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A nonautonomous predator–prey system with stage structure and double time delays ☆

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ABSTRACT

In the present paper we study a nonautonomous predator–prey model with stage structure and double time delays due to maturation time for both prey and predator. We assume that the immature and mature individuals of each species are divided by a fixed age, and the mature predator only attacks the immature prey. Based on some comparison arguments we discuss the permanence of the species. By virtue of the continuation theorem of coincidence degree theory, we prove the existence of positive periodic solution. By means of constructing an appropriate Lyapunov functional, we obtain sufficient conditions for the uniqueness and the global stability of positive periodic solution. Two examples are given to illustrate the feasibility of our main results.

1. Introduction

In the past several decades, the predator–prey systems play an important role in the modeling of multi-species population dynamics [1,2]. Many models of population growth were studied with time delays [3–10]. Some other age- and stage-structured models of various types (discrete and distributed time delays, stochastic, etc.) have been utilized [11–29]. In the pioneering work [29], a stage-structured model of population growth consisting of immature and mature individuals was proposed, where the stage-structure was modeled by the introduction of a constant time delay, reflecting a delayed birth of immature and a reduced survival of immature to their maturity. The model takes the form

$$\begin{cases}\dot{x}_i(t) = \alpha x_m(t) - \gamma x_i(t) - \alpha \mathrm{e}^{-\gamma\tau} x_m(t-\tau),\\ \dot{x}_m(t) = \alpha \mathrm{e}^{-\gamma\tau} x_m(t-\tau) - \beta x_m^2(t),\end{cases}$$

where x_i and $x_m(t)$ represent the immature and mature populations densities respectively, to model stage-structured population growth. There, $\alpha > 0$ represents the birth rate, $\gamma > 0$ is the immature death rate, $\beta > 0$ is the mature death and overcrowding rate, and τ is the time to maturity. The term $\alpha \mathrm{e}^{-\gamma\tau} x_m(t-\tau)$ represents the immature who were born at time $t-\tau$ and survive at time t with the immature death rate γ, and thus represents the transformation of immature to mature.

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In the classical predator–prey models it is usually assumed that each individual predator admits the same ability to feed on prey and each individual prey admits the same risk to be attacked by predator. However, considerable evidence shows that there are some species whose individuals have a life history that takes them through two stages, immature and mature, where immature predators are raised by their parents, and the rate they attack at prey and the reproductive rate can be ignored. Many periodic ratio-dependent predator–prey models with time delays and stage structure for both prey and predator were investigated and rich dynamics have been observed [30–39] and references therein. Many models assumed that predators only consume mature preys [18,20,24]. On the other hand, in the natural world, there are many species in which only immature individuals are consumed by their predators, and some other species in which more immature individuals than mature individuals are consumed by their predators. One typical example was described in [40], where *Chinese fire-bellied newt*, which is unable to feed on the mature *Rana chensinensis*, can only feed on the immature one. Based on this assumption, Zhang and his co-workers introduced an ordinary differential system and studied the global stability of the nonnegative equilibrium [17]. Moreover, biological or environmental parameters are naturally periodic subject to fluctuation in time. Effects of a periodically varying environment are important for evolutionary theory as the selective forces on systems in a fluctuating environment differ from those in a stable environment. Thus, the assumption on periodicity of parameters is a way of incorporating the periodicity of the environment (such as seasonal effects of weather, food supplies, mating habits and so forth). In [18], Xu and Wang incorporated the periodicity of the ecological and environmental parameters into a generalized system of (1) and assumed that the reproductive rate of predator during the immature stage is zero.

Mainly motivated by works [17,18,29,40], in this paper we consider the effects of time delays in combination with the periodicity of ecological and environmental parameters in the following stage-structured predator–prey system:

$$\begin{cases} x_1'(t) = a_1(t)x_2(t) - r_1(t)x_1(t) - a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds}x_2(t-\tau_1) - k_1(t)x_1(t)y_2(t), \\ x_2'(t) = a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds}x_2(t-\tau_1) - \beta_1(t)x_2^2(t), \\ y_1'(t) = a_2(t)x_1(t)y_2(t) - r_2(t)y_1(t) - a_2(t-\tau_2)e^{\int_{t-\tau_2}^{t} -r_2(s)ds} \cdot x_1(t-\tau_2)y_2(t-\tau_2), \\ y_2'(t) = a_2(t-\tau_2)e^{\int_{t-\tau_2}^{t} -r_2(s)ds}x_1(t-\tau_2)y_2(t-\tau_2) - \beta_2(t)y_2^2(t), \end{cases} \tag{1}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature and the mature prey at time t, respectively; $y_1(t)$ and $y_2(t)$ represent the densities of the immature and the mature predator population at time t, respectively; $a_1(t), a_2(t), r_1(t), r_2(t), \beta_1(t), \beta_2(t), k_1(t)$ are continuously positive periodic functions with period ω. The model is constructed under the following assumptions for both prey and predator.

(I) The prey population: the birth rate of the immature population is proportional to the existing mature population with a proportionality $a_1(t) > 0$; the death rate of the immature population is proportional to the existing immature population with a proportionality $r_1(t) > 0$; the death rate of the mature population is of a logistic nature, i.e., it is proportional to the square of the population with a proportionality $\beta_1(t) > 0$. The term

$$a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds}x_2(t-\tau_1)$$

represents the number of immature preys that were born at time $t-\tau_1$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. The predator population only feeds on the immature prey.

(II) The predator population: the death rate of the immature population is proportional to the existing immature population with a proportionality $r_2(t) > 0$; $k_1(t) > 0$ is the capturing rate of mature predator; $\frac{a_2(t)}{k_1(t)}$ is the rate of conversion of nutrients into the reproduction of the mature predator; the death rate of the mature population is of a logistic nature, i.e., it is proportional to the square of the population with a proportionality $\beta_2(t) > 0$. The term

$$a_2(t)e^{\int_{t-\tau_2}^{t} -r_2(s)ds}x_1(t-\tau_2)y_2(t-\tau_2)$$

represents the number of immature predators that were born at time $t-\tau_2$ which still survive at time t and are transferred from the immature stage to the mature stage at time t. It is assumed in model (1) that immature individual predators do not feed on prey and do not have the ability to reproduce.

In our study, we assume that the initial conditions of system (1) take the form:

$$\begin{cases} x_i(\theta) = \phi_i(\theta) > 0, \quad -\tau_1 \le \theta \le 0, i = 1, 2, \\ y_i(\theta) = \psi_i(\theta) > 0, \quad -\tau_2 \le \theta \le 0, i = 1, 2. \end{cases} \tag{2}$$

For the continuity of the initial conditions, we require

$$\begin{cases} x_1(0) = \displaystyle\int_{-\tau_1}^{0} a_1(\theta)\phi_2(\theta)e^{\int_0^{\theta}(r_1(s)+k_1(s)\psi_2(s))ds}d\theta, \\ y_1(0) = \displaystyle\int_{-\tau_2}^{0} a_2(\theta)e^{\int_0^{\theta} r_2(s)ds}\phi_1(\theta)\psi_2(\theta)d\theta. \end{cases} \tag{3}$$

Throughout this paper, for convenience of our statement, we adopt the notations:

$$\bar{f} = \frac{1}{\omega}\int_0^\omega f(t)\mathrm{d}t, \qquad f^L = \min_{t\in[0,\omega]} |f(t)|, \qquad f^M = \max_{t\in[0,\omega]} |f(t)|,$$

where f is a continuous ω-periodic function.

In the present paper, we present a qualitative analysis for the nonautonomous predator–prey system (1) by incorporating stage structures for both prey and predator and the periodicity of ecological and environmental parameters into the model. The rest of the paper is organized as follows. In the next section, we discuss the positivity of solutions and the permanence of system (1) under initial conditions (2) and (3). In Section 3, the existence of positive ω-periodic solutions of system (1) is shown by using Gaines and Mawhin's continuation theorem of coincidence degree theory. In Section 4, sufficient conditions are derived for the global stability of the positive ω-periodic solution of system (1) by constructing an appropriate Lyapunov functional. Two examples are given in Section 5 to illustrate the feasibility of our main results. A brief discussion is presented in Section 6.

2. Permanence

In many biological systems, the stability switches many times and the systems will eventually become unstable when time delays increase [41,42]. While for some other systems, for example [43–45], there will be no change in uniform persistence or permanence of systems even though the time delays change. Recently, uniform persistence or permanence concerning the long time survival of species population appears to be another important concept of stability from the viewpoint of mathematical ecology. This clearly necessitates a study of permanence in depth and of the modeling and analysis involved. Thus, in this section, we are looking for sufficient conditions that guarantee the permanence of system (1). Following [25], we use the definition of permanence of system and ultimately-bounded domain as follows:

Definition 1. The system $X'(t) = f(t, X_t(\theta)), t \geq 0, \theta \in [-\tau, 0], X \in \mathbb{R}^n$ is said to be permanence if, for any solution $X(t, \phi)$, there exists a constant $m > 0$ and $T = T(\phi)$, such that $X(t) > m$, for all $t > T$. The domain $D \in \mathbb{C}^n$ is said to be an ultimately-bounded domain, if D is a closed, bounded subset of $\mathbb{C}^n$, and there exists a constant $T = T(\phi)$, such that $X_t(\theta) \in D$, for all $t > T$.

Using a similar argument as in the proof of Theorem 2.1 in [19], one can see that solutions of system (1) with initial conditions (2) and (3) are positive for all $t > 0$. Moreover, we have

Theorem 1. *Solutions of system (1) with initial conditions (2) and (3) are ultimately bounded.*

Proof of Theorem 1. Suppose that $(x_1(t), x_2(t), y_1(t), y_2(t))^{\mathrm{T}}$ is an arbitrary positive solution of system (1) under conditions (2) and (3). We define

$$\rho(t) = a_2^M x_1(t) + a_2^M x_2(t) + k_1^L y_1(t) + k_1^L y_2(t).$$

Calculating the derivative of $\rho(t)$ along the solution of system (1), we have

$$\begin{aligned}\dot{\rho}(t) &= a_1(t)a_2^M x_2(t) - a_2^M r_1(t)x_1(t) - k_1^L r_2(t)y_1(t) - \beta_1(t)a_2^M x_2^2(t)\\ &\quad - (a_2^M k_1(t) - a_2(t)k_1^L)x_1(t)y_2(t) - k_1^L\beta_2(t)y_2^2(t),\\ &\leq -r\rho + a_2^M(a_1^M + r)x_2(t) - \beta_1^L a_2^M x_2^2(t) + k_1^L r y_2(t) - k_1^L\beta_2^L y_2^2(t),\end{aligned}$$

where $r = \min\{r_1^L, r_2^L\}$. Then there exists a positive number D such that

$$\dot{\rho} + r\rho \leq D,$$

which yields

$$\rho(t) < \frac{D}{r} + \left(\rho(0) - \frac{D}{r}\right)\mathrm{e}^{-rt}.$$

This implies that any positive solution of system (1) is ultimately bounded. So we completes the proof. □

Lemma 1 (*[19]*). *Consider the equation*

$$x'(t) = ax(t-\tau) - bx^2(t),$$

where $a, b, \tau > 0$, $x(t) > 0$ *for* $-\tau \leq \theta \leq 0$. *Then we have*

$$\lim_{t\to+\infty} x(t) = \frac{a}{b}.$$

Theorem 2. *System (1) with initial conditions (2) and (3) is permanent.*

Proof of Theorem 2. From the second equation of system (1), we have

$$\begin{aligned} x_2'(t) &= a_1(t-\tau_1)e^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)ds}x_2(t-\tau_1) - \beta_1(t)x_2^2(t), \\ &\leq a_1^M e^{-r_1^L\tau_1}x_2(t-\tau_1) - \beta_1^L x_2^2(t). \end{aligned}$$

By Lemma 1, there exists a value T_1, such that for any $\varepsilon > 0$ and $t \geq T_1$ we have

$$x_2(t) \leq \frac{a_1^M e^{-r_1^L\tau_1}}{\beta_1^L} + \epsilon := M_{12}. \tag{4}$$

By a direct computation, we have

$$x_1(t) = \int_{t-\tau_1}^{t} a_1(s)e^{\int_t^s r_1(m)+k_1(m)y_2(m)dm}x_2(s)ds.$$

Combining this with (4), for any $t \geq T_1$, gives

$$x_1(t) \leq \frac{a_1^M M_{12}}{r_1^L} := M_{11}. \tag{5}$$

Thus, for any $t \geq T_1$ we have

$$y_2'(t) \leq a_2^M e^{-r_2^L\tau_2}M_{11}y_2(t-\tau_2) - \beta_2(t)y_2^2(t).$$

By Lemma 1 again, there exists another value $T_2 > T_1$, such that for any $t \geq T_2$ we have

$$y_2(t) \leq \frac{a_2^M e^{-r_2^L\tau_2}M_{11}}{\beta_2^L} + \epsilon := M_{14}. \tag{6}$$

From the third equation of system (1), we get

$$y_1(t) = e^{\int_0^t -r_2(s)ds}\int_{t-\tau_2}^{t} a_2(s)e^{\int_0^s r_2(m)dm}x_1(s)y_2(s)ds. \tag{7}$$

Using (5) and (6), for any $t \geq T_2$, we have

$$y_1(t) \leq \frac{a_2^M M_{11}M_{14}(1-e^{-r_2^M\tau_2})}{r_2^L} := M_{13}.$$

By (6) and the second equation of system (1) for any $t \geq T_2$, we have

$$x_2'(t) \geq a_1^L e^{-r_1^M\tau_1 - k_1^M M_{14}\tau_1}x_2(t-\tau_1) - \beta_1^M x_2^2(t).$$

By Lemma 1, there exists a value $T_3 > T_2$, such that for any $t \geq T_3$ we have

$$x_2(t) \geq \frac{a_1^L e^{-r_1^M\tau_1 - k_1^M M_{14}\tau_1}}{\beta_1^M} - \epsilon := m_{12},$$

and

$$x_1(t) \geq \frac{a_1^L(1-e^{-r_1^L\tau_1})m_{12}}{r_1^M + k_1^M M_{14}} := m_{11}.$$

From the fourth equation of system (1), for any $t \geq T_3$, we have

$$y_2(t) \geq a_2^L e^{-r_2^M\tau_2}m_{11}y_2(t-\tau_2) - \beta_2^M y_2^2(t).$$

So there exists a value $T_4 > T_3$, such that for any $t \geq T_4$ we have

$$y_2(t) \geq \frac{a_2^L e^{-r_2^M\tau_2}m_{11}}{\beta_2^M} - \epsilon := m_{14}. \tag{8}$$

From (7) and (8) we get

$$y_1(t) \geq \frac{a_2^L m_{11} m_{14}(1 - e^{-r_2^L \tau_2})}{r_2^M} := m_{13}.$$

Repeating this process, we can obtain a more accurate bound of the solution of system (1) when t is sufficiently large. Namely, using (8) and the third equation of system (1), we have

$$x_2'(t) \leq a_1^M e^{-r_1^L \tau_1 - k_1^L m_{14} \tau_1} x_2(t - \tau_1) - \beta_1^L x_2^2(t).$$

By Lemma 1, there exists a value T_5 such that for any $\varepsilon > 0$ and $t \geq T_5$, we have

$$x_2(t) \leq \frac{a_1^M e^{-r_1^L \tau_1 - k_1^L m_{14} \tau_1}}{\beta_1^L} + \epsilon := M_{22},$$

and

$$x_1(t) \leq \frac{a_1^M M_{22}(1 - e^{-r_1^M \tau_1 - k_1^M M_{14} \tau_1})}{r_1^L + k_1^L m_{14}} := M_{21}.$$

So for any $t \geq T_5$, the fourth equation of (1) gives

$$y_2'(t) \leq a_2^M e^{-r_2^L \tau_2} M_{21} y_2(t - \tau_2) - \beta_2(t) y_2^2(t).$$

Then, there exists a value $T_6 > T_5$, such that for any $t \geq T_6$ we have

$$y_2(t) \leq \frac{a_2^M e^{-r_2^L \tau_2} M_{21}}{\beta_2^L} + \epsilon := M_{24}, \tag{9}$$

and

$$y_1(t) \leq \frac{a_2^M M_{21} M_{24}(1 - e^{-r_2^M \tau_2})}{r_2^L} := M_{23}.$$

By (9), for any $t \geq T_6$ we deduce

$$x_2'(t) \geq a_1^L e^{-r_1^M \tau_1 - k_1^M M_{24} \tau_1} x_2(t - \tau_1) - \beta_1^M x_2^2(t).$$

Lemma 1 tells us that there exists a value $T_7 > T_6$, such that for any $t \geq T_8$ we have

$$x_2(t) \geq \frac{a_1^L e^{-r_1^M \tau_1 - k_1^M M_{24} \tau_1}}{\beta_1^M} - \epsilon := m_{22},$$

$$x_1(t) \geq \frac{a_1^L (1 - e^{-r_1^L \tau_1}) m_{22}}{r_1^M + k_1^M M_{24}} := m_{21}.$$

From the fourth equation of system (1) we get

$$y_2(t) \geq a_2^L e^{-r_2^M \tau_2} m_{11} y_2(t - \tau_2) - \beta_2^M y_2^2(t).$$

So there exists a value $T_8 > T_7$, such that for any $t \geq T_8$ we have

$$y_2(t) \geq \frac{a_2^L e^{-r_2^M \tau_2} m_{21}}{\beta_2^M} - \epsilon := m_{24},$$

and

$$y_1(t) \geq \frac{a_2^L m_{21} m_{24}(1 - e^{-r_2^L \tau_2})}{r_2^M} := m_{23}.$$

Continuing the discussion in this manner, we will obtain eight sequences $\{M_{ni}\}$, $\{m_{ni}\}$ $(i = 1, 2, 3, 4)$ defined as follows:

$$M_{(n+1)2} = \frac{a_1^M e^{-r_1^L \tau_1 - k_1^L \tau_1 m_{n4}}}{\beta_1^L} + \epsilon, \qquad M_{(n+1)1} = \frac{a_1^M M_{(n+1)2}(1 - e^{-r_1^M \tau_1 - k_1^M \tau_1 M_{n4}})}{r_1^L + k_1^L m_{n4}},$$

$$M_{(n+1)4} = \frac{a_2^M e^{-r_2^L \tau_2} M_{(n+1)1}}{\beta_2^L} + \epsilon, \qquad M_{(n+1)3} = \frac{a_2^M M_{(n+1)1} M_{(n+1)4}(1 - e^{-r_2^M \tau_2})}{r_2^L},$$

$$m_{(n+1)2} = \frac{a_1^L e^{-r_1^M \tau_1 - k_1^M \tau_1 M_{(n+1)4}}}{\beta_1^M} - \epsilon, \qquad m_{(n+1)1} = \frac{a_1^L (1 - e^{-r_1^L \tau_1}) m_{(n+1)2}}{r_1^M + k_1^M M_{(n+1)4}},$$
$$m_{(n+1)4} = \frac{a_2^L e^{-r_2^M \tau_2} m_{n1}}{\beta_2^M} - \epsilon, \qquad m_{(n+1)3} = \frac{a_2^L (1 - e^{-r_2^L \tau_2}) m_{(n+1)1} m_{(n+1)4}}{r_2^M}. \tag{10}$$

Apparently, one can see that the sequences $\{M_{ni}\}$ $(i = 1, 2, 3, 4)$ are decreasing and the sequences $\{m_{ni}\}$ $(i = 1, 2, 3, 4)$ are increasing as n increases. Since $M_{ni} > 0$, we know that $\lim_{n\to+\infty} M_{ni}$ exists. Denote $\hat{M}_i = \lim_{n\to+\infty} M_{ni}$ $(i = 1, 2, 3, 4)$. Clearly, we have $m_{ni} < M_{ni}$. Hence, we know that $\lim_{n\to+\infty} m_{ni}$ exists too. Denote $\hat{m}_i = \lim_{n\to+\infty} m_{ni}$ $(i = 1, 2, 3, 4)$. From (10), we have

$$\hat{M}_2 = \frac{a_1^M e^{-r_1^L \tau_1 - k_1^L \tau_1 \hat{m}_4}}{\beta_1^L}, \qquad \hat{M}_1 = \frac{a_1^M \hat{M}_2 (1 - e^{-r_1^M \tau_1 - k_1^M \tau_1 \hat{M}_4})}{r_1^L + k_1^L \hat{m}_4}, \qquad \hat{M}_4 = \frac{a_2^M e^{-r_2^L \tau_2} \hat{M}_1}{\beta_2^L},$$
$$\hat{M}_3 = \frac{a_2^M (1 - e^{-r_2^M \tau_2}) \hat{M}_1 \hat{M}_4}{r_2^L}, \qquad \hat{m}_2 = \frac{a_1^L e^{-r_1^M \tau_1 - k_1^M \tau_1 \hat{M}_4}}{\beta_1^M}, \qquad \hat{m}_1 = \frac{a_1^L (1 - e^{-r_1^L \tau_1}) \hat{m}_2}{r_1^M + k_1^M \hat{M}_4},$$
$$\hat{m}_4 = \frac{a_2^L e^{-r_2^M \tau_2} \hat{m}_1}{\beta_2^M}, \qquad \hat{m}_3 = \frac{a_2^L (1 - e^{-r_2^L \tau_2}) \hat{m}_1 \hat{m}_4}{r_2^M}. \tag{11}$$

According to Definition 1, therefore we have completed the proof of Theorem 2. □

3. Existence of positive periodic solutions

In this section, we study the existence of positive periodic solutions of system (1). We start by quoting some well-known concepts and results introduced in [46] that will be utilized in this section. Suppose that both X and Y are real Banach spaces. Let $L : \operatorname{Dom} L \subset X \to Y$ be a linear mapping, and $N : X \to Y$ be a continuous mapping. L is called a Fredholm mapping of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < +\infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm mapping of index zero and there exist continuous projectors $P : X \to X$, and $Q : Y \to Y$ such that $\operatorname{Im} P = \operatorname{Ker} L$, $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$, then the restriction L_P of L to $\operatorname{Dom} L \cap \operatorname{Ker} P : (I - P)X \to \operatorname{Im} L$ is invertible. Denote the inverse of L_P by K_P. If Ω is an open bounded subset of X, the mapping N is called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\operatorname{Im} Q$ is isomorphic to $\operatorname{Ker} L$, there exists an isomorphism $J : \operatorname{Im} Q \to \operatorname{Ker} L$.

Before stating our result on positive ω-periodic solutions of system (1), we need the following technical lemma:

Lemma 2 ([46]). *Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Assume*

(a) *for each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$, $Lx \neq \lambda Nx$;*
(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$, $QNx \neq 0$;*
(c) $\deg\{JQN, \Omega \cap \operatorname{Ker} L, 0\} \neq 0$.

Then $Lx = Nx$ has at least one solution in $\bar{\Omega} \cap \operatorname{Dom} L$.

Theorem 3. *System (1) with initial conditions (2) and (3) has at least one strictly positive ω-periodic solution.*

Proof of Theorem 3. We rewrite system (1) as

$$\begin{cases} x_1(t) = \displaystyle\int_{t-\tau_1}^{t} a_1(s) e^{\int_t^s r_1(m) + k_1(m) y_2(m) dm} x_2(s) ds, \\ x_2'(t) = a_1(t - \tau_1) e^{\int_{t-\tau_1}^{t} -r_1(s) - k_1(s) y_2(s) ds} x_2(t - \tau_1) - \beta_1(t) x_2^2(t), \\ y_1(t) = \displaystyle\int_{t-\tau_2}^{t} a_2(s) e^{\int_t^s r_2(m) dm} x_1(s) y_2(s) ds, \\ y_2'(t) = a_2(t - \tau_2) y_2(t - \tau_2) e^{\int_{t-\tau_2}^{t} -r_2(s) ds} \displaystyle\int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s) e^{\int_{t-\tau_2}^{s} r_1(m) + k_1(m) y_2(m) dm} \cdot x_2(s) ds - \beta_2(t) y_2^2(t). \end{cases} \tag{12}$$

Consider a subsystem of system (12):

$$\begin{cases} x_2'(t) = a_1(t - \tau_1) e^{\int_{t-\tau_1}^{t} -r_1(s) - k_1(s) y_2(s) ds} x_2(t - \tau_1) - \beta_1(t) x_2^2(t), \\ y_2'(t) = a_2(t - \tau_2) y_2(t - \tau_2) e^{\int_{t-\tau_2}^{t} -r_2(s) ds} \displaystyle\int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s) e^{\int_{t-\tau_2}^{s} r_1(m) + k_1(m) y_2(m) dm} \cdot x_2(s) ds - \beta_2(t) y_2^2(t). \end{cases} \tag{13}$$

Let

$$u_1(t)=\ln[x_2(t)],\quad \text{and}\quad u_2(t)=\ln[y_2(t)]. \tag{14}$$

Substituting (14) into (13), we get

$$\begin{cases} u_1'(t)=a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t}-r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)}-\beta_1(t)\mathrm{e}^{u_1(t)},\\ u_2'(t)=a_2(t-\tau_2)\mathrm{e}^{u_2(t-\tau_2)-u_2(t)}\mathrm{e}^{\int_{t-\tau_2}^{t}-r_2(s)\mathrm{d}s}\displaystyle\int_{t-\tau_1-\tau_2}^{t-\tau_2}a_1(s)\mathrm{e}^{\int_{t-\tau_2}^{s}r_1(m)+k_1(m)\mathrm{e}^{u_2(m)}\mathrm{d}m}\cdot\mathrm{e}^{u_1(s)}\mathrm{d}s-\beta_2(t)\mathrm{e}^{u_2(t)}. \end{cases} \tag{15}$$

One can easily see that if system (15) has one ω-periodic solution $(u_1^*(t),\ u_2^*(t))^{\mathrm{T}}$, then $(x_2^*(t),\ y_2^*(t))^{\mathrm{T}}=(\mathrm{e}^{u_1^*(t)},\ \mathrm{e}^{u_2^*(t)})^{\mathrm{T}}$ is a positive ω-periodic solution of system (13). Thus, in what follows our goal is to show that system (15) has at least one ω-periodic solution.

To apply Lemma 2 to system (15) in a straightforward manner, we define

$$X=Y=\{(u_1(t),u_2(t))^{\mathrm{T}}\in C(\mathbb{R},\mathbb{R}^2):u_i(t+\omega)=u_i(t),\ i=1,2\},$$

and

$$\|(u_1(t),u_2(t))^{\mathrm{T}}\|=\max_{t\in[0,\omega]}|u_1(t)|+\max_{t\in[0,\omega]}|u_2(t)|,$$

where $|\cdot|$ denotes the Euclidean norm. It is easy to see that both X and Y are Banach spaces with the norm $\|\cdot\|$. Then we let

$$L:\mathrm{Dom}\,L\cap X\to X,\quad L(u_1(t),u_2(t))^{\mathrm{T}}=\left(\frac{\mathrm{d}u_1(t)}{\mathrm{d}t},\ \frac{u_2(t)}{\mathrm{d}t}\right)^{\mathrm{T}},$$

where $\mathrm{Dom}\,L=\{(u_1(t),u_2(t))^{\mathrm{T}}\in C(\mathbb{R},\mathbb{R}^2)\}$ and $N:X\to X$, $N\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}f_1(t)\\f_2(t)\end{bmatrix}$,

and

$$f_1(t)=a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t}-r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)}-\beta_1(t)\mathrm{e}^{u_1(t)},$$
$$f_2(t)=a_2(t-\tau_2)\int_{t-\tau_1-\tau_2}^{t-\tau_2}a_1(s)\mathrm{e}^{u_1(s)+u_2(t-\tau_2)-u_2(t)-\int_{t-\tau_2}^{t}r_2(n)\mathrm{d}n+\int_{t-\tau_2}^{s}r_1(m)+k_1(m)\mathrm{e}^{u_2(m)}\mathrm{d}m}\mathrm{d}s-\beta_2(t)\mathrm{e}^{u_2(t)}.$$

We define

$$P\begin{bmatrix}u_1\\u_2\end{bmatrix}=Q\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}\frac{1}{\omega}\int_0^{\omega}u_1(t)\mathrm{d}t\\ \frac{1}{\omega}\int_0^{\omega}u_2(t)\mathrm{d}t\end{bmatrix},\qquad \begin{bmatrix}u_1\\u_2\end{bmatrix}\in X=Y.$$

It is not difficult to verify that $\mathrm{Ker}\,L=\{x|x\in X,x=h,h\in R^2\}$, $\mathrm{Im}\,L=\{y|y\in Y,\int_0^{\omega}y(t)\mathrm{d}t=0\}$ is closed in Y, $\dim\mathrm{Ker}\,L=\mathrm{codim}\,\mathrm{Im}L=2$, and both P and Q are continuous projectors such that $\mathrm{Im}P=\mathrm{Ker}\,L$ and $\mathrm{Ker}\,Q=\mathrm{Im}\,L=\mathrm{Im}(I-Q)$. From the above we know that L is a Fredholm mapping of index zero. Furthermore, the inverse $K_P:\mathrm{Im}\,L\to\mathrm{Dom}\,L\cap\mathrm{Ker}\,P$ of L_P exists and has the form

$$K_P(y)=\int_0^t y(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^t y(s)\mathrm{d}s\mathrm{d}t.$$

Then $QN:X\to Y$ and $K_P(I-Q)N:X\to X$ are, respectively, given by

$$QNx=\begin{bmatrix}\frac{1}{\omega}\int_0^{\omega}f_1(t)\mathrm{d}t\\ \frac{1}{\omega}\int_0^{\omega}f_2(t)\mathrm{d}t\end{bmatrix},$$
$$K_P(I-Q)Nx=\int_0^t Nx(s)\mathrm{d}s-\frac{1}{\omega}\int_0^{\omega}\int_0^t Nx(s)\mathrm{d}s\mathrm{d}t-\left(\frac{t}{\omega}-\frac{1}{2}\right)\int_0^{\omega}Nx(s)\mathrm{d}s.$$

Apparently, QN and $K_P(I-Q)N$ are continuous.

In order to apply Lemma 2, we also need to find an appropriate open and bounded subset Ω. We look for Ω by the following two steps:

Step 1. To find estimates of $u_1(t)$ and $u_2(t)$, from the operator equation $Lx=\lambda Nx$, $\lambda\in(0,\ 1)$, we have

$$\frac{\mathrm{d}u_1(t)}{\mathrm{d}t}=\lambda[a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t}-r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)}-\beta_1(t)\mathrm{e}^{u_1(t)}],$$
$$\frac{\mathrm{d}u_2(t)}{\mathrm{d}t}=\lambda\left[\int_{t-\tau_1-\tau_2}^{t-\tau_2}a_1(s)\mathrm{e}^{u_1(s)+u_2(t-\tau_2)-u_2(t)-\int_{t-\tau_2}^{t}r_2(n)\mathrm{d}n+\int_{t-\tau_2}^{s}r_1(m)+k_1(m)\mathrm{e}^{u_2(m)}\mathrm{d}m}\mathrm{d}s\cdot a_2(t-\tau_2)-\beta_2(t)\mathrm{e}^{u_2(t)}\right]. \tag{16}$$

Suppose that $(u_1(t), u_2(t))^{\mathrm{T}} \in X$ is a solution of (16) for some $\lambda \in (0, 1)$. Integrating (16) over the interval $[0, \omega]$, we obtain

$$\int_0^{\omega} a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)}\mathrm{d}t = \int_0^{\omega} \beta_1(t)\mathrm{e}^{u_1(t)}\mathrm{d}t, \tag{17}$$

and

$$\int_0^{\omega} a_2(t-\tau_2)\mathrm{d}t \int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s)\mathrm{e}^{u_1(s)+u_2(t-\tau_2)-u_2(t)-\int_{t-\tau_2}^{t} r_2(n)\mathrm{d}n+\int_{t-\tau_2}^{s} r_1(m)+k_1(m)\mathrm{e}^{u_2(m)}\mathrm{d}m}\mathrm{d}s = \int_0^{\omega} \beta_2(t)\mathrm{e}^{u_2(t)}\mathrm{d}t. \tag{18}$$

Since $(u_1(t), u_2(t))^{\mathrm{T}} \in X$, there exist $\xi_i, \eta_i \in [0, \omega]$ such that

$$u_i(\xi_i) = \min_{t\in[0,\omega]} u_i(t), \qquad u_i(\eta_i) = \max_{t\in[0,\omega]} u_i(t), \quad i = 1, 2.$$

Multiplying the first equation of (16) by $\mathrm{e}^{u_1(t)}$ and integrating it over $[0, \omega]$, we have

$$\int_0^{\omega} a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)}\mathrm{d}t = \int_0^{\omega} \beta_1(t)\mathrm{e}^{2u_1(t)}\mathrm{d}t. \tag{19}$$

It follows from (19) that

$$\beta_1^L \int_0^{\omega} \mathrm{e}^{2u_1(t)}\mathrm{d}t \le \int_0^{\omega} a_1^M \mathrm{e}^{-r_1^L\tau_1}\mathrm{e}^{u_1(t-\tau_1)}\mathrm{d}t = \int_0^{\omega} a_1^M \mathrm{e}^{-r_1^L\tau_1}\mathrm{e}^{u_1(t)}\mathrm{d}t. \tag{20}$$

By using the inequality

$$\left(\int_0^{\omega} \mathrm{e}^{u_1(t)}\mathrm{d}t\right)^2 \le \omega \int_0^{\omega} \mathrm{e}^{2u_1(t)}\mathrm{d}t,$$

we derive from (20) that

$$\beta_1^L \left(\int_0^{\omega} \mathrm{e}^{u_1(t)}\mathrm{d}t\right)^2 \le a_1^M \omega \mathrm{e}^{-r_1^L\tau_1} \int_0^{\omega} \mathrm{e}^{u_1(t)}\mathrm{d}t,$$

which implies

$$\int_0^{\omega} \mathrm{e}^{u_1(t)}\mathrm{d}t \le \frac{a_1^M \omega \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L} \quad \text{and} \quad u_1(\xi_1) \le \ln \frac{a_1^M \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L}. \tag{21}$$

Combining (21) with (16) and (17), we deduce

$$\begin{aligned}\int_0^{\omega} |u_1'(t)|\mathrm{d}t &< \int_0^{\omega} [a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)\mathrm{e}^{u_2(s)}\mathrm{d}s}\mathrm{e}^{u_1(t-\tau_1)-u_1(t)} + \beta_1(t)\mathrm{e}^{u_1(t)}]\mathrm{d}t,\\ &= 2\int_0^{\omega} \beta_1(t)\mathrm{e}^{u_1(t)}\mathrm{d}t,\\ &\le \frac{2a_1^M \beta_1^M \omega \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L}.\end{aligned} \tag{22}$$

From (21) and (22) we also have

$$u_1(t) \le u_1(\xi_1) + \int_0^{\omega} |u_1'(t)|\mathrm{d}t \le \ln \frac{a_1^M \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L} + \frac{2a_1^M \beta_1^M \omega \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L} := C_1. \tag{23}$$

Multiplying the second equation of (16) by $\mathrm{e}^{u_2(t)}$ and integrating it over $[0, \omega]$, we have

$$\int_0^{\omega} a_2(t-\tau_2)\mathrm{d}t \int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s)\mathrm{e}^{u_1(s)+u_2(t-\tau_2)-\int_{t-\tau_2}^{t} r_2(n)\mathrm{d}n+\int_{t-\tau_2}^{s} r_1(m)+k_1(m)\mathrm{e}^{u_2(m)}\mathrm{d}m}\mathrm{d}s = \int_0^{\omega} \beta_2(t)\mathrm{e}^{2u_2(t)}\mathrm{d}t, \tag{24}$$

which implies that

$$\begin{aligned}\beta_2^L \int_0^{\omega} \mathrm{e}^{2u_2(t)}\mathrm{d}t &\le \int_0^{\omega} \frac{(a_1^M)^2 a_2^M \mathrm{e}^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L r_1^L} \mathrm{e}^{\frac{2a_1^M \beta_1^M \omega \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L}} \mathrm{e}^{u_2(t-\tau_1)}\mathrm{d}t,\\ &= \int_0^{\omega} \frac{(a_1^M)^2 a_2^M \mathrm{e}^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L r_1^L} \mathrm{e}^{\frac{2a_1^M \beta_1^M \omega \mathrm{e}^{-r_1^L\tau_1}}{\beta_1^L}} \mathrm{e}^{u_2(t)}\mathrm{d}t.\end{aligned} \tag{25}$$

Similarly, using (25) and the inequality

$$\left(\int_0^\omega e^{u_2(t)}dt\right)^2 \le \omega\int_0^\omega e^{2u_2(t)}dt,$$

we have

$$\beta_2^L\left(\int_0^\omega e^{u_2(t)}dt\right)^2 \le \frac{(a_1^M)^2a_2^M\omega e^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^Lr_1^L}e^{\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}}\int_0^\omega e^{u_2(t)}dt,$$

which implies

$$u_2(\xi_2) \le \frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}\ln\frac{(a_1^M)^2a_2^M e^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L\beta_2^Lr_1^L}. \quad (26)$$

By (16), (18) and (26), we have

$$\int_0^\omega |u_2'(t)|dt < 2\int_0^\omega \beta_2(t)e^{u_2(t)}dt \le \frac{2(a_1^M)^2a_2^M\omega\beta_2^M e^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L\beta_2^Lr_1^L}e^{\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}}, \quad (27)$$

which, together with (26), yields

$$u_2(t) \le u_2(\xi_2)+\int_0^\omega |u_2'(t)|dt \le \frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}\ln\frac{(a_1^M)^2a_2^M e^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L\beta_2^Lr_1^L}$$
$$+\frac{2(a_1^M)^2a_2^M\omega\beta_2^M e^{-r_1^L\tau_1-r_2^L\tau_2}}{\beta_1^L\beta_2^Lr_1^L}e^{\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}} := C_2. \quad (28)$$

Note that

$$\int_0^\omega a_1(t-\tau_1)e^{\int_{t-\tau_1}^t -r_1(s)-k_1(s)e^{u_2(s)}ds}e^{u_1(t-\tau_1)}dt = \int_0^\omega a_1(t)e^{\int_t^{t+\tau_1} -r_1(s)-k_1(s)e^{u_2(s)}ds}e^{u_1(t)}dt.$$

It follows from (19) that

$$\int_0^\omega \beta_1(t)e^{2u_1(t)}dt = \int_0^\omega a_1(t)e^{\int_t^{t+\tau_1} -r_1(s)-k_1(s)e^{u_2(s)}ds}e^{u_1(t)}dt \ge a_1^Le^{-r_1^M\tau_1-k_1^M\tau_1e^{C_2}}\int_0^\omega e^{u_1(t)}dt,$$

which yields

$$e^{u_1(\eta_1)} \ge \frac{a_1^Le^{-r_1^M\tau_1-k_1^M\tau_1e^{C_2}}}{\beta_1^M},$$

that is,

$$u_1(\eta_1) \ge \ln\frac{a_1^Le^{-r_1^M\tau_1-k_1^M\tau_1e^{C_2}}}{\beta_1^M}. \quad (29)$$

From (22) and (29) we obtain

$$u_1(t) \ge u_1(\eta_1)-\int_0^\omega |u_1'(t)|dt \ge \ln\frac{a_1^Le^{-r_1^M\tau_1-k_1^M\tau_1e^{C_2}}}{\beta_1^M}-\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L} := C_3. \quad (30)$$

Combining (30) with (23) yields

$$\max_{t\in[0,\omega]}|u_1(t)| < \max\left\{\left|\ln\frac{a_1^Me^{-r_1^L\tau_1}}{\beta_1^L}\right|+\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}, \left|\ln\frac{a_1^Le^{-r_1^M\tau_1-k_1^M\tau_1e^{C_2}}}{\beta_1^M}\right|+\frac{2a_1^M\beta_1^M\omega e^{-r_1^L\tau_1}}{\beta_1^L}\right\} := R_1. \quad (31)$$

Note that

$$\int_0^\omega a_2(t-\tau_2)dt\int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s)e^{u_1(s)+u_2(t-\tau_2)-\int_{t-\tau_2}^t r_2(n)dn+\int_{t-\tau_2}^s r_1(m)+k_1(m)e^{u_2(m)}dm}ds$$
$$=\int_0^\omega a_2(t)dt\int_{t-\tau_1}^t a_1(s)e^{u_1(s)+u_2(t)-\int_t^{t+\tau_2} r_2(n)dn+\int_t^s r_1(m)+k_1(m)e^{u_2(m)}dm}ds.$$

It follows from (24) that

$$\int_0^{\omega} \beta_2(t) e^{2u_2(t)} dt, = \int_0^{\omega} a_2(t) dt \int_{t-\tau_1}^{t} a_1(s) e^{u_1(s)+u_2(t)-\int_t^{t+\tau_2} r_2(n)dn+\int_t^s r_1(m)+k_1(m)e^{u_2(m)}dm} ds,$$
$$\geq \frac{a_1^L a_2^L e^{C_3 - r_2^M \tau_2}(1-e^{-r_1^L \tau_1})}{r_1^M + k_1^M e^{C_2}} \int_0^{\omega} e^{u_2(t)} dt,$$

which yields

$$u_2(\eta_2) \geq \ln \frac{a_1^L a_2^L e^{C_3 - r_2^M \tau_2}(1-e^{-r_1^L \tau_1})}{\beta_2^M (r_1^M + k_1^M e^{C_2})}. \tag{32}$$

We derive from (27) and (32) that

$$u_2(t) \geq u_2(\eta_2) - \int_0^{\omega} |u_2'(t)| dt \geq \ln \frac{a_1^L a_2^L e^{C_3 - r_2^M \tau_2}(1-e^{-r_1^L \tau_1})}{\beta_2^M (r_1^M + k_1^M e^{C_2})}$$
$$- \frac{2(a_1^M)^2 a_2^M \omega \beta_2^M e^{-r_1^L \tau_1 - r_2^L \tau_2}}{\beta_1^L \beta_2^L r_1^L} e^{\frac{2a_1^M \beta_1^M \omega e^{-r_1^L \tau_1}}{\beta_1^L}} := C_4.$$

This, together with (28), leads to

$$\max_{t\in[0,\omega]} |u_2(t)| < \max \left\{ \frac{2a_1^M \beta_1^M \omega e^{-r_1^L \tau_1}}{\beta_1^L} \cdot \left| \ln \frac{(a_1^M)^2 a_2^M e^{-r_1^L \tau_1 - r_2^L \tau_2}}{\beta_1^L \beta_2^L r_1^L} \right| \right.$$
$$+ \frac{2(a_1^M)^2 a_2^M \omega \beta_2^M e^{-r_1^L \tau_1 - r_2^L \tau_2}}{\beta_1^L \beta_2^L r_1^L} e^{\frac{2a_1^M \beta_1^M \omega e^{-r_1^L \tau_1}}{\beta_1^L}}, \left| \ln \frac{a_1^L a_2^L e^{C_3 - r_2^M \tau_2}(1-e^{-r_1^L \tau_1})}{\beta_2^M (r_1^M + k_1^M e^{C_2})} \right|$$
$$\left. + \frac{2(a_1^M)^2 a_2^M \omega \beta_2^M e^{-r_1^L \tau_1 - r_2^L \tau_2}}{\beta_1^L \beta_2^L r_1^L} e^{\frac{2a_1^M \beta_1^M \omega e^{-r_1^L \tau_1}}{\beta_1^L}} \right\} := R_2. \tag{33}$$

Clearly, R_1 and R_2 in (31) and (33) are independent of λ.

Step 2. To construct an appropriate open and bounded subset Ω, we denote $M = R_1 + R_2 + R_0$, where R_0 may be taken sufficiently large such that the unique solution $(u^*, v^*)^{\mathrm{T}}$ of the system of algebraic equations

$$\frac{1}{\omega} \int_0^{\omega} [a_1(t-\tau_1) e^{\int_{t-\tau_1}^{t} -r_1(s) - k_1(s) e^{u_2} ds} - \beta_1(t) e^{u_1}] dt = 0, \tag{34}$$
$$\frac{1}{\omega} \int_0^{\omega} \left[a_2(t-\tau_2) \int_{t-\tau_1-\tau_2}^{t-\tau_2} a_1(s) e^{u_1 - \int_{t-\tau_2}^{t} r_2(n)dn + \int_{t-\tau_2}^{s} r_1(m)+k_1(m)e^{u_2} dm} ds - \beta_2(t) e^{u_2} \right] dt = 0,$$

satisfies $\|(u^*, v^*)^{\mathrm{T}}\| = |u^*| + |v^*| < M$. Choose $\Omega = \{(u_1(t), u_2(t))^{\mathrm{T}} \in X : \|(u_1, u_2)^{\mathrm{T}}\| < M\}$, which means that the condition (a) in Lemma 2 is satisfied.

When $(u_1(t), u_2(t))^{\mathrm{T}} \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap \mathbb{R}^2$, $(u_1, u_2)^{\mathrm{T}}$ is a constant vector in $\mathbb{R}^2$ with $|u_1| + |u_2| = M$. Thus, we have

$$QN \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega} \int_0^{\omega} f_1(t) dt \\ \frac{1}{\omega} \int_0^{\omega} f_2(t) dt \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that the condition (b) in Lemma 2 is satisfied.

Take $J = I : \operatorname{Im} Q \to \operatorname{Ker} L$, $(u_1, u_2)^{\mathrm{T}} \to (u_1, u_2)^{\mathrm{T}}$. A direct calculation shows that

$$\deg(JQN(u_1, u_2)^{\mathrm{T}}, \ \Omega \cap \operatorname{Ker} L, (0,0)^{\mathrm{T}}) = 1,$$

where $(u_1^*, u_2^*)^{\mathrm{T}}$ is the unique solution of (34). So the condition (c) in Lemma 2 is satisfied too. In addition, one can easily see that the set $\{K^P(I-Q)Nx | x \in \bar{\Omega}\}$ is equicontinuous and uniformly bounded. By using the Arzela–Ascoli theorem, we see that $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Consequently, N is L-compact.

Now we have proved that Ω satisfies all the requirements in Lemma 2. This implies that system (15) has at least one ω-periodic solution. Thus, system (13) has at least one positive ω-periodic solution.

Let $(x_2^*(t),\ y_2^*(t))^{\mathrm{T}}$ be a positive ω-periodic solution of system (13). Then it is easy to verify that

$$x_1^*(t) = \int_{t-\tau_1}^{t} a_1(s)\mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2^*(m)\mathrm{d}m} x_2^*(s)\mathrm{d}s > 0,$$

and

$$y_1^*(t) = \int_{t-\tau_2}^{t} a_2(s)\mathrm{e}^{\int_t^s r_2(m)\mathrm{d}m} x_1^*(s)y_2^*(s)\mathrm{d}s > 0,$$

are also ω-periodic. Thus, $(x_1^*(t),\ x_2^*(t),\ y_1^*(t),\ y_2^*(t))^{\mathrm{T}}$ is a positive ω-periodic solution of system (1) under initial conditions (2) and (3). Therefore, we have completed the proof. □

4. Uniqueness and global stability

For many biological and ecological systems, the uniqueness and the stability of positive solution is naturally an interesting question [9,10,14,23,25,38,41,45,47,48]. In this section, we analyze the uniqueness and the global stability of positive periodic solution of system (1), and derive sufficient conditions that guarantee that system (1) has a unique positive ω-periodic solution which is globally stable.

Theorem 4. *Suppose that there exists a positive constant p such that*

$$\liminf_{t\to+\infty} A_i(t) > 0, \quad (i = 1, 2) \tag{35}$$

where

$$\begin{aligned} A_1(t) &= 2\beta_1(t)\hat{m}_2 - a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)\hat{m}_4\mathrm{d}m} \\ &\quad - p\int_{t+\tau_2}^{t+\tau_1+\tau_2} a_1(t)\hat{M}_4 a_2(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}\mathrm{e}^{\int_{s-\tau_2}^{t} r_1(m)+k_1(m)\hat{m}_4\mathrm{d}m}\mathrm{d}s, \\ A_2(t) &= 2p\beta_2(t)\hat{m}_4 - \int_t^{t+\tau_1} a_1(s-\tau_1)\hat{M}_2 k_1(t)\mathrm{d}s - a_2(t)p\mathrm{e}^{-\int_t^{t+\tau_2} r_2(m)\mathrm{d}m}\int_{t-\tau_1}^{t} a_1(s)\hat{M}_2\mathrm{e}^{\int_t^s r_1(n)+k_1(n)\hat{m}_4\mathrm{d}n}\mathrm{d}s \\ &\quad - \int_{t+\tau_2}^{t+\tau_1+\tau_2} pk_1(t)a_2(s-\tau_2)\hat{M}_2\hat{M}_4\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}\mathrm{d}s\int_{s-\tau_1-\tau_2}^{s-\tau_2} a_1(m)\mathrm{d}m, \end{aligned}$$

here $\hat{m}_i$ and $\hat{M}_i$ are defined as in (11). Then system (1) has a unique positive ω-periodic solution $(x_1^*(t),\ x_2^*(t),\ y_1^*(t),\ y_2^*(t))^{\mathrm{T}}$ which is globally stable.

Proof of Theorem 4. Suppose that $(x_1^*(t), x_2^*(t), y_1^*(t), y_2^*(t))^{\mathrm{T}}$ is a positive ω-periodic solution of system (1) with initial conditions (2) and (3). Let

$$\begin{aligned} V_1(t) &= |x_2(t) - x_2^*(t)| + \int_t^{t+\tau_1}\int_{s-\tau_1}^{t} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(u)|y_2(u) - y_2^*(u)|\mathrm{d}u\mathrm{d}s \\ &\quad + \int_{t-\tau_1}^{t} a_1(s)\mathrm{e}^{\int_s^{s+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(s) - x_2^*(s)|\mathrm{d}s. \end{aligned}$$

Calculating the upper right derivative of $V_1(t)$ along positive solutions of system (13), we get

$$\begin{aligned} D^+V_1(t) &= \mathrm{sgn}(x_2(t) - x_2^*(t))\{a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)\mathrm{d}s}x_2(t-\tau_1) \\ &\quad - a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2^*(s)\mathrm{d}s}x_2^*(t-\tau_1) - \beta_1(t)x_2^2(t) + \beta_1(t)x_2^{*2}(t)\} \\ &\quad + \int_t^{t+\tau_1} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(t)|y_2(t) - y_2^*(t)|\mathrm{d}s - \int_{t-\tau_1}^{t} a_1(t-\tau_1)x_2^*(t-\tau_1)k_1(u)|y_2(u) - y_2^*(u)|\mathrm{d}u \\ &\quad + a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t) - x_2^*(t)| \\ &\quad - a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t-\tau_1) - x_2^*(t-\tau_1)|, \\ &= \mathrm{sgn}(x_2(t) - x_2^*(t))\{a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)\mathrm{d}s}[x_2(t-\tau_1) - x_2^*(t-\tau_1)] \\ &\quad + a_1(t-\tau_1)x_2^*(t-\tau_1)[\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)\mathrm{d}s} - \mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2^*(s)\mathrm{d}s}] \\ &\quad - \beta_1(t)(x_2(t) + x_2^*(t))(x_2(t) - x_2^*(t))\} + \int_t^{t+\tau_1} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(t)|y_2(t) - y_2^*(t)|\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&- \int_{t-\tau_1}^{t} a_1(t-\tau_1)x_2^*(t-\tau_1)k_1(u)|y_2(u)-y_2^*(u)|\mathrm{d}u \\
&+ a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t)-x_2^*(t)| \\
&- a_1(t-\tau_1)\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t-\tau_1)-x_2^*(t-\tau_1)|, \\
\le\ & a_1(t-\tau_1)x_2^*(t-\tau_1)|\mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2(s)\mathrm{d}s} - \mathrm{e}^{\int_{t-\tau_1}^{t} -r_1(s)-k_1(s)y_2^*(s)\mathrm{d}s}| \\
&- \beta_1(t)(x_2(t)+x_2^*(t))|x_2(t)-x_2^*(t)| + \int_t^{t+\tau_1} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(t)|y_2(t)-y_2^*(t)|\mathrm{d}s \\
&- \int_{t-\tau_1}^{t} a_1(t-\tau_1)x_2^*(t-\tau_1)k_1(u)|y_2(u)-y_2^*(u)|\mathrm{d}u + a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t)-x_2^*(t)|.
\end{aligned}
$$

Clearly, $|\mathrm{e}^{-x}-\mathrm{e}^{-y}| \le |x-y|$ for arbitrary $x \ge 0$ and $y \ge 0$, which leads to

$$
\begin{aligned}
D^+V_1(t) \le\ & -\beta_1(t)(x_2(t)+x_2^*(t))|x_2(t)-x_2^*(t)| + \int_t^{t+\tau_1} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(t)|y_2(t)-y_2^*(t)|\mathrm{d}s \\
&+ a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m}|x_2(t)-x_2^*(t)|. \qquad (36)
\end{aligned}
$$

Let

$$
\begin{aligned}
V_2(t) =\ & \int_{t-\tau_2}^{t}\int_{s-\tau_1}^{s} a_2(s)\mathrm{e}^{-\int_s^{s+\tau_2} r_2(m)\mathrm{d}m}|y_2(s)-y_2^*(s)|a_1(m)\mathrm{e}^{\int_s^m r_1(n)+k_1(n)y_2(n)\mathrm{d}n} \\
&\times x_2(m)\mathrm{d}m\mathrm{d}s + |y_2(t)-y_2^*(t)| + \int_t^{t+\tau_1}\int_{s-\tau_1-\tau_2}^{t-\tau_2} a_2(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m} \\
&\times y_2^*(s-\tau_2)a_1(u)|x_2(u)-x_2^*(u)|\mathrm{e}^{\int_{s-\tau_2}^{u} r_1(m)+k_1(m)y_2(m)\mathrm{d}m}\mathrm{d}u\mathrm{d}s \\
&+ \int_t^{t+\tau_1}\int_{s-\tau_1-\tau_2}^{t-\tau_2} a_2(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}y_2^*(s-\tau_2)k_1(u)|y_2(u)-y_2^*(u)| \\
&\times \int_{s-\tau_1-\tau_2}^{s-\tau_2} a_1(m)x_2^*(m)\mathrm{d}m\mathrm{d}u\mathrm{d}s + \int_t^{t+\tau_2}\int_u^{u+\tau_1} a_1(u-\tau_2)a_2(s-\tau_2)y_2^*(s-\tau_2) \\
&\times |x_2(u-\tau_2)-x_2^*(u-\tau_2)|\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}\mathrm{e}^{\int_{s-\tau_2}^{u-\tau_2} r_1(n)+k_1(n)y_2(n)\mathrm{d}n}\mathrm{d}s\mathrm{d}u \\
&+ \int_t^{t+\tau_2}\int_u^{u+\tau_1} a_2(s-\tau_2)|y_2(u-\tau_2)-y_2^*(u-\tau_2)|y_2^*(s-\tau_2)k_1(u-\tau_2) \\
&\times \mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}\int_{s-\tau_1-\tau_2}^{s-\tau_2} a_1(m)x_2^*(m)\mathrm{d}m\mathrm{d}s\mathrm{d}u.
\end{aligned}
$$

Similarly, calculating the upper right derivative of $V_2(t)$ along positive solutions of system (13) gives

$$
\begin{aligned}
D^+V_2(t) \le\ & a_2(t)\mathrm{e}^{-\int_t^{t+\tau_2} r_2(m)\mathrm{d}m}|y_2(t)-y_2^*(t)|\int_{t-\tau_1}^{t} a_1(m)x_2(m)\mathrm{e}^{\int_t^m r_1(n)+k_1(n)y_2(n)\mathrm{d}n}\mathrm{d}m \\
&+ \int_{t+\tau_2}^{t+\tau_1+\tau_2} a_2(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}y_2^*(s-\tau_2)\mathrm{e}^{\int_{s-\tau_2}^{t} r_1(m)+k_1(m)y_2(m)\mathrm{d}m}\mathrm{d}s \\
&\times a_1(t)|x_2(t)-x_2^*(t)| + \int_{t+\tau_2}^{t+\tau_1+\tau_2} a_2(s-\tau_2)y_2^*(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}\int_{s-\tau_1-\tau_2}^{s-\tau_2} a_1(m) \\
&\times k_1(t)|y_2(t)-y_2^*(t)|x_2^*(m)\mathrm{d}m\mathrm{d}s - \beta_2(t)(y_2(t)+y_2^*(t))|y_2(t)-y_2^*(t)|. \qquad (37)
\end{aligned}
$$

We now define

$$V(t) = V_1(t) + pV_2(t), \qquad (38)$$

where p is a positive number. Then it follows from (36)–(38) that

$$
\begin{aligned}
D^+V(t) \le\ & |x_2(t)-x_2^*(t)|\Bigg[a_1(t)\mathrm{e}^{\int_t^{t+\tau_1} -r_1(m)-k_1(m)y_2(m)\mathrm{d}m} - \beta_1(t)(x_2(t)+x_2^*(t)) \\
&+ \int_{t+\tau_2}^{t+\tau_1+\tau_2} pa_2(s-\tau_2)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}y_2^*(s-\tau_2)a_1(t)\mathrm{e}^{\int_{s-\tau_2}^{t} r_1(m)+k_1(m)y_2(m)\mathrm{d}m}\mathrm{d}s\Bigg]
\end{aligned}
$$

$$+|y_2(t)-y_2^*(t)|\left[\int_t^{t+\tau_1} a_1(s-\tau_1)x_2^*(s-\tau_1)k_1(t)\mathrm{d}s - p\beta_2(t)(y_2(t)+y_2^*(t))\right.$$
$$+pa_2(t)\mathrm{e}^{-\int_t^{t+\tau_2} r_2(m)\mathrm{d}m}\int_{t-\tau_1}^{t} a_1(m)x_2(m)\mathrm{e}^{\int_t^m r_1(n)+k_1(n)y_2(n)\mathrm{d}n}\mathrm{d}m$$
$$\left.+\int_{t+\tau_2}^{t+\tau_1+\tau_2} pa_2(s-\tau_2)k_1(t)\mathrm{e}^{-\int_{s-\tau_2}^{s} r_2(m)\mathrm{d}m}y_2^*(s-\tau_2)\int_{s-\tau_1-\tau_2}^{s-\tau_2} a_1(m)x_2^*(m)\mathrm{d}m\mathrm{d}s\right].$$

By Theorem 2, for any $\varepsilon > 0$ there exists a value $T > 0$ such that when $t > T$ we have

$$\begin{cases}\hat{m}_2-\varepsilon < x_2(t) < \hat{M}_2+\varepsilon, & \hat{m}_2-\varepsilon < x_2^*(t) < \hat{M}_2+\varepsilon,\\ \hat{m}_4-\varepsilon < y_2(t) < \hat{M}_4+\varepsilon, & \hat{m}_4-\varepsilon < y_2^*(t) < \hat{M}_4+\varepsilon.\end{cases}$$

So for $t > T + 2\max\{\tau_1, \tau_2\}$ we derive that

$$D^+V(t) \le -(A_1(t)-\varepsilon)|x_2(t)-x_2^*(t)| - (A_2(t)-\varepsilon)|y_2(t)-y_2^*(t)|.$$

By (35), there exist positive constants α_1, α_2 and $T^* \ge T + 2\max\{\tau_1, \tau_2\}$ such that for $t \ge T^*$,

$$A_1(t) \ge \alpha_1 > 0, \qquad A_2(t) \ge \alpha_2 > 0.$$

Thus, for $t \ge T^*$ we have

$$D^+V(t) \le -\frac{\alpha_1}{2}|x_2(t)-x_2^*(t)| - \frac{\alpha_2}{2}|y_2(t)-y_2^*(t)|. \tag{39}$$

Integrating both sides of (39) on the interval $[T^*, t]$, we obtain that for $t \ge T^*$

$$V(t) + \frac{\alpha_1}{2}\int_{T^*}^t |x_2(s)-x_2^*(s)|\mathrm{d}s + \frac{\alpha_2}{2}\int_{T^*}^t |y_2(s)-y_2^*(s)|\mathrm{d}s \le V(T^*).$$

Hence, $V(t)$ is bounded on $[T^*, +\infty)$ and

$$\int_{T^*}^t |x_2(s)-x_2^*(s)|\mathrm{d}s < +\infty, \qquad \int_{T^*}^t |y_2(s)-y_2^*(s)|\mathrm{d}s < +\infty.$$

By Barbalat's Lemma [46], we conclude

$$\lim_{t\to+\infty}|x_2(t)-x_2^*(t)| = 0, \qquad \lim_{t\to+\infty}|y_2(t)-y_2^*(t)| = 0. \tag{40}$$

Since

$$x_1(t) = \int_{t-\tau_1}^t a_1(s)\mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2(m)\mathrm{d}m}x_2(s)\mathrm{d}s,$$
$$x_1^*(t) = \int_{t-\tau_1}^t a_1(s)\mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2^*(m)\mathrm{d}m}x_2^*(s)\mathrm{d}s,$$

we have

$$\begin{aligned}|x_1(t)-x_1^*(t)| &\le \int_{t-\tau_1}^t a_1(s)\mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2(m)\mathrm{d}m}|x_2(s)-x_2^*(s)|\mathrm{d}s\\ &\quad + \int_{t-\tau_1}^t a_1(s)x_2^*(s)|\mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2(m)\mathrm{d}m} - \mathrm{e}^{\int_t^s r_1(m)+k_1(m)y_2^*(m)\mathrm{d}m}|\mathrm{d}s\\ &\le \int_{t-\tau_1}^t a_1^M|x_2(s)-x_2^*(s)|\mathrm{d}s + \int_{t-\tau_1}^t a_1^M M_2\int_s^t k_1^M|y_2(m)-y_2^*(m)|\mathrm{d}m\mathrm{d}s,\end{aligned}$$

which, together with (40), yields

$$\lim_{t\to+\infty}|x_1(t)-x_1^*(t)| = 0.$$

By a closely similar argument, we can obtain

$$\lim_{t\to+\infty}|y_1(t)-y_1^*(t)| = 0.$$

Therefore, we have completed the proof of Theorem 4. □

5. Two examples

In this section, we give two examples to illustrate the feasibility of our main results.

Example 1. Consider the following delayed nonautonomous model:

$$\begin{cases} x_1'(t) = \left(2.1 + \dfrac{\sin(t)}{10}\right) x_2(t) - \left(0.2 + \dfrac{\sin(t)}{200}\right) x_1(t) - \left(2.1 + \dfrac{\sin(t-0.1)}{10}\right) \\ \qquad \times\ \mathrm{e}^{\int_{t-0.1}^{t} -\left(0.2+\frac{\sin(s)}{200}\right)-\left(\frac{1}{3}+\frac{\sin(s)}{20}\right)y_2(s)\mathrm{d}s} x_2(t-0.1) - \left(\dfrac{1}{3} + \dfrac{\sin(t)}{20}\right) x_1(t)y_2(t), \\ x_2'(t) = \left(2.1 + \dfrac{\sin(t-0.1)}{10}\right) \mathrm{e}^{\int_{t-0.1}^{t} -\left(0.2+\frac{\sin(s)}{200}\right)-\left(\frac{1}{3}+\frac{\sin(s)}{20}\right)y_2(s)\mathrm{d}s} x_2(t-0.1) - \left(1 + \dfrac{\sin(t)}{300}\right) x_2^2(t), \\ y_1'(t) = \left(0.3 + \dfrac{\cos(t)}{30}\right) x_1(t)y_2(t) - \left(\dfrac{1}{3} - \dfrac{\sin(t)}{50}\right) y_1(t) - \left(0.3 + \dfrac{\cos(t-1)}{30}\right) \\ \qquad \times\ \mathrm{e}^{\int_{t-1}^{t} -\left(\frac{1}{3}-\frac{\sin(s)}{50}\right)\mathrm{d}s} x_1(t-1)\, y_2(t-1), \\ y_2'(t) = \left(0.3 + \dfrac{\cos(t-1)}{30}\right) \mathrm{e}^{\int_{t-1}^{t} -\left(\frac{1}{3}-\frac{\sin(s)}{50}\right)\mathrm{d}s} x_1(t-1)\, y_2(t-1) - \left(3 - \dfrac{\sin(t)}{30}\right) y_2^2(t), \end{cases} \tag{41}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature and the mature prey population at time t, respectively; $y_1(t)$ and $y_2(t)$ represent the densities of the immature and the mature predator population at time t, respectively. By a direct computation, we have $A_1(t) \geq 0.5325$ and $A_2(t) \geq 1.2002$. That is, the (35) in Theorem 4 is satisfied. According to Theorem 4, system (41) has a unique positive ω-periodic solution $(x_1^*(t),\ x_2^*(t),\ y_1^*(t),\ y_2^*(t))^{\mathrm{T}}$ which is globally stable. With the initial conditions (0.3, 1.5, 0.002, 0.03), we use Matlab to illustrate the unique positive periodic solution of system (41) in Fig. 1. From Fig. 1(a) and (b) we can see that the solutions $x_1(t)$ and $x_2(t)$ are periodic. Since the amplitude of $y_1(t)$ and $y_2(t)$ is very small, we use phase portraits demonstrated in Fig. 1(c) and (d) to show the periodicity of $y_1(t)$ and $y_2(t)$. With the initial conditions (0.6, 2.5, 0.005, 0.035), we use Matlab to present the unique positive periodic solution of system (41) in Fig. 2(a) and (b).

Example 2. Now, we consider another example to illustrate the feasibility of our main results, which is concerning the prey zebra and the predator lion. We know that lions begin to prey by themselves at the age of two and the yearling fecundity of zebra is zero [52], so we assume $\tau_1 = \tau_2 = 2$. We also suppose that the birth rates of the predator and prey are periodic due to fluctuation in time and other parameters fixed in the following delayed nonautonomous model based on [52]:

$$\begin{cases} x_1'(t) = (3.001 + 3\sin(t))\, x_2(t) - 0.3x_1(t) - (3.001 + 3\sin(t-2)) \\ \qquad \times\ \mathrm{e}^{\int_{t-2}^{t}(-0.3-\frac{1}{30}y_2(s))\mathrm{d}s} x_2(t-2) - \dfrac{1}{30} x_1(t)y_2(t), \\ x_2'(t) = (3.001 + 3\sin(t-2)) \times \mathrm{e}^{\int_{t-2}^{t}(-0.3-\frac{1}{30}y_2(s))\mathrm{d}s} x_2(t-2) - 0.01x_2^2(t), \\ y_1'(t) = (0.01501 + 0.015\cos(t))\, x_1(t)y_2(t) - 0.4y_1(t) \\ \qquad - (0.01501 + 0.015\cos(t-2))\, \mathrm{e}^{\int_{t-2}^{t} -0.4\mathrm{d}s} x_1(t-2)\, y_2(t-2), \\ y_2'(t) = (0.01501 + 0.015\cos(t-2))\, \mathrm{e}^{\int_{t-2}^{t} -0.4\mathrm{d}s} x_1(t-2)\, y_2(t-2) - 0.1y_2^2(t), \end{cases} \tag{42}$$

where $x_1(t)$ and $x_2(t)$ denote the densities of the immature and the mature zebra population at time t, respectively; $y_1(t)$ and $y_2(t)$ represent the densities of the immature and the mature lion population at time t, respectively. It is easy to verify that the assumption in Theorem 3 is satisfied. According to our results, system (42) has at least one strictly positive ω-periodic solution. With the initial conditions (220*0.38, 220*0.62, 17, 9) and using Matlab, we can see positive periodic solutions $x_1(t), x_2(t), y_1(t)$ and $y_2(t)$ of system (42) in Fig. 3.

6. Discussion

As pointed in References [9,10], many consumer species go through two or more life stages as they proceed from birth to death. Some predator–prey models in the early literature ignore such reality and lump individuals into one single reproducing category which can be modeled by ordinary differential equations (ODEs). Unfortunately, such ODEs are only capable of generating simple equilibrium dynamics. In order to capture the oscillatory, persistent and stable behavior often observed in nature, various mechanisms are proposed. Such mechanisms include difference models and delay differential models [9,48,49]. In biological modeling of population growth, the standard Lotka–Volterra-type models are often used by ecologists to describe interactions between predator and prey populations. Recently, the traditional prey-dependent predator–prey models have been challenged by biologists [50,51] based on the fact that functional and numerical response over typical ecological time scales ought to depend on the densities of both prey and predator, especially when predators have to search, share or compete for food. Since the birth and the death rate may be affected by periodic factors such as the

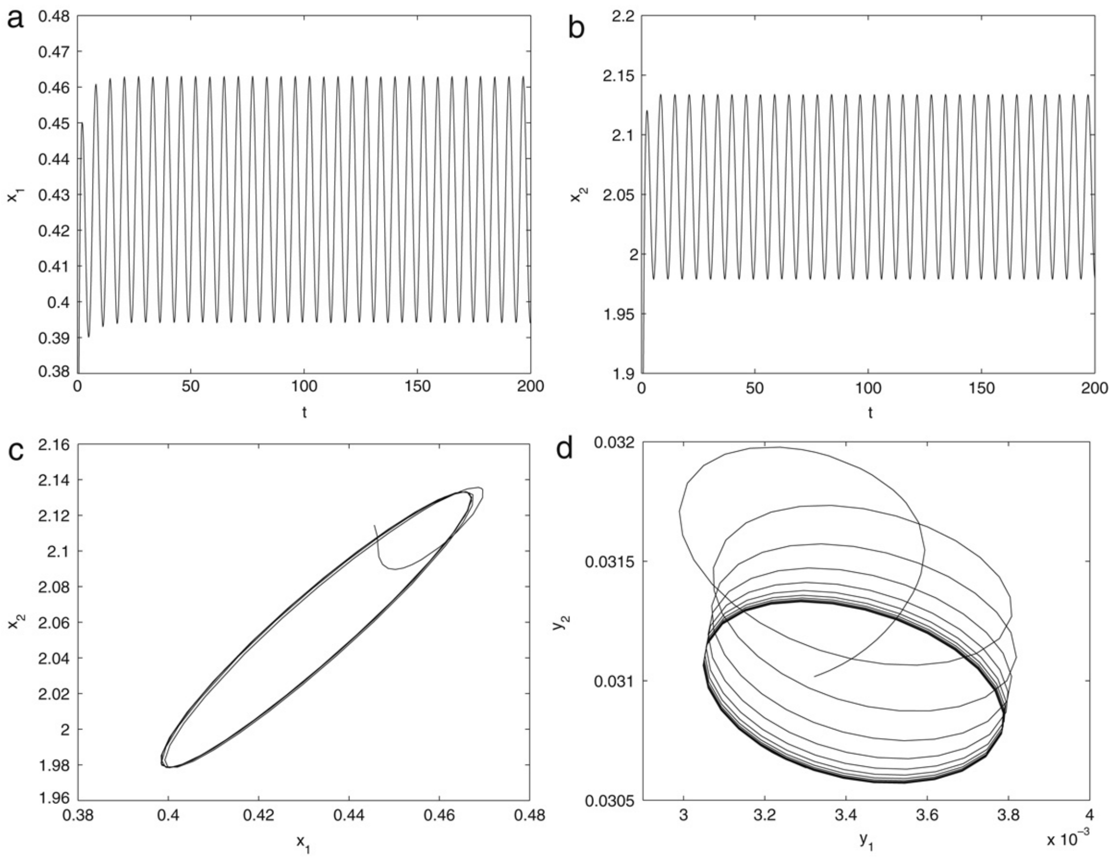

Fig. 1. Periodicity of the positive solution of system (41) with the initial condition (0.3, 1.5, 0.002, 0.03).

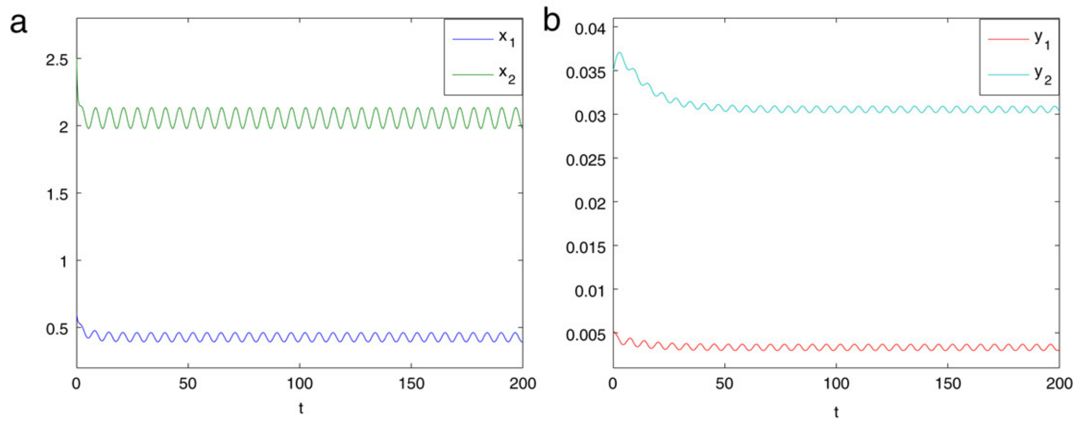

Fig. 2. Periodicity of the positive solution of system (41) with the initial condition (0.6, 2.5, 0.005, 0.035).

season switching and the change of climate, etc., it appears that a more suitable predator–prey model should be constructed by incorporating stage structures for both prey and predator and the periodicity of ecological and environmental parameters into the system.

Motivated by this fact and the important works [17,18,29,40], our main purpose of this paper is to analyze a two-species predator–prey nonautonomous model with stage structure for both prey and predator, in which there are two time delays due to the maturity for both immature prey and immature predator, respectively. Under certain initial conditions, the permanence of system (1) was investigated, and the existence of the positive ω-periodic solutions of system (1) was shown by using Gaines and Mawhin's continuation theorem of the coincidence degree theory. Sufficient conditions were obtained for the global stability of positive ω-periodic solutions of system (1) by constructing a suitable Lyapunov functional.

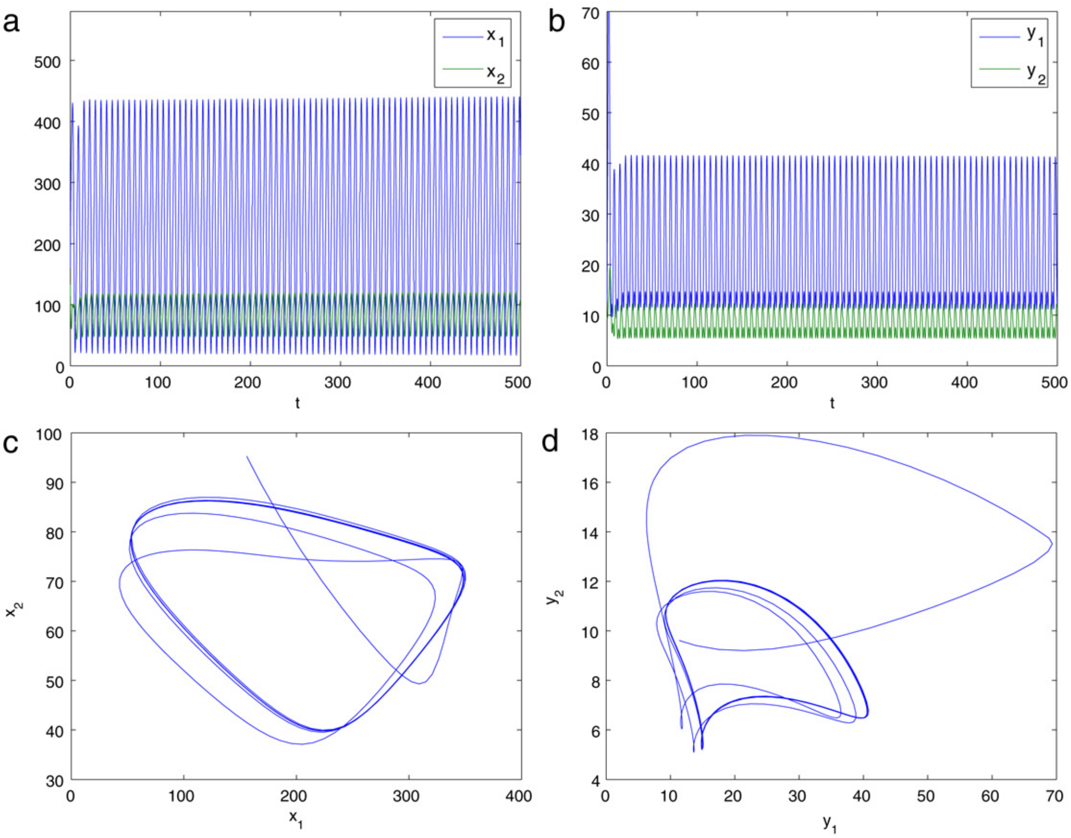

Fig. 3. Periodicity of the positive solution of system (42) with the initial condition $(220 * 0.38, 220 * 0.62, 17, 9)$.

When assumptions (I)–(II) and initial conditions change, more results on the dynamics of this model will be revealed in a forthcoming work, which should be submitted somewhere else.

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References

[1] M.P. Hassell, The Dynamics of Arthropod Predator–Prey Systems, Princeton University Press, NJ, 1978.
[2] S.R. Kerr, L.M. Dickie, The Biomass Spectrum: A Predator–Prey Theory of Aquatic Production, Columbia University Press, New York, 2001.
[3] R.M. May, G.R. Conway, M.P. Hassella, T.R.E. Southwood, Time delays, density dependence, and single-species oscillations, J. Anim. Ecol. 43 (1974) 747–770.
[4] W.S.C. Gurney, S.P. Blythe, R.M. Nisbet, Nicholson's glowflies revisited, Nature 287 (1980) 17–21.
[5] W.G. Aiellot, The existence of nonoscillatory solutions to a generalized, nonautonomous, delay logistic equation, J. Math. Anal. Appl. 149 (1990) 114–123.
[6] M.E. Fishera, B.S. Goh, Stability results for delayed-recruitment models in population dynamics, J. Math. Biol. 19 (1984) 147–156.
[7] H.I. Freedman, K. Gopalsamyg, Global stability in time-delayed single species dynamics, Bull. Math. Biol. 48 (1986) 485–492.
[8] S.Q. Ma, Z. Feng, Q.S. Lu, A two-parameter geometrical criteria for delay differential equations, Discrete Contin. Dyn. Syst. Ser. B 5 (2008) 397–413.
[9] Y. Kuang, Delay Differential Equations with Applications in Population Dynamics, Academic Press, Boston, 1993.
[10] A.C. Fowler, Mathematical Models in the Applied Sciences, in: Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1997.
[11] H.D. Landahl, B.D. Hanson, A three stage population model with cannibalism, Bull. Math. Biol. 37 (1975) 11–17.
[12] H.J. Barclay, P. Van Driessche, A model for a single species with two life history stages and added mortality, Ecol. Model. 11 (1980) 157–166.
[13] S.N. Wood, S.P. Blythe, W.S.C. Gurney, R.M. Nisbeti, Instability in mortality estimation schemes related to stage structured population data, Math. Med. Biol. 6 (1989) 47–68.
[14] F. Brauer, Z.E. Ma, Stability of stage-structured population models, J. Math. Anal. Appl. 126 (1987) 301–315.
[15] H.I. Freedman, J. Wu, Persistence and global asymptotic stability of single species dispersal models with stage structure, Quart. Appl. Math. 49 (1991) 351–371.
[16] L.M. Cai, X.Y. Song, Permanence and stability of a predator–prey system with stage structure for predator, J. Comput. Appl. Math. 201 (2007) 356–366.
[17] X.A. Zhang, L.S. Chen, A.U. Neumann, The stage-structured predator–prey model and optimal harvesting policy, Math. Biosci. 168 (2000) 201–210.

[18] R. Xu, Z.Q. Wang, Periodic solutions of a nonautonomous predator–prey system with stage structure and time delays, J. Comput. Appl. Math. 196 (2006) 70–86.
[19] X.Y. Song, L.S. Chen, Optimal harvesting and stability for a two-species competitive system with stage structure, Math. Biosci. 170 (2001) 173–186.
[20] Z.Q. Zhang, Periodic solutions of a predator–prey system with stage-structures for predator and prey, J. Math. Anal. Appl. 302 (2005) 291–305.
[21] J. Omari, S.A. Gourley, Dynamics of a stage-structured population model incorporating a state-dependent maturation delay, Nonlinear. Anal. 6 (2005) 13–33.
[22] S.A. Gourley, Y. Kuang, A stage structured predator–prey model and its dependence on maturation delay and death rate, J. Math. Biol. 49 (2004) 188–200.
[23] J. Omari, S.A. Gourley, Stability and traveling fronts in Lotka–Volterra competition models with stage structure, SIAM J. Appl. Math. 63 (2003) 2063–2086.
[24] L.M. Ou, G.L. Luo, Y.L. Jiang, Y.P. Li, The asymptotic behaviors of a stage-structured autonomous predator–prey system with time delay, J. Math. Anal. Appl. 283 (2003) 534–548.
[25] W. Wang, G. Mulone, F. Salemi, V. Salone, Permanence and stability of a stage-structured predator–prey model, J. Math. Anal. Appl. 262 (2001) 499–528.
[26] S.Q. Liu, M. Kouche, N. Tatar, Permanence extinction and global asymptotic stability in a stage structured system with distributed delays, J. Math. Anal. Appl. 301 (2005) 187–207.
[27] K.G. Magnusson, Destabilizing effect of cannibalism on a structured predator–prey system, Math. Biosci. 155 (1999) 61–75.
[28] W.G. Aiello, H.I. Freedman, J. Wu, Analysis of a model representing stage-structured population growth with state-dependent time delay, SIAM J. Appl. Math. 52 (1992) 885–869.
[29] W.G. Aiello, H.I. Freedman, A time-delay model of single-species growth with stage structure, Math. Biosci. 101 (1990) 139–153.
[30] E. Beretta, Y. Kuang, Global analyses in some delayed ratio-dependent predator–prey systems, Nonlinear. Anal. 32 (1998) 381–408.
[31] F. Berezovskaya, G. Karev, R. Arditi, Parametric analysis of the ratio-dependent predator–prey model, J. Math. Biol. 43 (2001) 221–246.
[32] D.M. Xiao, Z. Zhang, On the uniqueness and nonexistence of limit cycles for predator–prey systems, Nonlinearity 16 (2003) 1185–1201.
[33] O. Arino, A.E. abdllaoui, J. Mikram, J. Chattopadhyay, Infection in prey population may act as a biological control in ratio-dependent predator–prey models, Nonlinearity 17 (2004) 1101–1116.
[34] S.B. Hsu, T.W. Hwang, Y. Kuang, Global analysis of the MichaelisMenten ratio-dependent predator–prey system, J. Math. Biol. 42 (2001) 489–506.
[35] S.B. Hsu, T.W. Hwang, Y. Kuang, A ratio-dependent food chain model and its applications to biological control, Math. Biosci. 181 (2003) 55–83.
[36] C. Jost, O. Arino, R. Arditi, About deterministic extinction in ratio-dependent predator–prey models, Bull. Math. Biol. 61 (1999) 19–32.
[37] Y. Kuang, E. Beretta, Global qualitative analysis of a ratio-dependent predator–prey system, J. Math. Biol. 36 (1998) 389–406.
[38] R. Xu, L. Chen, Persistence and global stability for n-species ratio-dependent predator–prey system with time delays, J. Math. Anal. Appl. 275 (2002) 27–43.
[39] R. Xu, M.A.J. Chaplain, F.A. Davidson, Permanence and periodicity of a delayed ratio-dependent predator–prey model with stage structure, J. Math. Anal. Appl. 303 (2005) 602–621.
[40] S.C. Kats, L.B. Kats, C.B. Anzalone, Aggression by non-native crayfish deters breeding in California newts, Conserv. Biol. 11 (1997) 793–796.
[41] Z.E. Ma, Stability of predation models with time delays, Appl. Anal. 22 (1986) 169–192.
[42] S.Q. Ma, Q.S. Lu, Z. Feng, Double Hopf bifurcation for van der Pol–Duffing oscillator with parametric delay feedback control, J. Math. Anal. Appl. 338 (2008) 993–1007.
[43] W.D. Wang, Z.E. Ma, Harmless delays for uniform persistence, J. Math. Anal. Appl. 158 (1991) 256–268.
[44] H.F. Huo, Permanence and global attractivity of delay diffusive prey–predator systems with the Michaelis–Menten functional response, Comput. Math. Appl. 49 (2005) 407–416.
[45] J.R. Zhang, L.S. Chen, Permanence and global stability for two-species cooperative system with delays in two-patch environment, Math. Comput. Modeling. 32 (1996) 17–27.
[46] R.E. Gains, J.L. Mawhin, Coincidence Degree and Nonlinear Differential Equations, Spring, Berlin, 1977.
[47] A. Castro, M. Chhetri, R. Shivaji, Stability analysis of positive solutions to classes of reaction–diffusion systems, Differential Integral Equations 17 (2004) 391–406.
[48] M.M. May, Stability and Complexity in Model Ecosystems (Princeton Landmarks in Biology), Princeton University Press, NJ, 2001.
[49] W.W. Murdoch, C.J. Briggs, R.M. Nisbet, Consumer-Resource Dynamics, Princeton University Press, NJ, 2003.
[50] R. Arditi, H. Saiah, Empirical evidence of the role of heterogeneity in ratio-dependent consumption, Ecology 73 (1992) 1544–1551.
[51] A.P. Gutierrez, The physiological basis of ratio-dependent predator–prey theory: A metabolic pool model of Nicholsons blowflies as an example, Ecology 73 (1992) 1552–1563.
[52] M.G.L. Mills, T.M. Shenkt, predator–prey relationships: The impact of lion predation on wildebeest and zebra populations, J. Anim. Ecol. 61 (1992) 693–702.