\int_0^{s-t}\mu(\rho)d\rho} \beta^*(s)\, k(s-t)\, h(s-t)\, q(s)\, ds$$

$$- \lambda_\infty \int_t^{t+r_m} e^{-\int_0^{s-t}\mu(\rho)\,d\rho} \left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(s-t,s)} ds$$

$$q(r, t) = \int_t^{t+r_m-r} e^{-\int_r^{r+s-t}\mu(\rho)d\rho} \beta^*(s)\, k(r+s-t)\, h(r+s-t)\, q(s)\, ds$$

$$- \lambda_\infty \int_t^{t+r_m-r} e^{-\int_r^{r+s-t}\mu(\rho)d\rho} \left.\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r+s-t,s)} ds. \tag{10}$$

Under the assumption that

Assumption 1.

$$\int_0^{r_m} e^{-\int_0^r \mu(\rho)d\rho} \left|\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r,t)} dr < \infty, \qquad \forall t \in [0, \infty) \text{ a.e.}, \tag{11}$$

Eq. (10) has a unique solution and $q(r, t)$ is the mild solution of adjoint system

$$\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t}$$

$$= \mu(r)\, q(r, t) - \beta^*(t)\, k(r)\, h(r)\, q(t) + \lambda_\infty \frac{\partial L(p^*, \beta^*)}{\partial p},$$

$$q(0, t) = q(t). \tag{12}$$

Furthermore, if we assume

Assumption 2.

$$\lim_{t \to \infty} \int_0^{r_m} e^{-\int_0^r \mu(\rho)d\rho} \left|\frac{\partial L(p^*, \beta^*)}{\partial p}\right|_{(r,t)} dr = 0, \tag{13}$$

then there is a transversality condition

$$q(r, \infty) = 0. \tag{14}$$

THEOREM 2 (Minimum Principle). *Under Assumptions* 1 *and* 2, *the overtaking optimal control* (β^*, p^*) *satisfies*

$$\beta^*(t)\, H_\beta(\beta^*, p^*) = \max_{\beta_0 \leqslant \beta \leqslant \beta_1} \beta H_\beta(\beta^*, p^*), \qquad \forall t \in [0, \infty]\ a.e.,$$

where

$$H(\beta, p) = q(t)\,\beta(t) \int_{r_1}^{r_2} k(r)\,h(r)\,p(r, t)\,dr - \lambda_\infty L(p, \beta),$$

$$H_\beta(\beta^*, p^*) = \frac{\partial H(\beta^*, p^*)}{\partial \beta},$$

and $q(t)$ is the solution of the adjoint equation

$$\frac{\partial q(r, t)}{\partial r} + \frac{\partial q(r, t)}{\partial t} = \mu(r)\,q(r, t) - \beta^*(t)\,k(r)\,h(r)\,q(t) + \lambda_\infty \frac{\partial L(p^*, \beta^*)}{\partial p},$$

$$q(r, \infty) = 0,$$

$$q(0, t) = q(t),$$

where $\lambda_\infty \geqslant 0$, $q(t)$ are not both zero.

3. The Turnpike Property

In this part we investigate the asymptotic convergence properties of overtaking optimal trajectory. In the literature these are the so-called turnpike properties. We assume the following:

Assumption 3. $L(p(\cdot), \beta)$ satisfies the following growth condition: there exist $K_1 > 0$ and $K > 0$ such that

$$\|p(r)\|^2 + \beta^2 > K_1 \Rightarrow \int_0^{r_m} L(p(r), \beta)\,dr \geqslant K(\|p(r)\|^2 + \beta^2) \tag{15}$$

and $L(p(\cdot), \beta)$ is convex on $L^2(0, r_m) \times [\beta_0, \beta_1]$.

Assumption 4. There is an unique constant $\bar{c} \geqslant 0$, $\beta_0 \leqslant \bar{\beta} \leqslant \beta_1$ such that

$$\int_0^{r_m} L(\bar{c}e^{-\int_0^r \mu(\rho)\,d\rho}, \bar{\beta})\,dr = \min_{\substack{c \leqslant 0 \\ \beta_0 \leqslant \beta \leqslant \beta_1}} \int_0^{r_m} L(ce^{-\int_0^r \mu(\rho)\,d\rho}, \beta)\,dr. \tag{16}$$

We can now establish the weak turnpike theorem

Theorem 3. *Under Assumptions* 3 *and* 4 *if* $(\tilde{p}(r, t), \tilde{\beta}(t)) \in \mathscr{A}(p_0)$ *is such that*

$$\overline{\lim_{T \to \infty}} \int_0^T \int_0^{r_m} [L(\tilde{p}(r, t), \tilde{\beta}(t)) - L(\hat{p}(r), \bar{\beta})]\,dr\,dt = \alpha < \infty, \tag{17}$$

then necessarily

$$\frac{1}{T}\int_0^{r_m} \tilde{p}(r,t)\,dr \to \hat{p}(r),\ \frac{1}{T}\int_0^T \hat{\beta}(t)\,dt \to \bar{\beta}, \tag{18}$$

where $\hat{p}(r) = e^{-\int_0^r \mu(\rho)\,d\rho}$.

Proof. First we show that there exists a constant $\bar{M} > 0$ such that

$$\int_0^{r_m} \tilde{p}(r,t)\,dr \leqslant \bar{M}, \qquad \forall t \geqslant 0. \tag{19}$$

In fact, by (2) for $T > r_m$

$$\begin{aligned}\int_0^{r_m} \tilde{p}(r,T)\,dr &= \int_0^{r_m} \tilde{\beta}(T-r) \int_{r_1}^{r_2} k(s)\,h(s)\,\tilde{p}(s,T-r)\,ds\, e^{-\int_0^r \mu(\rho)\,d\rho}\,dr \\ &= \int_{T-r_m}^{T} \tilde{\beta}(t) \int_{r_1}^{r_2} k(r)\,h(r)\,\tilde{p}(r,t)\,dr e^{-\int_0^{T-t} \mu(\rho)\,d\rho}\,dt \\ &= \int_{T-r_m}^{T} \tilde{\beta}(t)\, e^{-\int_0^{T-t} \mu(\rho)\,d\rho} \int_0^{r_m} k(r)\,h(r)\,\tilde{p}(r,t)\,dr\,dt \\ &\leqslant M \int_{T-r_m}^{T} \int_0^{r_m} \tilde{p}(r,t)\,dr\,dt,\end{aligned}$$

where M is a constant. If $T_k \to \infty$ such that $\int_0^{r_m} \tilde{p}(r,T_k)\,dr \to \infty$, then the above expression says that

$$\int_{T_k - r_m}^{T_k} \int_0^{r_m} \tilde{p}(r,t)\,dr\,dt \to +\infty \qquad \text{as} \quad k \to \infty. \tag{20}$$

Using Jensen's inequality on L

$$\begin{aligned}&\frac{1}{r_m}\int_{T_k - r_m}^{T_k} \int_0^{r_m} L(\tilde{p}(r,t), \tilde{\beta}(t))\,dr\,dt \\ &\quad \geqslant \int_0^{r_m} L\left(\frac{1}{r_m}\int_{T_k - r_m}^{T_k} \tilde{p}(r,t)\,dt, \frac{1}{r_m}\int_{T_k - r_m}^{T_k} \tilde{\beta}(t)\,dt\right) dr \\ &\quad \geqslant K\left[\left\| \frac{1}{r_m}\int_{T_k - r_m}^{T_k} \tilde{p}(r,t)\,dt \right\|^2\right] \\ &\quad \to \infty, \qquad \text{as} \quad k \to \infty,\end{aligned}$$

i.e., $\lim_{k\to\infty} \int_{T_k - r_m}^{r_m} \int_0^{r_m} L(\tilde{p}(r,t), \tilde{\beta}(t))\,dr\,dt = +\infty$. This contradicts (17), and hence (19) holds.

Second, we show that there exists a constant $\hat{M}$ such that

$$\left\| \frac{1}{T} \int_0^T \tilde{p}(r, t)\, dt \right\| \leqslant \hat{M}, \qquad \forall T > 0. \tag{21}$$

Suppose the contrary, that there exists a sequence $\{T_k\}$, $T_k \to \infty$ such that

$$\left\| \frac{1}{T} \int_0^{T_k} \tilde{p}(r, t)\, dt \right\| \to +\infty, \qquad \text{as} \quad t \to \infty.$$

Using Jensen's inequality again on L

$$\begin{aligned} &\frac{1}{T_k} \int_0^{T_k} \int_0^{r_m} L(\tilde{p}(r, t), \tilde{\beta}(t))\, dr\, dt \\ &\quad \geqslant \int_0^{r_m} m_L \left(\frac{1}{T_k} \int_0^{T_k} \tilde{p}(r, t)\, dt, \frac{1}{T_k} \int_0^{T_k} \tilde{\beta}(t)\, dt \right) dr \\ &\quad \geqslant K \left[\left\| \frac{1}{T_k} \int_0^{T_k} \tilde{p}(r, t)\, dt \right\|^2 \right], \end{aligned}$$

we obtain

$$\begin{aligned} &K \left[\left\| \frac{1}{T_k} \int_0^{T_k} \tilde{p}(r, t)\, dt \right\|^2 \right] - \int_0^{r_m} L(\hat{p}(r), \beta_{cr})\, dr \\ &\quad \leqslant \frac{1}{T_k} \int_0^{T_k} \left[\int_0^{r_m} L(\tilde{p}(r, t), \tilde{\beta}(t)) - \int_0^{r_m} L(\hat{p}(r), \beta_{cr}) \right] dr\, dt. \end{aligned}$$

This contradicts (17) and so (21) holds.

Finally, for every $z(r) \in C^1(0, r_m)$, $z(r) = 0$ on (r_c, r_m), for some $r_c < r_m$, it can be shown that

$$\begin{aligned} &\langle \tilde{p}(r, t) - p_0(r), z(r) \rangle \\ &\quad = \int_0^t \tilde{p}(0, \tau)\, d\tau z(0) - \int_0^t \langle \mu(r)\, \tilde{p}(r, \tau), z(r) \rangle\, d\tau + \int_0^t \langle \tilde{p}(r, \tau), z'(r) \rangle\, d\tau \end{aligned} \tag{22}$$

so

$$\begin{aligned} &\frac{1}{T} \langle \tilde{p}(r, t) - p_0(r), z(r) \rangle \\ &\quad = \frac{1}{T} \int_0^T \tilde{\beta}(t) \int_{r_1}^{r_2} k(r)\, h(r)\, \tilde{p}(r, t)\, dr\, dt \cdot z(0) \\ &\qquad - \frac{1}{T} \int_0^T \langle \mu(r)\, \tilde{p}(r, \tau), z(r) \rangle\, d\tau + \frac{1}{T} \int_0^T \langle \tilde{p}(r, \tau), z'(r) \rangle\, d\tau. \end{aligned} \tag{23}$$

Suppose $(p^*(r), \beta^*)$ is a weak cluster point of the set

$$\left\{\left(\frac{1}{T}\int_0^1 \tilde{p}(r,t)\,dt, \frac{1}{T}\int_0^T \tilde{\beta}(t)\,dt\right)\right\}.$$

When T goes to infinite in (23), we have

$$\langle \mu(r)\,p^*(r), z(r)\rangle + \langle p^*(r), z'(r)\rangle = 0$$

for all $z(r) \in C_0^1(0, r)$. So $p^*(r) = ce^{-\int_0^r \mu(\rho)\,d\rho}$, $c \geqslant 0$. By (17), Jensen's inequality, and the continuity of $\int_0^{r_m} L(p(r), \beta)\,dr$, we see that

$$\int_0^{r_m} L(p^*(r), \beta^*)\,dr \leqslant \int_0^{r_m} L(\bar{p}(r), \bar{\beta})\,dr.$$

Therefore, by the uniqueness of $(\bar{p}, \bar{\beta})$, we have

$$p^*(r) = \bar{p}(r), \qquad \beta^* = \bar{\beta},$$

and this completes the proof.

Define the operator $\mathbf{A}: L^2(0, r_m) \to L^2(0, r_m)$ by

$$\begin{aligned} &\mathbf{A}\phi(r) = \phi'(r) + \mu(r)\,\phi(r), \\ &D(\mathbf{A}) = \{\phi(r) \mid \phi(r), \mathbf{A}\phi(r) \in L^2(0, r_m)\}; \end{aligned} \tag{24}$$

then it follows that

$$\begin{aligned} &\mathbf{A}^*\psi(r) = -\psi'(r) + \mu(r)\,\psi(r), \\ &D(\mathbf{A}^*) = \{\psi(r) \mid \psi(r), \mathbf{A}^*\psi(r) \in L^2(0, r_m)\}. \end{aligned} \tag{25}$$

By the assumptions already made on L, we know that there exists a $\psi(r) \in D(A^*)$ such that

$$\int_0^{r_m} L(\bar{p}(r), \bar{\beta})\,dr \leqslant \int_0^{r_m} L(p(r), \beta)\,dr - \langle p(r), \mathbf{A}^*\psi(r)\rangle$$
$$\text{for all} \quad p(r) \geqslant 0, \beta \in [\beta_0, \beta_1]. \tag{26}$$

Let $L_0(p(\cdot), \beta): L^2(0, r_m) \times \mathbb{R}^1 \to [0, \infty)$ be defined by

$$L_0(p(\cdot), \beta) = \begin{cases} \displaystyle\int_0^{r_m} L(p(r), \beta)\,dr - \int_0^{r_m} L(\bar{p}(r), \bar{\beta})\,dr - \langle p(r), \mathbf{A}^*\psi(r)\rangle \\ \qquad\qquad \text{for all} \quad p(r) \geqslant 0, \beta \in [\beta_0, \beta_1], \\ +\infty, \qquad \text{otherwise}. \end{cases} \tag{28}$$

Then $L_0(\bar{p}(\cdot), \bar{\beta}) = 0$. Furthermore, L_0 also satisfies the growth condition:

$$\| p(r)\|^2 + \beta^2 > K_2 \Rightarrow L_0(p(\cdot), \beta) \geqslant K(\| p(r)\|^2 + \beta^2). \tag{28}$$

LEMMA 1. *If an admissible pair* $(\tilde{p}(\cdot, \cdot), \tilde{\beta}(\cdot)) \in \mathscr{A}(p_0)$ *satisfies*

$$\int_0^\infty L_0(\tilde{p}(\cdot, t), \tilde{\beta}(t))\, dt < \infty \tag{29}$$

then necessarily $\| \tilde{p}(\cdot, t)\|$ *is bounded for* $t \geqslant 0$.

Proof. As in [5], we define

$$\Omega_T = \{ t \geqslant T \mid \| \tilde{p}(\cdot, t)\|^2 \geqslant K_2 \}$$

for each $T > 0$ and similar arguments show that

$$\lim_{T \to \infty} \operatorname{mes}(\Omega_T) = 0.$$

Choose $t > 1$ sufficiently large so that

$$\operatorname{mes}(\Omega_T) < 1.$$

Then for each $t \in \Omega_T$, there exists $h \in [0, 1]$ so that $t - h \notin \Omega_T$. Let $\sigma = t - h$, then

$$\begin{aligned} \tilde{p}(r, t) &= \tilde{p}(r, \sigma + h) = S(h)\, \tilde{p}(r, \sigma) \\ &= \begin{cases} \tilde{p}(r-h, \sigma)\, e^{-\int_{r-h}^{r} \mu(\rho)\, d\rho}, & r \geqslant h, \\ \beta(h-r) \displaystyle\int_{r_1}^{r_2} k(s)\, h(s)[S(h-r)\, \tilde{p}(s, \sigma)]\, ds e^{-\int_0^r \mu(\rho)\, d\rho}, & r < h. \end{cases} \end{aligned}$$

By this we can show easily that

$$\| \tilde{p}(\cdot, t)\| \leqslant M \| \tilde{p}(\cdot, \sigma)\|, \qquad M = \text{const}.$$

This is the desired result.

Remark. It can be shown that under the condition (17) and the assumption of Theorem 2 it follows that

$$\int_0^\infty L_0(\tilde{p}(\cdot, t), \tilde{\beta}(t))\, dt < \infty$$

and therefore $\| p(\cdot, t)\|$ is bounded for $t \geqslant 0$.

We introduce the set

$$G = \{ p(r) \in L^2(0, r_m) | \ni \beta \in [\beta_0, \beta_1] \text{ s.t. } L_0(p(\cdot), \beta) = 0 \} \tag{30}$$

and the following

DEFINITION 2. Let $\mathscr{F}$ be the family of all trajectories $p(r, \cdot) \geqslant 0$ such that

$$p(\cdot, t) \in G \text{ a.e.} \qquad \text{on } [0, \infty).$$

We say that G has property $\mathscr{G}$ (for convergence) if $p(\cdot, t) \overset{w}{\rightarrow} \bar{p}(\cdot)$ as $t \rightarrow \infty$ uniformly in $\mathscr{F}$.

The following results are true.

THEOREM 4. *Under Assumption* 4, *if* G *has the property* $\mathscr{G}$ *and if a feasible pair* $(\tilde{p}, \tilde{\beta})$ *is such that*

$$\int_0^\infty L_0(\tilde{p}(\cdot, t), \tilde{\beta}(t))\, dt < \infty, \tag{31}$$

then, necessarily, $\tilde{p}(\cdot, t)$ *converges weakly to* $\bar{p}(\cdot)$ *as* $t \rightarrow \infty$.

COROLLARY. *In addition to the hypotheses given in Theorem* 3, *let us suppose that there exists a pair* $(\tilde{p}, \tilde{\beta}) \in \mathscr{A}(p_0)$ *such that* (31) *holds; then if in the class of all bounded trajectories there exists an overtaking optimal solution, say* $(\hat{p}, \hat{\beta})$, *it follows that*

$$\lim_{t \rightarrow \infty} \hat{p}(\cdot . t) = \bar{p}(\cdot) \qquad \textit{in the weak sense.}$$

Remark. If the system (1) is controllable, i.e., there exist $\beta(t) \in U$ and $T > 0$, such that the corresponding trajectory $p(r, t)$ satisfy

$$p(r, T) = \bar{p}(r)$$

and define

$$(\tilde{p}(r, t), \tilde{\beta}(t)) = \begin{cases} (p(r, t), \beta(t)), & 0 \leqslant t \leqslant T, \\ (\bar{p}(r), \bar{\beta}), & t \geqslant T, \end{cases}$$

then condition (31) is satisfied.

3. EXISTENCE OF OVERTAKING OPTIMAL SOLUTIONS

Assumption 5. There exists $(\tilde{p}(r, t), \tilde{\beta}(t)) \in \mathscr{A}(p_0)$ such that

$$\int_0^\infty \int_0^{r_m} L(\tilde{p}(\cdot, t), \tilde{\beta}(t))\, dt < \infty. \tag{32}$$

THEOREM 5. *Under Assumption* 5 *there exists an overtaking optimal solution* $(\hat{p}, \hat{\beta})$.

Proof. Let

$$\Phi = \inf\left\{\int_0^\infty \int_0^{r_m} L(p(r,t), \beta(t))\, dr\, dt, (p, \beta) \in \mathscr{A}(p_0)\right\}.$$

By assumption, Φ is finite. Let $(p_n, \beta_n) \in \mathscr{A}(p_0)$ be a minimizing sequence. For any fixed $T > 0$, since $\beta_n(t) \in [\beta_0, \beta_1]$ for $t \geqslant 0$, we may extract, if necessary, a subsequence $\hat{\beta}(t)$ such that

$$\beta_n(t) \to \hat{\beta}(t) \qquad \text{weakly in } L^2(0, T).$$

Since $\{\beta(t) \mid \beta(t) \in [\beta_0, \beta_1] \text{ for } t \in [0, T] \text{ a.e.}\}$ is a closed convex subset of $L^2(0, T)$, it is weakly closed, and hence $\hat{\beta}(t) \in [\beta_0, \beta_1]$ for $t \in [0, T]$ a.e. By (2) $p_n(\cdot, t) \to \hat{p}(\cdot, t)$ weakly in $L^2(0, T; L^2(0, r_m)$ and $(\hat{p}, \hat{\beta}) \in \mathscr{A}(p_0)$. By convexity

$$\int_0^T \int_0^{r_m} L(p(r,t), \beta(t))\, dr\, dt$$

is weak l.s.c. over $L^2(0, T; L^2(0, r_m)) \times L^2(0, T)$. This shows that

$$\int_0^\infty \int_0^{r_m} L(\hat{p}(r,t), \hat{\beta}(t))\, dr\, dt \leqslant \Phi$$

and $(\hat{p}, \hat{\beta})$ is an overtaking optimal solution.

REFERENCES

1. W. L. CHAN AND GUO BAO ZHU, Optimal birth control of population dynamics, *J. Math. Anal. Appl.* **144** (1989), 532–552.
2. W. L. CHAN AND GUO BAO ZHU, Optimal birth control of population dynamics. II. Problems with free final time, phase constraints and mini-max costs, *J. Math. Anal. Appl.*, to appear.
3. M. BROKATE, Pontryagin's principle for control problems in age-dependent population dynamics, *J. Math. Biol.* **23** (1985), 75–102.
4. D. A. CARLSON, A. HAURIE, AND A. JABRANE, Existence of overtaking solutions to infinite dimensional control problems on unbounded time intervals, *SIAM J. Control Optim.* **25** (1987), 1517–1541.
5. J. SONG AND J.-Y. YU, "Population System Control," Springer-Verlag, New York, 1987.