Probabilistic analysis on the splitting-shooting method for image transformations

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Abstract

This paper explores a new topic in image processing where accuracy of images even in details is crucial, and adopts a new methodology dealing with discrete topics by continuous mathematics and numerical approximation. The key idea is that a pixel of images at different levels can be quantified by a greyness value, which can then be regarded as the mean of an integral of continuous functions over a small pixel region, and evaluated by numerical integration approximately. Hence, new treatments of approximate integration and new discrete algorithms of images have been developed. This paper also integrates different mathematics disciplines: numerical analysis, geometry, probability and statistics to discrete images that can be applied to many areas in computer sciences: image processing, computer graphics, computer vision, geometric added designs, and pattern recognition.

In this paper, new error analysis in terms of probability theory is explored for the popular splitting-shooting method (SSM) and the combination (CSIM) of the splitting-shooting-integrating methods proposed in [35–37], and convergence rates in probability of image greyness are proven to be $O_p(1/N^{15})$ higher than $O(1/N)$ reported in [45] by strict error analysis. Moreover, a new partial refinement technique of pixel partition is also proposed in this paper, to achieve the convergence rate $O_p(1/N^2)$ in probability for SSM. By the new study of this paper, the SSM can be applied to real images with 256 greyness levels. The numerical and graphical experiments are also provided to confirm the theoretical analysis made. Both the strict error bounds and the probabilistic error bounds with explicit constants are also derived for general $x$-norms as $x \geq 1$, and the countable probability inequalities for several sums of random variables are developed from probability theory, which are more suited to numerical computations. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper explores a new topic in image processing where accuracy of images in details is crucial, contrasted to image condense in [22, 25] where saving computer storage is important, and

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adopts a new methodology dealing with discrete topics by continuous mathematics and numerical approximation. The key idea is that a pixel of images at different levels can be quantified by a greyness value, which can then be regarded as the mean of an integral of continuous functions over a small pixel region, and evaluated by numerical integration approximately. Hence new treatments of approximate integration and new discrete algorithms of images have been developed.

In the book [35], the splitting-shooting method (SSM) for \( T \), the splitting-integrating method (SIM) for \( T^{-1} \), and their combination CSIM for a cycle \( T^{-1}T \) are provided systematically, and applied to many topics in image processing and pattern recognition. The advantage of SIM for \( T^{-1} \) and combination CSIM for \( T^{-1}T \) is no need of solving the nonlinear equations.

Let the standard image \( \hat{W} \) be distorted by \( T \), and then normalized by \( T^{-1} \) to itself,

\[
\hat{W} \xrightarrow{T} \hat{Z} \xrightarrow{T^{-1}} \hat{W},
\]

where the nonlinear transformation \( T \) is

\[
T:(\xi,\eta) \rightarrow (x,y), \quad x=x(\xi,\eta), \quad y=y(\xi,\eta),
\]

and the functions \( x(\xi,\eta) \) and \( y(\xi,\eta) \) are assumed to be known and explicit in this paper. By the numerical approaches for (1.1) shown the schematic steps of Fig. 1, the numerical methods are developed and samples of binary images are illustrated in Fig. 2.

In order to give a deep insight of SSM and CSIM, we have carried out a strict error analysis in [45]. The convergence rates of sequential greyness errors by SSM and CSIM are derived to be
O(1/N), when a 2D-pixel is split into $N \times N$ sub-pixels. Let us examine SSM and CSIM again due to their simplicity. In a large number of examples, numerical computations show that the performance of SSM and CSIM is always better than the theoretical results in [45]. In this paper, the effect of numerical cancellation involved in SSM has been considered by the central limit theorem in probability theory. By probabilistic analysis, it has been proven that a convergence rate $O(1/N^{1.5})$ in probability (denoted by $O_p(1/N^{1.5})$) can be achieved by SSM.

Furthermore, we also propose a new partial refinement technique of subpixels so as to gain the higher rate of $O_p(1/N^2)$ by SSM. By means of the study given in this paper, the new renovation of SSM and CSIM become more promising is image transformation. Of course, partial refinement needs to increase certain amount of computation, as a price to gain higher convergence rates. However, the amount of computation increased due to the partial refinement is only a small portion to the total amount of computation paid to the image transformation. As seen from the discussion in Section 5, to achieve the convergence rate of $O_p(1/N^2)$, the amount of extra CPU time is of the order $O(MN^{5/3})$. Thus, the total CPU time has the same order $O(MN^2)$ as that in [45, 46, 35], where $M$ is the number of image pixels.

This paper also integrates different mathematical disciplines: numerical analysis, geometry, probabilistic and statistical analysis to discrete images that can be applied to many areas in computer sciences: image processing, computer graphics, computer version, geometric computation, geometric added designs, and pattern recognition.

For easy references, we also collect the important notations that will be used often in our derivations later.

(1) Pixel greyness by SSM:

$$B_{ij} = \frac{1}{H^2} \int \int_{C_{ij}} b(x, y) \, dx \, dy,$$

(1.3)

$$\hat{B}_{ij} = \frac{1}{H^2} \int \int_{C_{ij}} \hat{b} \, dx \, dy = \frac{1}{H^2} \int \int_{\Omega_{ij}} \phi_{\mu} \, d\xi \, d\eta,$$

(1.4)
\[
\tilde{B}_{ij} = \tilde{B}_{ij}^{(N)} = \left( \frac{h}{H} \right)^2 \sum_{\mathcal{G}(2.20)} \hat{\phi}_\mu(\hat{G}) \mathcal{J}(\hat{G}),
\]
where \(\Omega_{ij} \xrightarrow{T} \square_{ij}\), and \(\hat{\phi}_\mu = \hat{\phi}_\mu(\xi, \eta)\), and \(\hat{\phi}_\mu(\xi, \mu)\) (\(\mu = 0, 1\)) are the piecewise constant and bilinear interpolatory functions.

(II) Pixel greyness errors in \(\alpha\)-norms by SSM:

\[
E_\alpha(\tilde{B}) = E_\alpha^{(N)}(\tilde{B}) = \left( \sum_{ij} \left| \tilde{B}_{ij}^{(N)} - B_{ij} \right|^\alpha M \right)^{1/\alpha}, \quad \alpha \geq 1,
\]

\[
\Delta E_\alpha(\tilde{B}) = \Delta E_\alpha^{(N)}(\tilde{B}) = \left( \sum_{ij} \left| \tilde{B}_{ij}^{(N)} - B_{ij}^{(N-1)} \right|^\alpha M \right)^{1/\alpha}, \quad \alpha \geq 1,
\]
where \(N = N_p = 2^p\), \(p = 0, 1, \ldots\)

2. The splitting-shooting method (SSM) and its combination

2.1. New view of discrete images

Since an image is composed of many digital pixels, we denote

\[
\tilde{W} = \{\tilde{W}_{ij}\}, \quad \tilde{Z} = \{\tilde{Z}_{ij}\},
\]
where \(W_{ij}\) and \(Z_{ij}\) are the pixels located at points \((i,j)\) and \((I,J)\) with the coordinates, \((\xi_i, \eta_j) = (iH, jH)\), \((\chi_i, \gamma_j) = (IH, JH)\), respectively, and \(H\) is the mesh resolution in an optical scanner.

Suppose that \(q\)-levels of pixel greyness are assigned to \(\tilde{W}_{ij}\) and \(\tilde{Z}_{ij}\). Also \(G_k (k = 1, 2, \ldots, q)\) denotes the \(k\)th level of greyness, and \(G_1\) and \(G_q\) are the whitest and the darkest greyness, respectively. We use the notation \("\tilde{W}_{ij} = \star\) at \(G_k (1 \leq k \leq q)\" to represent an image pixel \(W_{ij}\) at \((i,j)\) with the greyness level \(G_k\). In Step 1, we may transfer the pixels \(\tilde{W}_{ij}\) to greyness \(\Phi_{ij}\) by

\[
\Phi_{ij} = \frac{k - 1}{q - 1} \quad \text{if } \tilde{W}_{ij} = \star \text{ with the } k\text{th grey level } G_k, \quad 1 \leq k \leq q.
\]

Suppose that a pixel greyness \(\hat{\Phi}_{ij}\) under \(T^{-1}\) or \(T^{-1}T\) has been obtained by numerical methods; in Step 8 the normalized pixels can be assigned by

\[
\tilde{W}_{ij} = \begin{cases} 
\star \text{ at } G_q & \text{if } \hat{\Phi}_{ij} \geq 1 - \frac{1}{2(q-1)}, \\
\star \text{ at } G_k, \ k = 2, 3, \ldots, q - 1 & \text{if } \frac{k-1}{q-1} \leq \hat{\Phi}_{ij} < \frac{k-1}{q-1}, \\
\star \text{ at } G_k, \ k = 2, 3, \ldots, q - 1 & \text{if } \hat{\Phi}_{ij} < \frac{1}{2(q-1)}.
\end{cases}
\]

In Step 2, we choose the piecewise constant and bilinear interpolation of \(\phi(\xi, \eta)\):

\[
\hat{\phi}_0(\xi, \eta) \text{ in } \square_{ij}, \quad \hat{\phi}_1(\xi, \eta) \text{ in } \square_{ij},
\]
where the regions are

\[
\square_{ij} = \left\{ (\xi, \eta), \quad (i - \frac{1}{2})H < \xi \leq (i + \frac{1}{2})H, \right. \\
\left. (j - \frac{1}{2})H < \eta \leq (j + \frac{1}{2})H, \right\}
\]  

(2.5)

and

\[
\bar{\square}_{ij} = \left\{ (\xi, \eta), \quad iH < \xi \leq (i + 1)H, \right. \\
\left. jH < \eta \leq (j + 1)H, \right\}
\]  

(2.6)

Our numerical algorithms are based on the following interpretation on discrete images: A pixel \( \bar{W}_{ij} \) can be viewed as the representative over the pixel region \( \square_{ij} \); if so, then its greyness can be defined by an average of a continuous greyness function \( \phi(\xi, \eta) \) over \( \square_{ij} \):

\[
\Phi_{ij} = \frac{1}{H^2} \int_{\square_{ij}} \phi(\xi, \eta) \, d\xi \, d\eta.
\]  

(2.7)

Similarly, the greyness of \( \bar{Z}_{ij} \) is given by

\[
B_{ij} = \frac{1}{H^2} \int_{\bar{\square}_{ij}} b(x, y) \, dx \, dy,
\]  

(2.8)

where the pixel region

\[
\bar{\square}_{ij} = \left\{ (x, y), \quad (I - \frac{1}{2})H < x \leq (I + \frac{1}{2})H, \right. \\
\left. (J - \frac{1}{2})H < y \leq (J + \frac{1}{2})H, \right\}
\]  

(2.9)

and the continuous greyness function \( b(x, y) \) can be formed by the similar interpolation.

Note that the important viewpoints of (2.7) and (2.8) enable us to solicit the continuous mathematics, numerical integration, numerical analysis and probability analysis to discrete images. This is also a distinctive feature of this paper from many other literatures in image processing.

2.2. The splitting-shooting method for \( T \)

Denote by \( P \) and \( J \) the Jacobian matrix and its determinant

\[
P = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}, \quad J = \det(P).
\]  

(2.10)

We also assume

\[
0 < J_0 \leq J(\xi, \eta) \leq J_M,
\]  

(2.11)

where \( J_0 \) and \( J_M \) are positive constants independent of \( \xi \) and \( \eta \).

Next, we split the pixel region \( \square_{ij} \) of \( \bar{W}_{ij} \) into \( N \times N \) disjoint sub-regions:

\[
\square_{ij,kl} = \left\{ (\xi, \eta), \quad (i - \frac{1}{2})H + (k - 1)h < \xi \leq (i - \frac{1}{2})H + kh, \right. \\
\left. (j - \frac{1}{2})H + (l - 1)h < \eta \leq (j - \frac{1}{2})H + lh. \right\}
\]  

(2.12)
where $1 \leq k, l \leq N$, and $h$ is the edge length of $\Box_{ij, kl}$:

$$h = H/N.$$  \hfill (2.13)

Hence,

$$\Box_{ij} = \bigcup_{k,l=1}^{N} \Box_{ij, kl}. \hfill (2.14)$$

Let us first consider Step 3 in Fig. 1 for images through the transformation $T$ in (1.1). Noting that

$$\phi(\xi, \eta) = b(x(\xi, \eta), (\xi, \eta)),$$  \hfill (2.15)

we see that the greyness (2.8) under $\Omega_{ij} \rightarrow \Box_{ij}$ is reduced to

$$B_{ij} = \frac{1}{H^2} \int \int_{\Omega_{ij}} b(x, y) \, dx \, dy$$

$$= \frac{1}{H^2} \int \int_{\Omega_{ij}} \phi(\xi, \eta) f(\xi, \eta) \, d\xi \, d\eta. \hfill (2.16)$$

The splitting-shooting method is designed in [37] to seek the pixel greyness of the distorted image under $T$ by collecting the contributions made to $B_{ij}$ by all sub-pixels $\Box_{ij, kl}$ whose transformed centroid images belong to the pixel region $\Box_{ij}$. The greyness $B_{ij}$ can be evaluated by

$$B_{ij} \approx \tilde{B}_{ij} = \tilde{B}_{ij}^{(N)} = \left( \frac{h}{H} \right)^2 \sum_{\forall \text{Eq. (2.20)}} \hat{\phi}_\mu(\hat{G}) f(\hat{G}), \hfill (2.17)$$

where $\hat{\phi}_\mu$ is the $\mu$-order interpolation function of $\phi$, and $\hat{G}$ is the centroid of $\Box_{ij, kl}$ with the coordinates

$$\xi(\hat{G}) = (i - \frac{1}{2})H + (k - \frac{1}{2})h, \quad \eta(\hat{G}) = (j - \frac{1}{2})H + (l - \frac{1}{2})h.$$  \hfill (2.18)

Let $\hat{G} \rightarrow \hat{G}^*$, then

$$x(\hat{G}^*) = x(\xi(\hat{G}), \eta(\hat{G})), \quad y(\hat{G}^*) = y(\xi(\hat{G}), \eta(\hat{G})).$$  \hfill (2.19)

The pixel region $\Box_{ij}$, which the transformed centroid falls into, is located by $(I, J)$ where

$$I = \lfloor x(\hat{G}^*)/H + \frac{1}{2} \rfloor, \quad J = \lfloor y(\hat{G}^*)/H + \frac{1}{2} \rfloor.$$  \hfill (2.20)

Let us address the algorithm nature of the splitting-shooting method (SSM). The SSM consists of the following three processes (see Figs. 3 and 4):

(A) The splitting process (2.14) of a pixel region $\Box_{ij}$ into the uniform smaller sub-regions $\Box_{ij, kl}$, which is, in fact, as a pixel split into many smaller subpixels, or as if a bullet were split into many tiny pellets.

(B) The shooting process (2.19), in which each subpixel flies like a pellet shot along the trajectory of its center of gravity, then to fall into a pixel region $\Box_{ij}$ in $XOY$, defined by (2.20).

(C) The collecting process (2.17), in which each pixel at $(I, J)$ collects all flying subpixels fallen into $\Box_{ij}$, to obtain its approximate greyness.
We may imagine the SSM algorithm as if a hail of tiny pellets were shot over a target bird from the hunting gun, as illustrated in Figs. 3 and 4. When \( N = 1 \) (i.e., without splitting) the SSM leads to the traditional methods in [22, 25] as if the entire bullet were shot, to cause undesirably superfluous holes and blanks (as "measles") of enlargement images, see [35, 36]. From the above imagination, the splitting-shooting method is called. Note that the SSM is a kind of approximate integration, but it is different from the traditional integration rules in [16, 70]. Obviously, the SSM is more efficient and effective in image transformations than the traditional methods, so that it has found numerous important applications, particularly those in binary images (see [35]).

The simple collecting process (C) for the pixel at \((L, J)\) is purely based on the trajectory of the flying subpixel whose gravity center falls into \(\Box_{I, J}\). Hence, three cases occur as shown in Fig. 4: Case A: the entire subpixel falls into \(\Box_{I, J}\); Case B: the subpixel falls across the boundary of \(\Box_{I, J}\) with the inside gravity center; Case C: the subpixel falls across the boundary of \(\Box_{I, J}\) but with the outside gravity center. The errors resulting from Cases B and C are much larger than those from Case A. Consequently, the low convergence rate \(O(1/N)\) of sequential greyness is obtained by strict error analysis in [45], which is, indeed, a conservative estimation because we did not count cancellation of the over estimates in Case B and the under-estimates in Case C. In this paper, we adopt probability analysis for counting such a cancellation, to provide the sharp convergence rates in probability \(O_p(1/N^{1.5})\) and \(O_n(1/N^2)\) using partial refinement. To our surprise, discrepancy of
the numerical results and the probability analysis is so insignificant that we must believe that the real computational images of many pixels obey the statistics and probability laws. A remarkable consequence of this paper is that the SSM can also be applied to multi- (e.g., 256) level images, a great development from [34, 45].

2.3. The splitting-integrating method for $T^{-1}$

Let us introduce the splitting-integrating method for images through $T^{-1}$. In Step 6, we again choose the piecewise constant interpolation $(\mu = 0)$, $\hat{b}_0(x, y)$ in $\square_{ij}$ and the piecewise bilinear interpolation $(\mu = 1)$, $\hat{b}_1(x, y)$ in $\square_{ij}$, where $\square_{ij}$ is given in (2.9), and

$$\square_{ij} = \begin{cases} (x, y), & IH < x \leq (J+1)H, \\ JH < y \leq (J+1)H. \end{cases}$$

(2.21)

In Step 7, the greyness of the restored pixels $\Phi_{ij}$ can be evaluated by (2.7) and the centroid rule of integration.

$$\Phi_{ij} = \frac{1}{H^2} \int \int_{\square_{ij}} \phi(\xi, \eta) \, d\xi \, d\eta = \frac{1}{H^2} \sum_{k,l=1}^{N} \int \int_{\square_{ij,kl}} \phi(\xi, \eta) \, d\xi \, d\eta$$

$$= \frac{1}{H^2} \sum_{k,l=1}^{N} \int \int_{\square_{ij,kl}} b_\mu(x, y) \, d\xi \, d\eta$$

$$\approx \hat{\Phi}_{ij} = \hat{\Phi}_{ij}^{(N)} = \left( \frac{h}{H} \right)^2 \sum_{k,l=1}^{N} \hat{b}_\mu(x(\hat{G}^*), y(\hat{G}^*)),$$

(2.22)

where $x(\hat{G}^*)$ and $y(\hat{G}^*)$ are given in (2.19).

The splitting-integrating method in (2.22) is, indeed, the composite central rule of integration in [16, 70]. The sample splitting process (2.14) and the integrating process (2.22) are performed easily for the images under the inverse transformation $T^{-1}$. To our astonishment, there is no need of solving the nonlinear equations, as done in the traditional methods in image treatments! In fact, when only the nonlinear functions, $x = x(\xi, \eta)$ and $y = y(\xi, \eta)$ in (1.2), are known, the inverse transformation is defined by $T^{-1}:(x, y) \rightarrow (\xi, \eta)$. It seems that the nonlinear solutions of $(\xi, \eta)$ from $(x, y)$ are inevitable because the inverse functions $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ are unknown. Interestingly, the SIM bypasses such a nonlinear solution process skillfully, thus to greatly simplify the algorithms for normalization of distort images. Consequently, the SIM has found the profound applications, in particular those for pattern recognition. However, the integrand functions resulting from the piecewise interpolations in (2.4) have different regularities in different subregions, contrasted to the analysis in numerical integration of [16, 70], where the integrand regularity is uniform over the entire integration region. Although the central rule in (2.22) does not distinguish the non-uniform smoothness of the integrand, strict error analysis on the SSM is derived in [44, 45], also to reach the optimal convergence rates $O(1/N^2)$. Moreover, even the high convergence rates $O(1/N^3)$ and $O(1/N^4)$ in [43] can be achieved by means of spline functions of [63].
2.4. Combination (CSIM) of splitting-shooting-integrating methods for $T^{-1}T$

We combine SSM in Step 3 and CSIM in Step 7 to evaluate the images under $T^{-1}T$ (see Fig. 1). This forms the CSIM method.

Two cases of the cycle conversion will be discussed. Case II consists of 1–8 in Fig. 1. In Steps 4 and 5, the following converted relations are used:

$$
\hat{Z}_{ij} = \begin{cases} 
\ast & \text{with } G_q, \text{ if } \hat{B}_{ij} \geq 1 - \frac{1}{2(q-1)}, \\
\ast & \text{with } G_k, \text{ if } \frac{k-\frac{3}{2}}{q-1} \leq \hat{B}_{ij} < \frac{k-\frac{1}{2}}{q-1}, \\
\circ & \text{if } \hat{B}_{ij} < \frac{1}{2(q-1)},
\end{cases}
$$

(2.23)

and

$$
\hat{B}_{ij}^{\|} = \frac{k-1}{q-1} \text{ if } \hat{Z}_{ij} = \ast \text{ with } G_k.
$$

(2.24)

In fact, the greyness $B_{ij}$ will be changed under Steps 4 and 5 by

$$
\hat{B}_{ij}^{\|} = \begin{cases} 
1 & \text{if } \hat{B}_{ij} \geq 1 - \frac{1}{2(q-1)}, \\
\frac{k-1}{q-1} & \text{if } \frac{k-\frac{3}{2}}{q-1} \leq \hat{B}_{ij} < \frac{k-\frac{1}{2}}{q-1}, \\
0 & \text{if } \hat{B}_{ij} < \frac{1}{2(q-1)}.
\end{cases}
$$

(2.25)

Case I consists of Steps 1–3 and 6–8, where the greyness $B_{ij}$ after Step 3 will be used directly for $T^{-1}$ without any changes. The distorted image $\hat{Z}$ may be obtained from $\{\hat{B}_{ij}\}$, but no feedback step as in (2.24) is considered. A binary image under $T^{-1}T$ is illustrated in Fig. 2, where ‘*’ and ‘.’ denote the black and white pixels, and ‘+’ and ‘.’ the greyness in the ranges $(\frac{1}{2}, \frac{1}{4}]$ and $(\frac{1}{4}, 0)$, respectively.

The above algorithm CSIM is very simple and easy to carry out because no nonlinear equations are required, either. As a result, the CSIM has been widely applied to many areas of image processing and pattern recognition. Obviously, the errors of CSIM come mainly from those of both SSM and SIM. Hence the larger errors resulting from SSM will damage the efficiency of combination CSIM. In this paper, our efforts are paid to improve performance of SSM and so CSIM. However, the exposition of this paper focuses only on the SSM for saving space, because the similar description and analysis of combinations CSIM may follow [45, 46] easily.

3. Strict error analysis for the splitting-shooting method

Let the division number

$$
N_p = 2^p, \quad p = 0, 1, \ldots,
$$

(3.1)

then we define the generalized global mean greyness errors of $\hat{B}_{ij}$ in $\alpha$-norms:

$$
E_\alpha(\hat{B}) = E_\alpha^{(N_p)}(\hat{B}) = \left( \frac{\sum_p |\hat{B}_{ij}^{(N_p)} - B_{ij}|^\alpha}{M} \right)^{1/\alpha}, \quad \alpha \geq 1.
$$

(3.2)
As a special case, we have
\[ E_2(\tilde{B}) = E_2^{(N_p)}(\tilde{B}) = \left[ \sum_{ij} \frac{(\tilde{B}_{ij}^{(N_p)} - B_{ij})(\tilde{B}_{ij}^{(N_p-1)} - B_{ij})}{M} \right]^{1/2}, \]  
(3.3)
where \( M \) is the total number of nonempty pixels of \( \tilde{Z} \).

An error analysis for \( E_2^{(N_p)}(\tilde{B}) \) in (3.3) is given in a recent paper [45], in terms of the Sobolev norms. In this paper, we will estimate the bounds of \( E_2^{(N_p)}(\tilde{B}) \) in (3.2) directly, which is more useful in practice, especially the case where \( \alpha = 1 \). The constants in the error bounds beyond the order \( O(1/N^p) \) are also our concern. Hence, the analysis in this section can be regarded as a generalization to [45].

Define the sequential errors of pixel greyness of two consecutive numbers \( N_p \) and \( N_{p-1} \) in \( \alpha \)-norms:
\[ \Delta E_2(\tilde{B}) = \Delta E_2^{(N_p)}(\tilde{B}) = \left( \sum_{ij} \frac{|\tilde{B}_{ij}^{(N_p)} - \tilde{B}_{ij}^{(N_{p-1})}|^\alpha}{M} \right)^{1/\alpha}, \quad \alpha \geq 1. \]  
(3.4)
When \( \alpha = 1 \), we simply denote \( E(\tilde{B}) = E_1(\tilde{B}) \) and \( \Delta E(\tilde{B}) = \Delta E_1(\tilde{B}) \).

In order to given bounds for the above statistical errors, we shall first consider the error of pixel greyness.
\[ \varepsilon_{ij}^{(N)} = |B_{ij} - \tilde{B}_{ij}^{(N)}|, \]  
(3.5)
where \( B_{ij} \) and \( \tilde{B}_{ij}^{(N)} \) are given in (2.16) and (2.17), respectively. We note that the numerical quadrature (2.17) is different from the conventional methods in [16, 70].

3.1. Preliminary lemmas

Denote
\[ \square_{ij,kl} \rightarrow \square_{ij,kl}^*, \quad \hat{G} \rightarrow \hat{G}^*. \]  
(3.6)
There are three different cases where \( \square_{ij,kl} \) intersects with \( \square_{ij} \).

Case A: The whole image \( \square_{ij,kl}^* \) completely falls into \( \square_{ij} \). Then
\[ \exists (I,J) \text{ such that } \square_{ij,kl}^* \subseteq \square_{ij}. \]  
(3.7)

Case B: \( \square_{ij,kl}^* \) partially overlaps \( \square_{ij} \) and transformed centroid \( \hat{G}^* \) is located within \( \square_{ij} \). Then
\[ \exists (I,J) \text{ such that } \square_{ij,kl}^* \not\subseteq \square_{ij} \text{ and } \hat{G}^* \in \square_{ij}. \]  
(3.8)

Case C: \( \square_{ij,kl}^* \) partially overlaps \( \square_{ij} \) but \( \hat{G}^* \) is located outside \( \square_{ij} \). Then
\[ \exists (I,J) \text{ such that } \square_{ij,kl}^* \not\subseteq \square_{ij} \text{ and Area } (\square_{ij,kl}^* \cap \square_{ij}) \neq 0 \text{ and } \hat{G}^* \not\in \square_{ij}. \]  
(3.9)
Furthermore, a transformation \( T \) is said to be regular if the following two conditions are satisfied:
1. \( x(\xi,\eta), y(\xi,\eta) \in C^2(\Omega) \), where \( C^2(\Omega) \) denotes the space of all functions having continuous \( k \)-th order derivatives;
2. (2.11) holds true.
Also, we may construct a function $\phi \in C^2(\Omega)$ satisfying $\phi(iH, jH) = \Phi_{ij}$ by means of the cubic spline function [63]. Since $H$ is fixed and $0 \leq \Phi_{ij} \leq 1$ we can assume that

$$\phi \in C^2(\Omega).$$  \hspace{1cm} (3.10)

The maximal norms over $\Omega$ are defined by

$$\|v\|_{n, \infty, \Omega} = \max_{|x| \leq n} |D^2v|, \quad |v|_{n, \infty, \Omega} = \max_{|x| = n} |D^2v|.$$  

If the maximal norm $\|v\|_{n, \infty, \Omega}$ is finite, then we say that $v \in D^n(\Omega)$. Hence $C^n(\Omega) \subset D^n(\Omega)$. Then, the following lemma can be shown by elementary calculus.

**Lemma 3.1.** Let $\Box$ be a square in $\zeta \circ \eta$ with the edge length $h$, and $f \in D^2(\Box)$. Then the error bounds for the centroid rule of integration can be given as

$$\left| \int \int_{\Box} f \, d\zeta \, d\eta - f(G)h^2 \right| \leq \frac{1}{12} h^4 |f|_{2, \infty, \Box},$$  \hspace{1cm} (3.11)

where $G$ is the centroid of $\Box$.

**Lemma 3.2.** Let $\Box$ be a square with the boundary length $h$ (Fig. 3), and $\Box$ be split into two non-empty sub-regions $S_1$ and $S_2$ in Fig. 5. Also assume $f \in C(\Box)$, then

$$\int \int_{S_1} f \, d\zeta \, d\eta - f(G)h^2 = -\theta f(P_0^*)h^2 + O(h^4),$$  \hspace{1cm} (3.12)

and

$$\int \int_{S_2} f \, d\zeta \, d\eta = \theta f(P_0^*)h^2,$$  \hspace{1cm} (3.13)

where the centroid $G \in S_1$, but

$$G \notin S_2, \; P_0^* \in S_2, \; \theta \in (0, \frac{1}{2}), \; \text{and} \; h \ll 1.$$  \hspace{1cm} (3.14)
Proof. Based on the mean-value theorem of integrals, we have
\[
\int \int_{S_1} f \, d\xi \, d\eta = \int \int_{S_1} f(P_0^*) \, |S_2| = \theta f(P_0^*) h^2,
\]
where \( \theta = |S_2|/|\Box| < \frac{1}{2} \). Also from Lemma 3.1 and (3.15), we obtain
\[
\int \int_{S_1} f \, d\xi \, d\eta - f(G) h^2 = \left( \int \int_{\Box} f \, d\xi \, d\eta - f(G) h^2 \right) - \int \int_{S_1} f \, d\xi \, d\eta
\]
\[= -\theta f(P_0^*) h^2 + O(h^4). \]  
(3.16)
This completes the proof of Lemma 3.2. \( \square \)

Lemma 3.3. Let \( T \) be regular and \( N = 2^p = H/h \). Then the sub-pixel numbers \( N_A \) and \( N_{BJC} \) in Cases A and \( B \cup C \) have the following bounds:
\[
N_A \leq \frac{1}{\mathcal{J}_0} N^2, \quad N_{BJC} \leq \frac{2\sqrt{2}N}{|\lambda_{\min}(P)|}, \tag{3.17}
\]
where \( \mathcal{J}_0 \) is the lower bound of the Jacobian determinant, and \( \lambda_{\min}(P) \) is the minimal eigenvalue in magnitude of the Jacobian matrix \( P \), given in (2.10).

Proof. We have from (2.11), (3.6) and (3.7)
\[
H^2 = \int \int_{\Box, U} dx \, dy \geq \sum_{ij, kl \text{ Case A}} \int \int_{\Box_{ij, kl}} dx \, dy
\]
\[= \sum_{ij, kl \text{ Case A}} \int \int_{\Box_{ij, kl}} f \, d\xi \, d\eta \geq \mathcal{J}_0 \sum_{ij, kl \text{ Case A}} \sum |\Box_{ij, kl}| \geq \mathcal{J}_0 h^2 N_A.
\]
So \( N_A \leq H^2/(\mathcal{J}_0 h^2) = N^2/\mathcal{J}_0 \).
Next, since \( \Delta \xi = \Delta \eta = h \) and
\[
\begin{pmatrix}
\Delta x \\
\Delta y
\end{pmatrix} = P \begin{pmatrix}
\Delta \xi \\
\Delta \eta
\end{pmatrix} = h P \begin{pmatrix}
1 \\
1
\end{pmatrix}, \tag{3.18}
\]
where \( P \) is the Jacobian matrix given in (2.10), we have \( \sqrt{(\Delta x)^2 + (\Delta y)^2} \geq |\lambda_{\min}(P)| \sqrt{2} h \). Hence the transformed sub-pixels \( \Box_{ij, kl}^* \) have the maximal intersection length along \( x \) and \( y \), given by
\[
\text{length}_x(\Box_{ij, kl}^*) \geq \sqrt{2} |\lambda_{\min}(P)| h. \]
Then we obtain
\[
N_{BJC} = 2[N_x(B \cup C) + N_y(B \cup C)]
\]
\[\leq 2 \left[ \frac{H}{\text{length}_x(\Box_{ij, kl}^*)} + \frac{H}{\text{length}_y(\Box_{ij, kl}^*)} \right]
\leq \frac{4 H}{\sqrt{2} h |\lambda_{\min}(P)|} \geq 2\sqrt{2} N \frac{1}{|\lambda_{\min}(P)|}.
\]
This completes the proof of Lemma 3.3. \( \square \)
Lemma 3.4. Denote by $\lambda_1$ and $\lambda_2$ the two eigenvalues of the Jacobian matrix $P$. Then for the regular transformation $T$, there exist the constants $C_0$ and $C_1$ such that

$$0 < C_0 \leq |\lambda_i| \leq C_1, \quad i = 1, 2,$$

(3.19)

where $C_0$ and $C_1$ are two bounded constants.

Proof. Since the determinant $J = \lambda_1 \cdot \lambda_2$, we have $\lambda_1 \cdot \lambda_2 \leq J \leq C$ due to $x, y \in C^2(\Omega)$. If $\lambda_i$ and $\lambda_2$ are a pair of the conjugate eigenvalues,

$$J_0 < |\lambda_1|^2 = |\lambda_2|^2 = |J| \leq J_M < C.$$ 

So Eq. (3.19) holds true. Next, if $\lambda_i$ are real, they must have the same sign due to (2.11). Since $\lambda_1 + \lambda_2$ is the trace of matrix $P$, which is a continuous function on a compact set, we have

$$|\lambda_i| \leq |\lambda_1| + |\lambda_2| = |\lambda_1 + \lambda_2| = \left| \frac{\partial x}{\partial \xi} + \frac{\partial y}{\partial \eta} \right| \leq \left| \frac{\partial x}{\partial \xi} \right| + \left| \frac{\partial y}{\partial \eta} \right| < C.$$

Consequently, we have $|\lambda_i| = |J|/|\lambda_2| \geq J_0/C = C_0 > 0$. Similarly, $|\lambda_2| \geq C_0$. This completes the proof of Lemma 3.4.  

Comparing with the assumptions of regular transformations given in [45], we eliminate the quasi-uniform sub-pixels $\Box_{ij,kl}$. In fact, such a quasi-uniform assumption is guaranteed by Lemma 3.4. This is also an improvement in analysis of SSM.

3.2. Main results of error bounds

The interpolation function using (2.4) satisfies $\hat{\phi} \in D^\mu(\Omega)$, $\mu=0,1$. Let $N$ be even, then for $\mu=0,1$ we have

$$\hat{\phi}(\xi, \eta) \in C^2(\square_{ik,kl}).$$

(3.20)

Then, it follows from (3.6)–(3.9) that $\hat{\phi} \in C^2(\hat{\Omega}_{ij})$, $\hat{\phi} \in D^\mu(\partial \hat{\Omega}_{ij})$, where the collection of small pixel regions is

$$\hat{\Omega}_{ij} = \bigcup_{ij,kl, \text{Case A}} \Box_{ij,kl}, \quad \partial \hat{\Omega}_{ij} = \bigcup_{ij,kl, \text{Cases B,C}} \Box_{ij,kl}.$$ 

(3.21)

Now, we come to a position to explore the error bounds of the greyness obtained from SSM, in particular, the involved constants will be specified (refer to [45]).

Theorem 3.5. Let $T$ be regular, $N(=H/h)$ even, $\Phi_{ij}$ given, and (3.10) hold. Also assume

$$x(\xi, \eta), y(\xi, \eta) \in C^3(\Omega),$$

(3.22)
and the pixel greyness is computed in Step 3 by (2.17), the splitting-shooting method. Then for 
\( \mu = 0, 1 \) there exist the bounds

\[
\hat{e}_{ij} = |\tilde{B}_{ij} - \tilde{B}_{ij}^{(N)}| 
\leq e_{ff} = \frac{h^2}{12} \left( \frac{1}{2} \phi \mathcal{J} \right)_{2, \infty, \Omega_{ij}} + \sqrt{2} \frac{h}{H} \left( \frac{1}{\lambda_{\min}(P)} \right) |\phi \mathcal{J}|_{0, \infty, \overline{\Omega}_{ij}} + O(h^3),
\]  

(3.23)

where

\[
\tilde{B}_{ij} = \frac{1}{H^2} \int \int_{\Omega_{ij}} \tilde{b} \, dx \, dy = \frac{1}{H^2} \int \int_{\Omega_{ij}} \tilde{\phi}_{\mu} \mathcal{J} \, d\xi \, d\eta.
\]  

(3.24)

In (3.24), \( \Omega_{ij} \rightarrow \square_{ij} \), and \( \tilde{\phi}_{\mu} = \tilde{\phi}_{\mu}(\xi, \eta) \) (\( \mu = 0, 1 \)).

**Proof.** By considering different cases of \( \square_{ij} \) in (3.6)–(3.9), we have

\[
\hat{e}_{ij} = \frac{1}{H^2} \left| \int \int_{\Omega_{ij}} \tilde{\phi} \mathcal{J} \, d\xi \, d\eta - \left( \frac{h}{H} \right)^2 \sum_{v(2.20)} \tilde{\phi}(\tilde{G}) \mathcal{J}(\tilde{G}) \right| \leq R_1 + R_{II} + R_{III},
\]  

(3.25)

where

\[
R_1 = \frac{1}{H^2} \sum_{ij, kl \text{ Case A}} \left| \int \int_{\square_{ij} \cap \Omega_{ij}} \tilde{\phi} \mathcal{J} \, d\xi \, d\eta - h^2 \tilde{\phi}(\tilde{G}) \mathcal{J}(\tilde{G}) \right|,
\]  

(3.26)

\[
R_{II} = \frac{1}{H^2} \sum_{ij, kl \text{ Case B}} \left| \int \int_{\square_{ij} \cap \Omega_{ij}} \tilde{\phi} \mathcal{J} \, d\xi \, d\eta - h^2 \tilde{\phi}(\tilde{G}) \mathcal{J}(\tilde{G}) \right|,
\]  

(3.27)

\[
R_{III} = \frac{1}{H^2} \sum_{ij, kl \text{ Case C}} \left| \int \int_{\square_{ij} \cap \Omega_{ij}} \tilde{\phi} \mathcal{J} \, d\xi \, d\eta \right|.
\]  

(3.28)

By means of Lemma 3.1, (3.20) and (3.17) we have the bounds

\[
R_1 \leq \frac{1}{H^2} \sum_{ij, kl \text{ Case A}} \frac{h^4}{12} |\tilde{\phi} \mathcal{J}|_{2, \infty, \square_{ij}, kl} = \frac{1}{12} \sum_{ij, kl \text{ Case A}} \frac{h^4}{H^2} N_{\lambda} |\tilde{\phi} \mathcal{J}|_{2, \infty, \overline{\Omega}_{ij}} \leq \frac{1}{12} \frac{h^2}{J_0} |\tilde{\phi} \mathcal{J}|_{2, \infty, \overline{\Omega}_{ij}},
\]  

(3.29)

Next, from Lemmas 3.2 and 3.3, it follows that

\[
R_{II} + R_{III} \leq \frac{h^2}{2H^2} \sum_{ij, kl \text{ Cases B, C}} (|\tilde{\phi}|_{0, \infty, \square_{ij}, kl} + O(h^4))
\leq \frac{h^2}{2H^2} N_{\lambda}(\tilde{\phi}|_{0, \infty, \square_{ij}, kl} + O(h^4))
\]
Combining (3.29) and (3.30) yields the defined results (3.33). This completes the proof of Theorem 3.5, by noting that $|\lambda_{\min}(P)|$ has a positive lower bound following from Lemma 3.4. \hfill \Box

\section*{Theorem 3.6} Let all conditions in Theorem 3.5 hold true. There exist the bounds for the absolute errors (3.2) by Steps 2 and 3 in Fig. 1. Then for $\mu = 0, 1$

$$E_\mu(B) = E_\mu^N(B)$$

$$\leq E_M = \frac{1}{3} H^2 |b|_{2, \infty, S} + \frac{1}{12} \int_0 \left| \phi \right|_{2, \infty, \tilde{Q}} + \sqrt{2} \frac{h}{H} \frac{1}{|\lambda_{\min}(P)|} \left| \phi \right|_{0, \infty, \tilde{O}} + O(h^3),$$

where $\alpha \geq 1$ and $S$ is the image region of $\tilde{Z}$,

$$S = \bigcup_U \tilde{O}_U, \tilde{Q} = \bigcup_U \tilde{O}_U, \text{ and } \tilde{O} = \bigcap_U \tilde{O}_U.$$  \(\Box\)

\textbf{Proof.} From the definition in (3.2) and Minkowski’s inequality for $x \geq 1$,

$$E_\mu(B) = \left( \frac{1}{M} \sum_U |B| - \tilde{B}^U |^x \right)^{1/x}$$

$$\leq \left( \frac{1}{M} \sum_U |B| - \tilde{B}^U |^2 \right)^{1/2} + \left( \frac{1}{M} \sum_U |\tilde{B}^U |^2 \right)^{1/2},$$

where $B$ and $\tilde{B}^U$ are defined in (2.8) and (3.24), respectively.

Similarly by noting the piecewise interpolation $\tilde{B}^U$ of $b$, we obtain from calculus

$$\left| \int \int_U (\tilde{b}(x, y) - \tilde{B}^U(x, y)) \, dx \, dy \right| \leq d_\mu H^2 |b|_{2, \infty, \Box_U},$$

where $d_0 = \frac{1}{12}$, and $d_1 = \frac{1}{3}$. Hence, we have

$$\left( \sum_U |B| - \tilde{B}^U |^2 \right)^{1/2} \leq \left( \sum_U \left( \frac{1}{H^2} \left| \int \int_U (b(x, y) - \tilde{B}^U(x, y)) \, dx \, dy \right| \right) \right)^{1/2}$$

$$\leq \frac{1}{3} H^2 M^{1/2} |b|_{2, \infty, S}.$$  \(\Box\)

Third, by applying of Theorem 3.5 and (3.32), we obtain

$$\left( \sum_U |\tilde{B}^U |^x \right)^{1/2} \leq M^{1/2} \left\{ \frac{1}{12} \int_0 \left| \phi \right|_{2, \infty, \tilde{O}} + \sqrt{2} \frac{h}{H} \frac{1}{|\lambda_{\min}(P)|} \left| \phi \right|_{0, \infty, \tilde{O}} + O(h^3) \right\}.$$  \(\Box\)

The desired results (3.31) follows from (3.33)-(3.36). This completes the proof of Theorem 3.6.
Theorem 3.7. Let (3.1), \( N = 2^p = H/h \) and all conditions in Theorem 3.6 hold. Then for \( \mu = 0,1 \) the sequential errors (3.4) have the following bounds

\[
\Delta E_s^{(N)}(B) \leq \Delta E_{st} = \frac{5}{12} h^2 J_0 \|\hat{\phi}J\|_{L^2,\infty,\hat{\alpha}} + \frac{3\sqrt{2}}{H} \frac{h}{|\lambda_{\min}(P)|} \|\hat{\phi}\|_{L^2,\infty,\hat{\alpha}}, \quad \alpha \geq 1. \tag{3.37}
\]

Proof. From (3.4) we have Theorem 3.5

\[
\Delta E_s^{(N)}(B) = \left( \frac{1}{M} \sum_{U} |\tilde{B}^{(N)}_{U} - \tilde{B}^{(N-1)}_{U}|^2 \right)^{1/2}
\leq \left\{ \frac{1}{M} \sum_{U} |\tilde{B}^{(N)}_{U} - B_{U}|^2 \right\}^{1/2} + \left\{ \frac{1}{M} \sum_{U} |\tilde{B}^{(N-1)}_{U} - B_{U}|^2 \right\}^{1/2}
\leq \frac{1}{12} J_0 (h^2 + h_1^2) \|\hat{\phi}J\|_{L^2,\infty,\hat{\alpha}} + \frac{\sqrt{2}}{H} \frac{(h + h_1)}{|\lambda_{\min}(P)|} \|\hat{\phi}\|_{L^2,\infty,\hat{\alpha}}, \tag{3.38}
\]

where \( h = H/N_p, \) \( h_1 = H/N_{p-1} = H/N_{p/2} = 2h. \) The conclusion (3.36) follows; the proof of Theorem 3.7 is complete. \( \square \)

Based on Theorems 3.5 and 3.7, we have the following corollary.

Corollary 3.8. Let all conditions in Theorem 3.7 hold. Then when \( N \rightarrow \infty, \) there exist the asymptotic relations for \( \mu = 0,1 \)

\[
E_s^{(N)}(\tilde{B}) = O(H^2) + O\left(\frac{H^2}{N^2}\right) + O\left(\frac{1}{N}\right) \quad \text{and} \quad \Delta E_s^{(N)}(\tilde{B}) = O\left(\frac{H^2}{N^2}\right) + O\left(\frac{1}{N}\right), \tag{3.39}
\]

where \( \alpha \geq 1. \)

Although the convergence rates of the greyness under \( T \) obtained from the SSM are the same as Li [45], the constants involved in the convergence orders in Theorems 3.5–3.7 are specified and the proofs here are also simpler.

4. Probabilistic error analysis for the splitting-shooting method

The convergence rate established in Corollary 3.8 is based on the most conservative error analysis. It may only occur in the worst situation: either only the Case B or only the Case C for all sub-pixels. In this situation, the image transformation \( T \) must map straight lines parallel to axes in \( \xi \circ \eta \) to straight lines parallel to axes in \( XOY. \) This means that the transformation \( T \) must be a composition of a translation and an enlargement. However, such a simplest image transformation is not considered in practical image processing. Because the image transformation \( T \) of practical interest is factually nonlinear, the worst situation is in fact not realistic. Moreover, due to the cancellation of errors between Cases B and C, the real convergence rates should be much higher than what has been established in Section 3 and in [45, 35]. This has been shown by our tremendous examples.
of numerical computation. Our experience tells that the actual convergence rates in SSM and CSIM are very close to $O(N^{-1.5})$. This is a result surprisingly matching with a result of the law of large numbers and the central limit theorem in probability theory, provided that the errors for all sub-pixels of Cases B and C are independent, and identically and uniformly distributed over the interval $[-h^2/2, h^2/2]$.

In this section, we shall quote some basic results of limiting theorems in probability theory. Then, by assuming the errors caused in sub-pixels in Cases B and C are independent, identically and uniformly distributed over the interval $[-h^2/2, h^2/2]$, and by applying those limiting results of probability theory, we shall give a randomized error analysis on the performance of the SSM. By the duality CSIM of the positive image transform $T$ and its inverse transformation $T^{-1}$, the same results for SSM are true for the CSIM.

4.1. Probability inequalities for sum of independent random variables

The probabilistic evaluation of image greyness errors are discussed in Theorems 4.1–4.3 below under $T$ by the SSM and under $T^{-1}T$ by the CSIM. There occur three sums of independent random variables,

$$
\sum_{i=1}^{n} X_i, \left( \frac{1}{M} \sum_{i=1}^{M} |X_i|^2 \right)^{1/2}, \left( \frac{1}{M} \sum_{i=1}^{M} |X_i|^2 \right)^{1/2} + \left( \frac{1}{M} \sum_{i=1}^{M} |Y_i|^2 \right)^{1/2},
$$

where $\alpha \geq 1$, $X_i$ and $Y_i$ are independent variables with zero expectation and finite variances. We first discuss the most interesting case as $\alpha = 1$ and then in Section 4.3 as $\alpha > 1$. The first sum model of random variables are studied in many reports, e.g., [31, 57, 74], etc., and summarized in the recent monograph of Petrov [57]. Also probability analysis has been already applied to numerical methods (see [67]). In this section, however, we will derive rather different inequalities to evaluate errors in probability for all those three random models. Compared to the existing probability literatures, the new inequalities in this paper are countable with all the explicit constants, and then well suited to numerical computations. Moreover, the new inequalities given in Theorems 4.1–4.3, 4.7, 4.8 below are adapted not only to the limit theorems as done in many papers cited in [57], but also to small $n$ in practical application of image computation, i.e., transformations of binary images.

We now quote basic inequalities in probability theory [5, 57, 65]:

**Lemma 4.1** (Chebyshev inequality). Let $X$ be a random variable with finite variance $\sigma^2$. Then,

$$
P(|X - E(X)| \geq \varepsilon) \leq \frac{\sigma^2}{\varepsilon^2}, \quad \varepsilon > 0,
$$

where $E(X)$ denotes the expectation of $X$.

**Lemma 4.2** (The central limit theorem). Let $X_1, X_2, \ldots, X_n$ be independent random variables with Mean $= (1/n) \sum_{i=1}^{n} E(X_i)$, the bounded variance $V(X_i) = \sigma_i^2$ and $\sigma = \sqrt{\sum_{i=1}^{n} \sigma_i^2/n}$. Then for any
real $k$, the limit
\[
\lim_{n \to \infty} F_n = \lim_{n \to \infty} P \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left( \frac{X_i - \text{Mean}}{\sigma} \right) \leq k \right) = \Phi(k),
\]
where $\Phi(x)$ is the distribution function of the standard normal random variable:
\[
\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \, dt.
\]

Lemma 4.3 (Berry–Esseen inequality). Let $X_1, X_2, \ldots, X_n$ be independent random variables with zero expectations and finite variances $\sigma_1^2, \ldots, \sigma_n^2$. Write $V_n^2 = \sum_{i=1}^{n} \sigma_i^2 = n\sigma^2$. Denote by $F_n$ the distribution function of the normalized variable. Then,
\[
\sup_x |F_n(x) - \Phi(x)| \leq \tilde{c} V_n^{-3} \sum_{i=1}^{n} E|x_i|^3,
\]
where $\tilde{c}$ is an absolute constant, currently known as
\[
\frac{1}{\sqrt{2\pi}} < \tilde{c} < 0.7975.
\]

We now prove a lemma representing the function (4.3).

Lemma 4.4. The normal distribution function (4.3) is expressed by the simple functions:
\[
\Phi(x) = 1 - \sqrt{\frac{2}{\pi}} \int_{x + \sqrt{x^2 + \theta_0}}^{\infty} e^{-t^2/2} \, dt = 1 - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} > 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x}, \quad x > 0,
\]
where $8/\pi \leq \theta_0 < 4$, and $0 < \theta_1 < 1$.

Proof. From the formulas in [1, p. 298]:
\[
\frac{1}{x + \sqrt{x^2 + 2}} e^{x^2} \int_{x}^{\infty} e^{-r^2} \, dr \leq \frac{1}{x + \sqrt{x^2 + 4/\pi}}, \quad x \geq 0,
\]
we have
\[
\int_{x}^{\infty} e^{-r^2} \, dr = \frac{e^{-x^2}}{x + \sqrt{x^2 + \theta_2}},
\]
where $4/\pi \leq \theta_2 < 2$. Also it follows from (4.3) and (4.8)
\[
1 - \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-r^2/2} \, dr = \frac{1}{\pi} \int_{\sqrt{x}/\sqrt{2}}^{\infty} e^{-t^2} \, dt = \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{\pi x + \sqrt{x^2 + 2\theta_2}}.
\]
This is the first result in (4.6) if letting $\theta_0 = 2\theta_2$. The other result in (4.6) follows by using the following two inequalities
\[
x + \sqrt{x^2 + \theta_0} > 2x, \quad x + \sqrt{x^2 + \theta_0} \leq 2(x + 1), \quad \forall x > 0.
\]
This completes the proof of Lemma 4.8.
Now we prove an important inequality.

**Theorem 4.5.** Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with a common probability distribution function $F(x)$, and finite mean and variance be

$$E(X_i) = 0, \quad V(X_i) = \sigma^2, \quad \forall i \tag{4.10}$$

where $\sigma$ is a positive constant. Then, for any constant $k > 0$,

$$P(|S_n| < k \sigma \sqrt{n}) \geq 1 - P_{k,n} \tag{4.11}$$

where $S_n = \sum_{i=1}^{n} X_i$, and the explicit constant

$$P_{k,n} = \min \left\{ \frac{1}{k^2}, \sqrt{\frac{2}{\pi}} \frac{e^{-k^2/2}}{k} + \frac{1.6 \sigma |X_1|^3}{\sqrt{n}} \right\}. \tag{4.12}$$

**Proof.** Based on the Chebyshev inequality, we have

$$P \left( \left| \frac{1}{n} S_n \right| \geq \varepsilon \right) = P(|S_n| \geq n \varepsilon) \leq \frac{\sigma^2}{n \varepsilon^2}. \tag{4.13}$$

Let $\varepsilon = k \sigma / \sqrt{n}$, then $P(|S_n| \geq k \sigma \sqrt{n}) \leq 1/k^2$, i.e.,

$$P(|S_n| < k \sigma \sqrt{n}) \geq 1 - \frac{1}{k^2}. \tag{4.14}$$

Next, the central limit theorem

$$\lim_{n \to \infty} F_n = \lim_{n \to \infty} P \left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \text{Mean}}{\sigma} \leq k \right) = \Phi(k), \tag{4.15}$$

and the Berry–Esseen inequality leads to

$$|F_n - \Phi(k)| \leq \frac{c\sigma |X_1|^3}{\sigma^3 \sqrt{n}}. \tag{4.16}$$

Therefore, when Mean $= 0$, we have

$$P \left( \frac{1}{\sqrt{n}} \frac{S_n}{\sigma} \leq k \right) \geq \Phi(k) - \frac{c\sigma |X_1|^3}{\sigma^3 \sqrt{n}}, \tag{4.17}$$

and then

$$P(|S_n| < k \sigma \sqrt{n}) \geq 2\Phi(k) - 1 - \frac{2c\sigma |X_1|^3}{\sigma^3 \sqrt{n}}. \tag{4.18}$$

Also from (4.6),

$$1 - \Phi(x) < \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \quad x > 0. \tag{4.19}$$
This leads to
\[ \Phi(x) > 1 - \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x}. \] (4.20)

Hence, Eq. (4.18) is reduced to
\[ P(|S_n| \leq k\sigma\sqrt{n}) > 1 - \frac{\sqrt{2} e^{-k^2/2}}{\sqrt{\pi} k} - \frac{2\sigma E|X_1|^3}{\sigma^3 \sqrt{n}}. \] (4.21)

The desired results (4.12) follow from (4.14), (4.21) and (4.5), and the proof of Theorem 4.5 is complete. □

When \( k \) and \( n \) are given, the constant \( P_{k,n} \) in (4.12) can be evaluated directly for the known random models. Note that when \( n \) is small, and \( k \) (\( > 1 \)) is not large, the Chebyshev inequality offers better estimates of \( P_{k,n} \). This is why we combine both the Chebyshev and the Berry–Esseen inequalities. It is due to the rather small ranges of \( \bar{c} \) and \( \theta_1 \) in (4.5) and (4.6), the counted probability from Theorem 4.5 is meaningful and useful.

More sophisticated probability inequalities can be found in [57], which are mainly devoted to the limit theorems as \( n \to \infty \) and \( k \to \infty \). Unfortunately, those inequalities are not always countable because some bounded constants therein are still unknown, even in a certain range.

The global mean greyness errors in (3.2) are most significant in image computation, which involve the random models, ((1/M)\( \sum_{i=1}^{M} |X_i|^p \))\(^{1/p} \). We will first derive a probability inequality, Theorem 4.6 as \( z = 1 \), then apply it to provide the probabilistic error bounds.

**Theorem 4.6.** Let \( X_1, X_2, \ldots, X_M \) be independent random variables in the normal distribution with
\[ E(X_i) = 0, \quad V(X_i) = \sigma_i^2. \] (4.22)

Then for \( k > 0 \) there exist the probability bounds:
\[ P \left( \bar{X}_M \leq \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi - 2}{\pi}} \frac{k}{\sqrt{M}} \right) \sigma \right) > 1 - P_{k,M}, \] (4.23)

where
\[ \bar{X}_M = \frac{1}{M} \sum_{i=1}^{M} |X_i|, \quad \sigma = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \sigma_i^2}, \]
\[ P_{k,M} = \min \left\{ \frac{1}{k^2}, \frac{e^{-k^2/2}}{\sqrt{2\pi} k} + \frac{b_0}{\sqrt{M}} \right\}, \quad b_0 = (\max \bar{c}) \left( \frac{\pi}{\pi - 2} \right)^{1.5} \frac{2\sqrt{2}}{\sqrt{\pi}} = 5.81. \] (4.24)

**Proof.** The absolute values \( |X_i| \) are also independent random variables, with a common probability density function:
\[ f_i(x) = \sqrt{\frac{2}{\pi \sigma}} e^{-x^2/2\sigma^2}, \quad 0 \leq x < \infty, \quad \sigma > 0. \] (4.25)
From this, it is not difficult to derive their expectation and variance, given as

$$\text{Mean} = E(|X_1|) = \int_0^\infty xf_1(x) \, dx = \sqrt{\frac{2}{\pi}} \sigma,$$  \hspace{1cm} (4.26)

$$\sigma_1^2 = E((|X_1| - \text{Mean})^2) = \left(1 - \frac{2}{\pi}\right) \sigma^2.$$  \hspace{1cm} (4.27)

Also, we have

$$E(|X_1|^3) = \frac{2\sqrt{2}}{\sqrt{\pi}} \sigma^3.$$  \hspace{1cm} (4.28)

Based on the central limit theorem (4.15), we have

$$\lim_{M \to \infty} F_M^* = \lim_{M \to \infty} P \left( \frac{1}{\sqrt{M}} \sum_{i=1}^M \frac{|X_i| - \text{Mean}}{\sigma_1} \leq k \right) = \Phi(k).$$  \hspace{1cm} (4.29)

Also from (4.26) and (4.27),

$$F_M^* = P \left( \bar{X}_M \leq \text{Mean} + \frac{k\sigma_1}{\sqrt{M}} \right) = P \left( \bar{X}_M \leq \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi - 2}{\pi} \frac{k}{\sqrt{M}}} \right) \sigma \right).$$  \hspace{1cm} (4.30)

Next, by the Berry–Esseen inequality (4.16) and (4.27), (4.28), we obtain

$$|F_M^* - \Phi(k)| \leq \frac{\varepsilon E|X_1|^3}{\sigma_1^3 \sqrt{M}} = b_0 \frac{1}{\sqrt{M}}.$$  \hspace{1cm} (4.31)

where $b_0$ is given in (4.24). Combining (4.29)–(4.31) yields

$$P \left( \bar{X}_M \leq \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi - 2}{\pi} \frac{k}{\sqrt{M}}} \right) \sigma \right) \geq \Phi(k) - \frac{b_0}{\sqrt{M}}.$$  \hspace{1cm} (4.32)

Also following the proofs in Theorem 4.1, we apply (4.14) to get

$$P \left( \bar{X}_M < \text{Mean} + \frac{k\sigma_1}{\sqrt{M}} \right) > 1 - \frac{1}{k^2}.$$  \hspace{1cm} (4.33)

Hence, we obtain from (4.32) and (4.33) and Lemma 4.3

$$P \left( \bar{X}_M \leq \left( \sqrt{\frac{2}{\pi}} + \sqrt{\frac{\pi - 2}{\pi} \frac{k}{\sqrt{M}}} \right) \sigma \right) \geq 1 - P_{k,M},$$  \hspace{1cm} (4.34)

where

$$P_{k,M} = \min \left\{ \frac{1}{k^2}, 1 - \Phi(k) + \frac{b_0}{\sqrt{M}} \right\} = \min \left\{ \frac{1}{k^2}, \frac{1}{\sqrt{2\pi} \frac{1}{k}} + \frac{b_0}{\sqrt{M}} \right\}.$$  \hspace{1cm} (4.35)

This completes the proof of Theorem 4.6. □
To evaluate the probabilistic sequential errors (3.4) as $x=1$, we now consider the sum of independent random variables

$$Z_M = \frac{1}{M} \sum_{i=1}^{M} (|X_i| + |Y_i|),$$

(4.36)

and prove the following theorem.

**Theorem 4.7.** Let $X_i$ and $Y_i$, $i = 1, 2, \ldots, M$, be independent random variables in the normal distribution with zero expectation and bounded variance:

$$E(X_i) = 0, \quad E(Y_i) = 0, \quad V(X_i) = \sigma^2_i(X), \quad V(Y_i) = \sigma^2_i(Y).$$

Then, there exist the probabilistic error bounds

$$P \left( Z_M \leq \sqrt{\frac{2}{\pi}} (\sigma_x + \sigma_y) + \sqrt{\frac{\pi - 2}{\pi}} \frac{k}{\sqrt{M}} \sqrt{\sigma^2_x + \sigma^2_y} \right) \geq 1 - P_{k,M},$$

(4.37)

where $P_{k,M}$ is given by (4.24), and

$$\sigma_x = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \sigma^2_i(X)}, \quad \sigma_y = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \sigma^2_i(Y)}.$$

**Proof.** Write $\text{Mean}(|X|) = (1/M) \sum_{i=1}^{M} |X_i|$ and

$$(\sigma^*)^2 = \frac{1}{2M} \sum_{i=1}^{M} \{(|X_i| - \text{Mean}(X))^2 + (|Y_i| - \text{Mean}(Y))^2\}.$$

Applying the central limit theorem to $X_i$ and $Y_i$, we obtain

$$\lim_{M \to \infty} F^{**}_{2M} = \Phi(k),$$

(4.38)

where

$$F^{**}_{2M} = P \left( \frac{1}{\sqrt{2M} \sigma^*} \sum_{i=1}^{M} \{(|X_i| - \text{Mean}(X)) + (|Y_i| - \text{Mean}(Y))\} \leq k \right).$$

By noting the definition of $Z_M$ in (4.36), we obtain from (4.26) and (4.27)

$$F^{**}_{2M} = P \left( Z_M \leq \text{Mean}(|X|) + \text{Mean}(|Y|) + \frac{\sqrt{2k\sigma^*}}{\sqrt{M}} \right)$$

$$= P \left( Z_M \leq \sqrt{\frac{2}{\pi}} (\sigma_x + \sigma_y) + \sqrt{\frac{\pi - 2}{\pi}} \frac{k}{\sqrt{M}} \sqrt{\sigma^2_x + \sigma^2_y} \right).$$

(4.39)
Next, applying the Berry–Esseen inequality to $X_i$ and $Y_i$ leads to
\[
|P_{2M}^{**} - \Phi(k)| \leq \frac{\bar{c} \sum_{i=1}^{M} (E|X_i|^3 + E|Y_i|^3)}{(2M\sigma^*)^{3/2}}.
\]
From (4.27) and (4.28),
\[
|P_{2M}^{**} - \Phi(k)| \leq \frac{\bar{c}M^{2/3}}{M^{3/2} \left(\frac{n-2}{n}\right)^{1/2}} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{M}} (\sigma_x^2 + \sigma_y^2)^{3/2} = \bar{c} \left(\frac{\pi}{\pi-2}\right) 2\sqrt{2} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{M}} (\sigma_x^2 + \sigma_y^2)^{3/2} \\
\leq (\max \bar{c}) \left(\frac{\pi}{\pi-2}\right) 2\sqrt{2} \frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{M}} = b_0 \frac{1}{\sqrt{M}}.
\]
In (4.40), we have used the following inequality $\sigma_x^3 + \sigma_y^3 \leq (\sigma_x^2 + \sigma_y^2)^{3/2}$. By following the proof of Theorem 4.6, (4.39) and (4.40), the desired results (4.37) are obtained. This completes the proof of Theorem 4.7.

4.2. Probabilistic error bounds in 1-norms

Now, we are in position to establish the probabilistic error bounds of $|\hat{B}_U - \hat{B}_U^{(N)}|$ and $E_1(\hat{B})$ in (3.2) and $\Delta E_1(\hat{B})$ in (3.4) for the simple case $\alpha = 1$.

Assume the factor $\theta$ in Lemma 3.2 is a random variable. In fact, for sub-pixels $\Box_{ij,kl} \rightarrow \Box_{ij,kl}^+$ in Case B, the contribution to $B_{ij}$ of $\Box_{ij,kl}$ is over-estimated, and for these sub-pixels in Case C, the contribution of $\Box_{ij,kl}$ to $B_{ij}$ is ignored. Denote
\[
\delta = \pm \theta = \pm |S_i|/h^2 \in (-\frac{1}{2}, \frac{1}{2}).
\]
As the rounding errors, we may assume that the random variable $\delta = \pm \theta$ of $\Box_{ij,kl}$ satisfies a uniform distribution. Then we have
\[
E(\delta) = 0, \quad \sigma_0^2 = V(\delta) = \frac{1}{12}, \quad E(|\delta|^3) = \frac{1}{32}.
\]
We have the following lemma by means of Theorem 4.4.

**Lemma 4.8.** Assume that the random variable $\delta = \pm \theta_{ij,kl}$ satisfies the uniform distribution on $(-\frac{1}{2}, \frac{1}{2})$. Then, when $N$ is large and $H$ is small, there exist the bounds for $\mu = 0,1$

\[
P \left( |R_{II,III}| \leq \frac{\sqrt{2}}{\sqrt{6} \sqrt{k_{\min}(P)}} \left( \frac{h}{H} \right)^{1.5} |\hat{\phi} f|_{0, \infty, \partial Q_{ij}} \right) \geq 1 - P_{k,N},
\]
where
\[
R_{II,III} = \left( \frac{1}{H^2} \sum_{ij,kl} \int_{C_{ij,kl}^* \cap C_{ij}} \hat{\phi} f \, d\xi \, d\eta - h^2 \hat{\phi}(\hat{G}) \hat{f}(\hat{G}) \right) + \frac{1}{H^2} \sum_{ij,kl} \int_{C_{ij,kl} \cap C_{ij}} \hat{\phi} f \, d\xi \, d\eta,
\]
(4.43)
and

\[
\bar{\theta}_{k,N} = \min \left\{ \frac{1}{k^2}, \sqrt{\frac{2}{\pi}} \frac{e^{-k^2/2}}{k} + \frac{a_0}{\sqrt{N}} \sqrt{\lambda_{\min}(P)} \right\},
\]

(4.45)

\[
a_0 = 1.6 \times \frac{3}{4} \times \frac{\sqrt{3}}{\sqrt{2\sqrt{2}}} = 1.2.
\]

(4.46)

**Proof.** From Lemma 3.2, we have

\[
R_{II+III} = \frac{1}{H^2} \sum_{ij,kl} \theta_{ij,kl} h^2 |\hat{\phi} \mathcal{J}|_{ij,kl} + \frac{1}{H^2} \sum_{ij,kl} (-\theta_{ij,kl}) h^2 |\hat{\phi} \mathcal{J}|_{ij,kl}.
\]

(4.47)

Since, when \(N\) is large and \(H\) is small,

\[
|\hat{\phi} \mathcal{J}|_{ij,kl} \approx |\hat{\phi} \mathcal{J}|_{0,\infty, \delta_{ij}}, \quad \forall ij,kl.
\]

(4.48)

We obtain

\[
R_{II+III} \leq \Pi \frac{h^2}{H^2} |\hat{\phi} \mathcal{J}|_{0,\infty, \delta_{ij}},
\]

(4.49)

where

\[
\Pi_{ij} = \sum_{ij,kl} \delta_{ij,kl}, \quad \delta_{ij,kl} = \pm \theta_{ij,kl}.
\]

(4.50)

Based on the assumption on the random variable \(\delta = \delta_{ij,kl}\), we have from Theorem 4.5 and Lemma 3.3, and (4.42)

\[
P(|\Pi| \leq k \sigma \sqrt{N_{BUC}}) = P \left( |\Pi_{ij}| \leq \frac{k}{\sqrt{12}} \sqrt{\frac{2\sqrt{2N}}{\lambda_{\min}(P)}} \right) \geq 1 - \bar{\theta}_{k,N_{BUC}}. \]

(4.51)

This implies from (4.49) and \(N = H/h\) that

\[
P \left( |R_{II+III}| \leq \frac{\sqrt{2}}{\sqrt{6}} \frac{k}{\sqrt{\lambda_{\min}(P)}} \left( \frac{h}{H} \right)^{1.5} |\hat{\phi} \mathcal{J}|_{0,\infty, \delta_{ij}} \right) \geq 1 - \bar{\theta}_{k,N_{BUC}}. \]

(4.52)

The constant \(\bar{\theta}_{k,N}\) in (4.45) is obtained again from Theorem 4.5 and Lemma 3.3 and (4.42). This completes the proof of Lemma 4.8. \(\square\)

Now we prove a main theorem.

**Theorem 4.9.** Under the conditions in Theorem 3.1 and Lemma 4.8, the greyness errors have the following probabilistic error bounds:

\[
P(|\hat{B}_{ij} - \hat{B}^{(N)}_{ij}| \leq \varepsilon_1) \geq 1 - \bar{\theta}_{k,N},
\]

(4.53)
\[ P(\varepsilon_1 < |\tilde{B}_{IJ} - \tilde{B}_{IJ}^{(N)}| \leq \varepsilon_{st}) \leq \tilde{P}_{k,N}, \quad (4.54) \]

where \( \varepsilon_{st} \) is given in Theorem 3.5 and

\[ \varepsilon_1 = \frac{1}{12} \frac{h^2}{\mathcal{F}_0} |\hat{\phi}_f|_{\infty,2,\tilde{\alpha}_t} + \frac{\sqrt{2}}{6} \frac{k}{\lambda_{\min}(P)} \left( \frac{h}{H} \right)^{1.5} |\hat{\phi}_f|_{0,\infty,\tilde{\alpha}_t}. \quad (4.55) \]

**Proof.** Since

\[ |\tilde{B}_{IJ} - \tilde{B}_{IJ}^{(N)}| \leq R_1 + |R_{II+III}|, \quad (4.56) \]

the first bounds (4.53) are obtained from (3.29) and Lemma 4.8. The second bounds (4.54) follows from Theorem 3.5 immediately.

Now we prove the another important theorem by means of Theorem 4.6.

**Theorem 4.10.** Let the conditions in Theorem 4.9 hold. Assume that the random variables \((R_{II+III})_U\) in (4.44) are normally distributed. Then, for \( \mu = 0, 1 \) there exist bounds in probability for the absolute errors (3.2) as \( \alpha = 1 \):

\[ P(E_1^{(N)}(\tilde{B}) \leq \varepsilon_2) \geq 1 - P_{k,M}, \quad (4.57) \]
\[ P(\varepsilon_2 \leq E_1^{(N)}(\tilde{B}) \leq E_{st}) \leq P_{k,M}, \quad (4.58) \]

where \( M \) is the pixel number of the image \( Z \), \( E_{st} \) is given in Theorem 3.6 and

\[ \varepsilon_2 = \frac{1}{3} H^2 |b|_{2,\infty,S} + \frac{1}{12} \frac{h^2}{\mathcal{F}_0} |\hat{\phi}_f|_{2,\infty,\hat{a}} \]
\[ + \left( \frac{2}{\sqrt{\pi}} + \sqrt{\frac{\pi - 2}{\pi}} \frac{k}{\sqrt{M}} \right) \frac{\sqrt{2}}{\sqrt{6} \lambda_{\min}(P)} \left( \frac{h}{H} \right)^{1.5} |\hat{\phi}_f|_{0,\infty,\hat{a}}. \quad (4.59) \]

**Proof.** From (3.33) and (3.25),

\[ E_1(\tilde{B}) \leq \varepsilon_A + \tilde{X}_M, \quad (4.60) \]

where

\[ \varepsilon_A = \frac{1}{M} \sum_{IJ} \{|B_{IJ} - \tilde{B}_{IJ}| + (R_I)_U\}, \quad \tilde{X}_M = \frac{1}{M} \sum_{IJ} |R_{II+III}|_{IJ}. \]

From Theorem 3.6 we have

\[ \varepsilon_A \leq \frac{1}{3} H^2 |b|_{2,\infty,S} + \frac{1}{12} \frac{h^2}{\mathcal{F}_0} |\hat{\phi}_f|_{2,\infty,\hat{a}}. \quad (4.61) \]

Based on the normality assumption, for the random variables \((R_{II+III})_U\) we have

\[ E((R_{II+III})_U) = 0, \quad V((R_{II+III})_U) = \sigma^2_{U}. \]
where
\[ \sigma_{ij} = \frac{\sqrt{2}}{\sqrt{6} \lambda_{\min}(P)} \left( \frac{h}{H} \right)^{1.5} \phi_j |_{0, \infty, \partial \Omega} \]

Hence, we have
\[ \sigma = \sqrt{\frac{1}{M} \sum_{i,j} \sigma_{ij}^2} = b_1 \left( \frac{h}{H} \right)^{1.5}, \tag{4.62} \]

where the constant \( b_1 \) has the bounds
\[ b_1 \leq \frac{\sqrt{2}}{\sqrt{6} \lambda_{\min}(P)} \phi_j |_{0, \infty, \partial \Omega}. \tag{4.63} \]

The desired results (4.57) are obtained from Theorem 4.6 and (4.62), and Eq. (4.58) follows from (4.57) and Theorem 3.6. This completes the proof of Theorem 4.10. \( \Box \)

Since \( \sqrt{\sigma_x^2 + \sigma_y^2} \leq \sigma_x + \sigma_y \), we obtain the following theorem directly from Theorem 4.7 where the sum of independent of random variables are suited to estimate (3.4) as \( \alpha = 1 \).

**Theorem 4.11.** Let (3.1), \( N = 2^p = H/h \) and all conditions in Theorem 4.10 hold. Then for \( \mu = 0,1 \) there exist the probabilistic error bounds for the sequential errors (3.4) as \( \alpha = 1 \):
\[ P(\Delta E_1^{(N)}(\tilde{B}) \leq \varepsilon_3) \geq 1 - P_{k,M}, \quad P(\varepsilon_3 < \Delta E_1^{(N)}(\tilde{B}) \leq \Delta E_m) \leq P_{k,M}, \]

where \( P_{k,M} \) and \( \Delta E_m \) are given in Theorem 4.6 and Theorem 3.7,
\[ \varepsilon_3 = \frac{5}{12} h^2 \phi_0 |_{0, \infty, \partial \Omega} + b_3 \left( \frac{2}{\sqrt{\pi}} + \left( \frac{\pi - 2}{\pi} + \frac{k}{\sqrt{M}} \right) \left( \frac{h}{H} \right)^{1.5} \phi_j |_{0, \infty, \partial \Omega} \right) \]
and \( b_3 = \sqrt{2}(1 + 2\sqrt{2})/\sqrt{6} = 1.85 \).

First we note that in Theorems 4.9–4.11, the conservative error bounds in Section 3 may occur only in a very small possibility. For instance, in our computational samples in Section 6, \( N \) varies from 1 to 256, but \( M \) is rather large, as \( M \approx 10^3 - 10^5 \). When \( k = 3 \), we have
\[ \tilde{P}_{k,N} = \min \left\{ \frac{0.11, 1.2 \sqrt{\lambda_{\min}(P)}}{\sqrt{N}} \right\}, \quad P_{k,M} = \min \left\{ \frac{0.11, 5.83}{\sqrt{M}} + 0.0015 \right\}. \]

We may increase \( k \) to decrease \( \tilde{P}_{k,N} \) and \( P_{k,M} \). Based on Theorems 4.9–4.11, we conclude that
\[ |\tilde{B}_{ij} - \tilde{B}_{ij}^{(N)}| = O(N^{-3/2}) \quad \text{in probability}, \tag{4.64} \]
which will be simply denoted by \( O_p(N^{-3/2}) \) in the remainder of this paper. Similarly,
\[ |E_1^{(N)}(\tilde{B})| = O(H^2) + O_p(N^{-3/2}), \quad |\Delta E_1^{(N)}(\tilde{B})| = O_p(N^{-3/2}). \tag{4.65} \]
The assumption of the normal distribution is requested in Theorems 4.10 and 4.11, which have been verified perfectly in the computation given in Section 6.2 below. More justification on the statistical assumption is all provided there.

For simplicity, only the convergence rates $O_p(N^{-3/2})$ of image greyness are discussed. In fact the almost sure convergence rates $O(N^{-3/2}\sqrt{\log N})$ a.s. can be achieved as in [3], based on Bernstein inequality [31, 57, 74], Bikels inequality and a renovated Bikels inequality in [5, 57]. Details appear elsewhere. Although the Bernstein inequality is also countable, when $k$ and $n$ are not large, the probability evaluated is not better than that evaluated by Theorem 4.5. However, the same orders of convergence rates as in (4.64) and (4.65) can also be achieved by other probability inequalities in [57, 65].

4.3. Further analysis on probability error bounds in $\alpha (\geq 1)$ norms

In this subsection, we discuss the general sums of random variables

\[
\tilde{X}_{M,\alpha} = \left( \frac{1}{M} \sum_{i=1}^{M} |X_i|^\alpha \right)^{1/\alpha}, \quad \alpha \geq 1
\]  
(4.66)

\[
\tilde{Y}_{M,\alpha} = \left( \frac{1}{M} \sum_{i=1}^{M} |X_i|^\alpha \right)^{1/\alpha} + \left( \frac{1}{M} \sum_{i=1}^{M} |Y_i|^\alpha \right)^{1/\alpha}, \quad \alpha \geq 1.
\]  
(4.67)

The cases of $\alpha = 1$ are discussed in Sections 4.1 and 4.2. First we obtain the following lemma by calculus.

Lemma 4.12. Let the random variable $X$ be a normal distribution with zero expectation, then

\[
E(|X|^\alpha) = \frac{1}{\sqrt{\pi}}(\sqrt{2\sigma})^\alpha \Gamma \left( \frac{\alpha + 1}{2} \right), \quad \alpha \geq 1,
\]  
(4.68)

where $\Gamma(z)$ is the Gamma function defined by

\[
\Gamma(z) = \int_0^\infty t^{z-1}e^{-t} \, dt, \quad z > 0.
\]  
(4.69)

Now we prove an important theorem.

Theorem 4.13. Let $X_1, X_2, \ldots, X_M$ be independent random variables in the normal distribution with

\[
E(X_i) = 0, \quad V(X_i) = \sigma_i, \quad \sigma = \sqrt{\frac{1}{M} \sum_{i=1}^{M} \sigma_i^2}.
\]

Then for any $k > 0$, there exist the probability bounds for (4.66)

\[
P \left( \tilde{X}_{M,\alpha} \leq \sqrt{2\sigma} \left( \frac{1}{\pi} \Gamma \left( \frac{\alpha + 1}{2} \right) + \beta_\alpha \frac{k}{\sqrt{M}} \right)^{1/\alpha} \right) > 1 - P_{k,M,\alpha},
\]  
(4.70)
where $\alpha \geq 1$,

$$P_{k,M,z} = \min \left\{ \frac{1}{k^2}, \frac{k}{\sqrt{2\pi k}} + \frac{k}{\sqrt{M}} \right\},$$  \hspace{1cm} (4.71)$$

$$\beta_z = \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{2\alpha + 1}{2} \right) - \frac{1}{\pi} \Gamma^2 \left( \frac{\alpha + 1}{2} \right),$$  \hspace{1cm} (4.72)$$

$$\gamma_z = \frac{c}{\sqrt{\pi}} \Gamma \left( \frac{3\alpha + 1}{2} \right) / \beta_z^{1.5}.$$  \hspace{1cm} (4.73)$$

**Proof.** By applying the central limit theorem and the Berry–Esseen inequality to the random variables $|X_i|^z$, we have

$$\lim_{M \to \infty} F^*_{M,z} = \Phi(k),$$  \hspace{1cm} (4.74)$$

$$|F^*_{M,z} - \Phi(k)| \leq \frac{\delta E(|X|^3)}{\sigma^2 \sqrt{M}},$$  \hspace{1cm} (4.75)$$

where $\sigma^2$ is the variance of $|X|^z$, and

$$F^*_{M,z} = P \left( \frac{1}{\sqrt{M}} \left( \sum_{i=1}^{M} |X_i|^z - \frac{E(|X|^z)}{\sigma_z} \right) \leq k \right).$$  \hspace{1cm} (4.76)$$

Hence, it follows from (4.76)

$$F^*_{M,z} = P \left( \frac{1}{\sqrt{M}} \sum_{i=1}^{M} |X_i|^z \leq E(|X|^z) + \frac{k\sigma_z}{\sqrt{M}} \right) = P \left( \tilde{X}_{M,z} \leq \left( E(|X|^z) + \frac{k\sigma_z}{\sqrt{M}} \right)^{1/\alpha} \right),$$  \hspace{1cm} (4.77)$$

where $E(|X|^z)$ is given in Lemma 4.12. Next since the variance $\sigma^2 = E(|X|^2) - [E(|X|^z)]^2$, we have from Lemma 4.12,

$$\sigma^2 = (\sqrt{2}\sigma)^2 \left( \frac{1}{\sqrt{\pi}} \Gamma \left( \frac{2\alpha + 1}{2} \right) - \frac{1}{\pi} \Gamma^2 \left( \frac{\alpha + 1}{2} \right) \right).$$  \hspace{1cm} (4.78)$$

Also from Lemma 4.12,

$$E(|X|^3) = \frac{1}{\sqrt{\pi}} (\sqrt{2}\sigma)^3 \Gamma \left( \frac{3\alpha + 1}{2} \right).$$  \hspace{1cm} (4.79)$$

By following the proof of Theorem 4.6, the desired probability bounds (4.70) are obtained. \f

**Theorem 4.14.** Let all conditions in Theorem 4.7 hold. Then for any $k > 0$, there exist the probability bounds for (4.67)

$$P \left( \tilde{Y}_{M,z} \leq 2^{(z-1)/2} \sqrt{2(z + 1)} \right) \left\{ \frac{1}{\Gamma \left( \frac{\alpha + 1}{2} \right)} + \beta_z \frac{k}{\sqrt{M}} \right\}^{1/\alpha} \geq P_{k,M,z},$$  \hspace{1cm} (4.80)$$

where $P_{k,M,z}$ and $\beta_z$ are given in Theorem 4.13, and $\sigma_x$ and $\sigma_y$ in Theorem 4.7.
Proof. From the Holder inequality as $\alpha \geq 1$, $x^{1/\alpha} + y^{1/\alpha} \leq 2^{1-1/\alpha}(x + y)^{1/\alpha} = 2^{(\alpha-1)/\alpha}(x + y)^{1/\alpha}$, we first apply Theorem 4.13 to the joining random variables $|X|^\alpha$ and $|Y|^\alpha$, and then obtain (4.80) by following the arguments in Theorem 4.7. This completes the proof of Theorem 4.14. \hfill \Box

Now we apply Theorems 4.13 and 4.14 to the errors (3.2) and (3.4), to obtain the following theorems without proofs.

Theorem 4.15. Let all conditions in Theorem 4.10 hold. Then for $\mu = 0, 1$,

$$P(E^{(N)}(\tilde{B}) \leq \epsilon_{4, x}) \geq 1 - P_{k, M, x}, \quad P(\epsilon_{4, x} \leq E^{(N)}(\tilde{B}) \leq E_{s1}) \leq P_{k, M, x},$$

where $\alpha \geq 1$, and

$$\epsilon_{4, x} = \frac{1}{3} H^2 |b|_{t, \infty, \Omega}^2 + \frac{1}{12} \hat{\phi} |J_0|_{t, \infty, \Omega}^2 \hat{\phi} + \beta_{z, k} \frac{k}{\lambda_{\min}(P)} \left( \frac{h}{H} \right)^{1.5} \hat{\phi} |J_{0, \infty, \Omega}^2 \hat{\phi}$$

and $P_{k, M, x}$ and $\beta_{z}$ are given in (4.71) and (4.72).

Theorem 4.16. Let all conditions in Theorem 4.11 hold. Then for $\mu = 0, 1$, then

$$P(\Delta E^{(N)}_x(\tilde{B}) \leq \epsilon_{5, x}) \geq 1 - P_{k, M, x}, \quad P(\epsilon_{5, x} \leq \Delta E^{(N)}_x(\tilde{B}) \leq \Delta E_{s1}) \leq P_{k, M, x},$$

where $\alpha \geq 1$, and

$$\epsilon_{5, x} = \frac{5}{12} H^2 |b|_{t, \infty, \Omega}^2 + b_{4, x} \left( \frac{1}{\lambda_{\min}(P)} \right) \frac{1}{\lambda_{\min}(P)} \left( \frac{h}{H} \right)^{1.5} \hat{\phi} |J_{0, \infty, \Omega}^2 \hat{\phi}$$

with $b_{4, x} = \frac{3\sqrt{2}}{\sqrt{3}} (1 + 2\sqrt{2}) 2^{(x-1)/2}$.

When $x$ is integer, the constants $\beta_{z}$ and $\gamma_{z}$ in (4.72) and (4.73) are explicit. By noting the formulas of the Gamma functions, $\Gamma(x + 1) = x \Gamma(x)$ and $\Gamma(1/2) = \sqrt{\pi}$, we obtain the countable values of $\beta_{z}$ and $\gamma_{z}$.

1. When $x$ is odd, $x = 2l + 1, l = 0, 1, \ldots$,

$$\beta_{2l+1} = \frac{(4l + 1)!!}{2(2l+1)} - \frac{1}{\pi} \left( \frac{1}{2} \right)^{2}, \quad \gamma_{2l+1} = \frac{\tilde{\epsilon}}{\sqrt{\pi}} (3l + 1)! / \beta_{2l+1}^{1.5}.$$

2. When $x$ is even, $x = 2l, l = 1, 2, \ldots$,

$$\beta_{2l} = \frac{1}{4l} \left( (4l - 1)!! - (2l - 1)!! \right)^2, \quad \gamma_{2l} = \frac{\tilde{\epsilon}}{\sqrt{\pi}} \left( (4l - 1)!! - (2l - 1)!! \right)^{1.5} / \left( (2l - 1)!! \right)^{1.5}.$$

Here the notation $(2l + 1)! = (2l - 1) \cdot (2l - 1) \cdots 3 \cdot 1$. For the important cases, $x = 1, 2$, we have

$$\beta_1 = \frac{1}{2} - \frac{1}{\pi}, \quad \gamma_1 = b_0; \quad \beta_2 = \frac{1}{2}, \quad \gamma_2 = \frac{15\tilde{\epsilon}}{2\sqrt{2\pi}} \leq 2.39,
where \( b_0 \) is given in (4.24). It is easy to see that the probability bounds of \( E_1(\tilde{B}) \) and \( \Delta E_1(\tilde{B}) \) are special cases as \( \alpha = 1 \). Since \( E_2(\tilde{B}) \) and \( \Delta E_2(\tilde{B}) \) are useful in computation (see [45]), we give the following corollary.

**Corollary 4.17.** Let all conditions in Theorems 4.13–4.16 hold. When \( \alpha = 2 \),

\[
P \left( \bar{X}_{M,2} \leq \sigma \left( \frac{1}{\sqrt{\pi}} + \frac{k}{\sqrt{M}} \right)^{1/2} \right) > 1 - P_{k,M,2},
\]

\[
P \left( \bar{Y}_{M,2} \leq \sqrt{2} (\sigma_x + \sigma_y) \left( \frac{1}{\sqrt{\pi}} + \frac{k}{\sqrt{M}} \right)^{1/2} \right) > 1 - P_{k,M,2},
\]

and

\[
P (E_2^{(N)}(\tilde{B}) \leq \varepsilon_{4,2}) \geq 1 - P_{k,M,2}, \quad P (\Delta E_2^{(N)}(\tilde{B}) \leq \varepsilon_{5,2}) \geq 1 - P_{k,M,2},
\]

where

\[
\varepsilon_{4,2} = \frac{1}{3} H^2 |b|_{2,\infty,3} + \frac{h^2}{12} \left| \hat{\phi} \mathcal{F}_{0,\infty,\tilde{\alpha}} - \frac{\sqrt{2}}{\sqrt{3}} \left( \frac{1}{\sqrt{\pi}} + \frac{k}{\sqrt{M}} \right)^{1/2} \frac{1}{|\lambda_{\min}(P)|} \left( \frac{h}{H} \right)^{1.5} \right| \hat{\phi} \mathcal{F}_{0,\infty,\bar{\alpha}},
\]

\[
\varepsilon_{5,2} = \frac{5}{12} h^2 \left| \hat{\phi} \mathcal{F}_{0,\infty,\tilde{\alpha}} + b_{4,2} \right| \left( \frac{1}{\sqrt{\pi}} + \frac{k}{\sqrt{M}} \right)^{1/2} \frac{1}{|\lambda_{\min}(P)|} \left( \frac{h}{H} \right)^{1.5} \right| \hat{\phi} \mathcal{F}_{0,\infty,\bar{\alpha}},
\]

where \( b_{4,2} = (\sqrt{2} \sqrt{2}/\sqrt{3})(1 + 2\sqrt{2}) = 3.72 \).

It is interesting to note that when \( \alpha \) changes, the probability errors and bounds vary, but not their strict error bounds in Section 3.

5. Partial refinements of pixel partitions in SSM

5.1. Partial refinement technique

To achieve the desired convergence rates \( O_p(1/N^2) \), we adopt the new partial refinement techniques proposed in this section, accompanied with brief analysis. Simulation results for the new techniques and their performances will be discussed in the last section. It is worth pointing out that to achieve \( O_p(1/N^2) \), higher than \( O_p(1/N^{1.5}) \), the additional CPU time needed is moderate. Based on the analysis on Sections 3 and 4, the smaller greyness errors with the convergence rate \( O(1/N^2) \) come from Case A, than that with \( O(1/N) \) or \( O_p(1/N^{1.5}) \) from Cases B \( \cup \) C. To reduce the global greyness errors, we had better balance the two resource errors from Case A and Cases B \( \cup \) C. Consequently, we should distinguish them, and choose a smaller boundary length \( h_1 \) of sub-pixels in Cases B \( \cup \) C, i.e.,

\[
h_1 < h.
\]

Then we are able to reduce the global greyness errors to \( O_p(1/N^2) \). An illustration of partial refinement is shown in Fig. 6. Classification of sub-pixels into Case A and Cases B \( \cup \) C is easy to be
Fig. 6. A sketch diagram of the refinement partition of pixels.

done by checking whether or not the four corner points of \( \Box_{ij,kl} \) fall into the same region \( \Box_{ij} \) (see (2.20)).

For simplicity, let \( \alpha = 1 \). By following the lines in Sections 3 and 4, we obtain the following theorem.

**Theorem 5.1.** Let \( h \) and \( h_1 \) be the boundary lengths of pixels \( \Box_{ij,kl} \) in Case A and Cases B \( \cup \) C respectively. Assume all conditions in Theorem 4.9 hold, then

\[
P(|\hat{B}_{ij} - \hat{B}_{ij}^{(N)}| \leq \varepsilon_1^*) \geq 1 - P_{k,N} \tag{5.2}
\]

and

\[
P(\varepsilon_2^* < |\hat{B}_{ij} - \hat{B}_{ij}^{(N)}| \leq \varepsilon_2^* ) \leq 1 - P_{k,N}, \tag{5.3}
\]

where \( P_{k,N} \) is given in (4.12),

\[
\varepsilon_1^* = \varepsilon_1(h, h_1) = \frac{1}{12} \int_0^h |\hat{\phi} \mathcal{J}|_{\infty,2,\bar{\alpha}_0} + \frac{\sqrt{2}}{6} \sqrt{k} \frac{h_1}{\lambda_{\min}(P)} \left( \frac{h_1}{H} \right)^{1.5} |\hat{\phi} \mathcal{J}|_{0,\infty,\bar{\alpha}_0}, \tag{5.4}
\]

\[
\varepsilon_2^* = \frac{1}{12} \int_0^h |\hat{\phi} \mathcal{J}|_{\infty,2,\bar{\alpha}_0} + \frac{\sqrt{2}h_1}{H} \left( \frac{1}{\lambda_{\min}(P)} \right) |\hat{\phi} \mathcal{J}|_{0,\infty,\bar{\alpha}_0}. \tag{5.5}
\]

We take as an example the errors in (5.2). Since the same conclusions can be made if using those for \( E_2(\hat{B}) \) and \( \Delta E_2(\hat{B}) \). Now, let us choose \( h_1 \) in Cases B \( \cup \) C to balance \( O(h^2) \) in Case A. Based on Theorem 5.1, let

\[
h_1^{1+\beta} = O(h^2), \quad \beta = 0 \text{ for (5.5) or } \frac{1}{2} \text{ for (5.4)}, \tag{5.6}
\]
where \( h = H/N \) and \( h_1 = H/(N \times N_1) \). The integer number \( N_1 (\geq 1) \) is the refinement number for the sub-pixels in Cases \( B \cup C \). Hence, we have from (5.6) \( N_1 = O(N^{(1-\beta)/(1+\beta)}) \), when \( \beta = 0 \) and \( \beta = \frac{1}{2} \). Then

\[
N_i = O(N) \quad \text{and} \quad N_i = O(N^{1/3}). \tag{5.7}
\]

Once the sub-pixels \( \Box_{i,j,k} \) are recognized to be as in Cases \( B \) and \( C \), they will again divided into \( N_1 \) sub-pixels, e.g., \( N = 4 \) and \( N_1 = 4 \) in Fig. 6.

Next, let us count flops for CPU time needed by the refinement technique. The total number of flops is

\[
\text{Tot.flops} = F_{\text{case } A} + F_{\text{cases } B \cup C} = O(N^2) + O(N_{B \cup C}) = O(N^2) + O(N \times N^2). \tag{5.8}
\]

Hence, when \( N_1 = O(N) \),

\[
\text{Tot.flops} = O(N^2) + O(N^3) \tag{5.9}
\]

and when \( N_1 = O(N^{1/3}) \),

\[
\text{Tot.flops} = O(N^2) + O(N^{5/3}). \tag{5.10}
\]

Note that for the refinement as \( N_1 = O(N^{1/3}) \), order \( O(N^{5/3}) \) of the flops in Cases \( B \cup C \) is lower than \( O(N^2) \) in Case \( A \). We write down this important result as a corollary.

**Corollary 5.2.** Let all the conditions in Theorems 5.1 and 4.9–4.11 hold, and assume the refinement number \( N_1 = O(N^{1/3}) \), there exist the asymptotes:

\[
|\bar{B}_{i,j} - \tilde{B}_{i,j}^{(N)}| = O_p(1/N^2), \quad |\Delta E_1^{(N)}(B)| = O_p(1/N^2), \tag{5.11}
\]

\[|E_1^{(N)}(B)| = O(H^2) + O_p(1/N^2). \tag{5.12}
\]

When \( N_1 = O(N) \), it can be seen from (5.8) and (5.10) that the convergence rate is only \( O_p(1/N^{4/3}) \) per \( O(N^2) \) flops. Hence such a choice of \( N_1 \) should be avoided in practice.

The important alternative of \( N_1 \) is \( N_1 = O(N^{1/2}) \). Since \( \text{Tot.flops} = O(N^2) \), we have found the better convergence rates in strict error analysis:

\[
|\Delta E_1^{(N)}(B)| = O(h_1) = O(h^{1.5}) = O\left(\frac{1}{N^{1.5}}\right), \tag{5.13}
\]

those in probability

\[
|\Delta E_1^{(N)}(B)| = O_p(1/N^2). \tag{5.14}
\]

A summary of the above important results is given in Table 1, for three types of refinements by

\[
(N,N_1) = (N,N), \quad (N,\sqrt{N}), \quad (N,\sqrt{N}), \tag{5.15}
\]

where \( h = H/h_1 \), \( h_1 = H/N \times N_1 \), and \( N_1 \) is the number of local refinement for Cases \( B \) and \( C \). A significant contribution of this paper is that the convergence rates raise from \( O(1/N) \) in [45] to the convergence rate \( O_p(1/N^2) \) in probability.
Table 1
Summary on convergence rates of sequential greyness by SSM using the three refinement techniques

<table>
<thead>
<tr>
<th>Refinements</th>
<th>CPU time</th>
<th>Strict error analysis</th>
<th>Probabilistic error analysis</th>
<th>Computational results</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Conv. rate per (O(N^2)) flops</td>
<td>Conv. rate in prob. per (O(N^2)) flops</td>
<td>Conv. rate in prob. per (O(N^2)) flops</td>
</tr>
<tr>
<td>(N)</td>
<td>(N_1)</td>
<td>Flops</td>
<td>Conv. rate in prob.</td>
<td>Conv. rate in prob.</td>
</tr>
<tr>
<td>(N)</td>
<td>(N)</td>
<td>(O(N^3))</td>
<td>(O(N^{1/2}))</td>
<td>(O_p\left(\frac{1}{N^{1/2}}\right))</td>
</tr>
<tr>
<td>(N)</td>
<td>(\sqrt{N})</td>
<td>(O(N^2))</td>
<td>(O(N^{1/2}))</td>
<td>(O_p\left(\frac{1}{N^{1/2}}\right))</td>
</tr>
<tr>
<td>(N)</td>
<td>(\sqrt{N})</td>
<td>(O(N^2))</td>
<td>(O(N^{1/2}))</td>
<td>(O_p\left(\frac{1}{N^{1/2}}\right))</td>
</tr>
</tbody>
</table>

5.2. Error analysis on CSIM

As in Section 2.4, the SSM using the partial refinement technique may join the splitting-integrating method again to form the CSIM for \(T^{-1}T\). Since the convergence rate \(O(1/N^2)\) is proven in [45] for the splitting-integrating method for the images under \(T^{-1}\) as \(\mu = 1\), the final convergence rate \(O_p(1/N^2)\) can also maintain by CSIM as \(\mu = 1\). The detailed proofs are omitted here because they may follow Sections 3 and 4 and [45].

To choose this section, we merely discuss CSIM in Case II, where the evaluation \(\Delta B_U = |\tilde{B}^U_{II} - \tilde{B}_U|\) where \(\tilde{B}^U_{II}\) is given in (2.25), involving the rounding effect. In fact, the rounding errors wherein are as a random variable on a uniform distribution on \((-1/2(q-1), 1/2(q-1))\). Hence, from (4.42) and Theorem 4.5 we have the following corollary.

Corollary 5.3. The greyness errors by CSIM in Case II, resulting from (2.25) have the bounds

\[
P\left(\frac{1}{M} \sum_U \Delta B_U \leq \frac{k}{\sqrt{12M(q-1)}}\right) \geq 1 - P_{k,M}. \tag{5.16}
\]

Corollary 5.3 implies that when \(M\) is large, the differences between Cases I and II in CSIM are insignificant, having the error order is \(O_p(1/(\sqrt{M}(q-1)))\), much better than \(O(1/(q-1))\) in [45].

6. Numerical and graphical experiments

In this section, we carry out the numerical and graphical experiments to verify the convergence rates \(O_p(1/N^{15})\), and \(O_p(1/N^2)\) by using the refinement technique. To support the assumption of random variables in Cases B \(\cup\) C, we will also evaluate in Section 6.2 their errors by using the solutions of the advanced SSM in [50].

6.1. Binary images

For simplicity, we first choose binary images and the quadratic transformation in [45], since binary images have tremendous applications in image processing and pattern recognition (see [22, 25, 35]).
Table 2
The sequential errors $\Delta E$ and their rates as well as the absolute errors $E$ under $T$ and $T^{-1}T$ by SSM and CSIM in Cases I and II as $\mu = 0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T$</th>
<th>$T^{-1}T$</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta E$</td>
<td>Ratio</td>
<td>$\Delta E$</td>
<td>Ratio</td>
</tr>
<tr>
<td>2</td>
<td>0.9501</td>
<td>/</td>
<td>0.9607</td>
<td>/</td>
</tr>
<tr>
<td>4</td>
<td>0.2811</td>
<td>3.38</td>
<td>0.2637</td>
<td>3.65</td>
</tr>
<tr>
<td>8</td>
<td>0.8396$\times 10^{-1}$</td>
<td>3.35</td>
<td>0.6356$\times 10^{-1}$</td>
<td>4.15</td>
</tr>
<tr>
<td>16</td>
<td>0.3245$\times 10^{-1}$</td>
<td>2.59</td>
<td>0.2602$\times 10^{-1}$</td>
<td>2.44</td>
</tr>
<tr>
<td>32</td>
<td>0.1200$\times 10^{-1}$</td>
<td>2.70</td>
<td>0.8470$\times 10^{-2}$</td>
<td>3.07</td>
</tr>
<tr>
<td>64</td>
<td>0.4084$\times 10^{-2}$</td>
<td>2.94</td>
<td>0.2656$\times 10^{-2}$</td>
<td>3.19</td>
</tr>
<tr>
<td>128</td>
<td>0.1232$\times 10^{-2}$</td>
<td>3.31</td>
<td>0.8602$\times 10^{-3}$</td>
<td>3.09</td>
</tr>
<tr>
<td>256</td>
<td>0.4113$\times 10^{-3}$</td>
<td>3.00</td>
<td>0.3457$\times 10^{-3}$</td>
<td>2.49</td>
</tr>
</tbody>
</table>

Table 3
The sequential errors $\Delta E$ and their rates as well as the absolute errors $E$ under $T$ and $T^{-1}T$ by SSM and CSIM in Cases I and II as $\mu = 1$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T$</th>
<th>$T^{-1}T$</th>
<th>Case I</th>
<th>Case II</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta E$</td>
<td>Ratio</td>
<td>$\Delta E$</td>
<td>Ratio</td>
</tr>
<tr>
<td>2</td>
<td>0.8352</td>
<td>/</td>
<td>0.5051</td>
<td>/</td>
</tr>
<tr>
<td>4</td>
<td>0.2546</td>
<td>3.28</td>
<td>0.1282</td>
<td>3.94</td>
</tr>
<tr>
<td>8</td>
<td>0.7607$\times 10^{-1}$</td>
<td>3.35</td>
<td>0.3839$\times 10^{-1}$</td>
<td>3.34</td>
</tr>
<tr>
<td>16</td>
<td>0.2946$\times 10^{-1}$</td>
<td>2.58</td>
<td>0.1509$\times 10^{-1}$</td>
<td>2.54</td>
</tr>
<tr>
<td>32</td>
<td>0.1093$\times 10^{-1}$</td>
<td>2.70</td>
<td>0.5116$\times 10^{-2}$</td>
<td>2.95</td>
</tr>
<tr>
<td>64</td>
<td>0.3757$\times 10^{-2}$</td>
<td>2.91</td>
<td>0.1587$\times 10^{-2}$</td>
<td>3.22</td>
</tr>
<tr>
<td>128</td>
<td>0.1169$\times 10^{-2}$</td>
<td>3.21</td>
<td>0.4383$\times 10^{-3}$</td>
<td>3.62</td>
</tr>
<tr>
<td>256</td>
<td>0.3877$\times 10^{-3}$</td>
<td>3.02</td>
<td>0.1698$\times 10^{-3}$</td>
<td>2.58</td>
</tr>
</tbody>
</table>

Tables 2 and 3 list the greyness errors and the ratios of sequential errors for $\mu = 0, 1$ in Cases I and II. For the errors in (3.2) and (3.4), we only compute the sequential errors for the most important case as $\alpha = 1$. For (3.1), $(N_p/(N_{p-1}))^{1.5} = 2.83$. Most of the ratios in Tables 2 and 3 are close to 2.83. A clear view is also illustrated in Fig. 7; this perfectly verifies the order, $O_p(1/N^{1.5})$.

To test $O_p(1/N^2)$ we choose three types of refinements

$$((N,N_1) = (N,N), (N,\sqrt{N}), (N,\sqrt{N}).$$

The results are listed in Tables 4–11 for $\mu = 0, 1$. In these tables, we also provide the values of

$$pd^2 = \sqrt{\frac{1}{M} \sum_{Cases B_{BC}} \delta_{ij,kl}^2}, \quad \delta_{ij,kl} = \left( \frac{h_i}{H} \right)^2 \phi(G) J(G)_{ij,kl},$$

$$$$

(6.1)
The curves of $\Delta E$, $\Delta E_1(\tilde{B})$ as $\alpha = 1$ and $pd_2$ are drawn in Figs. 7–11. The following asymptotic relations can be found:

$$
\Delta E = \Delta E_1(\tilde{B}) = O_p(1/N^3), \quad \text{pd}_2 = O_p(1/N^3) \quad \text{for } (N,N_1) = (N,N),
$$

$$
\Delta E = \Delta E_1(\tilde{B}) = O_p(1/N^{2.5}), \quad \text{pd}_2 = O_p(1/N^{2.5}) \quad \text{for } (N,N_1) = (N,\sqrt{N}),
$$

$$
\Delta E = \Delta E_1(\tilde{B}) = O_p(1/N^2), \quad \text{pd}_2 = O_p(1/N^2) \quad \text{for } (N,N_1) = (N,\sqrt{N}).
$$
Table 5
The values of \(p^2\), the sequential errors \(\Delta E\) and their ratios, as well as the absolute errors \(E\) under \(T\) and \(T^{-1}T\) by SSM and CSIM in Case I as \(\mu = 0\) by the refinement \((N, N_1) = (N, N)\)

<table>
<thead>
<tr>
<th>Refin.</th>
<th>(T)</th>
<th>(T^{-1}T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>(N_1)</td>
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</tr>
<tr>
<td>2</td>
<td>/</td>
<td>0.9501 /</td>
</tr>
<tr>
<td>4</td>
<td>/</td>
<td>0.2715 3.50</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>0.2855 \cdot 10^{-1} 9.50</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>0.5614 \cdot 10^{-2} 5.09</td>
</tr>
<tr>
<td>32</td>
<td>6</td>
<td>0.1194 \cdot 10^{-2} 4.76</td>
</tr>
<tr>
<td>64</td>
<td>8</td>
<td>0.2021 \cdot 10^{-3} 5.91</td>
</tr>
</tbody>
</table>

Table 6
The values of \(p^2\), the sequential errors \(\Delta E\) and their ratios, as well as the absolute errors \(E\) under \(T\) and \(T^{-1}T\) by SSM and CSIM in Case I as \(\mu = 0\) by the refinement \((N, N_1) = (N, \sqrt{N})\)

<table>
<thead>
<tr>
<th>Refin.</th>
<th>(T)</th>
<th>(T^{-1}T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>(N_1)</td>
<td>(\Delta E)</td>
</tr>
<tr>
<td>2</td>
<td>/</td>
<td>0.9501 /</td>
</tr>
<tr>
<td>4</td>
<td>/</td>
<td>0.2811 3.38</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>0.8004 \cdot 10^{-1} 3.51</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>0.1200 \cdot 10^{-1} 6.67</td>
</tr>
<tr>
<td>32</td>
<td>3</td>
<td>0.3623 \cdot 10^{-2} 3.31</td>
</tr>
<tr>
<td>64</td>
<td>4</td>
<td>0.6479 \cdot 10^{-3} 5.59</td>
</tr>
</tbody>
</table>

Table 7
The values of \(p^2\), the sequential errors \(\Delta E\) and their ratios, as well as the absolute errors \(E\) under \(T\) and \(T^{-1}T\) by SSM and CSIM in Case I as \(\mu = 1\) by the refinement \((N, N_1) = (N, N)\)

<table>
<thead>
<tr>
<th>Refin.</th>
<th>(T)</th>
<th>(T^{-1}T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(N)</td>
<td>(N_1)</td>
<td>(\Delta E)</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.7483 11.2</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>0.6697 \cdot 10^{-1} 11.2</td>
</tr>
<tr>
<td>8</td>
<td>8</td>
<td>0.5906 \cdot 10^{-2} 7.01</td>
</tr>
<tr>
<td>16</td>
<td>16</td>
<td>0.1098 \cdot 10^{-2} 8.71</td>
</tr>
<tr>
<td>32</td>
<td>32</td>
<td>0.1791 \cdot 10^{-3} 6.13</td>
</tr>
</tbody>
</table>

Eq. (6.5) is coincident with the analysis in Sections 4 and 5, and Eqs. (6.3) and (6.4) have even better performance. The computational results (6.3)-(6.5) are also listed in Table 1 for comparison with theoretical analysis.

For binary images, the errors \(\Delta E\) at \(N = 8\) in Tables 2 and 3 are in the range (0.038,0.084). Hence, \(N = 4\) is good enough for practical application (see [35]).
Table 8
The values of \( \Delta E \), the sequential errors and their ratios, as well as the absolute errors \( E \) under \( T \) and \( T^{-1}T \) by SSM and CSIM in Case I as \( \mu = 1 \) by the refinement \((N, N_1) = (N, \sqrt{N})\)

<table>
<thead>
<tr>
<th>Refin.</th>
<th>( T )</th>
<th>( T^{-1}T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>( N_1 )</td>
<td>( \Delta E )</td>
</tr>
<tr>
<td>2</td>
<td>/</td>
<td>0.7639</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>0.2361</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>0.2456 ( \cdot 10^{-1} )</td>
</tr>
<tr>
<td>16</td>
<td>4</td>
<td>0.4931 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>32</td>
<td>6</td>
<td>0.1083 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>64</td>
<td>8</td>
<td>0.1796 ( \cdot 10^{-3} )</td>
</tr>
</tbody>
</table>

Table 9
The values of \( \Delta E \), the sequential errors and their ratios, as well as the absolute errors \( E \) under \( T \) and \( T^{-1}T \) by SSM and CSIM in Case I as \( \mu = 1 \) by the refinement \((N, N_1) = (N, \sqrt{N})\)

<table>
<thead>
<tr>
<th>Refin.</th>
<th>( T )</th>
<th>( T^{-1}T )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>( N_1 )</td>
<td>( \Delta E )</td>
</tr>
<tr>
<td>2</td>
<td>/</td>
<td>0.7639</td>
</tr>
<tr>
<td>4</td>
<td>/</td>
<td>0.2426</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>0.6698 ( \cdot 10^{-1} )</td>
</tr>
<tr>
<td>16</td>
<td>2</td>
<td>0.1039 ( \cdot 10^{-1} )</td>
</tr>
<tr>
<td>32</td>
<td>3</td>
<td>0.3153 ( \cdot 10^{-2} )</td>
</tr>
<tr>
<td>64</td>
<td>4</td>
<td>0.5793 ( \cdot 10^{-3} )</td>
</tr>
</tbody>
</table>

6.2. Probability distribution function

To explore the statistical nature, we evaluate the real errors in Cases B \( \cup \) C

\[
e_{ij} = B_{ij} \big|_{SSM} - B_{ij} \big|_{SSM},
\]

where \( B_{ij} \big|_{SSM} \) are the greyness solutions obtained by the advanced splitting-shooting method in [50] using partition techniques to reach the convergence rate \( O(1/N^2) \). The statistical results are obtained as follows.

1. When \( \mu = 0 \), \( M = 1037 \), \( E(e_{ij}) = -0.1394 \times 10^{-4} \), and

\[
E(\sigma) = \sqrt{\frac{1}{M-1} \sum_{ij} e_{ij}^2} = 0.0913.
\]

2. When \( \mu = 1 \), \( M = 1194 \), \( E(e_{ij}) = 0.1625 \times 10^{-6} \), and \( E(\sigma) = 0.08219 \).

We also count the number of \( e_{ij} \) belonging to the following sub-intervals

\[
0.2l + 0.1 \leq \frac{e_{ij}}{\sigma} \leq 0.2(l + 1) + 0.1, \quad l = 0, \pm 1, \pm 2, \ldots
\]

The values \( l \), \( l/M \) and the integration \( \Phi(x) \) and listed in Tables 10 and 11 for \( \mu = 0, 1 \).
Let us compare the computed function $\tilde{\Phi}(x)$ with the true values $\Phi(x)$, which are cited from Blake [8]. Hence, we can find the following:

\[
\max_i (\Delta \Phi(x_i)) = 0.033, \quad \bar{\sigma} = 0.0099 \quad \text{as} \quad \mu = 0 \quad \text{at} \quad N = 4,
\] (6.9)
Table 11
The distribution of greyness errors $\epsilon_{ij}$ in Cases B $\cup$ C or $\mu = 1$ at $N = 4$

<table>
<thead>
<tr>
<th>$i$</th>
<th>$x$</th>
<th>$l$</th>
<th>$l/M$</th>
<th>$\Phi(x)$</th>
<th>$\Phi_{(x)}^{(x)}$</th>
<th>$\Phi(x) - \Phi_{(x)}^{(x)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-20</td>
<td>-3.90</td>
<td>2</td>
<td>0.00168</td>
<td>0.00168</td>
<td>0.00002</td>
<td>0.00166</td>
</tr>
<tr>
<td>-19</td>
<td>-3.70</td>
<td>2</td>
<td>0.00168</td>
<td>0.00335</td>
<td>0.00005</td>
<td>0.00330</td>
</tr>
<tr>
<td>-18</td>
<td>-3.50</td>
<td>0</td>
<td>0.00000</td>
<td>0.00335</td>
<td>0.00011</td>
<td>0.00324</td>
</tr>
<tr>
<td>-17</td>
<td>-3.30</td>
<td>3</td>
<td>0.00251</td>
<td>0.00586</td>
<td>0.00023</td>
<td>0.00563</td>
</tr>
<tr>
<td>-16</td>
<td>-3.10</td>
<td>2</td>
<td>0.00168</td>
<td>0.00754</td>
<td>0.00048</td>
<td>0.00706</td>
</tr>
<tr>
<td>-15</td>
<td>-2.90</td>
<td>0</td>
<td>0.00000</td>
<td>0.00754</td>
<td>0.00097</td>
<td>0.00657</td>
</tr>
<tr>
<td>-14</td>
<td>-2.70</td>
<td>0</td>
<td>0.00000</td>
<td>0.00754</td>
<td>0.00187</td>
<td>0.00567</td>
</tr>
<tr>
<td>-13</td>
<td>-2.50</td>
<td>5</td>
<td>0.00419</td>
<td>0.01173</td>
<td>0.00347</td>
<td>0.00826</td>
</tr>
<tr>
<td>-12</td>
<td>-2.30</td>
<td>7</td>
<td>0.00586</td>
<td>0.01759</td>
<td>0.00621</td>
<td>0.01138</td>
</tr>
<tr>
<td>-11</td>
<td>-2.10</td>
<td>4</td>
<td>0.00335</td>
<td>0.02094</td>
<td>0.01072</td>
<td>0.01022</td>
</tr>
<tr>
<td>-10</td>
<td>-1.90</td>
<td>8</td>
<td>0.00670</td>
<td>0.02764</td>
<td>0.01786</td>
<td>0.00978</td>
</tr>
<tr>
<td>-9</td>
<td>-1.70</td>
<td>7</td>
<td>0.00586</td>
<td>0.03350</td>
<td>0.02872</td>
<td>0.00478</td>
</tr>
<tr>
<td>-8</td>
<td>-1.50</td>
<td>7</td>
<td>0.00586</td>
<td>0.03936</td>
<td>0.04457</td>
<td>-0.00521</td>
</tr>
<tr>
<td>-7</td>
<td>-1.30</td>
<td>9</td>
<td>0.00754</td>
<td>0.04690</td>
<td>0.06681</td>
<td>-0.01991</td>
</tr>
<tr>
<td>-6</td>
<td>-1.10</td>
<td>22</td>
<td>0.01843</td>
<td>0.06533</td>
<td>0.09680</td>
<td>-0.03147</td>
</tr>
<tr>
<td>-5</td>
<td>-0.90</td>
<td>34</td>
<td>0.02848</td>
<td>0.09380</td>
<td>0.13567</td>
<td>-0.04187</td>
</tr>
<tr>
<td>-4</td>
<td>-0.70</td>
<td>38</td>
<td>0.03183</td>
<td>0.12563</td>
<td>0.18406</td>
<td>-0.05843</td>
</tr>
<tr>
<td>-3</td>
<td>-0.50</td>
<td>75</td>
<td>0.06281</td>
<td>0.18444</td>
<td>0.24196</td>
<td>-0.05352</td>
</tr>
<tr>
<td>-2</td>
<td>-0.30</td>
<td>62</td>
<td>0.05193</td>
<td>0.24037</td>
<td>0.30854</td>
<td>-0.06817</td>
</tr>
<tr>
<td>-1</td>
<td>-0.10</td>
<td>99</td>
<td>0.08291</td>
<td>0.32328</td>
<td>0.38209</td>
<td>-0.05881</td>
</tr>
<tr>
<td>0</td>
<td>0.10</td>
<td>320</td>
<td>0.26801</td>
<td>0.59129</td>
<td>0.53983</td>
<td>0.05146</td>
</tr>
<tr>
<td>1</td>
<td>0.30</td>
<td>125</td>
<td>0.10469</td>
<td>0.69598</td>
<td>0.61791</td>
<td>0.07807</td>
</tr>
<tr>
<td>2</td>
<td>0.50</td>
<td>78</td>
<td>0.06533</td>
<td>0.76131</td>
<td>0.69146</td>
<td>0.06985</td>
</tr>
<tr>
<td>3</td>
<td>0.70</td>
<td>72</td>
<td>0.06030</td>
<td>0.82161</td>
<td>0.75804</td>
<td>0.06357</td>
</tr>
<tr>
<td>4</td>
<td>0.90</td>
<td>52</td>
<td>0.04355</td>
<td>0.86516</td>
<td>0.81594</td>
<td>0.04922</td>
</tr>
<tr>
<td>5</td>
<td>1.10</td>
<td>35</td>
<td>0.02931</td>
<td>0.89447</td>
<td>0.86433</td>
<td>0.03014</td>
</tr>
<tr>
<td>6</td>
<td>1.30</td>
<td>29</td>
<td>0.02429</td>
<td>0.91876</td>
<td>0.90320</td>
<td>0.01556</td>
</tr>
<tr>
<td>7</td>
<td>1.50</td>
<td>20</td>
<td>0.01675</td>
<td>0.93551</td>
<td>0.93319</td>
<td>0.00232</td>
</tr>
<tr>
<td>8</td>
<td>1.70</td>
<td>22</td>
<td>0.01843</td>
<td>0.95394</td>
<td>0.95543</td>
<td>-0.00149</td>
</tr>
<tr>
<td>9</td>
<td>1.90</td>
<td>15</td>
<td>0.01256</td>
<td>0.96650</td>
<td>0.97128</td>
<td>-0.00478</td>
</tr>
<tr>
<td>10</td>
<td>2.10</td>
<td>8</td>
<td>0.00670</td>
<td>0.97320</td>
<td>0.98214</td>
<td>-0.00894</td>
</tr>
<tr>
<td>11</td>
<td>2.30</td>
<td>5</td>
<td>0.00419</td>
<td>0.97739</td>
<td>0.98928</td>
<td>-0.01189</td>
</tr>
<tr>
<td>12</td>
<td>2.50</td>
<td>9</td>
<td>0.00754</td>
<td>0.98492</td>
<td>0.99379</td>
<td>-0.00887</td>
</tr>
<tr>
<td>13</td>
<td>2.70</td>
<td>5</td>
<td>0.00419</td>
<td>0.98911</td>
<td>0.99653</td>
<td>-0.00742</td>
</tr>
<tr>
<td>14</td>
<td>2.90</td>
<td>2</td>
<td>0.00168</td>
<td>0.99079</td>
<td>0.99813</td>
<td>-0.00734</td>
</tr>
<tr>
<td>15</td>
<td>3.10</td>
<td>2</td>
<td>0.00168</td>
<td>0.99246</td>
<td>0.99903</td>
<td>-0.00657</td>
</tr>
<tr>
<td>16</td>
<td>3.30</td>
<td>2</td>
<td>0.00168</td>
<td>0.99414</td>
<td>0.99952</td>
<td>-0.00538</td>
</tr>
<tr>
<td>17</td>
<td>3.50</td>
<td>1</td>
<td>0.00084</td>
<td>0.99497</td>
<td>0.99977</td>
<td>-0.00480</td>
</tr>
<tr>
<td>18</td>
<td>3.70</td>
<td>2</td>
<td>0.00168</td>
<td>0.99665</td>
<td>0.99989</td>
<td>-0.00324</td>
</tr>
<tr>
<td>19</td>
<td>3.90</td>
<td>1</td>
<td>0.00084</td>
<td>0.99749</td>
<td>0.99995</td>
<td>-0.00246</td>
</tr>
<tr>
<td>20</td>
<td>4.10</td>
<td>3</td>
<td>0.00251</td>
<td>1.00000</td>
<td>0.99998</td>
<td>0.00002</td>
</tr>
</tbody>
</table>

$M = 1194$, Mean $= 0.1625 \times 10^{-6}$, $E(\sigma) = 0.08219$ and $\sigma^2 = V(\epsilon_{ij})$.

$$\max_i (\Delta \Phi(x_i)) = 0.078, \quad \bar{\sigma} = 0.034 \quad \text{as } \mu = 1 \text{ at } N = 4,$$

$$\sigma^2 = V(\Delta \Phi(x_i)). \quad \text{(6.10)}$$

where $\Delta \Phi(x_i) = \Phi(x) - \Phi(x)$ and $\sigma^2 = V(\Delta \Phi(x_i))$. 
Table 12
The errors of greyness and pixels and their ratios of girl-images under $T$ and $T^{-1}T$ by SSM and CSIM in Case 1 as $\mu = 0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>$T$ Sequential $\Delta E$</th>
<th>Ratio</th>
<th>PE</th>
<th>Ratio</th>
<th>$T^{-1}T$ Sequential $\Delta E$</th>
<th>Ratio</th>
<th>PE</th>
<th>Ratio</th>
<th>$T^{-1}T$ Absolute $E$</th>
<th>PE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.6524</td>
<td>166.1</td>
<td>/</td>
<td>0.1809</td>
<td>69.37</td>
<td>0.0951</td>
<td>22.97</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.2105</td>
<td>53.67</td>
<td>3.10</td>
<td>0.8679</td>
<td>22.24</td>
<td>2.09</td>
<td>9.75</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>0.5488 · 10$^{-1}$</td>
<td>3.84</td>
<td>14.00</td>
<td>3.10</td>
<td>0.2455 · 10$^{-1}$</td>
<td>3.54</td>
<td>6.27</td>
<td>3.55</td>
<td>0.03085</td>
<td>7.92</td>
</tr>
<tr>
<td>16</td>
<td>0.2171 · 10$^{-1}$</td>
<td>5.53</td>
<td>2.53</td>
<td>0.9437 · 10$^{-2}$</td>
<td>2.60</td>
<td>2.41</td>
<td>2.60</td>
<td>0.02864</td>
<td>7.39</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>0.8273 · 10$^{-2}$</td>
<td>2.11</td>
<td>2.63</td>
<td>0.3752 · 10$^{-2}$</td>
<td>2.52</td>
<td>0.95</td>
<td>2.53</td>
<td>0.02843</td>
<td>7.36</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>0.3182 · 10$^{-2}$</td>
<td>0.81</td>
<td>2.60</td>
<td>0.1490 · 10$^{-2}$</td>
<td>2.52</td>
<td>0.37</td>
<td>2.56</td>
<td>0.02830</td>
<td>7.32</td>
<td></td>
</tr>
<tr>
<td>128</td>
<td>0.1362 · 10$^{-2}$</td>
<td>0.35</td>
<td>2.63</td>
<td>0.7578 · 10$^{-3}$</td>
<td>0.18</td>
<td>2.07</td>
<td>0.02830</td>
<td>7.33</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 8. Curves of pd2 and the sequential errors $\Delta E$ versus $N$ by SSM and CSIM in Case 1 as $\mu = 1$ by the refinement $(N, N_1) = (N, N)$.

On the other hand, based on the Berry–Esseen theory (4.16) again, and the estimates on $E|X_1|^3$ given in (4.28) we obtain

$$\max_x |\Phi(x) - \Phi(x)| \leq \frac{\varepsilon E|X_1|^3}{\sigma \sqrt{M}} = \frac{\varepsilon 2\sqrt{2}}{\sqrt{\pi \sqrt{M}}} \leq \frac{1.28}{\sqrt{M}}.$$
Hence, \( \max_x |\Phi(x) - \Phi(x)| = 0.040 \) as \( \mu = 0 \) and \( 0.037 \) as \( \mu = 1 \), being very close to (6.9) and (6.10), indeed. Such a consistency implies a justification of our probability assumptions in Section 4. Below let us provide more arguments on these assumptions.

It seems doubtful that the errors caused by the sub-pixels on Cases B and C are independently distributed, because everything in the image processing is deterministic. However, due to the non-linearity of image transformation and due to the irregularity of errors from large number of sub-pixels, the behavior of the errors is very close to the assumption made in our randomized error analysis. In Tables 10 and 11, the cumulative distribution of the normalized average errors and that of the standard normal random variables are given in columns 5 and 6, and their differences are listed in the last column. The differences show that, even though the division number \( N \) is as small as 4, the goodness of fitting is so surprisingly nice that we have to give up our original attempt to present these results by graphics, because we cannot see any difference between the two curves of the cumulative distribution functions in our graphing. In our numerical examples with higher division number \( N \), the goodness of fitting is even nicer than the case of \( N = 4 \) and thus is not presented.

6.3. 256 Greyness level images

One purpose of this paper is application of SSM and CSIM to multi-level images, e.g., those with 256 greyness levels. We choose the standard girl-image with \( 256 \times 256 \) pixels in Fig. 12. By using the same quadratic transformation in Section 6.1, we obtain the numerical errors, and list them in Tables 12–17. The girl-images have 33886 non—"white" pixels in the original of Fig. 12. It
can be seen that there also exist the convergence rates $O_p(1/N^{1.5})$, and $O_p(1/N^2)$ by the refinement $(N, N_1) = (N, \sqrt[3]{N})$.

The ratios of greyness errors are very close to the ratios of pixel errors (denoted simply PE in the Tables 12–17). Also the maximal discrepancy of pixel errors between Cases I and II in Step 8 are

$$\max \text{Abs}\{|\text{PE}_{\text{Case I}} - |\text{PE}_{\text{Case II}}\| = 0.05,$$

which is much smaller than $\frac{1}{2}$. Note that the probabilistic error analysis in Step 5 between Cases I and II is also given in Section 5.2.

Finally, we also provide in Figs. 13–16 the girl-images under $T$ by SSM and $T^{-1}T$ by CSIM, and CSIM using the refinement $(N, N_1) = (N, \sqrt[3]{N})$, where the pixel errors in SSM are less than one greyness level. Evidently, the new refinement techniques may save the CPU time due to the smaller division number $N$ used.

6.4. Concluding remarks

To close this paper, let us make a few important remarks.

1. In this paper, as a generalization of [45] the general error bounds $E_a(B)$ and $\Delta E_a(B)$ with $a \geq 1$ of greyness by SSM and CSIM are estimated with explicit constants, although the same convergence
rate $O(1/N)$ is obtained as in [45]. Based on new probability inequalities, the convergence rate $O_p(1/N^{1.5})$ is proven for SSM and then for CSIM as $\mu = 1$ by following this paper and [45]. This is a significant contribution on error analysis on the discrete algorithms, compared to [36, 45, 46].

2. To achieve the goals in Remark 1, we have derive the countable probability inequalities, given in Theorems 4.5–4.7, 4.13 and 4.14 for different sums of independent random variables, related to $E_x(B)$ and $\Delta E_x(B)$ as $x \geq 1$. Also Theorems 4.9–4.11, 4.15 and 4.16 display clearly by the explicit possibility evaluation that the conservative estimates as $O(1/N)$ is very rare in practice, and that the better convergence rate $O_p(1/N^{1.5})$ occurs in most of time.

3. The new partial refinements of pixel partition are proposed, thus to raise the convergence rates. Based on the theoretical analysis (also see Table 1), we may achieve

$$O_p(1/N^{1.5}) \text{ by } (N, N_1) = (N, \sqrt{N}) \quad \text{and} \quad O_p(1/N^2) \text{ by } (N, N_1) = (N, \sqrt[3]{N}).$$  \hfill (6.11)

However, from the empirical results we also obtain

$$O_p(1/N^{2.5}) \text{ by } (N, N_1) = (N, \sqrt{N}).$$  \hfill (6.12)

Hence, in computation we may choose $N_1$ as

$$N_1 = O(N^\beta), \quad \frac{1}{3} \leq \beta \leq \frac{1}{2}. \hfill (6.13)$$
4. Note that $O_p(1/N^2)$ can be obtained by SSM using the refinement techniques for both $\mu = 0$ and $\mu = 1$. However, the error bounds of greyness by the splitting-integrating method (SIM) for $T^{-1}$ are $O(1/N^{\mu+1})$. When $\mu = 1$, $O_p(1/N^2)$ is also gained. However, when $\mu = 0$, only $O(1/N)$ is obtained. Hence we may combine the SSM using the refinement and SIM using the partition technique in [50], to reach $O_p(1/N^2)$ as well.

5. The SSM and CSIM have already been applied to binary images or the images with 16 greyness levels in [35]. By virtue of the study in this paper, they can be applied for 256 greyness levels; the successful samples are also provided in this paper. The slight discrepancy of pixel errors by CSIM between Cases I and II suggests that either of these two cases be used in application in 256-greyness level images. For the images with 256 greyness levels, the computed pixel and greyness errors are very coincident with each other. In fact, the pixel errors in Tables 14–17 can be computed directly from the greyness errors multiplying 255. This provides a justification of our analytical approaches, by which, the continuous greyness errors are our main concern; and the discrete pixel errors are naturally consequences.

6. In summary, the new probabilistic error analysis is explored to prove the convergence rates $O_p(1/N^2)$ in probability of the image greyness and pixels, obtained by the splitting-shooting method and combination CSIM as $\mu = 1$. This is a significant development, compared to the convergence rates $O(1/N)$ reported in the recent paper [45]. Note that the discrete methods SSM and CSIM do not involve in the nonlinear solutions for nonlinear image transformations. However,
SSM and CSIM are well suited not only to binary images as shown in [36, 35], but also to multiple (e.g., 256) level images as illustrated in this paper. Hence we may adopt SSM and CSIM to facilitate image transformation, such as perspective transformation, harmonic transformation, etc. in [35].

7. Finally, let us briefly address our research on discrete images. It is expected that the 21st century may become the era of computer engineering and computer sciences. Hence, the applications of
### Table 13

The errors of greyness and pixels and their ratios of girl-images under \( T \) and \( T^{-1}T \) by SSM and CSIM in Case II as \( \mu = 0 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Sequential ( \Delta E )</th>
<th>Ratio</th>
<th>PE</th>
<th>Ratio</th>
<th>Absolute ( E )</th>
<th>PE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.1801</td>
<td></td>
<td>46.11</td>
<td></td>
<td>0.09115</td>
<td>22.07</td>
</tr>
<tr>
<td>4</td>
<td>( 0.8647 \cdot 10^{-1} )</td>
<td>2.08</td>
<td>22.11</td>
<td>2.09</td>
<td>0.03827</td>
<td>9.73</td>
</tr>
<tr>
<td>8</td>
<td>( 0.2448 \cdot 10^{-1} )</td>
<td>3.53</td>
<td>6.25</td>
<td>3.53</td>
<td>0.03064</td>
<td>7.83</td>
</tr>
<tr>
<td>16</td>
<td>( 0.9362 \cdot 10^{-2} )</td>
<td>2.61</td>
<td>2.39</td>
<td>2.61</td>
<td>0.02831</td>
<td>7.26</td>
</tr>
<tr>
<td>32</td>
<td>( 0.3615 \cdot 10^{-2} )</td>
<td>2.59</td>
<td>0.93</td>
<td>2.58</td>
<td>0.02810</td>
<td>7.21</td>
</tr>
<tr>
<td>64</td>
<td>( 0.1184 \cdot 10^{-2} )</td>
<td>3.05</td>
<td>0.32</td>
<td>2.72</td>
<td>0.02792</td>
<td>7.18</td>
</tr>
<tr>
<td>128</td>
<td>( 0.3647 \cdot 10^{-3} )</td>
<td>3.25</td>
<td>0.12</td>
<td>2.73</td>
<td>0.02790</td>
<td>7.18</td>
</tr>
</tbody>
</table>

### Table 14

The errors of greyness and pixels and their ratios of girl-images under \( T \) and \( T^{-1}T \) by SSM and CSIM in Case I as \( \mu = 1 \)

<table>
<thead>
<tr>
<th>( N )</th>
<th>Sequential ( \Delta E )</th>
<th>Ratio</th>
<th>PE</th>
<th>Ratio</th>
<th>Absolute ( E )</th>
<th>PE</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.6294</td>
<td></td>
<td>160.5</td>
<td></td>
<td>0.1237</td>
<td>31.59</td>
</tr>
<tr>
<td>4</td>
<td>0.2012</td>
<td>3.13</td>
<td>51.30</td>
<td>3.13</td>
<td>0.5803 \cdot 10^{-1}</td>
<td>2.13</td>
</tr>
<tr>
<td>8</td>
<td>( 0.5231 \cdot 10^{-1} )</td>
<td>3.85</td>
<td>13.34</td>
<td>3.85</td>
<td>0.1784 \cdot 10^{-1}</td>
<td>3.25</td>
</tr>
<tr>
<td>16</td>
<td>( 0.2079 \cdot 10^{-1} )</td>
<td>2.52</td>
<td>5.30</td>
<td>2.52</td>
<td>0.6755 \cdot 10^{-2}</td>
<td>2.64</td>
</tr>
<tr>
<td>32</td>
<td>( 0.7902 \cdot 10^{-2} )</td>
<td>2.63</td>
<td>2.01</td>
<td>2.64</td>
<td>0.2772 \cdot 10^{-2}</td>
<td>2.44</td>
</tr>
<tr>
<td>64</td>
<td>( 0.2927 \cdot 10^{-2} )</td>
<td>2.70</td>
<td>0.76</td>
<td>2.69</td>
<td>0.1071 \cdot 10^{-2}</td>
<td>2.59</td>
</tr>
<tr>
<td>128</td>
<td>( 0.1064 \cdot 10^{-2} )</td>
<td>2.75</td>
<td>0.27</td>
<td>2.75</td>
<td>0.4813 \cdot 10^{-3}</td>
<td>2.23</td>
</tr>
</tbody>
</table>

Image processing and computer vision will grow up much faster than ever. In order to carry out their computations correctly and efficiently, applied mathematics and computational mathematics will play an extremely role. Two excellent examples in image computing are known as the fast Fourier transformation in [19, 51, 17, 56, 73, 76, 77] and the wavelets in [69, 15, 12]. The digital images and patterns are discrete. We recall, however, that numerical methods are a discipline to deal with the continuous subjects (e.g., the solutions of ordinary and partial differential equations) by discrete approaches (e.g., the finite algebraic equations). Also the Lagrange, Hermite and spline interpolations may amend discrete data into continuous functions. Our research on this area over the last decade is to modify and to renovate the existing numerical methods in [2, 4, 7, 16, 63, 68, 70], which have been proving to be successful to mechanics and physics, to suit well image treatments. Consequently, a number of amazing results have been obtained. Several efficient discrete algorithms, such as SSM, SIM, CSIM, CIIM, etc., have been developed and many fascinating applications have been found, see [27, 35–42]. Since 1993, our efforts have
Table 15
The errors of greyness and pixels and their ratios of girl-images under $T$ and $T^{-1}T$ by SSM and CSIM in Case II as $\mu = 1$

<table>
<thead>
<tr>
<th>$N$</th>
<th>Sequential</th>
<th></th>
<th></th>
<th>Absolute</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta E$</td>
<td>Ratio</td>
<td>PE</td>
<td>Ratio</td>
</tr>
<tr>
<td>0</td>
<td>0.1233</td>
<td>/</td>
<td>31.47</td>
<td>/</td>
</tr>
<tr>
<td>4</td>
<td>0.5761 $\cdot 10^{-1}$</td>
<td>2.14</td>
<td>14.76</td>
<td>2.14</td>
</tr>
<tr>
<td>8</td>
<td>0.1765 $\cdot 10^{-1}$</td>
<td>3.26</td>
<td>4.50</td>
<td>3.26</td>
</tr>
<tr>
<td>16</td>
<td>0.6624 $\cdot 10^{-2}$</td>
<td>2.66</td>
<td>1.69</td>
<td>2.62</td>
</tr>
<tr>
<td>32</td>
<td>0.2638 $\cdot 10^{-2}$</td>
<td>2.51</td>
<td>0.67</td>
<td>2.51</td>
</tr>
<tr>
<td>64</td>
<td>0.8310 $\cdot 10^{-2}$</td>
<td>3.17</td>
<td>0.22</td>
<td>3.08</td>
</tr>
<tr>
<td>128</td>
<td>0.2411 $\cdot 10^{-3}$</td>
<td>3.45</td>
<td>0.07</td>
<td>3.16</td>
</tr>
</tbody>
</table>

Table 16
The errors of greyness and pixels and their ratios of girl-images under $T$ and $T^{-1}T$ by SSM using the refinement $(N, N_1) = (N, \sqrt{N})$ and CSIM in Case II as $\mu = 0$

<table>
<thead>
<tr>
<th>Refin.</th>
<th>$T$</th>
<th>$T^{-1}T$</th>
<th></th>
<th></th>
<th>Absolute</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\Delta E$</td>
<td>Ratio</td>
<td>PE</td>
<td>Ratio</td>
<td>$E$</td>
</tr>
<tr>
<td>2 /</td>
<td>0.6524</td>
<td>/</td>
<td>166.1</td>
<td>/</td>
<td>0.4656</td>
</tr>
<tr>
<td>4 /</td>
<td>0.2105</td>
<td>3.10</td>
<td>63.61</td>
<td>3.10</td>
<td>0.1949</td>
</tr>
<tr>
<td>8 /</td>
<td>0.4995 $\cdot 10^{-1}$</td>
<td>4.25</td>
<td>12.61</td>
<td>4.26</td>
<td>0.2669 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>16 /</td>
<td>0.8217 $\cdot 10^{-2}$</td>
<td>6.02</td>
<td>2.10</td>
<td>6.00</td>
<td>0.9564 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>32 /</td>
<td>0.2151 $\cdot 10^{-2}$</td>
<td>3.10</td>
<td>0.71</td>
<td>3.15</td>
<td>0.1857 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>64 /</td>
<td>0.4428 $\cdot 10^{-3}$</td>
<td>5.99</td>
<td>0.22</td>
<td>3.20</td>
<td>0.4276 $\cdot 10^{-2}$</td>
</tr>
</tbody>
</table>

Table 17
The errors of greyness and pixels and their ratios of girl-images under $T$ and $T^{-1}T$ by SSM using the refinement $(N, N_1) = (N, \sqrt{N})$ and CSIM in Case I as $\mu = 1$

<table>
<thead>
<tr>
<th>Refin.</th>
<th>$T$</th>
<th>$T^{-1}T$</th>
<th></th>
<th></th>
<th>Absolute</th>
</tr>
</thead>
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<tr>
<td></td>
<td>$\Delta E$</td>
<td>Ratio</td>
<td>PE</td>
<td>Ratio</td>
<td>$E$</td>
</tr>
<tr>
<td>2 /</td>
<td>0.6295</td>
<td>/</td>
<td>160.5</td>
<td>/</td>
<td>0.4468</td>
</tr>
<tr>
<td>4 /</td>
<td>0.2012</td>
<td>3.13</td>
<td>51.30</td>
<td>3.13</td>
<td>0.1874</td>
</tr>
<tr>
<td>8 /</td>
<td>0.4716 $\cdot 10^{-1}$</td>
<td>4.27</td>
<td>12.03</td>
<td>4.27</td>
<td>0.2579 $\cdot 10^{-1}$</td>
</tr>
<tr>
<td>16 /</td>
<td>0.7901 $\cdot 10^{-2}$</td>
<td>5.97</td>
<td>2.01</td>
<td>5.97</td>
<td>0.9250 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>32 /</td>
<td>0.2595 $\cdot 10^{-2}$</td>
<td>3.05</td>
<td>0.66</td>
<td>3.05</td>
<td>0.1797 $\cdot 10^{-2}$</td>
</tr>
<tr>
<td>64 /</td>
<td>0.5706 $\cdot 10^{-3}$</td>
<td>4.55</td>
<td>0.15</td>
<td>4.55</td>
<td>0.4138 $\cdot 10^{-3}$</td>
</tr>
</tbody>
</table>
been paid to analyze the mathematical nature of these discrete algorithms, and then to explore more effective and advanced numerical algorithms, see [43–50]. Error analysis of Ciarlet [13] and Strang and Fix [68] is a main tool; probability analysis now is also employed. In this paper, different mathematical discipline: numerical methods, analysis, geometry in [10, 14, 26, 52, 53, 55, 61, 62, 64], statistics and probability, are all integrated together to study the image transformations.
Fig. 14(a). The girl-images by SSM and CSIM in Case 1 as $\mu = 1$ at $N = 64$.

Fig. 14(b). The girl-images by SSM and CSIM in Case 1 as $\mu = 1$ at $N = 128$.

successfully, which can be applied to many areas in computer sciences: image processing in [17, 30, 32, 56, 58, 73, 77], computer graphics in [11, 21, 22, 26, 33, 55, 59, 71, 75], computer vision in [6, 28, 29, 66, 76], geometric aided designs in [6, 11, 24, 33, 55, 71], and pattern recognition in [18, 23, 24, 34, 53, 54, 56, 60, 72, 78].
Fig. 15(a). The girl-images by SSM using the refinement \((N, N_1) = (N, \sqrt{N})\) and CSIM in Case II as \(\mu = 0\) at \(N = 32\).

Fig. 15(b). The girl-images by SSM using the refinement \((N, N_1) = (N, \sqrt{N})\) and CSIM in Case II as \(\mu = 0\) at \(N = 64\).

Acknowledgements

We are grateful to the referee for his/her valuable suggestions, and also thank to Tian-Xiang Huang in preparing this manuscript. This work was supported in part by National Science Council of Taiwan.
Fig. 16(a). The girl-images by SSM using the refinement $N - \sqrt{N}$ and CSIM in Case I as $\mu = 1$ at $N = 32$.

Fig. 16(b). The girl-images by SSM using the refinement $(N, N_1) = (N, \sqrt{N})$ and CSIM in Case II as $\mu = 1$ at $N = 64$.

References


