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Nuclear Physics B 893 (2015) 501–511

NUCLEAR PHYSICS B

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Isotropic Lifshitz critical behavior from the functional renormalization group

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Received 22 December 2014; accepted 11 February 2015

Available online 19 February 2015

Editor: Hubert Saleur

Abstract

The Lifshitz critical behavior for a single component field theory is studied for the specific isotropic case in the framework of the Functional Renormalization Group. Lifshitz fixed point solutions of the flow equation, derived by using a Proper Time regulator, are searched at lowest and higher order in the derivative expansion. Solutions are found when the number of spatial dimensions d is contained within the interval $5.5 < d < 8$.

1. Introduction

Lifshitz critical points represent a particular class of tricritical points on the phase diagram, characterized by the coexistence of a disordered phase with vanishing order parameter, a homogeneous ordered phase with finite constant order parameter, and a modulated phase where the order parameter shows a periodic structure with finite wave vector, and a typical realization is observed in a ferromagnet with three phases: paramagnetic, ferromagnetic and helicoidal or sinusoidal.

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http://dx.doi.org/10.1016/j.nuclphysb.2015.02.013

This idea was first introduced in [1], by a generalization of the usual Landau–Ginzburg ϕ^4 model where the space coordinates of a d dimensional space are separated in parallel and orthogonal components, respectively spanning an m and a $d-m$ dimensional space, and the terms involving derivatives with respect to one or to the other set of coordinates in the action are treated differently. Then, the usual kinetic term with the square gradient of the field in one set of coordinates can be kept finite while the square gradient related to the other set of coordinates is suppressed, so that the dynamics of the terms with four powers of the gradient of the field becomes essential.

The anisotropy introduced between the two sets of coordinates generates a multicritical point, namely the Lifshitz point, whose universal behavior is determined by the three parameters (d, m, N) (N indicates the number of components of the field ϕ), and which shows a peculiar critical behavior that requires two different anomalous dimensions and two critical exponents, related to two different correlation lengths, in order to describe the two-point correlation function. The interest in the Lifshitz points is due to the large variety of systems that present this kind of critical behavior, such as magnetic systems like the MnP compound or the so-called ANNNI model, but also polymer mixtures, liquid crystals, high-T_c superconductors. For reviews on this subject see [2–4].

More recently the Lifshitz critical behavior has found application in a theory of gravity formulated on the basis of a conjectured anisotropy between time and space coordinates that reduces the ultraviolet pathologies of the theory [5–7]. In addition, a duality between an $O(N)$ model at an isotropic Lifshitz point and a higher spin gravity theory has been proposed in [8,9], while models involving effects of Lorentz invariance violation due to a Lifshitz anisotropy are discussed in [10,11].

The critical properties of the Lifshitz point were originally studied in the framework of the ϵ-expansion [1], and the hard task of evaluating the free propagator for generic m and d, made the calculation of the $O(\epsilon^2)$ corrections a very difficult problem which was eventually solved in [12–14]. Analogous difficulties appeared in the computation of the critical properties at large N with the relative $O(1/N)$ corrections [15].

Among the possible configurations of d and m for the Lifshitz critical point there is one, namely the case $m = d$, in which the isotropy is recovered again. Then, all the space coordinates have the same critical behavior, which however is different from the standard case where the kinetic term for all the space coordinates is quadratic in the gradient, $O(\partial^2)$, while for $m = d$ it is quartic, $O(\partial^4)$. The interest in the isotropic $m = d$ case was primarily motivated for its application to the mixtures of a homopolymer blend and diblock copolymer for which a Lifshitz point is predicted by mean field theory, and the measurement of critical exponents was performed in [16]. The ϵ expansion in this case is realized along the diagonal $m = d$ with $\epsilon = 8 - d$ [1,17]. More recently, a numerical approach by Monte Carlo simulations indicates that the isotropic Lifshitz points could be destroyed upon inclusion of fluctuations [18]. Therefore, it is certainly of interest to analyze the problem by means of a different non-perturbative approach, suitable to study systems at criticality, namely the Functional Renormalization Group (FRG).

The FRG approach, [19–21], consists of a functional differential flow equation for the running effective action which provides a description of the physics at an energy scale k that, in turn, runs from a large ultraviolet scale, where the bare action is defined, down to small scales and eventually to $k = 0$, where the running action becomes equal to the standard effective action. The flow equation is the result of the progressive integration of the fluctuations with momentum contained in an infinitesimal interval centered around k, so that, when $k = 0$, all fluctuations have been integrated out. In practice, the integration of the fluctuations is performed by introducing a particular cut-off function that selects the desired interval of modes and, clearly, the flow equation carries an explicit dependence on the specific cut-off employed. Many reviews are available on the various formulations of the FRG flow equation and on its numerous applications [22–26].

The idea of applying the FRG equation to the study of the Lifshitz critical behavior is not new, as it is implemented in [27], where the Lifshitz fixed point and the main critical exponents are evaluated in the uniaxial case $m = 1$, and in [28], where the specific case of $N = 3$ and $m = 1$ is analyzed. Both papers show that the FRG approach is suitable to investigate these critical properties, avoiding some technical difficulties encountered within other non-perturbative approaches. However the FRG has not yet been used in a numerical study of the isotropic case $m = d$.

Therefore, in this paper we focus on the isotropic $m = d$ problem and make use of the Proper Time (PTRG) version of the FRG equation. This is a flow equation originally derived from a proper-time regularization of the one-loop effective action, [29–31] that can equally be obtained from generalized Callan–Symanzik flows [32], or, more generally can be derived in the framework of the background field flows [33,34]. This equation, that has previously been used for studies of phase transitions [32,30,31,35–38], spontaneous symmetry breaking and tunneling phenomena [39–41], gravity [42,43], has the advantage of being accurate and rapidly converging in the determination of the critical properties of the theory and therefore suitable to approach the problem considered here. In addition, as the PTRG has been used in [38] to study the Ising universality class to fourth order in the derivative expansion, thus including the $O(\partial^4)$ terms, we can take advantage of using the formalism already developed in [38] to study the Lifshitz critical point for the one component field theory, $N = 1$, in the isotropic limit $m = d$.

In Section 2, we recall the essential properties of the Lifshitz critical behavior; in Section 3 the PTRG and the structure of the corresponding flow equations are outlined, while the numerical results are discussed in Section 4. Conclusions are summarized in Section 5.

2. Lifshitz critical behavior

The general form of a d-dimensional action, suitable to investigate on the occurrence of a tricritical Lifshitz point is

$$\Gamma[\phi] = \int \mathrm{d}^{d-m}x_\perp \, \mathrm{d}^m x_\parallel \left\{ W_\parallel \, (\partial_\parallel^2 \phi)^2 + W_\perp \, (\partial_\perp^2 \phi)^2 + \frac{Z_\parallel}{2}(\partial_\parallel \phi)^2 + \frac{Z_\perp}{2}(\partial_\perp \phi)^2 + V(\phi) \right\} \tag{1}$$

where in general $\phi(x)$ is an N-component vector field, although here we shall focus on the single component field theory with $N = 1$. The potential V is a generic function of the field, while the coordinates x are decomposed in parallel, $x_\parallel$, and orthogonal, $x_\perp$, components, that respectively belong to an m-dimensional and a $(d - m)$-dimensional subspace which possess two different scaling behaviors. In fact, at mean field level, one observes that, by keeping $Z_\parallel > 0$, a vanishing and a non-vanishing minimum of the potential V respectively correspond to disordered and ordered phase, while for $Z_\parallel < 0$, a critical value of the minimum of V separates the disordered phase from a modulated phase with an oscillating ground state, so that these three phases meet at the point characterized by $Z_\parallel = 0$ and by the vanishing of the minimum of V.

As a consequence one expects that this configuration corresponds to the tricritical Lifshitz fixed point and, as $Z_\parallel = 0$, the role of the term $(1/2)Z_\parallel(\partial_\parallel \phi)^2$ is now played by the term

$W_\parallel(\partial_\parallel^2\phi)^2$ and therefore the scaling of the parallel and orthogonal coordinates must be different. This leads to the introduction of two different anomalous dimensions, η_{l2} and η_{l4}, to fully describe the scaling of the two point functions, $\Gamma^{(2)}(q_\perp \to 0, q_\parallel = 0) \sim q_\perp^{2-\eta_{l2}}$ and $\Gamma^{(2)}(q_\perp = 0, q_\parallel \to 0) \sim q_\parallel^{4-\eta_{l4}}$. Accordingly, two different correlation lengths with two critical indexes are required at criticality.

It is natural to associate the two sets of coordinates with two scales: $\kappa_\perp$, $\kappa_\parallel$, and introduce the anomalous dimensions through the field renormalization: $Z_\perp \propto \kappa_\perp^{-\eta_{l2}}$ and $W_\parallel \propto \kappa_\parallel^{-\eta_{l4}}$. If one connects the two scales by the anisotropy parameter θ: $\kappa_\parallel = \kappa_\perp^\theta$, then consistency in the scaling of the two field renormalizations in Eq. (1) requires:

$$\theta = \frac{2-\eta_{l2}}{4-\eta_{l4}}. \tag{2}$$

The scaling dimension of all the other operators are directly read from Eq. (1) and can be expressed for instance in terms of $\kappa_\perp$. Then, as already seen, the dimensions of $Z_\perp$ and $W_\parallel$ are $[-\eta_{l2}]$ and $[-\theta\eta_{l4}]$, while the dimension of the field ϕ is

$$D_\phi^{(m)} = \frac{d-m+\theta(m-4+\eta_{l4})}{2}, \tag{3}$$

and those of $Z_\parallel$, $W_\perp$, V are respectively: $[\theta(2-\eta_{l4})]$, $[-(2+\eta_{l2})]$, $[d+\theta m-m]$.

One immediately notices that the above scaling dimensions are rather different from those observed for instance at the Wilson–Fisher fixed point which are very close to the canonical dimensions because the anomalous dimension in that case turns out to be very small. The anisotropic scaling pointed out above is instead realized in proximity of the Lifshitz critical point, if it exists. In other words, this scaling occurs only if a fixed point solution (Lifshitz fixed point) of the corresponding FRG flow equations is found.

In this case, it is interesting to notice that, while the scaling $Z_\perp$ and $W_\parallel$ depends on the sign of η_{l2} and η_{l4}, the parameter $Z_\parallel$, which vanishes at the critical point at the mean field level, is in fact a relevant parameter according to its scaling dimension and, on the contrary, $W_\perp$ is irrelevant. Therefore one expects the full fixed point solution to be unstable with respect to small perturbations of $Z_\parallel$ around its fixed point value.

The particular isotropic case is easily obtained by requiring that no orthogonal coordinate is present, which means that the above equations must be simplified by setting $m = d$, $Z_\perp = W_\perp = 0$ and $\eta_{l2} = 0$. Then we are left with parallel coordinates only and we can define with no ambiguity: $Z \equiv Z_\parallel$, $W \equiv W_\parallel$ and $\eta \equiv \eta_{l4}$. It is also convenient to reexpress the scaling dimensions in terms of the orthogonal scale $k \equiv \kappa_\parallel = \kappa_\perp^\theta$, instead of $\kappa_\perp$, in order to absorb θ in the scale parameter. Then, the scaling dimensions of W, Z, V become $[-\eta]$, $[2-\eta]$, $[d]$ and, from Eq. (3), the dimension of ϕ is $D_\phi = D_\phi^{(m=d)}$:

$$D_\phi = \frac{d-4+\eta}{2}. \tag{4}$$

The isotropic case resembles the standard analysis where, in addition to the standard $O(\partial^2)$ kinetic term, an additional quartic term, $O(\partial^4)$, is added to the action. However, in the standard analysis the quartic term is irrelevant and the quadratic is marginal, while here, according to the different scaling, the quartic terms is marginal and the quadratic is relevant and the occurrence of a Lifshitz fixed point directly depends on the interplay of these two parameters.

3. PTRG flow equation

Once the scaling of the various operators is set, one has to look for fixed point solutions of the FRG equations and, as already anticipated, we make use here of the PTRG flow equation

$$k\,\frac{\partial\Gamma_k}{\partial k}=-\frac{1}{2}\mathrm{Tr}\int_0^\infty\frac{\mathrm{d}s}{s}\,k\frac{\partial g_k}{\partial k}\,\exp\left(-s\frac{\delta^2\Gamma_k}{\delta\phi\delta\phi}\right)\tag{5}$$

with the specific choice of the step function as a regulator, properly adjusted for our quartic in the momentum two-point function: $g_k=\Theta\left(1-2Wk^4s\right)$. A specific ansatz for the scale dependent action Γ_k is already given in Eq. (1) where in general it is assumed that all parameters $W_\parallel$, $W_\perp$, $Z_\parallel$, $Z_\perp$, V depend both on the field ϕ and on the running scale k, so that the renormalization effects are encoded in the flow of these parameters with k. In addition, as we are interested in studying the isotropic case, we must set $m=d$ and discard the parallel coordinates subspace in the action and therefore, the full flow equation (5) in the approximation scheme of the derivative expansion, [44], is reduced to a set of three coupled partial differential equations for V, W, Z.

The derivation of the flow equations with terms involving four field derivatives is rather long, but we can easily adapt the flow equations derived in [38] to study the Ising universality class at order $O(\partial^4)$ in the derivative expansion, with the only change related to the different scaling behavior of the various parameters at the Lifshitz critical point. Therefore, by following [38], and after rescaling the field and V, W, Z by their scaling dimensions: $\phi=k^{D_\phi}\,x$, $W(k,\phi)=k^{-\eta}\,w(k,x)$, $Z(k,\phi)=k^{2-\eta}\,z(k,x)$, $V(k,\phi)=k^d\,v(k,x)$, the three flow equations read:

$$k\partial_k v-d\,v+D_\phi\,x\partial_x v=\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\left(-\frac{a_0}{2w}\right)}\tag{6}$$

$$k\partial_k w+\eta w+D_\phi\,x\partial_x w=-\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\left(-\frac{a_0}{2w}\right)}\,K_w\tag{7}$$

$$k\partial_k z-(2-\eta)z+D_\phi\,x\partial_x z=-\int\frac{\mathrm{d}^d p}{(2\pi)^d}\,\mathrm{e}^{\left(-\frac{a_0}{2w}\right)}\,K_z\tag{8}$$

where D_ϕ is given in Eq. (4), the parameter $a_0=\partial_x^2 v+z\,p^2+2w\,p^4$ in the exponential stems from the two-point function. K_w, K_z are polynomials in the loop momentum variable p respectively up to order p^{20} and p^{14} with coefficient functions depending on v, w, z and their first and second derivatives with respect to the rescaled field x. The kernels K_w, K_z encode all the interactions among operators coming from the derivative terms of the action and they have very long expressions which we do not report here.

Finally, a fixed point corresponds to a k-independent solution, $v^*(x)$, $w^*(x)$, $z^*(x)$, of the flow equations, (6), (7), (8). Then, in the search for fixed points, the first term in each of the flow equations (6), (7), (8), involving a derivative with respect to k, must be discarded and one is left with three coupled second order ordinary differential equations. In the scheme of the derivative expansion, the lowest order approximation, known as Local Potential approximation (LPA), is realized by solving Eq. (6) for V and keeping fixed $w^*=1/2$ and $\eta=0$ and $z^*=0$. At the next order the kinetic term is turned on but, while usually this amounts to turning on the $O(\partial^2)$ terms and treating the $O(\partial^4)$ as a subleading correction, in the Lifshitz case the leading kinetic term involves w^*, the coefficient of the $O(\partial^4)$ operator. Therefore, after discussing the LPA, we shall first study the coupled equations (6) and (7) by keeping $z=0$ and, as a final step, we shall release the constraint $z=0$ and consider the full set (6), (7), (8).

4. Results

The resolution of the set of Eqs. (6), (7), (8) requires a proper number of boundary conditions. In fact, symmetry properties of the action require vanishing of the derivatives of the solution with respect to the field x at $x = 0$: $v^{*\prime}(0) = w^{*\prime}(0) = z^{*\prime}(0) = 0$ and, in addition, the overall normalization of the action is set by taking $w^*(0) = 1/2$. Then it is important to look at the asymptotic behavior at large $x \gg 1$.

We first consider the case of positive $D_\phi > 0$ in Eq. (4) and focus on the LPA, i.e. we fix $w^* = 1/2$ and $\eta = 0$ and $z^* = 0$ together with the boundary $v^{*\prime}(0) = 0$ and observe that the right hand side of Eq. (6), in the limit of large x, is exponentially suppressed as long as $v^{*\prime\prime}(x)$ diverges in this limit. Therefore, from the left hand side of Eq. (6) it is easy to check that the potential diverges at large x as $v(x) \sim x^{d/D_\phi}$ as long as $D_\phi > 0$. Incidentally, we notice that the case with a divergent potential with power $0 < d/D_\phi \leq 2$, such that its second derivative vanishes or tends to a finite value and the exponential does not suppress the right hand side of Eq. (6), is excluded because it would require $d < 0$.

This power-law divergent behavior of the potential $v^*(x)$ at large x puts a very strong constraint on Eq. (6). In fact, only a discrete number of values $v^*(0)$ produces a solution with no singularity at any finite x. Any other different value of the boundary $v^*(0)$ yields a solution that is singular at some finite x. By retaining only those solutions that are regular on the full x axis, we find at most a discrete set of fixed point solutions. Remarkably, the same kind of structure is encountered and discussed in [44,45] for the standard problem which corresponds to setting $m = 0$ in the action (1) (and the anomalous dimension is η_{l2}, while $\eta_{l4} = 0$), the only difference being the scaling dimension of the field:

$$D_\phi^{m=0} = \frac{(d - 2 + \eta_{l2})}{2}. \tag{9}$$

In [45], the only non-singular (and non-gaussian) solution found for $2 < d < 4$ is the Wilson–Fisher fixed point.

We now include the other differential equations (7), (8) with the related boundaries $w^{*\prime}(0) = z^{*\prime}(0) = 0$ $w^*(0) = 1/2$ and again with $D_\phi > 0$. As in the case of the LPA, where the asymptotic behavior of the solution selects a discrete number of values $v^*(0)$, here the same asymptotic structure allows for a discrete number of solutions, i.e. of values of the three parameters $v^*(0)$, $z^*(0)$ and η.

Let us now examine the structure of the equations when $D_\phi < 0$. In this case the potential is no longer divergent at large x and it is expected to converge to a finite value. As a consequence, in the right hand side of Eq. (6) the exponential at large x tends to 1 and this determines unambiguously the limiting value of the potential for $x \to \infty$: $v^*(x) \to \bar{v}$. Then, even the subleading vanishing term in the potential at large x is determined from Eq. (6), up to a constant factor α: $v^*(x) \sim \bar{v} + \alpha\, x^{d/D_\phi}$. The different asymptotic behavior of the potential drastically modifies the spectrum of the solutions from discrete to continuous. In fact, in the LPA when $D_\phi < 0$, one finds different non-singular solutions of Eq. (6) for each value assigned to the boundary $v^*(0)$ (in all cases the second boundary, $v^{*\prime}(0) = 0$, is to be enforced), i.e. a line of fixed points is observed, parametrized by the value of $v^*(0)$. As an example, three fixed potentials, $v^*(x)$, obtained in the LPA at $d = 3$ are shown in Fig. 1 for three different values of $v^*(0)$.

If instead $D_\phi > 0$ i.e. if $d > 4$, we find, by numerical resolution of Eq. (6), only one non-gaussian solution at each fixed d, and the corresponding $v^{*\prime\prime}(0)$ is displayed in Fig. 2 (blue triangles pointing upward). $v^{*\prime\prime}(0)$ grows monotonically in the full interval $4 < d < 8$ although it is only partially visible in Fig. 2.

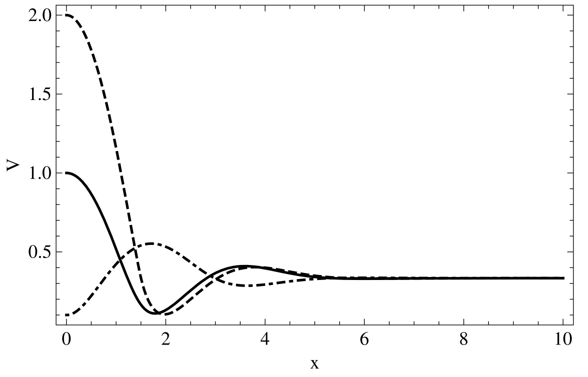

Fig. 1. Solutions of Eq. (6) in the LPA at $d = 3$ and with boundaries $v^*(0) = 1$ (solid), $v^*(0) = 2$ (dashed), $v^*(0) = 0.1$ (dot-dashed), respectively. Note the asymptotic behavior, constant for any initial condition.

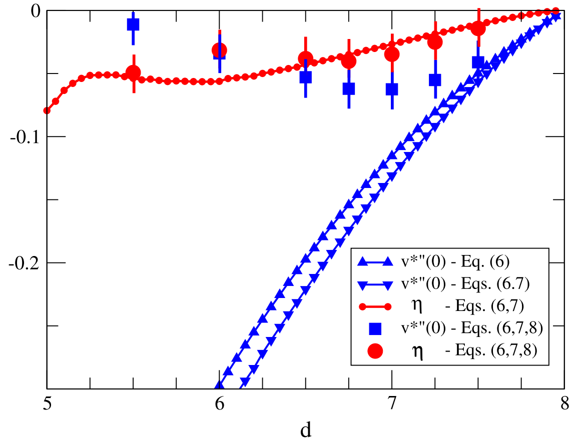

Fig. 2. $v^{*\prime\prime}(0)$ computed in the LPA (blue triangles pointing upward), by solving Eqs. (6), (7) (blue triangles pointing downward) and by solving Eqs. (6), (7), (8), (blue squares). Anomalous dimension η as obtained from Eqs. (6), (7) (small red circles), and from Eqs. (6), (7), (8) (large red circles). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

It is very interesting to look at the upper and lower critical dimensions in this problem. As discussed in [1], in the anisotropic case with $m \neq d$, the upper critical dimension is

$$d_u(m) = 4 + \frac{m}{2}. \tag{10}$$

Eq. (10) in the isotropic case with $m = d$ becomes $d_u = 8$, while when $m = 0$ the known result $d_u = 4$ is recovered. The value of the upper critical dimension is also obtained by requiring that there must be at least one relevant interaction operator in the potential in order to have a non-gaussian fixed point. In fact, after expanding the potential in powers of the field, the smallest interaction operator is $\lambda\phi^4$ (cubic and other odd powers are excluded because of the symmetry of the action) and λ is relevant if its scaling dimension is positive. Then, by recalling the dimension of the field, Eq. (3), and the definition of θ, Eq. (2), it is easy to find that the dimension of λ is positive if $d < 4 + m(1 - \theta) - 2\eta_{l2}$. The upper limit, apart from the small corrections due to η_{l2}

and η_{l4}, coincides with Eq. (10). This result confirms our numerical findings in the LPA (where the anomalous dimension is neglected), which show the presence of an isotropic Lifshitz point only below $d = 8$. We shall see that this result holds even in the higher order approximation, where the anomalous dimension turns out to be zero at $d = 8$.

Let us now turn to the lower critical dimension. It is known [4], that in the case of an N-component vector field with symmetry $O(N)$, the lower critical dimension that marks the limit below which the Goldstone fluctuations destroy long range order, is

$$d_l^{O(N)} = 2 + \frac{m}{2}, \tag{11}$$

while for the Ising case, $N = 1$ and $m = 0$, the lower critical dimension becomes:

$$d_l^{Ising} = 1. \tag{12}$$

One immediately realizes that the asymptotic behavior of the fixed point solution of the FRG equations is strictly connected to the lower critical dimension. In fact, as discussed above for the Ising isotropic case with $m = d$ and $N = 1$, the nature of the solutions of Eqs. (6), (7), (8), essentially depends on the sign of D_ϕ, but the same argument could be repeated for the $O(N)$ theory with $m < d$, as the left hand side of the equations which determines the asymptotic behavior of the solution, remains substantially unchanged. Therefore, a discrete spectrum is obtained only if $D_\phi^m > 0$ which, according to Eq. (3) and Eq. (2), gives

$$d > 2 + m(1 - \theta) - \eta_{l2}, \tag{13}$$

and this is in agreement with Eq. (11) if η_{l2} and η_{l4} are neglected.

As a check, we can restrict ourselves to the case $m = 0$ where $\eta_{l4} = 0$ and we can make use of already known results on η_{l2} which, as could be expected, is a function of d. By using FRG techniques, in [46] it is shown that $\eta_{l2} = 0$ for $d \le 2$ in the $O(N)$ theory. Therefore, Eq. (13) reduces to $d > 2$ and we find full agreement with Eq. (11) at $m = 0$ because of the vanishing of the anomalous dimension for $d \le 2$. Again with $m = 0$ but for the Ising case, $N = 1$, a positive $\eta_{l2} > 0$ is obtained for $d \ge 2$ in [46] which indicates that the right hand side in Eq. (13) is now smaller than 2. Unfortunately in [46] η_{l2} is not computed for $d < 2$ and we can only deduce that Eq. (13) is fulfilled for d strictly smaller than 2, which is not in contradiction with Eq. (12), although full matching would require to show that $(1 - \eta_{l2}) \to 0^+$ when $(d - 1) \to 0^+$.

By going back to the isotropic Lifshitz case at $m = d$, we see that, with the help of Eq. (2) and by recalling the definition introduced above, $\eta = \eta_{l4}$, Eq. (13) reduces to $d > 4 - \eta$. This means that in the LPA, where the approximation $\eta = 0$ is used, a change in the spectrum of the solution from discrete to continuous occurs at $d = 4$, as verified in the numerical analysis illustrated above. Then, in the approximation beyond the LPA for $N = 1$, the anomalous dimension turns out to be negative, $\eta < 0$, with the implication that the change in the spectrum occurs at a larger value $d > 4$, and, accordingly, even the lower critical dimension becomes larger than 4.

These examples clearly show the relation between the number of dimensions $\hat{d}$ at which $v^*(x)$ changes its asymptotic behavior from divergent to finite and the lower critical dimension d_l. In fact, since a physically meaningful fixed point must exist above d_l and at the same time such solutions are absent for $d < \hat{d}$, one concludes either that $d_l = \hat{d}$ or, at least, $\hat{d}$ represents a lower bound for d_l.

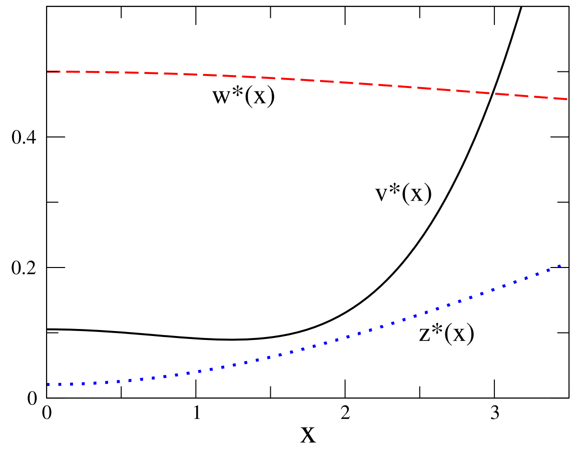

Fig. 3. Solution of the coupled flow equations (6), (7), (8), $v^*(x)$ (black solid), $w^*(x)$ (red dashed) and $z^*(x)$ (blue dotted), computed at $d = 7.5$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Finally, we illustrate the numerical resolution of the FRG equations beyond LPA in two steps. The first one consists in solving the two Eqs. (6), (7) by keeping $z^* = 0$, i.e. by reducing the kinetic part of the action to the $O(\partial^4)$ term, with no $O(\partial^2)$ contribution. The second step is the resolution of the full set of Eqs. (6), (7), (8), including the effects of the $O(\partial^2)$ term which, as already noticed above, is a relevant operator and therefore its role in determining the fixed point solution is essential.

We found a single solution of the coupled equations (6), (7), i.e. a Lifshitz critical point, when d is in the range $5 \le d < 8$. In this approximation the anomalous dimension η is treated as a parameter to be determined in the resolution of the coupled equations. Our finding for η is plotted in Fig. 2 (small red dots) together with the corresponding value of $v^{*\prime\prime}(0)$ (blue triangles pointing downward). The latter shows a small correction with respect to the LPA case, while the former is negative with a non-monotonic behavior and tends to zero when $d \to 8^-$. Once η is determined, we can go back to the definition of upper and lower critical dimension. As anticipated, $\eta = 0$ at $d = 8$ and therefore $d = 8$ coincides with the upper critical dimension. On the other hand, Eq. (13) now gives $d > 4 - \eta \gtrsim 4$, because $\eta < 0$, and $|\eta| \approx O(10^{-1})$.

The final step involves the resolution of the three coupled equations, (6), (7), (8), and in this case the numerical analysis is much more demanding and the accuracy of the results is reduced by a residual dependence on the endpoint of the integration range of the field x. Therefore we solved the equations only for a few values of d. A plot of the solution obtained at $d = 7.5$ is reported in Fig. 3 while the anomalous dimension η (large red circles) and the second derivative of the potential $v^{*\prime\prime}(0)$ (blue squares) are reported in Fig. 2 together with the estimated error on these quantities.

The plots in Fig. 2 show the importance of including all relevant parameters, such as z^* in the determination of the Lifshitz fixed point. In fact, while turning on z does not change the order of magnitude of the anomalous dimension, one observes a drastic change in $v^{*\prime\prime}(0)$ if compared to the previous approximations. In addition, no solution was found at $d = 5$, which suggests that the Lifshitz fixed point is effectively destroyed by the fluctuations induced by z when d is decreased, although we cannot exclude a possible failure in the numerical search of the solution.

5. Conclusions

In conclusion, we have studied the isotropic Lifshitz critical behavior for a single component field theory, i.e. with $m = d$ and $N = 1$, by means of the PTRG flow equations. In particular we solved the fixed point equations first in the lowest order approximation, the LPA, and then in the first and second order of the derivative expansion, by including fluctuations associated to the $O(\partial^4)$ and to the $O(\partial^2)$ operators.

From the constraints on the asymptotic structure of the solution, already in the LPA it is evident that a single physically meaningful fixed point solution can be obtained only for $d > 4$ which can be related to the lower critical dimension and, when the constraint coming from of the upper critical dimension is also included, one gets $4 < d < 8$. This, on one hand, supports the Monte Carlo analysis performed in [18] at $d = 3$ but, on the other hand, strongly questions the reliability of the results on the Lifshitz critical behavior observed in [16] at $d = 3$.

The numerical analysis performed by including the parameter z^*, which is a relevant operator that strongly influences the structure of the solution, shows the existence of a Lifshitz point within the interval $5.5 < d < 8$, and the anomalous dimension η determined at this critical points is always negative and $|\eta| << 1$. In particular, no evidence of a solution in $d \leq 5$ has been found.

A final comment concerns the importance of extending this analysis to the $O(N)$ theory. In fact, if the Lifshitz critical point survives down to $d = 4$ (with $\eta = 0$ at $d = 4$), then, the lower critical dimension $d_l^{O(N)} = 4$ would play for the Lifshitz case the same role of $d = 2$ for the standard critical behavior of the $O(N)$ theory.

Acknowledgements

The authors thank A. Codello and H. Diehl for e-mail correspondence.

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