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ERRATUM

TO: Marta C. BUNGE, TOPOS THEORY AND SOUSLIN'S HYPOTHESIS

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Harry Dole (Department of Mathematics, The University of Chicago) has pointed out to me that my Lemma 4.1 [1, p. 182] is false, as a simple counter example shows. He has kindly provided a modification of the proof of that portion of Theorem 3.3 [1, p. 177] which subsumes Lemma 4.1 in its formulation, as well as a corrected form of Lemma 4.5.2 [1, p. 186]. These corrections are given below. I am extremely grateful to Professor Saunders Mac Lane for having discussed my paper in his seminar at Chicago and to Harry Dole for having so generously corrected the errors in it. The same notations as used in [1] are kept throughout.

3.3. Theorem. *Let* $\boldsymbol{P}$ *be a partially ordered set. Then, in* $\mathrm{Sh}_{\neg\neg}(\mathbf{P})$, *the following are equivalent*:

(i) $B \overset{u}{\rightarrowtail\!\!\!\twoheadrightarrow} \hat{\omega}_1$ *is cofinal*;
(ii) B *is uncountable.*

It is only required to reprove (i) $\Rightarrow$ (ii). Let B be cofinal but $\mathrm{Epi}\,(\hat{\omega}_1, B) \neq 0$. Then, there exists and epi $p \times \hat{\omega} \overset{k}{\twoheadrightarrow} p \times B$. We want to obtain a contradiction, by defining some $\omega \overset{f}{\to} \omega_1$, with f cofinal. For $q \geq p$ define θ_q as in the following diagram

$$\begin{tikzcd}
\hat{\omega}(q) \arrow[r, two heads] & B(q) \arrow[r] & \hat{\omega}_1(q) \\
\omega = \omega(q) \arrow[u, tail] \arrow[urr, "\theta_q"'] & & \omega_1 \arrow[u, tail]
\end{tikzcd}$$

Fix $n \in \omega$. For any $\alpha \in \omega_1$, let $A_\alpha = \{q \in P \mid q \geq p \ \&\ \theta_q(n) = \alpha\}$. Let $B_n = \{A_\alpha \mid A_\alpha \neq \emptyset\}$. Then, $B_n \neq \emptyset$, by density $\Delta\omega_1 \to \hat{\omega}_1$. By the axiom of choice, let Z_n be the range of a choice function into B_n. We will show that Z_n is countable by the ω-chain condition on $\boldsymbol{P}$. To show that $\boldsymbol{P}$ satisfies the ω-chain condition, Tierney's Lemma 2 in [2, p. 32] need only be modified to show that any partial map that extends two trees can be made into a tree. Now, let $q, q' \in Z_n$ be such that $q \neq q'$ and suppose that there exists $\bar{p} \in P$ such that $\bar{p} \geq q \ \&\ \bar{p} \geq q'$. We have a contradiction because $\theta_q(n) \neq \theta_{q'}(n)$ and the following commutative diagram

$$\begin{tikzcd}
 & \hat{\omega}_1(\bar{p}) & \\
\hat{\omega}_1(q) \arrow[ur] & \omega_1 \arrow[l] \arrow[r] \arrow[u, tail] \arrow[ul, phantom, "="] \arrow[ur, phantom, "="] & \hat{\omega}_1(q').
\arrow[ul]
\end{tikzcd}$$

Thus, Z_n is countable. Define $f:\omega \to \omega_1$ as $f(n)=\sup\{\theta_q(n) | q \in Z_n\}$. We show f cofinal: Let $\alpha \in \omega_1$. By cofinality of B, there exists $q \geqslant p$, there exists $\beta \geqslant \alpha$ and there exists $q \xrightarrow{y} B$, such that

$$\begin{tikzcd}
q \arrow[r, "y"] \arrow[d] \arrow[dr, phantom, "="] & B \arrow[d] \\
1 \arrow[r, "\hat{\beta}"] & \hat{\omega}_1
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
 & B(q) \arrow[d] \\
1 \arrow[ur, "y_q"] \arrow[r, "\hat{\beta}_q"'] \arrow[r, phantom, shift left=3, "="] & \hat{\omega}_1(q).
\end{tikzcd}$$

Let $z \in \hat{\omega}(q)$ be such that $z \mapsto y_q$ via $\hat{\omega}(q) \twoheadrightarrow B(q)$. But, there exists $q' \geqslant q$ such that $z_{q'} \in \omega$ and $z_{q'} \mapsto y_{q'}$. Moreover, there exists $r \geqslant q'$ such that $\hat{\beta}_r \in \omega_1$. Note: $z_r \mapsto y_r \mapsto \hat{\beta}_r = \beta \in \omega_1$. Thus $f(z_r) \geqslant \beta$, and we have the desired contradiction.

4.5.2. Lemma. *Let $B \overset{u}{\rightarrowtail} \hat{\omega}_1$ be a chain of $(\hat{\omega}_1,\ a(R))$. Then, either $B=0$ or* $\operatorname{Epi}(\hat{\omega}, B) \neq 0$.

Proof. Assume $B \neq 0$ and $\operatorname{Epi}(\hat{\omega}, B)=0$. Let $p \in \boldsymbol{P}$ be such that $B(p) \neq 0$. Set $W=\{q \in \boldsymbol{P} | q \geqslant p \ \ \& \ \ \exists \beta \in q^* \text{ s.t. } q \Vdash \hat{\beta} \in u\}$. Define $f: W \to \omega_1$, as $f(q)= \max\{\beta \in q^* \,|\, q \Vdash \hat{\beta} \in u\}$. Note that the image of f is cofinal since B is cofinal. Index image (f) as β_i, where $\beta_i < \beta_j$ for $i<j$. By the axiom of choice, pick q_j such that $f(q_j)=\beta j$. Extend q_j to $\bar{q}_j$ by adding the node β_{j+1} such that $\bar{q}_j(\beta_j, \beta_{j+1})=1$. Let $\{\gamma_j\}_{j \in I}$ be an ordered cofinal set disjoint from $\{\beta_j\}_{j \in I}$ such that $\gamma_j \geqslant \max(\bar{q}_j^*)$. Extend $\bar{q}_j$ to r_j by adding node γ_j such that $r_j(\beta_j, \gamma_j)=1$ but $r_j(\beta_{j+1}, \gamma_j)=0$. Define $V \rightarrowtail \Delta\omega_1$ as $V(q)=\{\gamma_j \,|\, q \geqslant r_j\}$. Then, $V \rightarrowtail \Delta\omega_1$ is an antichain since given $\gamma_j, \gamma_k \in V(q)$ where $j<k$, $q(\gamma_j, \gamma_k)=0$. Thus, we obtain $a(V) \rightarrowtail \hat{\omega}_1$ is an antichain and cofinal by construction. This contradicts Lemma 4.5.1 [1, p. 185].

References

[1] Marta C. Bunge, Topos Theory and Souslin's Hypothesis, J. Pure Appl. Algebra 4 (1974) 159–187.

[2] Miles Tierney, Sheaf Theory and the Continuum Hypothesis, in: Toposes, Algebraic Geometry and Logic, Lecture Notes in Mathematics 274 (Springer Verlag, Berlin, Heidelberg, New York, 1972).