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The chromatic numbers of double coverings of a graph

Dongseok Kim, Jaeun Lee[1]

Department of Mathematics, Kyungpook National University, Taegu 702-201, Republic of Korea
Department of Mathematics, Yeungnam University, Kyongsan 712-749, Republic of Korea

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Abstract

If we fix a spanning subgraph H of a graph G, we can define a chromatic number of H with respect to G and we show that it coincides with the chromatic number of a double covering of G with co-support H. We also find a few estimations for the chromatic numbers of H with respect to G.

Keywords: Chromatic numbers; Double coverings

1. Introduction

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The cardinality of a set X is denoted by $|X|$. Throughout the paper, we assume all graphs are finite and simple.

The aim of this article is to find adequate formulae for the chromatic numbers of covering graphs. The chromatic number $\chi(G)$ of a graph G is the smallest number of colors needed to color the vertices of G so that no two adjacent vertices share the same color. Since the exploratory paper by Dirac [5], the chromatic number has been in the center of graph theory research. Its rich history can be found in several articles [9,15]. The concept of covering graphs is relatively new [7,8]. Its precise definition can be given as follows. For a graph G, we denote the set of all vertices adjacent to $v \in V(G)$ by $N(v)$ and call it the *neighborhood* of a vertex v. A graph $\widetilde{G}$ is called a *covering* of G with a projection $p : \widetilde{G} \to G$, if there is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. In particular, if p is two-to-one, then the projection $p : \widetilde{G} \to G$ is called a *double covering* of G. Some structures or properties of graphs work nicely with covering graphs. The characteristic polynomials of a covering graph $\widetilde{G}$ and its base graph G have a strong relation [6,12,13]. The enumeration of non-isomorphic covering graphs has been well studied [10,12]. Amit, Linial, and Matousek find the asymptotic behavior of the chromatic numbers of n-fold coverings without considering isomorphic types [1]. We will relate the chromatic number and double covering graphs as follows.

A *signed graph* is a pair $G_\phi = (G, \phi)$ of a graph G and a function $\phi : E(G) \to \mathbb{Z}_2$, $\mathbb{Z}_2 = \{1, -1\}$. We call G the *underlying graph* of G_ϕ and ϕ the *signing* of G. A signing ϕ is in fact a $\mathbb{Z}_2$-voltage assignment of G, which was defined by Gross and Tucker [7]. It is known [7,8] that every double covering of a graph G can be constructed as follows:

[1] The author was supported in part by Com^{2}MaC-KOSEF(R11-1999-054).
E-mail addresses: dongseok@knu.ac.kr (D. Kim), julee@yu.ac.kr (J. Lee).

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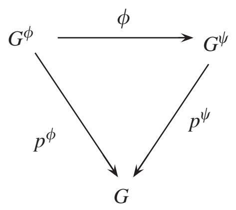

Fig. 1. Commuting diagram of isomorphic coverings.

let ϕ be a signing of G. The double covering G^ϕ of G derived from ϕ has the following vertex set $V(G^\phi)$ and edge set $E(G^\phi)$,

$$V(G^\phi) = \{v_g | v \in V(G) \text{ and } g \in \mathbb{Z}_2\},$$

$$E(G^\phi) = \{(u_g, v_{\phi(u,v)g}) | (u, v) \in E(G), g \in \mathbb{Z}_2\}.$$

Two double coverings $p^\phi : G^\phi \to G$ and $p^\psi : G^\psi \to G$ are *isomorphic* if there exists a graph isomorphism $\phi : G^\phi \to G^\psi$ such that the diagram in Fig. 1 commutes.

For a spanning subgraph H of G, colorings f and g of H are *compatible* in G if for each edge $(u, v) \in E(G) - E(H)$, $f(u) \neq g(v)$ and $f(v) \neq g(u)$. The smallest number of colors such that H has a pair of compatible colorings is called the *chromatic number of H with respect to G* and denoted by $\chi_G(H)$. Since $(f|_H, f|_H)$ is a pair of compatible colorings of H for any spanning subgraph H of G and any coloring f of G, one can find $\chi_G(H) \leqslant \chi(G)$ for any spanning subgraph H of G. We remark that $\chi_G(G) = \chi(G)$ for any graph G, and that $\chi_G(\mathcal{N}_{|V(G)|}) = 2$ if G has at least one edge, where $\mathcal{N}_n$ is the null graph on n vertices.

In Section 2, we recall some basic properties. We show that the chromatic numbers of double coverings of a given graph can be computed from the number $\chi_G(H)$ for any spanning subgraph H of G. In Section 3, we will estimate the number $\chi_G(H)$. We discuss a generalization to n-fold covering graphs in Section 4.

2. Basic properties

Let ϕ be a signing of G. We define the *support* of ϕ by the spanning subgraph of G whose edge set is $\phi^{-1}(-1)$, and denoted by $spt(\phi)$. Similarly, we define the *co-support* of ϕ by the spanning subgraph of G whose edge set is $\phi^{-1}(1)$, and denoted by $\mathrm{cospt}(\phi)$. Any spanning subgraph H of G can be described as a co-support $\mathrm{cospt}(\phi)$ of a signing ϕ of G. Let ϕ_H be the signing of G with $\mathrm{cospt}(\phi_H) = H$. Let f and g be compatible $\chi_G(H)$-colorings of H. We define a function

$$h : V(G^\phi) \to \{1, 2, \ldots, \chi_G(H)\}$$

by $h(v_1) = f(v)$ and $h(v_{-1}) = g(v)$ for each $v \in V(G)$. Then, by the compatibility of f and g, h is a $\chi_G(H)$-coloring of G^ϕ. Hence, $\chi(G^\phi) \leqslant \chi_G(H)$. Conversely, let h be a $\chi(G^\phi)$-coloring of G^ϕ. We define two $\chi(G^\phi)$-colorings f and g of H by $f(v) = h(v_1)$ and $g(v) = h(v_{-1})$ for each $v \in V(G)$. Then f and g are compatible because h is a coloring of G^ϕ. Hence, $\chi_G(H) \leqslant \chi(G^\phi)$. Now, we have the following theorem.

Theorem 1. *Let H be a spanning subgraph of a graph G. Then*

$$\chi_G(H) = \chi(G^{\phi_H}),$$

where ϕ_H is the signing of G with $\mathrm{cospt}(\phi_H) = H$.

It is not hard to see that the graph G in Fig. 2 has two non-isomorphic connected double coverings. We exhibit spanning subgraphs H_1, H_2 of G corresponding to two non-isomorphic connected covering graphs of G and their chromatic numbers with respect to G in Fig. 3.

For a subset $X \subset V(G)$ and for a spanning subgraph H of G, let H_X denote a new spanning subgraph of G defined as follow: Two vertices in X or in $V(G) - X$ are adjacent in H_X if they are adjacent in H, while a vertex in X and a

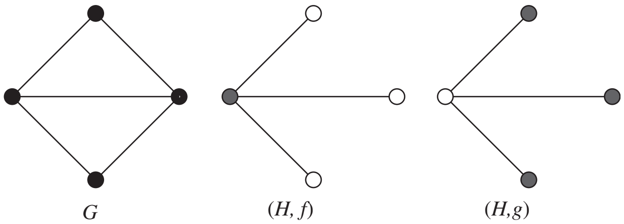

Fig. 2. A spanning subgraph H of G with $\chi_G(H) = 2$.

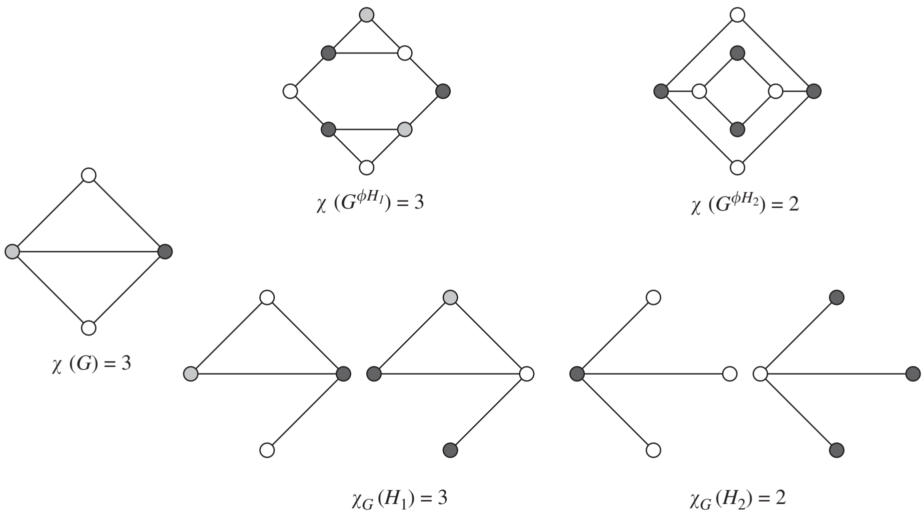

Fig. 3. $\chi_G(H)$ and the chromatic number of double coverings G^{ϕ_H} of a graph G.

vertex in $V(G) - X$ are adjacent in H_X if they are adjacent in G but not adjacent in H, i.e., they are adjacent in the complement $\bar{H}(G)$ of H in G. Two spanning subgraphs H and K of G are *Seidel switching equivalent* in G if there exists a subset $X \subset V(G)$ such that $H_X = K$. Clearly, the Seidel switching equivalence is an equivalence relation on the set of spanning subgraphs of G, and the equivalence class $[H]$ of a spanning subgraph H of G is $\{H_X : X \subset V(G)\}$.

For a signing $\phi : E(G) \to \mathbb{Z}_2$ and for any $X \subset V(G)$, let ϕ_X be the signing obtained from ϕ by reversing the sign of each edge having exactly one end point in X. If $\psi = \phi_X$ for some $X \subset V(G)$ then ϕ and ψ are said to be *switching equivalent* [4].

It is clear that for a subset $X \subset V(G)$ and for a spanning subgraph H of G, $H_X = \mathrm{cospt}((\phi_H)_X)$. By a slight modification of the proof of Corollary 4 [11], we obtain the following theorem.

Theorem 2. *Let G be a graph. Let H, K be spanning subgraphs of G. Then the following statements are equivalent.*

(1) *Two graphs H and K are Seidel switching equivalent.*
(2) *Two signings ϕ_H and ϕ_K are switching equivalent.*
(3) *Two double coverings G^{ϕ_H} and G^{ϕ_K} of G are isomorphic as coverings.*

The following corollary follows easily from Theorem 1 and 2.

Corollary 3. *Let H and K be two spanning subgraphs of a graph G. If they are switching equivalent, then*

$$\chi_G(H) = \chi_G(K).$$

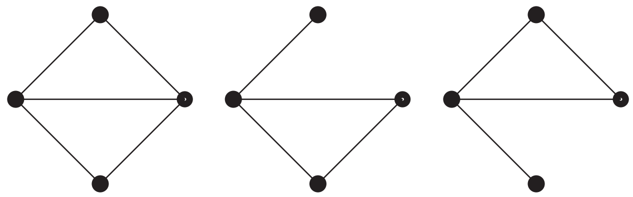

Fig. 4. Two non-switching equivalent subgraphs H, K in G with $\chi_G(H)=\chi_G(K)$.

The converse of Corollary 3 is not true in general. We have provided two non-switching equivalent spanning subgraphs H, K in G with $\chi_G(H)=\chi_G(K)=3$ in Fig. 4.

For a coloring f of H, let $\mathscr{I}_f$ be the number of colors in $\{1,2,\ldots,\chi(H)\}$ such that the preimage $f^{-1}(i)$ is independent in $\bar{H}(G)$ (and hence, also in G).

Corollary 4. *Let G be a connected graph and let H be a spanning subgraph of G. Then*

$$\max_{K\in[H]}\{\chi(K)\}\leqslant\chi_G(H)\leqslant\min_{K\in[H],f}\{\chi(G),2\chi(K)-\mathscr{I}_f\},$$

where f runs over all $\chi(K)$-colorings of K.

Proof. It is clear that $\chi(H)\leqslant\chi_G(H)\leqslant\chi(G)$. Let f be a $\chi(H)$-coloring of H such that

$$\{i\,|\,f^{-1}(i)\text{ is independent in }G\}=\{\chi(H),\chi(H)-1,\ldots,\chi(H)-\mathscr{I}_f+1\}.$$

We define a function $g:V(H)\to\{1,2,\ldots,2\chi(H)-\mathscr{I}_f\}$ as follows: For a vertex v in $V(H)$,

$$g(v)=\begin{cases} f(v) & \text{if } \chi(H)-\mathscr{I}_f+1\leqslant f(v)\leqslant\chi(H),\\ f(v)+\chi(H) & \text{otherwise.}\end{cases}$$

Then g is a coloring of H, and f and g are compatible. Now, the corollary comes from Corollary 3.

For a partition $\mathscr{P}=\{V_1,V_2,\ldots,V_k\}$ of the vertex set $V(G)$ of G, we define a new simple graph $G/\mathscr{P}$ as follows: the vertex set of $G/\mathscr{P}$ is $\{V_1,V_2,\ldots,V_k\}$ and there is an edge between two vertices V_i and V_j in $G/\mathscr{P}$ if and only if there exist two vertices $v_i\in V_i$ and $v_j\in V_j$ such that v_i and v_j are adjacent in G where $i\neq j$. We call $G/\mathscr{P}$ the *quotient graph* associated with a partition $\mathscr{P}$. For a subset S of $V(G)$, let $G[S]$ be the subgraph of G whose vertex set S and whose edge set is the set of those edges of G that have both ends in S. We call $G[S]$ the *subgraph induced by S*. □

Corollary 5. *Let $\mathscr{P}=\{V_1,V_2,\ldots,V_k\}$ be a partition of the vertex set of a connected graph G and let $H=\cup_{i=1}^k G[V_i]$ be the disjoint union of the induced subgraphs $G[V_i]$. If $G/\mathscr{P}$ is bipartite, then $\chi(G)=\chi_G(H)$.*

Proof. Let $X=\{[v_{i_1}],[v_{i_2}],\ldots,[v_{i_k}]\}$ be a part of the bipartition of the vertex set of the bipartite graph $G/\mathscr{P}$. Then $H_{\cup_{j=1}^k V_{i_j}}=G$. Then, the corollary follows from Corollary 3 or 4. □

The following theorem finds a necessary and sufficient condition for the bipartiteness of covering graphs.

Theorem 6 (*Archdeacon et al.* [2]). *Let G be a non-bipartite graph with a generating voltage assignment ν in $\mathscr{A}$ which derives the covering graph $\tilde{G}$. Then $\tilde{G}$ is bipartite if and only if there exists a subgroup $\mathscr{A}_e$ of index two in $\mathscr{A}$ such that for every cycle C, $\nu(C)\in\mathscr{A}_e$ if and only if the length of C is even.*

It is obvious that $\chi_G(H)=1$ if and only if G is a null graph. In Theorem 7, we find a necessary and sufficient condition of $\chi_G(H)=2$.

Theorem 7. *Let G be a connected graph having at least one edge and let H be a spanning subgraph of G. Then* $\chi_G(H) = 2$ *if and only if either G is bipartite or* $H \in [\mathcal{N}_{|V(G)|}]$, *where* $\mathcal{N}_{|V(G)|}$ *is the null graph on* $|V(G)|$ *vertices.*

Proof. Let G be a bipartite graph and H be a spanning subgraph of G. Then there exists a graph K in the switching class $[H]$ of H in G such that K has at least one edge. By Corollary 4, $\chi_G(H) = 2$. We recall that G itself is a spanning subgraph of G and $\chi_G(G) = \chi(G)$. Therefore, if a graph G has at least one edge, then G is bipartite if and only if $\chi_G(H) = 2$ for any spanning subgraph H of G.

If G is not a bipartite graph, there exist a signing ϕ of G such that G^ϕ is bipartite [7]. It follows the connectedness of G that there exists a subset Y of $V(G)$ such that $\operatorname{cospt}(\phi_Y)$ is connected. Since G^ϕ and G^{ϕ_Y} are isomorphic, by Theorem 2, G^{ϕ_Y} is bipartite. We note that $\operatorname{cospt}(\phi_Y)$ is isomorphic to a subgraph of G^{ϕ_Y} and hence it is bipartite. Let e be an edge of G such that one end is in Y and the other is in $V(G) - Y$. If $\phi_Y(e) = -1$, then there exists an even cycle which contains the edge e as the only edge whose value under ϕ_Y is -1. It follows from Theorem 6 that G^{ϕ_Y} is not bipartite. This is a contradiction. It implies that for an edge e of G, $\phi_Y(e) = 1$ if and only if one end of e is in Y and the other is in $V(G) - Y$. Let X be a part of the bipartition of $\operatorname{cospt}(\phi_Y)$, i.e., every edge e in $\operatorname{cospt}(\phi_Y)$ has one end in X and the other end in $V(G) - X$. Then $\operatorname{cospt}((\phi_Y)_X) = \mathcal{N}_{|V(G)|}$. Notice that $\operatorname{cospt}((\phi_Y)_X) = \operatorname{cospt}(\phi_Z)$, where $Z = (Y - X) \cup (X - Y)$. Hence H is switching equivalent to $\mathcal{N}_{|V(G)|}$. It completes the proof of theorem. □

3. Computations of $\chi_G(H)$

In this section, we aim to estimate the number $\chi_G(H)$ for any spanning subgraph H of G. Let H be a spanning subgraph of G, and let $F_1, F_2, \ldots, F_k$ be the components of the complement $\bar{H}(G)$ of H in G. Then $\mathscr{P}_{\bar{H}} = \{V(F_1), V(F_2), \ldots, V(F_k)\}$ is a partition of the vertex set $V(H) = V(G)$. Now, we consider a $\chi(H/\mathscr{P}_{\bar{H}})$-coloring c of the quotient graph $H/\mathscr{P}_{\bar{H}}$. Then c induces a partition $\mathscr{P}_c = \{c^{-1}(1), \ldots, c^{-1}(\chi(H/\mathscr{P}_{\bar{H}}))\}$ of the vertex set $H/\mathscr{P}_{\bar{H}}$. By composing the quotient map : $G \to H/\mathscr{P}_{\bar{H}}$ and $c : H/\mathscr{P}_{\bar{H}} \to \{1, 2, \ldots, \chi(H/\mathscr{P}_{\bar{H}})\}$, we obtain a partition of H and by slightly abusing notation we denoted it identically $\mathscr{P}_c$. One can notice that each vertex of $H/\mathscr{P}_c$ can be considered as a union of the vertex sets $V(F_1), \ldots, V(F_k)$. For each $i = 1, \ldots, \chi(H/\mathscr{P}_{\bar{H}})$, let $H_c(i) = H[c^{-1}(i)]$, where we consider $c^{-1}(i)$ as a subset of $V(H) = V(G)$. A coloring f of H *respects the coloring c* of $H/\mathscr{P}_{\bar{H}}$ if $|f(H_c(i))| = \chi(H_c(i))$ and $f(H_c(i)) \cap f(H_c(j)) = \emptyset$ for any $1 \leqslant i \neq j \leqslant \chi(H/\mathscr{P}_{\bar{H}})$. For a coloring f which respects c, let $\mathscr{I}_f(i)$ be the number of colors in $\{i_1, i_2, \ldots, i_{\chi(H_c(i))}\}$ such that the vertex set $f^{-1}(i_k)$ is independent in $\bar{H}(G)$ and let $\mathscr{D}_f(i) = \chi(H_c(i)) - \mathscr{I}_f(i)$ for each $i = 1, \ldots, \chi(H/\mathscr{P}_{\bar{H}})$. Let

$$\Delta_S = \max\left\{0,\ 2\max\{s \mid s \in S\} - \sum_{s\in S} s\right\}$$

for any subset S of natural numbers.

Theorem 8. *Let G be a connected graph and let H be a spanning subgraph of G. Then*

$$\chi_G(H) \leqslant \min_c \left\{ \sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})} \chi(H_c(i)) + \Delta_{\{\chi(H_c(i)) \mid i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})\}},\right.$$
$$\left. \sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})} \chi(H_c(i)) + \min_f \left\{\Delta_{\{\mathscr{D}_f(i) \mid i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})\}}\right\} \right\},$$

where c runs over all $\chi(H/\mathscr{P}_{\bar{H}})$*-colorings of* $H/\mathscr{P}_{\bar{H}}$ *and f runs over all colorings of H which respect c.*

Proof. Let c be a $\chi(H/\mathscr{P}_{\bar{H}})$-coloring of $H/\mathscr{P}_{\bar{H}}$ and let f be a coloring of H which respects c.

First, we want to show that

$$\chi_G(H) \leqslant \sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})} \chi(H_c(i)) + \Delta_{\{\chi(H_c(i)) \mid i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})\}}.$$

Without loss of generality, we may assume that $\chi(H_c(1))\geqslant\chi(H_c(2))\geqslant\cdots\geqslant\chi(H_c(\chi(H/\mathscr{P}_{\bar{H}})))$. Let the image

$$f(V(H_c(i)))=\left\{\sum_{j=1}^{i-1}\chi(H_c(j))+1,\ldots,\sum_{j=1}^{i}\chi(H_c(j))\right\}$$

and

$$\ell=\Delta_{\{\chi(H_c(i))\}\}|i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})\}}.$$

Then

$$\ell=\max\left\{\chi(H_c(1))-\sum_{i=2}^{\chi(H/\mathscr{P}_{\bar{H}})}\chi(H_c(i)),0\right\}.$$

We define $g:V(H)\to\{1,2,\ldots,n+\ell\}$ by $g(v)=f(v)-\chi(H_c(1))$, where

$$n=\sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})}\chi(H_c(i))$$

and the arithmetic is done by modulo $n+\ell$. Then g is a coloring of H. Since $f(V(H_c(i)))\cap g(V(H_c(i)))=\emptyset$ and each edge in $E(G)-E(H)=E(\bar{H}(G))$ connects two vertices in $H_c(i)$ for some $i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})$, we can see that f and g are compatible. Hence,

$$\chi_G(H)\leqslant\sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})}\chi(H_c(i))+\ell.$$

Next, we want to show that

$$\chi_G(H)\leqslant\sum_{i=1}^{\chi(H/\mathscr{P}_{\bar{H}})}\chi(H_c(i))+\Delta_{\{\mathscr{D}_f(i)|i=1,2,\ldots,\chi(H/\mathscr{P}_{\bar{H}})\}}.$$

In general, we may assume that $\mathscr{D}_f(1)\geqslant\mathscr{D}_f(2)\geqslant\cdots\geqslant\mathscr{D}_f(\chi(H/\mathscr{P}_{\bar{H}}))$. Now, we aim to define another coloring g of H such that f and g are compatible. To do this, first for the vertices v of H such that the set $f^{-1}(f(v))$ is independent in $\bar{H}(G)$, we define $g(v)=f(v)$. Next, by using a method similar to the first case, we can extend the function g to whole graph H so that f and g are compatible colorings of H.

Finally, by taking the minimum value among all $\chi(H/\mathscr{P}_{\bar{H}})$-coloring c of $H/\mathscr{P}_{\bar{H}}$ and all coloring f of H which respect c, we have the theorem. □

The following example shows the upper bound in Theorem 8 is sharp.

Example 9. Let m,n be integers with $2\leqslant m\leqslant n$. Let K_{m-1} be the complete graph on $m-1$ vertices $v_1,\ldots,v_{m-1}$. Let H_m be a spanning subgraph of K_n obtained by adding $n-m+1$ isolated vertices $v_m,\ldots,v_n$ to K_{m-1}. Then $\chi_{K_n}(H_m)=m$.

Proof. To show $m\leqslant\chi_{K_n}(H_m)$, we set $X=V(K_{m-1})$. Then $\chi((H_m)_X)=m$ and hence $m\leqslant\chi_{K_n}(H_m)$ by Corollary 4. We can show that $\chi_{K_n}(H_m)\leqslant m$ by using two methods which are contained in the proof of Theorem 8. For the first method, we replace H_m by $(H_m)_X$. We observe that $\overline{(H_m)_X}(K_n)=K_{n-m+1}\cup\{v_1,\ldots,v_{m-1}\}$ and $(H_m)_X/\mathscr{P}_{(\bar{H}_m)_X}=K_m$. Let c be a (m)-coloring of $(H_m)_X/\mathscr{P}_{\bar{H}_X}$ such that $V((H_m)_c(i))=\{v_i\}$ for each $i=1,2,\ldots,m-1$ and $V((H_m)_c(m+1))=\{v_m,\ldots,v_n\}$. We note that $\chi((H_m)_c(i))=1$ for each $i=1,2,\ldots,m$. Since $\Delta_{\{1,1,\ldots,1\}}=0$, by Theorem 8, we have $\chi_{K_n}(H_m)=\chi_{K_n}((H_m)_X)\leqslant m$. For the second method, let c be the trivial coloring of $H_m/\mathscr{P}_{\overline{H_m}}=K_1$ and let f

be a $(m-1)$-coloring of H_m such that $f(v_i)=i$ for each $i=1,2,\ldots,m-1$ and $f(v_m)=f(v_{m+1})=\cdots=f(v_n)=1$. Then f respects c and

$$\mathscr{D}_f(1)=\chi((H_m)_c(1))-\mathscr{D}_f(1)=\chi(H_m)-\mathscr{D}_f(1)=(m-1)-(m-2)=1.$$

Since $\Delta_{\{1\}}=2-1=1$, by Theorem 8, we have $\chi_{K_n}(H_m)\leqslant m$. □

Example 9 can be generalized to the following corollary.

Corollary 10. *Let H be a complete m-partite graph which is a spanning subgraph of K_n. Then $\chi_{K_n}(H)=m$.*

Proof. We observe that the complement $\bar{H}(K_n)$ of H is also a spanning subgraph of K_n having at least k components of which each vertex set is a subset of a part of H. It is not hard to show that $H/\mathscr{P}_{\bar{H}}$ is also a complete m-partite graph and $\chi(H_c(i))=1$ for each $i=1,\ldots,m$. By Theorem 8, $\chi_G(H)\leqslant m$. Since $\chi(H)=m$, by Corollary 4, it completes the proof. □

If each component of a spanning subgraph H of a graph G is a vertex induced subgraph, we can have an upper bound of the chromatic number of H induced by G which is simpler than that in Theorem 8.

Theorem 11. *Let $\mathscr{P}=\{V_1,V_2,\ldots,V_k\}$ be a partition of the vertex set of a connected graph G. Let H be a disjoint union of the induced subgraphs $G[V_1],G[V_2],\ldots,G[V_n]$. Then we have*

$$\max_{V_i,V_j}\{\chi(G[V_i\cup V_j])\}\leqslant\chi_G(H)\leqslant\max_{V_i,V_j}\{\chi(G[V_i])+\chi(G[V_j])\},$$

where V_i and V_j runs over all pairs of adjacent vertices in $G/\mathscr{P}$.

Proof. Let V_i and V_j be two adjacent vertices in $G/\mathscr{P}$. Then $G[V_i\cup V_j]$ is a subgraph of H_{V_i}. By Corollary 4, $\chi(G[V_i\cup V_j])\leqslant\chi(H_{V_i})\leqslant\chi_G(H)$ and hence

$$\max\{\chi(G[V_i\cup V_j])|V_i \text{ is adjacent to } V_j \text{ in } G/\mathscr{P}\}\leqslant\chi_G(H).$$

For the second inequality, let

$$M=\max\{\chi(G[V_i])+\chi(G[V_j])|V_i \text{ is adjacent to } V_j \text{ in } G/\mathscr{P}\}.$$

By the definition of M, there exist s, t and M-coloring $f:V(H)\to\{1,2,\ldots,M\}$ of H such that

$$\chi(G[V_s])+\chi(G[V_t])=M$$

and

$$f(G[V_i])=\{1,2,\ldots,\chi(G[V_i])\}$$

for each $i=1,2,\ldots,k$. We note that f may not be surjective. We define another M-coloring g of H such that $g(G[V_i])=\{M,M-1,\ldots,M-\chi(G[V_i])+1\}$. Now, we aim to show that f and g are compatible. Let uv be an edge of G which is not in $E(H)$. Now, by the construction of $G/\mathscr{P}$, then there exist i and j such that $u\in V_i$ and $v\in V_j$. By the definition of $G/\mathscr{P}$, V_i is adjacent to V_j in $G/\mathscr{P}$. If $f(V_i)\cap g(V_j)\neq\emptyset$, then, by the construction of f and g, $M<\chi(G[V_i])+\chi(G[V_j])$. This contradicts the hypothesis of M. Thus, $f(V_i)\cap g(V_j)=\emptyset$. Similarly, we can see that $g(V_i)\cap f(V_j)=\emptyset$. Therefore, $f(u)\neq g(v)$ and $g(u)\neq f(v)$, i.e., f and g are compatible. It completes the proof. □

By Corollary 5 and Theorem 11, we have the following corollaries.

Corollary 12. *Let $\mathscr{P}=\{V_1,V_2,\ldots,V_k\}$ be a partition of the vertex set of a connected graph G. If $G/\mathscr{P}$ is bipartite, then*

$$\max_{V_i,V_j}\{\chi(G[V_i\cup V_j])\}\leqslant\chi(G)\leqslant\max_{V_i,V_j}\{\chi(G[V_i])+\chi(G[V_j])\},$$

where V_i and V_j runs over all pairs of adjacent vertices in $G/\mathscr{P}$.

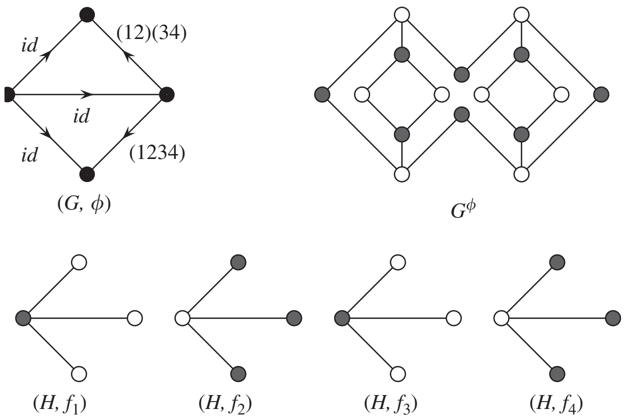

Fig. 5. A permutation voltage assignment ϕ of G, compatible colorings of $H = \mathrm{cospt}(\phi)$ with $\chi_G(H) = 2$ and its corresponding 4-fold covering graph G^{ϕ}.

Corollary 13. *Let H be a spanning subgraph of a connected graph G such that H has k components $H_1, H_2, \ldots, H_k$ with $\chi(H_i) \geqslant \chi(H_{i+1})$ for each $i = 1, 2, \ldots, k-1$.*

(1) *If the complement $\bar{H}(G)$ of H in G is the complete k partite graph, then we have $\chi_G(H) = \chi(H_1) + \chi(H_2)$.*
(2) *If each component of H is the complete graph, i.e, $H_i = K_{\ell_i}$ for each $i = 1, 2, \ldots, k$. Then we have $\chi_G(H) \leqslant \ell_1 + \ell_2$. In particular, if G is the complete graph K_n, then we have $\chi_G(H) = \ell_1 + \ell_2$.*

Proof. We observe that, in any case, each component H_i of H is an induced subgraph $G[V(H_i)]$ for each $i = 1, 2, \ldots, k$, and $\mathscr{P} = \{V(H_i) | i = 1, 2, \ldots, k\}$ forms a partition of G.

(1) If the complement $\bar{H}(G)$ is the complete k partite graph, then $H/\mathscr{P}$ is the complete graph K_k, i.e., each pair of vertices in $H/\mathscr{P}$ is adjacent in $H/\mathscr{P}$. Since $\chi(G[V(H_1) \cup V(H_2)]) = \chi(H_1) + \chi(H_2)$ and $\max\{\chi(G[V(H_i)]) + \chi(G[V(H_j)]) \,|\, 1 \leqslant i \neq j \leqslant k\} = \chi(H_1) + \chi(H_2)$, by Theorem 11, we have $\chi_G(H) = \chi(H_1) + \chi(H_2)$.
(2) If $H_i = K_{\ell_i}$ for each $i = 1, 2, \ldots, k$, then, By Theorem 11, we have $\chi_G(H) \leqslant \ell_1 + \ell_2$. If G is the complete graph K_n, then $\bar{H}(G)$ of H in G is the complete k partite graph, by (1), we have $\chi_G(H) = \ell_1 + \ell_2$. □

4. Further remarks

4.1. Existence of a spanning subgraph H of G with $\chi_G(H) = m$ for any m with $2 \leqslant m \leqslant \chi(G)$

For $n \geqslant 2$, and for any spanning subgraph H of a complete graph K_n, we have $2 \leqslant \chi_{K_n}(H) \leqslant n$. For converse, we showed that for any integer m, between 2 and n, there exists a spanning subgraph H_m of K_n such that $\chi_{K_n}(H_m) = m$ in Example 9. One can ask this can be extended to an arbitrary connected graph. Let G be connected graph. For any m with $2 \leqslant m \leqslant \chi(G)$, let H be an m-critical subgraph of G, that is, $\chi(H) = m$ and for any proper subgraph S of H $\chi(S) < m$. Let $\tilde{H}$ be a spanning subgraph of G obtained by adding $|V(G)| - |V(H)| + 1$ isolated vertices to $H - v$ for some v in $V(H)$. By Theorem 11, $\chi_G(\tilde{H}) = m$.

4.2. n-Fold covering graphs

For n-fold covering graphs, let S_n denote a symmetric group on n elements $\{1, 2, \ldots, n\}$. Every edge of a graph G gives rise to a pair of oppositely directed edges. We denote the set of directed edges of G by $D(G)$. By e^{-1} we mean the reverse edge to an edge e. Each directed edge e has an initial vertex i_e and a terminal vertex t_e. A *permutation voltage assignment* ϕ on a graph G is a map $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The *permutation derived graph* G^{ϕ} is defined as follows: $V(G^{\phi}) = V(G) \times \{1, \ldots, n\}$, and for each edge $e \in D(G)$ and

$j \in \{1, \ldots, n\}$ let there be an edge (e, j) in $D(G^{\phi})$ with $i_{(e,j)} = (i_e, j)$ and $t_{(e,j)} = (t_e, \phi(e)j)$. The natural projection $p_{\phi} : G^{\phi} \to G$ is a covering. In [7,8], Gross and Tucker showed that every n-fold covering $\tilde{G}$ of a graph G can be derived from a voltage assignment.

Let H be a spanning subgraph of a graph G which is the co-support of ϕ, *i. e.*, $V(H) = V(G)$ and $E(H) = \phi^{-1}(\text{id})$, where id is the identity element in S_n and for $E(H)$, we identify each pair of oppositely directed edges of $\phi^{-1}(\text{id})$. Then our chromatic number of H respect G naturally extends as follows; colorings $f_1, f_2, \ldots, f_n$ of H are *compatible* if for $e^+ = (u, v) \in D(G) - D(H)$, $f_i(u) \neq f_{\phi((u,v)(i)}(v)$ for $i = 1, 2, \ldots, n$. The smallest number of colors such that H has an n-tuple of compatible colorings is called the nth *chromatic number of H with respect to G* and denoted by $\chi_G(H)$. Unlike two fold coverings, estimations of the nth chromatic numbers of H with respect to G are not easy. The asymptotic behavior of the chromatic numbers of non-isomorphic n-fold coverings could be very fascinating compare to the result by Amit et al. [1].

We conclude the discussion with an example. It is easy to see that all odd-fold covering graphs of the graph G in Fig. 2 have chromatic number 3. We provide a 4-fold covering graph induced by the coloring ϕ in Fig. 5 together with 4 compatible colorings f_1, f_2, f_3 and f_4 of the spanning subgraph $H = \text{cospt}(\phi)$.

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