

Pseudocomplemented Semilattices, Boolean Algebras, and Compatible Products¹

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Pseudocomplemented semilattices are studied here from an algebraic point of view, stressing the pivotal role played by the pseudocomplements and the relationship between pseudocomplemented semilattices and Boolean algebras. Following the pattern of semiprime ring theory, a notion of Goldie dimension is introduced for complete pseudocomplemented lattices and calculated in terms of maximal uniform elements if they exist in abundance. Products in lattices with 0-element are studied and questions about the existence and uniqueness of compatible products in pseudocomplemented lattices, as well as about the abundance of prime elements in lattices with a compatible product, are discussed. Finally, a Yoood decomposition theorem for topological rings is extended to complete pseudocomplemented lattices. © 2001 Academic Press

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INTRODUCTION

A pseudocomplemented semilattice is a (meet) semilattice S having a least element 0 and is such that, for each x in S , there exists a largest element x^\perp such that $x \wedge x^\perp = 0$. In spite of what the name could suggest, a complemented lattice need not be pseudocomplemented, as is easily seen by considering the lattice of all subspaces of a vector space of dimension greater than one.

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When dealing with pseudocomplemented semilattices two important examples are in order: (i) the lattice of all (two-sided) ideals of a semiprime ring R , where the pseudocomplement of an ideal I of R is given by its usual annihilator, and (ii) the lattice of all open subsets of a topological space X , where the pseudocomplement of an open subset G is its exterior. These two examples are sources of inspiration for the theory of pseudocomplemented lattices. On one hand, the mapping $x \rightarrow x^\perp$ retains the usual properties of the annihilators; on the other hand, the mapping $x \rightarrow x^{\perp\perp}$, when applied to an open subset G of X , produces the least regular open subset, $G^{\perp\perp} = \text{int}(G)$, containing G . Properties of these two mappings are collected in (1.7).

Inspired by the theory of Banach algebras [BoG] (or topological rings [Y]), we pay attention to two important classes of pseudocomplemented semilattices, namely, annihilator semilattices and dual semilattices. A pseudocomplemented semilattice S is annihilator if $x \neq 1$ implies $x^\perp \neq 0$, with $1 := 0^\perp$ being the largest element of S , and S is said to be dual if $x = x^{\perp\perp}$ for all $x \in S$. Dual semilattices are annihilator but the converse is not true, as seen by considering the nonmodular pentagon (Johnson gave in [J] an example of a commutative semisimple Banach algebra whose lattice of closed ideals is annihilator but not dual). Moreover, a pseudocomplemented lattice is annihilator if, and only if, it is complemented (2.1).

In (2.4) we list different characterizations of dual semilattices. Some of them are well-known in the lattice setting, as is the fundamental one, due to Frink [Fr1, Fr2], stating that dual semilattices are precisely Boolean algebras. Others were proved by Yood [Y] for lattices of closed ideals of topological rings.

More generally, Frink showed in [Fr2] that for every pseudocomplemented semilattice S the set $\mathcal{B}(S)$ of all pseudocomplements of S has a structure of Boolean algebra (the Boolean algebra associated to the lattice of the open sets of a topological space X consists precisely of the regular open subsets of X). This key fact in the theory of pseudocomplemented semilattices can be obtained also from the dual characterization of Boolean algebras cited above.

The relationship between a pseudocomplemented semilattice and its associated Boolean algebra can be illuminated by studying uniform elements. Among other results, we prove that any uniform element u of a pseudocomplemented semilattice S is contained in a *unique* maximal uniform element, namely, $u^{\perp\perp}$. Moreover, the maximal uniform elements of S are precisely the atoms of $\mathcal{B}(S)$. Thus, the abundance of uniform elements in a pseudocomplemented semilattice S is equivalent to the atomicity of its associated Boolean algebra; in fact, $\mathcal{B}(S)$ is then isomorphic to the powerset $\mathcal{P}(M)$, where M is the set of the maximal uniform elements of S . This result, proved in (2.9), has the consequence that the

Boolean algebra of a finite pseudocomplemented lattice L is the power set of the set of the atoms of L [ChM, (3.5)].

In Section 3 we study the Goldie dimension of a complete pseudocomplemented lattice. As in the classical case of the lattice of two-sided ideals of a semiprime ring (see [L, p. 334]), we relate the Goldie dimension to annihilators. Thus, our approach to Goldie dimension is different from that in [GPu1] (see also [GPu2] and [GPu3]), where the Goldie dimension of modular lattices generalizing the dimension of modules is studied.

Let α be a cardinal number. A complete pseudocomplemented lattice L will be said to have *Goldie dimension* α ($\dim L = \alpha$) if L contains an independent subset X of cardinal α and, for every independent subset Y of L , $\text{card } Y \leq \alpha$. Since L and $\mathcal{B}(L)$ have the same meet operations, it is not difficult to see that (3.3) L has Goldie dimension if, and only if, so does $\mathcal{B}(L)$. In this case, $\dim L = \dim \mathcal{B}(L)$. A sufficient condition for a complete pseudocomplemented lattice L to have Goldie dimension is the abundance of uniform elements, that is, (3.5) if any nonzero annihilator of a complete pseudocomplemented lattice L contains a uniform element, then L has Goldie dimension equal to $\text{card } M$, where M is the set of all maximal uniform elements of L . In particular, if L is atomic, then $\dim L = \text{card } A$, where A is the set of the atoms of L . The existence of an independent subset of uniform elements whose join is essential is not, however, a necessary condition for a complete pseudocomplemented lattice to have Goldie dimension (3.6). Complete pseudocomplemented lattices L having finite Goldie dimension are characterized in (3.7) as those satisfying the chain conditions on annihilators. Then $\mathcal{B}(L)$ is the powerset of an n -element set, where n is the number of maximal uniform elements of L . Finally, we show in (3.9) that a complete pseudocomplemented lattice is a finite Boolean algebra if, and only if, it has finite Goldie dimension and this coincides with its length.

By a pseudomultiplicative lattice we mean a lattice L with 0-element and which is endowed with a product xy satisfying the following conditions for all $x, y, z \in L$; (4.1) $x \leq y \Rightarrow zx \leq zy$ and $xz \leq yz$, (4.2) $xy \leq x \wedge y$, and (4.3) $x(y \vee z) \leq xy \vee xz$. Any lattice L with 0-element is trivially pseudomultiplicative for the product defined by $xy = 0$ for all $x, y \in L$. On the other hand, any distributive lattice can be equipped with a nice product by taking $xy = x \wedge y$. Other examples of pseudomultiplicative lattices are the lattice of ideals of an algebraic system (as a nonassociative ring or quadratic Jordan algebra), the lattice of closed ideals of a topological algebraic system, and the lattice of normal subgroups of a group. It should be noted that for the lattice of the ideals of a quadratic Jordan algebra J , the lattice product is given by $B * C = U_B C$ (B and C ideals of J), where, as usual, $x \mapsto U_x$ denotes the Jordan U -operator. It was pre-

cisely the nonlinearity of the quadratic Jordan product which motivated us to adopt the nonsymmetrical distributivity condition (4.3).

To avoid trivialities, we restrict ourselves to products which are compatible with the meet in the sense that $xy = 0 \Leftrightarrow x \wedge y = 0$, equivalently, $x^2 = 0 \Rightarrow x = 0$. A pseudomultiplicative lattice whose product is compatible will be called semiprime. Regarding compatible products, two questions should be considered. Does a pseudocomplemented lattice necessarily have a compatible product? And, in such a case, is this compatible product necessarily unique? With respect to the second question, we prove in (4.15) that a pseudocomplemented lattice is distributive if, and only if, it has a (unique) idempotent product; in fact, this product coincides with the meet. Two relevant examples of lattices with an idempotent product are the lattice of ideals of a von Neumann regular ring and the lattice of closed ideals of a C^* -algebra. However, a distributive pseudocomplemented lattice can have two different compatible products, although this anomaly disappears by considering Boolean algebras (4.18). In (4.21) we give a partial answer to the first question showing that a complete pseudocomplemented lattice which is atomic can be equipped with a compatible (even commutative and associative) product.

Let L be a pseudomultiplicative lattice with a greatest element 1 and a product denoted by xy . Defining prime elements as usual, we have that if the meet of the set of the prime elements of L is 0, then the lattice is semiprime. The converse holds for the lattice of ideals of a semiprime ring. However, this does not remain true in general (5.1). Thus, it seems natural to define a pseudomultiplicative lattice to be strongly semiprime if the meet of its prime elements is 0.

Kaplansky noted that in any semiprime nonassociative ring the intersection of the prime ideals is 0. In fact, he provided a proof of this result which could also be applied to groups, with normal subgroups playing the role of ideals and commutators as the product. This led him to formulate the question for complete quasimultiplicative lattices such that any nonzero element contains a nonzero compact element, thus generalizing in fact a theorem by Keimel for complete algebraic semiprime multiplicative lattices. On the other hand, Rosický pointed out that the unit interval $[0, 1]$ is a strongly semiprime complete multiplicative lattice which however is not algebraic and extended Keimel's theorem to continuous multiplicative lattices, covering the example of the unit interval. These results (Kaplansky's and Rosický's) remain true under slightly more general conditions that allow us to include lattices of ideals of algebraic systems which are not necessarily linear (5.2). To conclude this section we obtain, as a consequence of the relationship between prime elements and uniform elements (5.6), that any complete pseudocomplemented lattice having enough uni-

form elements and endowed with a compatible product is strongly semiprime (5.7).

A topological ring R is said to be decomposable if it is the closure of the direct sum of its minimal closed ideals (each of which is then a topologically simple topological ring). Yood proved in [Y, (2.6)] that R is decomposable if, and only if, the intersection of its closed prime ideals is 0 and every proper closed (two-sided) ideal of R has nonzero annihilator. As a consequence of this result he obtained an improvement of some standard decomposition theorems (see [BoG, Sm, and To]). Later, Fernández López and Rodríguez Palacios [FR] considered this decomposability question in the setting of complete normed nonassociative algebras. In lattice terms, a complete normed nonassociative algebra is decomposable if, and only if, the complete lattice of its closed ideals is strongly semiprime and annihilator. This lattice point of view allowed the authors of [FR] to provide a nonassociative version of Yood's theorem, obtaining some decomposition theorems for complete normed alternative and Jordan algebras, and a new proof of a structure theorem for the nonassociative H^* -algebra which had been proved previously in [CuR, Theorem 1]. As can be expected from the point of view adopted in [FR], it is possible to give a purely lattice version of Yood's theorem. This is the aim of (6.3).

The reader is referred to [B, CrD, and Gr] for basic notions on lattices. For nonassociative algebras and Jordan systems, we adopt as a general reference the books [Z] and [Lo], for C^* -algebras [Di], and for JB -algebras [H]. Finally, for general accounts on JB^* -triples see [R] and [Ru].

1. PSEUDOCOMPLEMENTED SEMILATTICES

Recall that a partially ordered set (S, \leq) in which any pair of elements a and b of S has a *meet* (or infimum), $a \wedge b$, is called a (*meet*) *semilattice*. If, besides, there exists the *join* (or supremum), $a \vee b$, of any pair of elements a and b , then S is said to be a *lattice*. A semilattice in which every subset has a meet is actually a *complete lattice*.

The notion of the annihilator is by far one of the most important in semiprime ring theory and can be considered in semilattices without reference to any product. Following [B, p. 125], a *pseudocomplemented semilattice* (for short, a *p-semilattice*) is a semilattice S with a least element 0 and a unary operation (the *pseudocomplementation*) $x \rightarrow x^\perp$ such that, for each $x \in S$,

$$x \wedge y = 0 \Leftrightarrow y \leq x^\perp, \quad y \in S,$$

which is equivalent to saying that for each $x \in S$ there exists a largest element $x^\perp \in S$ such that $x \wedge x^\perp = 0$. Note that any p-semilattice has a greatest element $1 = 0^\perp$.

Pseudocomplemented semilattices satisfying the Descending Chain Condition (in particular, finite p-semilattices) are in fact lattices (*p-lattices*); any pair of elements x and y has the upper bound 1 and hence a least upper bound. In Section 2 we will give an example of a p-semilattice which is not a lattice, but first some examples of p-lattices are collected.

EXAMPLES. The following list of examples of pseudocomplemented semilattices will be enlarged in Section 4.

(1.1) The complete lattice of all (two-sided) ideals of a semiprime ring R is pseudocomplemented, with the pseudocomplement I^\perp of an ideal I of R being the usual annihilator of I . The same is true for the complete lattice of all closed ideals of a semiprime topological ring.

(1.2) Any distributive lattice L which is finite or, more generally, which satisfies the Ascending Chain Condition is pseudocomplemented. For $x \in L$, take x^\perp to be maximal among all $y \in L$ such that $x \wedge y = 0$. It follows from distributivity that x^\perp is actually maximum. Therefore, $x \rightarrow x^\perp$ defines a pseudocomplementation on L .

(1.3) Let (X, T) be a topological space. Then T is a complete lattice with the join as the union and the meet as the interior of the intersection; i.e., $\bigvee A_\alpha = \bigcup A_\alpha$ and $\bigwedge A_\alpha = \text{int}(\bigcap A_\alpha)$ for any family $\{A_\alpha\}$ of open subsets of X . This lattice is pseudocomplemented with the orthogonal of an open subset A of X being its exterior: $A^\perp = \text{ext}(A)$.

(1.4) An easy example of a p-lattice which is not modular is given by the pentagon L_5 , which can be described as the power set of a two-element set $\{x, y\}$ with an additional element z such that $\{x\} < z < \{x, y\}$. Note that $\{x\}^\perp = z^\perp = \{y\}$ and $\{y\}^\perp = z$.

(1.5) A semilattice S with 0-element will be said to have nonzero *core* if it contains a nonzero element c such that $c \leq x$ for every nonzero element x of S . Every semilattice with nonzero core is trivially a p-semilattice, with $x^\perp = 0$ for every nonzero element x of L . It should be noted that any semilattice S becomes a semilattice with nonzero core by adding at most two new elements, say 0 and c , such that $0 < c < x$ for all $x \in S$. Therefore, the class of p-semilattices is very large.

(1.6) Any element x of a p-semilattice yields two p-semilattices, the *principal ideal* $[0, x]$, and the *quotient* $[x^\perp, 1]$, where the pseudocomplements are given respectively by $y^{\perp x} := y^\perp \wedge x$ ($y \leq x$) and $y^{\perp x} := (y \wedge x)^\perp$, for all $y \geq x^\perp$.

Recall that a lattice L is said to be *complemented* if it has a least element 0 and a greatest element 1, and each of its elements has a

complement; i.e., for each $a \in L$, there exists $a' \in L$ such that $a \vee a' = 1$ and $a \wedge a' = 0$. In spite of what the name could suggest, complemented lattices are not necessarily pseudocomplemented. The lattice of the subspaces of a vector space of dimension greater than one is complemented, but not pseudocomplemented. However, any distributive complemented lattice L is pseudocomplemented, with x^\perp being the *unique* complement of x in L .

Pseudocomplementations retain most of the properties of the annihilators of ideals in a semiprime ring (for this reason, sometimes we will refer to x^\perp as the *annihilator* of x). Some of the well-known properties of the pseudocomplementation are listed below.

PROPOSITION 1.7. *Let S be a pseudocomplemented semilattice, $x, y \in S$, and $\{y_\alpha\}$ be a family of elements of S . Then*

- (1) $x \leq y \Rightarrow y^\perp \leq x^\perp$.
- (2) $x \leq x^{\perp\perp}$.
- (3) $x^\perp = x^{\perp\perp\perp}$.
- (4) $(x \wedge y)^{\perp\perp} = x^{\perp\perp} \wedge y^{\perp\perp}$.
- (5) *If both $\bigvee_\alpha y_\alpha$ and $\bigwedge_\alpha y_\alpha^\perp$ exist, then $(\bigvee_\alpha y_\alpha)^\perp = \bigwedge_\alpha y_\alpha^\perp$.*
- (6) *If both $\bigvee_\alpha y_\alpha^\perp$ and $\bigvee(x^\perp \wedge y_\alpha^\perp)$ exist, then $x^\perp \wedge (\bigvee_\alpha y_\alpha^\perp)^{\perp\perp} = (\bigvee(x^\perp \wedge y_\alpha^{\perp\perp}))^{\perp\perp}$.*

Proof. Statements (1) and (2) are clear and prove that the operation $x \rightarrow x^\perp$ is a *symmetric Galois connection* [B, p. 125]. Statement (3) follows from (1) and (2) and gives as a consequence, together with (1) and (2), that the operation $x \rightarrow x^{\perp\perp}$ is a *closure operation*. The proof of (4) is not trivial and can be found in [Fr2, (18)], while (5) was proved in [So1, (1.1)]. Therefore we only need to prove (6), which already announces the relationship between p-semilattices and Boolean algebras. Its proof is based on the following standard technique, $x^\perp = y^\perp$ iff, for all $a \in S$, $x^\perp \leq a^\perp$ is equivalent to $y^\perp \leq a^\perp$.

$$\begin{aligned}
 x^\perp \wedge (\bigvee_\alpha y_\alpha^\perp)^{\perp\perp} &= x^\perp \wedge (\bigwedge_\alpha y_\alpha^{\perp\perp})^\perp \leq a^\perp \\
 &\Leftrightarrow x^\perp \wedge (\bigwedge_\alpha y_\alpha^{\perp\perp})^\perp \wedge a^{\perp\perp} = 0 \\
 &\Leftrightarrow x^\perp \wedge a^{\perp\perp} \leq (\bigwedge_\alpha y_\alpha^{\perp\perp})^{\perp\perp} = (\bigvee_\alpha y_\alpha^\perp)^\perp = \bigwedge_\alpha y_\alpha^{\perp\perp} \\
 &\Leftrightarrow x^\perp \wedge a^{\perp\perp} \leq y_\alpha^{\perp\perp} \forall \alpha \Leftrightarrow x^\perp \wedge a^{\perp\perp} \wedge y_\alpha^\perp = 0 \forall \alpha \\
 &\Leftrightarrow x^\perp \wedge y_\alpha^\perp \leq a^\perp \forall \alpha \\
 &\Leftrightarrow (\bigvee_\alpha (x^\perp \wedge y_\alpha^\perp))^{\perp\perp} \leq a^{\perp\perp\perp} = a^\perp.
 \end{aligned}$$

■

Let S be a semilattice having a least element 0 . A nonzero element $u \in S$ is an *atom* if $0 \leq x \leq u$ implies $x = 0$ or $x = u$, for all $x \in S$. As a consequence of (1.7), we obtain

COROLLARY 1.8. *Let S be a pseudocomplemented semilattice, and let A be the set of all its atoms. If X, Y are subsets of A such that both $\bigvee X$ and $\bigvee Y$ do exist, then $X \subset Y$ if, and only if, $(\bigvee Y)^\perp \leq (\bigvee X)^\perp$.*

Proof. Clearly, for X, Y subsets of A , $X \subset Y$ implies $\bigvee X \leq \bigvee Y$ and hence $(\bigvee Y)^\perp \leq (\bigvee X)^\perp$. Conversely, if $(\bigvee Y)^\perp \leq (\bigvee X)^\perp$ then $X \subset Y$. Otherwise, take an element $x \in X$ which is not in Y . Then $x \leq (\bigvee Y)^\perp$ by (1.7.5). But $x \not\leq (\bigvee X)^\perp$, since $x \leq (\bigvee X)^\perp \leq x^\perp$ would imply $x = 0$, which is a contradiction. ■

2. PSEUDOCOMPLEMENTED SEMILATTICES AND BOOLEAN ALGEBRAS

Borrowing terminology from the theory of Banach algebras [BoG] (or topological rings [Y]), we will say that a p-semilattice S is *annihilator* if $x^\perp \neq 0$ for all $x \neq 1$ in S . A p-lattice need not be annihilator, as seen by considering the three-element chain $C_2 = \{0, x, 1\}$.

Principal ideals of annihilator semilattices do not inherit in general the annihilator condition. Following the notation of (1.4), the nonmodular pentagon L_5 is annihilator, but the principal ideal $[\emptyset, z]$ is a three-element chain and therefore is not annihilator. However, the annihilator condition is inherited by the quotient determined by a pseudocomplement: If S is an annihilator semilattice, then $[x^\perp, 1]$ is annihilator for every $x \in S$. Indeed, if $y \geq x^\perp$ is such that $y^{\perp x} = (y \wedge x)^\perp = x^\perp$, then $y^\perp = 0$, so $y = 1$ and $[x^\perp, 1]$ is annihilator.

Annihilator semilattices which are also lattices will be called *annihilator lattices*. There exist annihilator semilattices which are not lattices. Consider the partially ordered subset S obtained as the disjoint union $S = \mathcal{P}(A) \cup C_\infty$, where $\mathcal{P}(A)$ is the power set of the set $A = \{a, b, c\}$ and $C_\infty: x_1 > x_2 > \dots$ is an infinite decreasing chain satisfying the following conditions: $\{a, b\} > x_1$, $\{a\} < x_n$, and $\{b\} < x_n$ (for all n). Then S is an annihilator semilattice which is not a lattice; the pair of elements $\{a\}$ and $\{b\}$ does not have a join in S .

The following result, whose proof was the aim of [Lb], is actually a consequence of (1.7.5). See also [ChM, (4.2)] where it is stated for *finite* p-lattices.

(2.1) A pseudocomplemented lattice L is annihilator if, and only if, it is complemented.

Suppose first that L is complemented, and let $x \neq 1$. Take a complement x' of x . Then $1 = x \vee x'$ implies $x' \neq 0$, and $x \wedge x' = 0$ implies that $0 \neq x' \leq x^\perp$, so L is annihilator. Conversely, if L is annihilator we have by (1.7.5) that $(x \vee x^\perp)^\perp = x^\perp \wedge x^{\perp\perp} = 0$, which implies $x \vee x^\perp = 1$. ■

Another question concerning annihilators is when the greatest element 1 of a complete p-lattice L is the join of the atoms of L . In fact, many structure theorems in Banach algebras or topological rings can be stated in these terms (see [Y] and [FR]). Although, as we will see in the last section, the suitable setting to deal with this question is that of the lattices endowed with a product, we can already give a first approach without reference to any kind of product.

Recall that a semilattice S with a least element 0 is said to be *atomic* if, for every nonzero element $x \in S$, the principal ideal $[0, x]$ contains an atom. Semilattices satisfying the descending chain condition, and in particular finite semilattices, are clearly atomic. Note that a complete p-lattice is atomic if, and only if, $(\bigvee A)^\perp = 0$, where A is the set of its atoms.

PROPOSITION 2.2. *For a complete pseudocomplemented lattice L the following conditions are equivalent:*

- (i) *The greatest element 1 is the join of the subset of the atoms of L .*
- (ii) *L is annihilator and atomic.*

Proof. (i) \Rightarrow (ii). Let $x \neq 1$ in L . Then there exists an atom $a \in A$ such that $a \not\leq x$, so $a \wedge x = 0$ and hence $x^\perp \neq 0$, which proves that L is annihilator. To show that L is atomic, let $x \neq 0$ in L . Then $[0, x]$ does contain at least an atom, since otherwise we have by (1.7.5) that $x \leq (\bigvee A)^\perp = 1^\perp = 0$.

(ii) \Rightarrow (i). Since L is atomic, $(\bigvee A)^\perp = 0$. Hence $\bigvee A = 1$ because L is annihilator. ■

The reader is again referred to [ChM, (4.2)] where the above equivalence is proved for finite p-lattices.

Going on with the terminology of topological rings [Y], a p-semilattice S will be said to be *dual* if $x^{\perp\perp} = x$ for all $x \in S$. By (1.7.3), a p-semilattice S is dual if, and only if, the pseudocomplementation $x \rightarrow x^\perp$ is injective. Hence it is clear that dual semilattices are annihilator: $x^\perp = 0 = 1^\perp \Rightarrow x = 1$. But the converse is false, as seen by considering the nonmodular pentagon (1.4).

As usual [CrD, p. 35], by a *Boolean algebra* we mean a complemented distributive lattice. Frink gave in [Fr1] the following concise characterization of Boolean algebras, which can also be found in [La, (1.1.2)].

(2.3) A semilattice S with a least element 0 is a Boolean algebra if, and only if, it has a unary operation $x \rightarrow x'$ such that $x \wedge y' = 0 \Leftrightarrow x \leq y$.

It is clear that any dual semilattice $(L, \wedge, 0, \perp)$ satisfies the condition (2.3) by taking $y' = y^\perp$, so it is a Boolean algebra. However, we will not use this fact in the proof of the following result, which provides several characterizations of Boolean algebras.

THEOREM 2.4. *For a semilattice L , the following conditions are equivalent:*

- (i) L is an annihilator lattice which is modular.
- (ii) L is pseudocomplemented and $[0, x]$ is annihilator for all $x \in L$.
- (iii) L is dual.
- (iv) L is a pseudocomplemented lattice with $a \vee b = (a^\perp \wedge b^\perp)^\perp$.
- (v) L is a Boolean algebra.

Moreover, for a complete lattice L the following conditions are equivalent:

- (vi) L is an atomic Boolean algebra.
- (vii) L is a pseudocomplemented lattice with $x = \bigvee \{a \in A : a \leq x\}$ for every $x \in L$, where A denotes the set of the atoms of L .
- (viii) L is isomorphic to the powerset $\mathcal{P}(A)$.

Proof. (i) \Rightarrow (ii). Let $x, y \in L$ with $y \leq x$. If $y^{\perp x} = y^\perp \wedge x = 0$, then it follows from (2.1) and modularity that $x = x \wedge 1 = x \wedge (y \vee y^\perp) = y$, so $[0, x]$ is annihilator.

(ii) \Rightarrow (iii). Let $x \in L$. Since $[0, x^{\perp\perp}]$ is annihilator and $x \leq x^{\perp\perp}$ by (1.7.2), $x^\perp \wedge x^{\perp\perp} = 0$ implies $x = x^{\perp\perp}$. Thus, L is dual.

(iii) \Rightarrow (iv). For $a, b \in L$, $a^\perp \wedge b^\perp \leq a^\perp \Rightarrow a \leq a^{\perp\perp} \leq (a^\perp \wedge b^\perp)^\perp$, and similarly $b \leq (a^\perp \wedge b^\perp)^\perp$. Thus $(a^\perp \wedge b^\perp)^\perp$ is an upper bound of the pair of elements a and b . Now, if $x \in L$ is another upper bound of a and b , then $x^\perp \leq a^\perp \wedge b^\perp$ and so $(a^\perp \wedge b^\perp)^\perp \leq x^{\perp\perp} = x$. Therefore, $(a^\perp \wedge b^\perp)^\perp$ is the join of the pair a, b .

(iv) \Rightarrow (v). By taking $a = b$ in (iv), we have that L is dual and hence annihilator, as already pointed out. So it is complemented by (2.1). Now it follows from (1.7.6) that it is distributive.

(v) \Rightarrow (i). Given $x \in L$, write x' to denote the *unique* complement of x . If $x \wedge y = 0$ for some $y \in L$, then $y = y \wedge 1 = y \wedge (x \vee x') = y \wedge x'$ implies $y \leq x'$. Thus, L is pseudocomplemented with $x^\perp = x'$. Now, $x^\perp = 0$ implies $x = x \vee x^\perp = 1$, so L is annihilator and is modular since it is distributive.

Suppose now that L is a complete lattice and let A denote the set of its atoms.

(vi) \Rightarrow (vii). By (v) \Rightarrow (ii), $[0, x]$ is annihilator, and since it is also complete and atomic, $x = \bigvee \{a \in A : a \leq x\}$ by (2.2).

(vii) \Rightarrow (viii). Consider the map $f: \mathcal{P}(A) \rightarrow L$ given by $f(X) := \bigvee X$. Clearly, f is onto and by (1.8) is one-to-one with f and f^{-1} order-preserving. Thus, f is a lattice isomorphism.

(viii) \Rightarrow (vi) is trivial. ■

Remarks 2.5. The equivalence (ii) \Leftrightarrow (iii) is due to Yood for the lattice of closed ideals of topological rings [Y, (2.9)]. On the other hand, Johnson gave in [J] an example of a commutative semisimple Banach algebra whose lattice of closed ideals is annihilator but not dual. Note that this lattice cannot be modular by (i) \Leftrightarrow (iii) of (2.4). Thus, while lattices of ideals are modular, lattices of closed ideals need not be modular. Finally, it is well known [CrD, (4.6)] that a complete Boolean algebra L is atomic if, and only if, it is the power set of the set A of its atoms.

For a p-semilattice S , write $\mathcal{B}(S) = \{x^\perp : x \in S\}$. By (1.7.3), $y \in \mathcal{B}(S) \Leftrightarrow y = y^{\perp\perp}$.

THEOREM 2.6 (Frink–Glivenko). *Let S be a pseudocomplemented semilattice.*

(i) *$\mathcal{B}(S)$ is a Boolean algebra with the original determination of the meet operation, the Boolean complement of an element being its pseudocomplement, and the join of a pair of elements x and y given by $x \nabla y = (x^\perp \wedge y^\perp)^\perp$.*

(ii) *If S is a complete lattice, then the Boolean algebra $\mathcal{B}(S)$ is also complete.*

(iii) *The mapping $x \rightarrow x^{\perp\perp}$ from S onto $\mathcal{B}(S)$ preserves meets, the 0-element, pseudocomplements, and joins when they exist.*

Comments on the Proof. Part (i) of the above theorem was proved for complete distributive lattices by Glivenko and in its full generality by Frink in [Fr2, Theorem 1]. Both proofs use special axiomatizations of Boolean algebras (as (2.3)) to get around the difficulty of proving distributivity. A direct proof of (i) is given in [Gr]. Another one is as follows: For $x, y \in \mathcal{B}(S)$ we have by (1.7.4) that $x \wedge y = x^{\perp\perp} \wedge y^{\perp\perp} = (x \wedge y)^{\perp\perp} \in \mathcal{B}(S)$. Therefore, $\mathcal{B}(S)$ is a semilattice which is clearly dual for the pseudocomplementation inherited from S . Hence, it is a Boolean algebra by (iii) \Rightarrow (v) of (2.4), with the join of a pair of elements x and y given by $x \nabla y = (x^\perp \wedge y^\perp)^\perp$.

Parts (ii) and (iii) were first proved by Glivenko under more restrictive conditions (see [B, p. 130]) and later extended by Frink [Fr2, Theorem 2] to the present form.

Remarks 2.7. A classical example of the Boolean algebra associated to a pseudocomplemented lattice is provided by the annihilator ideals of a semiprime ring (see [La, p. 111]). Another classical example is given by the regular open subsets of a topological space (X, T) , where $A \in T$ is regular if, and only if, $A = \text{int}(\overline{A})$. By [PC, p. 60], the Boolean algebra $\mathcal{B}(T)$ associated to the pseudocomplemented lattice T of (1.3) consists of the regular open subsets of X .

The relationship between a p-semilattice S and its associated Boolean algebra $\mathcal{B}(S)$ can be illuminated by studying uniform elements. Let S be a semilattice with 0-element. An element e of S is said to be *essential* if $e \wedge x \neq 0$ for any nonzero element $x \in S$. A nonzero element u of S is called *uniform* if any nonzero element $x \leq u$ is essential in the semilattice $[0, u]$. Some of the results proved in the next proposition about uniform elements were already considered in [FG2, (3.1)]. Nevertheless, they are included here for completeness.

PROPOSITION 2.8. *Let S be a pseudocomplemented semilattice.*

- (1) *If $u \in S$ is uniform, so is any nonzero element $x \leq u$.*
- (2) *A nonzero element u of S is uniform if, and only if, its annihilator u^\perp is maximal among all annihilators x^\perp , with $0 \neq x \in S$.*
- (3) *For each uniform element u of S , there exists a unique maximal uniform element v of S such that $u \leq v$, namely, $v = u^{\perp\perp}$.*
- (4) *The maximal uniform elements of S are precisely the atoms of $\mathcal{B}(S)$.*
- (5) *Maximal uniform elements are mutually orthogonal, i.e., if u and v are maximal uniform with $u \neq v$, then $u \wedge v = 0$.*
- (6) *If S is atomic, so is $\mathcal{B}(S)$. Moreover, the mapping $a \rightarrow a^{\perp\perp}$ is a bijection from the set A of the atoms of S onto the set M of its maximal uniform elements.*

Proof. Part (1) is clear.

(2) Suppose that u is a uniform element of S . We claim first that $b^\perp = u^\perp$ for every nonzero element $b \leq u$. Since $b \leq u$ we have $u^\perp \leq b^\perp$ by (1.7.1). On the other hand, if $b^\perp \not\leq u^\perp$ then $b^\perp \wedge u \neq 0$, and hence $b \wedge b^\perp = b \wedge (u \wedge b^\perp) \neq 0$ by the uniformity of u , which is a contradiction, so the claim is proved. Now let c be a nonzero element of L such that $u^\perp \leq c^\perp$. If $c \wedge u = 0$ then $c \leq u^\perp \leq c^\perp$ and hence $c = 0$, which is a contradiction, so $c \wedge u \neq 0$. Then we have by the first part of the proof that $c^\perp \leq (c \wedge u)^\perp = u^\perp$, and hence $c^\perp = u^\perp$, so u^\perp is maximal.

Suppose now that u^\perp is maximal among all the annihilators d^\perp , d a nonzero element of S . If b is a nonzero element of S such that $b \leq u$, then $u^\perp \leq b^\perp$ and hence $u^\perp = b^\perp$ by the maximality of u^\perp . Now for any

other nonzero element c of S such that $c \leq u$, we have that $b \wedge c \neq 0$. For, if $b \wedge c = 0$ then $c \leq b^\perp = u^\perp$, and hence $c \leq u^\perp \wedge u = 0$, which is a contradiction.

(3) Let $u \in S$ be uniform. By (1.7.2), $u \leq u^{\perp\perp}$. Moreover, $u^{\perp\perp}$ is uniform because, by (1.7.3), $u^{\perp\perp\perp} = u^\perp$, which is maximal by (2). Now let b be a uniform element of S such that $u \leq b$. Then, by (1.7.1), $b^\perp \leq u^\perp$ which implies $b^\perp = u^\perp$ by the maximality of b^\perp . Hence, $b \leq b^{\perp\perp} = u^{\perp\perp}$, as required.

(4) It follows from (2) and (3) that any maximal uniform element of S is of the form $u = u^{\perp\perp}$, where u^\perp is maximal annihilator or, equivalently, maximal in $\mathcal{B}(S) \setminus \{1\}$. Now, by duality of $\mathcal{B}(S)$, u^\perp is maximal iff $u = u^{\perp\perp}$ is minimal, i.e., an atom in $\mathcal{B}(S)$.

(5) It follows from (4) since meet operations are the same in S as in $\mathcal{B}(S)$, and different atoms are mutually orthogonal.

(6) Let b be a nonzero element of $\mathcal{B}(S)$. Then there exists an atom a of S such that $a \leq b$. Hence $a^{\perp\perp} \leq b^{\perp\perp}$ with $a^{\perp\perp}$ a maximal uniform element of S by (3) (since atoms are clearly uniform) and therefore an atom of $\mathcal{B}(S)$ by (4). Since atoms are clearly uniform, we have by (3) that $a^{\perp\perp}$ is maximal uniform for every $a \in A$, so $a \rightarrow a^{\perp\perp}$ defines a mapping from A to M . If $a^{\perp\perp} = b^{\perp\perp}$ for two atoms a and b , then $a^\perp = b^\perp$ by (1.7.3) and hence $a = b$. For $a \neq b \Rightarrow a \leq b^\perp = a^\perp$, which is a contradiction. Therefore, the mapping is an injection. Finally, given $u \in M$ there exists $a \in A$ such that $a \leq u$, and hence $a^{\perp\perp} \leq u^{\perp\perp} = u$, which implies $a^{\perp\perp} = u$ since $a^{\perp\perp}$ is maximal uniform by (3). ■

PROPOSITION 2.9. *For a complete pseudocomplemented lattice L the following conditions are equivalent:*

- (i) *The Boolean algebra $\mathcal{B}(L)$ is atomic.*
- (ii) *$\mathcal{B}(L)$ is isomorphic to the power set $\mathcal{P}(M)$, where M is the set of the maximal uniform elements of L .*
- (iii) *For any nonzero annihilator x^\perp there exists a uniform element u of L such that $u \leq x^\perp$.*
- (iv) *$\bigvee M$ is an essential element of L ; equivalently, $(\bigvee M)^\perp = 0$.*

Proof. (i) \Rightarrow (ii). By (2.6(ii)), $\mathcal{B}(L)$ is complete; and since it is also atomic, it is isomorphic to the power set of the set of its atoms by (2.4). However, the atoms of $\mathcal{B}(L)$ are precisely the maximal uniform elements of L (2.8.4).

(ii) \Rightarrow (iii). Clearly $\mathcal{B}(L)$ is atomic, and hence it follows, again by (2.8.4), that for any nonzero annihilator x^\perp there exists $u \in M$ such that $u \leq x^\perp$.

(iii) \Rightarrow (iv). If $(\vee M)^\perp \neq 0$ then there exists a uniform element u of L such that $u \leq (\vee M)^\perp$, and hence, by (1.7), $u^{\perp\perp} \leq (\vee M)^{\perp\perp\perp} = (\vee M)^\perp$, with $u^{\perp\perp}$ maximal uniform by (2.8.3), which is a contradiction. Thus, $(\vee M)^\perp = 0$.

(iv) \Rightarrow (i). Let $0 \neq b \in \mathcal{B}(L)$. Since $(\vee M)^\perp = 0$, it follows from (1.7.5) that $b \wedge u \neq 0$ for some $u \in M$. Set $v := b \wedge u$. By (2.8.1), v is uniform, with $v^{\perp\perp} \leq b^{\perp\perp} = b$. But $v^{\perp\perp}$ is maximal uniform by (2.8.3) and therefore an atom in $\mathcal{B}(L)$ by (2.8.4). Therefore, $\mathcal{B}(L)$ is atomic. ■

Remarks 2.10. (1) Complete pseudocomplemented lattices satisfying the equivalent conditions of (2.9) occur in Goldie theory and local Goldie theory (see [FG1, Theorem 3]).

(2) In [ChM, (3.5)], it was shown that the Boolean algebra of a finite p-lattice L is isomorphic to the powerset of the set of the atoms of L . Actually, this result remains true for any atomic complete p-lattice and can be derived from (2.9). Indeed, by (2.8.6), $\mathcal{B}(L)$ is atomic and the mapping $a \mapsto a^{\perp\perp}$ is a bijection from the set A of the atoms of L onto the set M of the maximal uniform elements of L . Now we apply (2.9)(i) \Rightarrow (ii).

(3) Another interesting special case of (2.9) occurs when L is a complete p-lattice satisfying the Ascending Chain Condition. Then its associated Boolean algebra $\mathcal{B}(L)$ satisfies both chain conditions and hence it is atomic. In the next section we will consider again this question in relation to the Goldie dimension of a complete pseudocomplemented lattice.

What can be said about the Boolean algebra of an annihilator lattice? An element $c \neq 1$ in a semilattice S with a greatest element 1 is called a *coatom* if $c \leq x$ implies $x = c$ or $x = 1$ for $x \in S$.

PROPOSITION 2.11. *If S is an annihilator semilattice, then S and $\mathcal{B}(S)$ have the same coatoms.*

Proof. If $c \neq 1$ is a coatom in S , then $c^\perp \neq 0$ implies $c \leq c^{\perp\perp} \neq 1$ and hence $c = c^{\perp\perp}$ is a maximal annihilator, i.e., a coatom in $\mathcal{B}(S)$. Conversely, let x^\perp be a coatom in $\mathcal{B}(S)$. If $x^\perp \leq y$ for some $y \in S$, then $x^\perp = x^{\perp\perp\perp} \leq y^{\perp\perp}$ implies $y^{\perp\perp} = x^\perp$ or $y^{\perp\perp} = 1$. If the first, $y \leq y^{\perp\perp} = x^\perp$; if the second, $y^{\perp\perp} = 1$, and hence $y^\perp = y^{\perp\perp\perp} = 0$, so $y = 1$. Thus, x^\perp is a coatom in S . ■

Remarks. (2.12) If S is a p-semilattice which is not annihilator, then the situation can be completely different. Let E denote the complete pseudocomplemented lattice defined by the usual (euclidean) topology of the real numbers \mathbb{R} (see (1.3) and (2.7)). It is clear that the coatoms of E are precisely the complements of the one-element sets. However, $\mathcal{B}(E)$ does not contain atoms, and therefore it has no coatoms. Note also that E

is not annihilator; the exterior of the complement of a single element set is empty.

(2.13) A classical result in lattice theory asserts (see [CrD, p. 89]) that a *finite distributive lattice can be embedded in a direct product of finitely many finite chains*. Let us approach this question from an algebraic point of view, that is, in terms of pseudocomplemented lattices. If L is a chain we are finished, so we may assume that L is not a chain. Let x_0 be the minimum element of L which is covered by at least two elements (note that such an element does exist). Clearly, $L = [0, x_0] \cup [x_0, 1]$ where $[0, x_0]$ is a chain, so without loss of generality we can suppose $x_0 = 0$. Since L is pseudocomplemented (1.2) and finite, we have by (2.9(iv)) that the join of the maximal uniform elements of L , say, $u_1 \vee \cdots \vee u_n$, is essential; equivalently, $\bigwedge_i u_i^\perp = 0$. Let ϕ be the mapping of L into the direct product of the lattices $L_i := [u_i^\perp, 1]$ defined by $\phi(x) = (x \vee u_1^\perp, \dots, x \vee u_n^\perp)$. It is clear that ϕ is order-preserving. Moreover, it follows from distributivity that $\phi(x) \leq \phi(y)$ implies $x \leq y$. Indeed, if $x \vee u_i^\perp \leq y \vee u_i^\perp$ for $1 \leq i \leq n$, then

$$x = x \vee 0 = x \vee \left(\bigwedge_i u_i^\perp \right) = \bigwedge_i (x \vee u_i^\perp) \leq \bigwedge_i (y \vee u_i^\perp) = y.$$

Finally, since L is not a chain, $n > 1$ and the cardinal of each L_i is less than the cardinal of L . Hence the proof follows by induction.

3. GOLDIE DIMENSION FOR COMPLETE PSEUDOCOMPLEMENTED LATTICES

Let L be a complete lattice with least element 0. A set X of *nonzero* elements of L is said to be *independent* if $x \wedge \bigvee (X \setminus \{x\}) = 0$ for all $x \in X$.

PROPOSITION 3.1. *Let L be a complete pseudocomplemented lattice. For a set X of nonzero elements of L the following conditions are equivalent:*

- (i) X is independent.
- (ii) X is finitely independent; i.e., every finite subset of X is independent.
- (iii) The elements of X are mutually orthogonal.

In this case, any n -element subset $\{x_1, x_2, \dots, x_n\}$ of X gives rise to a chain $0 < x_1 < x_1 \vee x_2 < \cdots < x_1 \vee \cdots \vee x_n$ of length n .

Proof. (i) \Rightarrow (ii) and (ii) \Rightarrow (iii) are trivial. To prove that (iii) \Rightarrow (i), let X be a set of mutually orthogonal elements of L . Given x in X , $x \leq y^\perp$ for all $y \in X \setminus \{x\}$, and hence, by (1.7.5), $x \leq \bigwedge_{y \neq x} y^\perp = (\bigvee_{y \neq x} y)^\perp = (\bigvee(X \setminus \{x\}))^\perp$; equivalently, $x \wedge \bigvee(X \setminus \{x\}) = 0$.

The last part is a direct consequence of (ii). ■

Clearly, different atoms are mutually orthogonal, and the same is true for maximal uniform elements of a complete p-lattice by (2.8.5). Hence it follows from (3.1) that we have

COROLLARY 3.2. *Let L be a complete pseudocomplemented lattice. If X is a subset of atoms or of maximal uniform elements, then X is independent.*

Let α be a cardinal number. A complete pseudocomplemented lattice L will be said to have *Goldie dimension* α ($\dim L = \alpha$) if L contains an independent subset X of cardinal α and, for every independent subset Y of L , $\text{card } Y \leq \alpha$.

Since L and $\mathcal{B}(L)$ have the same meet operations, it follows from (3.1) that any independent subset of $\mathcal{B}(L)$ remains independent in L . On the other hand, the mapping $x \rightarrow x^{\perp\perp}$ preserves orthogonality (1.7.4) and therefore also independence. Hence it is easy to derive the following result.

PROPOSITION 3.3. *Let L be a complete pseudocomplemented lattice. Then L has Goldie dimension if, and only if, so does $\mathcal{B}(L)$. In this case, $\dim L = \dim \mathcal{B}(L)$.*

The following result, providing a sufficient condition for a complete p-lattice to have Goldie dimension, is very useful.

PROPOSITION 3.4. *Let L be a complete pseudocomplemented lattice. If L contains an independent subset U of uniform elements whose join is essential, then L has Goldie dimension equal to $\text{card } U$.*

Proof. Let X be an independent subset of L . Since $\bigvee U$ is essential, we have by (1.7.5) that for each x in X there exists a uniform element u in U such that $u \wedge x \neq 0$. Put $U_x = \{u \in U : u \wedge x \neq 0\}$. Then the U_x are mutually disjoint: if $u \in U_x \cap U_y$ (for $x, y \in X$, $x \neq y$) then $u \wedge x \neq 0$, $u \wedge y \neq 0$ with $(u \wedge x) \wedge (u \wedge y) \leq x \wedge y = 0$, which contradicts the uniformity of u . Let c be a choice function of the family of all U_x . By the above, the mapping $x \rightarrow c(U_x)$ is an injection from X into U , so $\text{card } X \leq \text{card } U$. ■

COROLLARY 3.5. *Let L be a complete pseudocomplemented lattice. If L satisfies the equivalent conditions of (2.9) then L has Goldie dimension equal to $\text{card } M$, where M is the set of the maximal uniform elements of L . In*

particular, if L is atomic then $\dim L = \text{card} A$, where A is the set of the atoms of L .

Proof. By (3.2), M is an independent subset, and since L satisfies the equivalent conditions of (2.9), $\vee M$ is essential. Hence $\dim L = \text{card} M$ by (3.4). If L is atomic, then $\dim L = \text{card} M = \text{card} A$ by (2.8.6). ■

The existence of an independent subset of uniform elements whose join is essential is not a necessary condition for a complete p-lattice to have Goldie dimension, as the following example shows.

EXAMPLE 3.6. Let E be the complete pseudocomplemented lattice defined by the usual (euclidean) topology of the real numbers \mathbb{R} (1.3). Note first that E has no uniform elements (indeed, as pointed out in (2.12), $\mathcal{B}(E)$ does not contain any atom and therefore E has no uniform elements by (2.8.3) and (2.8.6)). However, E has Goldie dimension, with $\dim E = \aleph_0$. Clearly, E has a countable independent subset. On the other hand, any independent subset of E has cardinal $\leq \aleph_0$: Let $\{O_\lambda: \lambda \in \Lambda\}$ be an independent subset of (nonempty) open subsets of \mathbb{R} . For each $\lambda \in \Lambda$, $O_\lambda \cap \mathbb{Q} \neq \emptyset$ and we can take a choice function of the family of the $O_\lambda \cap \mathbb{Q}$. Then the mapping $\lambda \rightarrow c(O_\lambda \cap \mathbb{Q})$ is an injection from Λ into \mathbb{Q} .

However, the finite Goldie dimension is equivalent to the chain conditions on annihilators.

THEOREM 3.7. *For a complete pseudocomplemented lattice L the following conditions are equivalent:*

- (i) L has finite Goldie dimension.
- (ii) L does not contain infinite chains of annihilators.
- (iii) $\mathcal{B}(L)$ is finite.

In this case, the Goldie dimension, say $\dim L = n$, of L , equal to the number of maximal uniform elements of L , coincides with the annihilator length of L (length of the largest chain of annihilators), and $\mathcal{B}(L)$ is the power set of an n -element set.

Proof. (i) \Rightarrow (ii). It suffices to see that to each chain of annihilators $1 = a_0^\perp > a_1^\perp > \dots > a_n^\perp$ of length n we can associate an independent subset $\{x_1, \dots, x_n\}$ of cardinal n as follows. For each $1 \leq k \leq n$, set $x_k := a_{k-1}^\perp \wedge a_k$. Since $a_{k-1}^\perp > a_k^\perp$, $x_k \neq 0$. Moreover, for $x_k \neq x_j$, say $1 \leq j < k \leq n$, we have

$$x_j \wedge x_k \leq a_j \wedge a_{k-1} \leq a_j \wedge a_j^\perp = 0,$$

which proves that $\{x_1, \dots, x_n\}$ is an independent set by (3.1).

(ii) \Rightarrow (iii). $\mathcal{B}(L)$ satisfies both ascending and descending chain condition. Then, by (2.9), (i) \Rightarrow (ii), $\mathcal{B}(L)$ is isomorphic to the power set of a finite set, and therefore it is finite.

(iii) \Rightarrow (i). By (3.3), $\dim L = \dim \mathcal{B}(L)$ is finite.

Suppose now that L has finite Goldie dimension, say $\dim L = \dim \mathcal{B}(L) = n$. As seen in the proof of (i) \Rightarrow (ii), any chain of annihilators has length $\leq n$. Conversely, suppose that we are given an independent subset $\{x_1, \dots, x_n\}$ of L . We construct a chain of annihilators, $1 = a_0^\perp > a_1^\perp > \dots > a_n^\perp$, as follows.

Clearly, $x_1^\perp \neq 1$ since $x_1 \neq 0$, and $x_1^\perp \geq (x_1 \vee x_2)^\perp$. But x_1^\perp is actually greater than $(x_1 \vee x_2)^\perp$. For if $(x_1 \vee x_2)^\perp = x_1^\perp$, then $x_1 \wedge x_2 = 0$ implies $x_2 \leq x_1^\perp = (x_1 \vee x_2)^\perp \leq x_2^\perp$, which is a contradiction since $x_2 \neq 0$. By putting $a_0 = 0$, $a_1 = x_1$, and $a_2 = (x_1 \vee x_2)^\perp$, we obtain the chain of annihilators $1 = a_0^\perp > a_1^\perp > a_2^\perp$, and we can go on by recurrence. ■

EXAMPLE 3.8. The following example of the Boolean algebra of a pseudocomplemented lattice of finite Goldie dimension is taken from [L, p. 338]. Let K be a field and R the (commutative) semiprime ring $K[x_1, \dots, x_n]$ with the defining relation $x_1 \cdots x_n = 0$. It is not difficult to see that the pseudocomplemented lattice L of all ideals of R has finite Goldie dimension n , and consequently $\mathcal{B}(L)$ is the Boolean algebra of all subsets of an n -element set.

A semilattice S has *finite length* n ($\text{length } S = n$) if it has a chain of length n and every chain of S has length at most n . By (3.7), any complete p-lattice L of finite length has finite Goldie dimension, with $\dim L \leq \text{length } L$. But the converse is not true, as can be seen by considering an infinite chain.

For a finite Boolean algebra L , $\dim L = \text{length } L = n$, where n is the number of atoms of L (2.4). As we next show, this equality characterizes in fact the finite Boolean algebras inside the class of all p-lattices.

THEOREM 3.9. *A complete pseudocomplemented lattice L is a finite Boolean algebra if, and only if, L has finite length and $\dim L = \text{length } L$.*

Proof. Clearly, a finite Boolean algebra satisfies the above condition. For the converse it suffices to prove, by (2.4)(vii) \Rightarrow (viii), that each $x \in L$ is a joint of atoms. Let A denote the set of the atoms of L and suppose that $\bigvee \{a \in A : a \leq x\} < x$ for some $x \in L$. Write $A = \{a_1, \dots, a_m, a_{m+1}, \dots, a_n\}$, where $a_i \leq x$ for $1 \leq i \leq m$ and $a_j \wedge x = 0$ for $m < j \leq n$. We have that $\dim L = n$ by (3.5) and, by the last part of (3.1),

$$\begin{aligned} 0 < a_1 < \dots < a_1 \vee \dots \vee a_m < x < x \vee a_{m+1} \\ < \dots < x \vee a_{m+1} \vee \dots \vee a_n \end{aligned}$$

is a chain of length $n + 1$, which is a contradiction. Therefore, L is a finite Boolean algebra. ■

(3.10) It was stated in [CrD, p. 27] that *if a distributive lattice L has finite length n , then L has at most 2^n elements.*

We will give a proof of this result involving pseudocomplemented lattices (we are allowed to do this since such lattices are pseudocomplemented by (1.2)). The proof goes by induction on n . Clearly, a lattice of length ≤ 1 has at most two elements. Now, the induction step is a consequence of the following trivial observation.

Let L be a p -lattice and let a be an atom in L . Then L is the disjoint union of the subsets $[0, a^\perp]$ and $[a, 1]$.

Return to the case of a distributive lattice L of finite length n . By the above and the fact that both of the (distributive) lattices $[0, a^\perp]$ and $[a, 1]$ have length less than n , it follows from the induction hypothesis that

$$\text{card } L = \text{card}[0, a^\perp] + \text{card}[a, 1] \leq 2^{n-1} + 2^{n-1} = 2^n.$$

Remarks 3.11. A pseudocomplemented lattice L with finite length is not necessarily finite. Consider a complete lattice L with infinite atoms and length 2 that clearly does exist. Then enlarge it by adding a least element. The lattice \tilde{L} thus obtained is pseudocomplemented and has length 3 (see (1.5)).

4. COMPATIBLE PRODUCTS

All the semilattices considered in this section will have a least element 0. By a *product* on a semilattice S we mean a binary operation $(x, y) \rightarrow xy$ on S satisfying the following conditions:

$$(4.1) \quad \text{If } x \leq y \text{ then } zx \leq zy \text{ and } xz \leq yz \text{ for all } z \in S.$$

$$(4.2) \quad xy \leq x \wedge y \text{ for all } x, y \in S.$$

If $S = L$ is actually a lattice, we will require a further condition:

$$(4.3) \quad x(y \vee z) \leq xy \vee xz.$$

Lattices endowed with a product satisfying (4.1), (4.2), and (4.3) will be called *pseudomultiplicative lattices*. It follows from (4.1) and transitivity that

$$(4.4) \quad \text{if } x \leq y \text{ and } z \leq v, \text{ then } xz \leq yv.$$

We also note that, by (4.1), (4.3) is equivalent to

$$(4.5) \quad x(y \vee z) = xy \vee xz.$$

Moreover, by (4.2),

$$(4.6) \quad x0 = 0 = 0x.$$

As will be seen in the next examples, the notion of the pseudomultiplicative lattice is quite general, although we will be especially interested in semiprime lattices.

(4.7) Let S be a semilattice with a product xy . Then the following conditions are equivalent:

- (i) $xy = 0 \Leftrightarrow x \wedge y = 0$, for all $x, y \in S$.
- (ii) $x^2 := xx = 0 \Rightarrow x = 0$, for all $x \in S$.

Proof. (i) \Rightarrow (ii) is clear: $x^2 = 0$ implies $x = x \wedge x = 0$. (i) \Rightarrow (ii): In general, we have by (4.2) that $x \wedge y = 0$ implies $xy = 0 = yx$. Also, by (4.4), $(x \wedge y)^2 \leq xy$. Thus, if $xy = 0$ then $(x \wedge y)^2 = 0$ and hence $x \wedge y = 0$. ■

A product xy satisfying the above equivalent conditions will be said to be *compatible*. A pseudomultiplicative lattice whose product is compatible will be called *semiprime*.

EXAMPLES (4.8) *Elementary examples.* Any lattice can be regarded as a pseudomultiplicative lattice in a trivial way, i.e., by defining $xy = 0$ for all $x, y \in L$.

On the opposite side, every distributive lattice L is a pseudomultiplicative lattice for the product defined as the meet, $xy := x \wedge y$, for all $x, y \in L$. Note that this last product is clearly compatible.

Suppose now that L is a lattice with nonzero core c . Then L becomes a pseudomultiplicative lattice for the product defined by $xy = c$ whenever x, y are nonzero, and $xy = 0$ otherwise. Clearly, the product thus defined is compatible.

(4.9) *Multiplicative lattices.* In relation to a problem to which we will pay attention later, Kaplansky considered in [K, p. 245] complete lattices L endowed with a product, xy , satisfying (4.2), (4.5), and

$$(4.5.1) \quad (y \vee z)x = yx \vee zx.$$

It is easy to see that these lattices are pseudomultiplicative. We will use the term *quasi-multiplicative* to designate this particular kind of pseudomultiplicative lattice, avoiding the word “multiplicative” that has been used previously to mean lattices with a product subject to more restrictive conditions.

A complete lattice L is said to be *multiplicative* if it has a multiplication that is commutative, associative, and distributive over arbitrary joins, and

the greatest element acts as a multiplicative identity. Multiplicative lattices are pseudomultiplicative. Indeed, let $x, y, z \in L$. If $x \leq y$ then it follows from the distributivity of the product that $zy = z(y \vee x) = zy \vee zx$, so $zx \leq zy$. Similarly, $xz \leq yz$, and therefore the product satisfies (4.1). To prove (4.2), we have by (4.1) that, for arbitrary $x, y \in L$, $xy \leq x1 = x$, and similarly, $xy \leq 1y = y$, so $xy \leq x \wedge y$, as required. The reader is referred to [D] for results on multiplicative lattices.

(4.10) *The lattice of ideals of an algebraic system.* Following [FG2], by an *algebraic system* we mean any of the following algebraic structures: a nonassociative (algebra, triple, or pair) or a quadratic Jordan system (quadratic Jordan algebra, Jordan triple system, or Jordan pair). As shown in Examples 1–4 of [FG2], the complete lattice of ideals of an algebraic system becomes a pseudomultiplicative lattice for a natural and suitable product. For instance, let A be a nonassociative algebra over an arbitrary ring of scalars and let X, Y be nonempty subsets of A . Denote by $[XY]$ the ideal of A generated by the linear span of all products xy , $x \in X$ and $y \in Y$. Now let B and C be ideals of A . Define the product $B * C := [BC]$. Then the lattice $\mathcal{A}(A)$ of all ideals of A , endowed with the $*$ -product, is a pseudomultiplicative lattice. Note that if A is actually an associative (even alternative) algebra, then $B * C$ is merely BC . If J is a quadratic Jordan algebra, the product of two ideals B, C is given by $U_B C$, where, as usual, $x \mapsto U_x$ denotes the Jordan U -operator.

While the lattice product of a “linear” algebraic system (nonassociative algebra, triple, or pair) is actually distributive over arbitrary joins, this does not remain true for quadratic Jordan systems: Our decision to adopt the nonsymmetrical distributivity condition (4.3) was motivated precisely by the nonlinearity of the quadratic Jordan product $B * C := U_B C$ in the first variable. However, quadratic Jordan systems satisfy the weak version of (4.5.1)

$$(4.5.2) \quad (y \vee z)x \leq yx \vee zx \vee y \wedge z \wedge x,$$

which is a direct consequence of the fundamental Jordan identity [Lo, JP3]. Clearly, an algebraic system is semiprime if, and only if, the lattice of its ideals is semiprime.

(4.11) *The lattice of closed ideals of a topological algebraic system.* By a *topological algebraic system* we mean any algebraic system A endowed with a topology which makes continuous the algebraic operations of A . Let A be a topological algebraic system and let B, C be closed ideals of A . Denote now the lattice product of two ideals in $\mathcal{A}(A)$ by juxtaposition and by $S \rightarrow \bar{S}$ the closure operation. By the continuity of the algebraic operations, we have that $B\bar{C} \leq \overline{BC}$. Hence it is clear that $B * C := \overline{BC}$ defines a structure of pseudomultiplicative lattice on the lattice of closed ideals,

$\overline{\mathcal{A}}(A)$, of A . Again, a topological algebraic system which is semiprime has a semiprime lattice of closed ideals.

(4.12) *The lattice of normal subgroups of a group.* Following [S], we can see that the complete lattice $\mathcal{N}(G)$ of normal subgroups of a group G is a pseudomultiplicative lattice for the product defined as the commutator $[B, C]$ (subgroup of G generated by all the commutators $[b, c]$, $b \in B$, $c \in C$), with B, C normal subgroups of G . Such a lattice $\mathcal{N}(G)$ will be semiprime if, and only if, the group G does not contain nontrivial normal subgroups that are abelian.

(4.13) The nonmodular pentagon L_5 considered in (1.4) is a semiprime lattice for the product defined by the condition $z^2 = x$. Note that L_5 with this product is not a multiplicative lattice.

Later we will see that Example 4.13 is a particular case of a general process endowing any finite (even atomic) pseudocomplemented lattice with a compatible product.

The examples described above provide a source of pseudocomplemented lattices where the pseudocomplementation is generally given in terms of a compatible product. This is clear for a lattice with nonzero core (4.8), for the lattice of ideals (closed ideals) of a *semiprime* algebraic (topological algebraic) system (4.10 and 4.11), for the lattice of normal subgroups of a *semiprime* group (where semiprime here means an absence of nontrivial abelian normal subgroups) (4.12), and for the pentagon given in (4.13). In all these cases, the reason for the existence of a pseudocomplementation is that these lattices are complete and the product is *infinitely distributive*, i.e., satisfying

$$(4.5.3) \quad x \left(\bigvee_{\alpha} y_{\alpha} \right) = \bigvee_{\alpha} (xy_{\alpha})$$

for every x in L and every family $\{y_{\alpha}\}$ of elements of L . In general, we have

(4.14) Every complete semiprime lattice satisfying (4.5.3) is pseudocomplemented, where the pseudocomplement, x^{\perp} , of an element x is given by the supremum of all $y \in L$ annihilating x ; i.e., $xy = 0$.

It should be noted that complete pseudomultiplicative lattices satisfying (4.5.3) were studied in [FG2] under the name of *algebraic lattices*. But later the authors learned that the term algebraic lattice had been used previously in lattice theory with a different meaning.

Two other questions regarding compatible products should be considered. Does a pseudocomplemented lattice necessarily have a compatible product? And, in such a case, is this compatible product necessarily unique?

As said in (4.8), a lattice L which is distributive has a natural compatible product, namely, that defined as the meet $xy := x \wedge y$, $x, y \in L$. Note that this product is *idempotent*, that is, $x^2 = x$ for all $x \in L$. The converse is also true.

PROPOSITION 4.15. (1) *A lattice L is distributive if, and only if, it has a (unique) idempotent product; in fact, such a product coincides with the meet. Hence, if L is complete, then L is infinitely distributive if, and only if, the product satisfies (4.5.3).*

(2) *A lattice is a Boolean algebra if, and only if, it is annihilator and has an idempotent product.*

Proof. (1) Suppose that L is a lattice endowed with an idempotent product xy . For all $x, y \in L$ we have, by (4.4) and (4.2), $x \wedge y = (x \wedge y)^2 \leq xy \leq x \wedge y$, so $xy = x \wedge y$. Hence, L is distributive by (4.5).

(2) Clearly, any Boolean algebra satisfies these conditions, by (2.4). Suppose then that L is an annihilator lattice with an idempotent product. Then L is distributive by (1) and complemented by (2.1) and therefore is a Boolean algebra. ■

It is well known that the lattice of ideals of a von Neumann regular algebra [GoW, (7.3)] and the lattice of closed ideals of a C^* -algebra are infinitely distributive (in fact, the latter is the open set lattice of a topological space [Di, (3.2.2)]). As we will see next, both results can be derived from (4.15).

By a C^* -system we will mean any of the following algebraic structures: an associative or alternative C^* -algebra, a JB^* -algebra, a JB^* -triple, or a JB -algebra (see [Di, R, Ru, H] for definitions and basic results). It is known that in all these cases the lattice of closed ideals is idempotent (even algebraically idempotent) for the corresponding lattice product. A unified proof of this fact can be given by taking into account that any C^* -system can be regarded as a (real or complex) JB^* -triple. Now apply that every element in a JB^* -triple has a cubic root (see [MoR, (1.2)]). Also, the lattice of ideals of a von Neumann regular algebraic system is idempotent. Hence we obtain from (4.15)

COROLLARY 4.16. *The lattice of closed ideals of a C^* -system and the lattice of ideals of a von Neumann regular algebraic system are infinitely distributive.*

A distributive pseudocomplemented lattice can have two different compatible products. We only need to consider the three-element chain $C_2 = \{0, x, 1\}$, which is pseudocomplemented and has two different compatible products, namely, the one defined as the meet (as in every finite distributive lattice) and the second given by the fact that this lattice has a

core (4.8). As will be seen now, this anomaly disappears if we restrict ourselves to Boolean algebras. For this, two further properties on pseudocomplements will be required.

(4.17) Let S be a p-semilattice with a compatible product xy . For $x, y \in S$, we have

- (1) $(x^2)^\perp = x^\perp$ and
- (2) $(xy)^\perp = (x \wedge y)^\perp$.

Proof. Part 1 follows as in [FG2, 2.2(vi)], while Part 2 is a direct consequence of Part 1. Indeed, $xy \leq x \wedge y$ implies $(x \wedge y)^\perp \leq (xy)^\perp$. Conversely, $(x \wedge y)^2 \leq xy$ implies $(xy)^\perp \leq (x \wedge y)^{2\perp} = (x \wedge y)^\perp$ by Part 1. ■

COROLLARY 4.18. *A Boolean algebra L has a unique compatible product xy ; that given by the meet.*

Proof. Since Boolean algebras are dual lattices (2.4), the proof follows from (4.17.2). ■

(4.19) Let L be a complete lattice and let A denote the set of the atoms of L . Consider the mapping $\text{supp}: L \rightarrow \mathcal{P}(A)$ defined as $\text{supp}(x) := \{a \in A: a \leq x\}$. It is easy to verify that this mapping has the following properties for all $x, y_\alpha, y \in L$:

- (1) $\text{supp}(x \wedge y) = \text{supp}(x) \cap \text{supp}(y)$.
- (2) $\bigcup \text{supp}(y_\alpha) \leq \text{supp}(\bigvee y_\alpha)$.
- (3) $\text{supp}(0) = \emptyset$.
- (4) $\text{supp}(1) = A$.

If L is pseudocomplemented, we have

- (5) $\bigcup \text{supp}(y_\alpha) = \text{supp}(\bigvee y_\alpha)$ and
- (6) $\text{supp}(x \wedge (\bigvee y_\alpha)) = \bigcup \text{supp}(x \wedge y_\alpha)$.

Indeed, let $a \in L$ be such that a does not belong to $\bigcup \text{supp}(y_\alpha)$. Then $a \wedge y_\alpha = 0$ for all y_α , so $\bigvee y_\alpha \leq a^\perp$, and hence $a \notin \text{supp}(\bigvee y_\alpha)$, which proves (5). Now (6) follows from (1) and (5).

(4.20) Let L be a complete pseudocomplemented lattice. The *socle mapping* $\text{soc}: L \rightarrow L$ is defined by $\text{soc}(x) = \bigvee \text{supp}(x)$ for all $x \in L$. This mapping satisfies among others the following properties:

- (1) $\text{soc}(x) \leq x$.
- (2) $x \leq y \Rightarrow \text{soc}(x) \leq \text{soc}(y)$.
- (3) $\text{soc}(\text{soc}(x)) = \text{soc}(x)$.

$$(4) \quad x \leq y \Leftrightarrow \text{soc}(y)^\perp \leq \text{soc}(x)^\perp.$$

(5) $[0, \text{soc}(1)]$ is a complete annihilator lattice which is atomic.

The properties (1), (2), and (3) are clear; (4) follows from (1.8); and (5) is a consequence of (2.2).

The following result is a partial answer to the first question formulated before. In particular, it proves that finite pseudocomplemented lattices have a compatible product.

THEOREM 4.21. *Let L be a complete pseudocomplemented lattice which is atomic. Then L becomes a semiprime lattice for the (commutative and associative) product defined by $xy = \text{soc}(x \wedge y)$.*

Proof. Let us show that $xy = \text{soc}(x \wedge y)$ is a compatible product satisfying the required properties. Indeed, (4.1) and (4.2) follow from (4.20.2) and (4.20.1) respectively, and (4.3) is a consequence of (4.19.6). Finally, it is clear by atomicity of L that $xy = 0$ iff $x \wedge y = 0$, which completes the proof. ■

It follows from (4.18) that for an atomic complete Boolean algebra L the product defined in (4.21) is the unique compatible product on L and therefore coincides with the meet; i.e., $x \wedge y = \text{soc}(x \wedge y)$ for all $x, y \in L$. If L is merely annihilator instead of dual, we still have a partial uniqueness for compatible products. Indeed,

(4.22) Let L be an atomic complete annihilator lattice. For every compatible product xy on L , $x^2 = \text{soc}(x)$.

Proof. It follows from (4.17.1) that $\text{soc}(x) = \text{soc}(x^2) \leq x^2$ and from (2.2) that $1 = \text{soc}(x) \vee \text{soc}(x^\perp)$. Hence, $x^2 \leq x1 = x(\text{soc}(x) \vee \text{soc}(x^\perp)) = x \text{soc}(x) \leq \text{soc}(x)$. Therefore, $x^2 = \text{soc}(x)$. ■

Remarks 4.23. The partial uniqueness proved in (4.22) is no longer true without the annihilator condition. Consider, as in the comment after (4.16), a three-element chain. On the other hand, it is possible to give an example of an atomic complete annihilator lattice with two different compatible products. To construct such an example add three new elements, say y, z, v , to the power set, $\mathcal{P}(A)$, of the set $A = \{a, b, c\}$ satisfying the relations (i) $0 < y < \{b\}$, (ii) $\{a, b\} < z < A$, and (iii) $\{b, c\} < v < A$. It is easy to see that the lattice L thus obtained is annihilator and, since it is finite, is atomic and complete. Therefore, by (4.21), we have a compatible product on L defined by taking the join of the common support of two elements. It is still possible to define a new compatible product on L such that $zv = \{b\} \neq y = \text{soc}(z \wedge v)$.

5. PRIME ELEMENTS IN PSEUDOMULTIPLICATIVE LATTICES

Let L be a pseudomultiplicative lattice with a greatest element 1 and product denoted by xy . An element $p \neq 1$ in L will be called *prime* if $xy \leq p$ implies $x \leq p$ or $y \leq p$, for $x, y \in L$. It is straightforward that if the meet of the set of the prime elements of L is the least element 0, then the product is compatible. The converse is true for the lattice of ideals of a semiprime ring. However, this does not remain true in general.

(5.1) Following (2.12), the complete Boolean algebra $(\mathcal{B}(E), \Delta, \nabla)$ of the regular open subsets of \mathbb{R} , with its unique compatible product $BC = B \Delta C = B \cap C$, is a semiprime lattice with no prime elements.

Proof. Let P be a regular open subset of \mathbb{R} with $P \neq \mathbb{R}$. Then P cannot be the complementary of a single element of \mathbb{R} , so there exist two real numbers x and y , say $x < y$, which are not in P . Take $z \in \mathbb{R}$ such that $x < z < y$ and consider the open regular subsets $B = (-\infty, z)\nabla P$, $C = (z, +\infty)\nabla P$. By distributivity of the Boolean operations, $BC = B \Delta C = B \cap C = P$, but neither B nor C is contained in P . Thus, P is not prime. ■

From now on, by a *strongly semiprime* lattice we will mean a pseudomultiplicative lattice such that the meet of its prime elements is 0. As we have just seen, strongly semiprime lattices are semiprime, but the converse is not true in general.

In his paper [K], Kaplansky noted that in any semiprime nonassociative ring the intersection of the prime ideals is 0. In fact, he provided a proof of this result which could also be applied to groups, with normal subgroups playing the role of ideals and commutators as the product (see (4.12)). This led him to formulate the question for complete quasimultiplicative lattices (see (4.9)) such that any nonzero element contains a nonzero compact element, thus generalizing in fact a theorem by Keimel [Ke] for complete algebraic semiprime multiplicative lattices. On the other hand, Rosický pointed out in [Ro, Example 1] that the unit interval $[0, 1]$ is a strongly semiprime complete multiplicative lattice which, however, is not algebraic (in fact, the unique compact element in $[0, 1]$ is 0) and extended Keimel's theorem to continuous multiplicative lattices, covering the example of the unit interval.

These results (Kaplansky's and Rosický's) remain true under slightly more general conditions that allow one to include lattices of ideals of algebraic systems which are not necessarily linear, i.e., quadratic Jordan systems. For completeness we include the proof.

Following [Ro], an element u in a complete lattice L is said to be *below* $x \in L$, written $u \ll x$, if whenever there is a chain $\{y_\alpha\}$ with $x \leq \bigvee y_\alpha$, then $u \leq y_\alpha$ for some y_α . It is clear that $u \ll x \Rightarrow u \leq x$. Note that an element $c \in L$ is *compact* iff $c \ll c$.

THEOREM 5.2. *Let L be a complete semiprime lattice whose product also satisfies (4.5.2) and is such that for any nonzero element $x \in L$ there exists a nonzero $u \in L$ such that $u \ll x$. Then L is strongly semiprime.*

Proof. We only need to see that for every $0 \neq x \in L$ there exists a prime element $p \in L$ such that $x \not\leq p$. By hypothesis, there exists a nonzero element f_1 with $f_1 \ll x$. With f_n at hand, take f_{n+1} to be a nonzero element such that $f_{n+1} \ll f_n^2$. By using Zorn's lemma, there exists $p \in L$ being maximal with respect to the property $f_n \not\leq p$ for all n . We claim that p is prime. Suppose on the contrary that we have $y, z \in L$ such that $yz \leq p$ with $y \not\leq p$ and $z \not\leq p$. Taking $y' = y \vee p$ and $z' = z \vee p$, we have that $y' > p$ and $z' > p$, and hence, by the maximality of p , there exist $f_n \leq y'$ and $f_m \leq z'$ for some n, m , say, $m \geq n$. Then $f_{m+1} \ll f_m^2 \leq y'z'$. But, by (4.2), (4.3), and (4.5.2),

$$y'z' = (y \vee p)(z \vee p) = (y \vee p)z \vee (y \vee p)p \leq yz \vee p \leq p,$$

which is a contradiction since $f_n \not\leq p$ for all n . Therefore p is prime. Moreover, $x \not\leq p$, since otherwise $f_1 \ll x \leq p \Rightarrow f_1 \leq p$, which again is a contradiction. ■

It follows from (1.7.5) that any atom in a complete pseudocomplemented lattice is compact. Hence, we obtain as a consequence of (5.2) and (4.21),

COROLLARY 5.3. (1) *Any atomic complete lattice with a compatible product satisfying (4.5.2) is strongly semiprime. In particular:*

(2) *Any atomic complete pseudocomplemented lattice is strongly semiprime for the product $xy = \text{soc}(x \wedge y)$ defined in (4.21).*

Remarks 5.4. (1) As pointed out in (4.10), algebraic systems satisfy the condition (4.5.2), so it follows from (5.2) that any semiprime algebraic system is strongly semiprime. The reader could consult different versions of this result for lattices of ideals of nonassociative algebraic systems, such as alternative rings [Z, Proof of Theorem 6, p. 162], quadratic Jordan algebras [Th], or quadratic Jordan triple systems [Mc].

(2) It is natural to ask whether for a semiprime topological algebraic system A the intersection of its closed prime ideals is 0. It seems that the answer to this question is unknown even if A is a commutative Banach algebra. Nevertheless, a positive answer has been given by Somerset [So1] for semiprime Banach algebras whose lattice of closed ideals is a countably

generated continuous lattice. A concrete class of Banach algebras to which this applies is the class of TAF-algebras [So2].

Our next aim is to study prime elements in pseudocomplemented lattices endowed with a compatible product.

LEMMA 5.5. *Let L be a pseudocomplemented lattice with a compatible product, and let $x \in L$ be such that $x^\perp \neq 1$. Then x^\perp is prime if, and only if, it is a maximal annihilator. In this case, x^\perp is a minimal prime.*

Proof. Suppose first that $x^\perp \neq 1$ is prime. If $x^\perp \leq y^\perp$ for some $y \in L$, we have by the primeness of x^\perp that $0 = yy^\perp \leq x^\perp$ implies $y \leq x^\perp$ or $y^\perp \leq x^\perp$. If it is the first, $y \leq x^\perp \leq y^\perp$ implies $y = 0$, and hence $y^\perp = 1$; if the second, $x^\perp = y^\perp$. Thus x^\perp is maximal annihilator. Conversely, assume that x^\perp is a maximal annihilator, and let $a, b \in L$ be such that $ab \leq x^\perp$. Then, by (4.1) and (4.2),

$$(1) \quad a(bx) \leq (ab) \wedge x = 0.$$

On the other hand, $bx \leq x$ implies $x^\perp \leq (bx)^\perp$ and hence, by the maximality of x^\perp , $x^\perp = (bx)^\perp$ or $(bx)^\perp = 1$. If the first, $x^\perp = (bx)^\perp$ implies by (1) that $a \leq x^\perp$; if the second, $(bx)^\perp = 1$ implies $bx \leq (bx)^{\perp\perp} = 1^\perp = 0$, so $b \leq x^\perp$. Therefore x^\perp is prime.

Suppose finally that x^\perp satisfies the above equivalent conditions, and let p be a prime element such that $p \leq x^\perp$. As before, $0 = xx^\perp \leq p$ implies $x \leq p$ or $x^\perp \leq p$. In the first case, $x = 0$, which is a contradiction since $x^\perp \neq 1$. Thus $x^\perp \leq p$, which proves that x^\perp is minimal prime. ■

The following proposition, which generalizes a well-known result for lattices of ideals of semiprime rings (see [L, (11.41)]), provides a way of producing prime elements by means of uniform elements. Note that it also completes the statement of (2.8).

PROPOSITION 5.6. *Let L be a pseudocomplemented lattice with a compatible product. For a nonzero element $u \in L$, the following conditions are equivalent:*

- (i) u^\perp is a maximal annihilator,
- (ii) u^\perp is a minimal prime,
- (iii) u^\perp is prime,
- (iv) u is uniform.

Proof. (i) \Rightarrow (ii) follows from (5.5), and (ii) \Rightarrow (iii) is trivial. Finally, (iii) \Rightarrow (iv) and (iv) \Rightarrow (i) follow from (2.8.2) and (5.5). ■

COROLLARY 5.7. *Let L be a complete pseudocomplemented lattice with a compatible product. If L satisfies the equivalent conditions of (2.9), then L is strongly semiprime. In particular, this is true if L has finite Goldie dimension.*

Proof. By (1.7.5), $\bigwedge u_\alpha^\perp = (\bigvee u_\alpha)^\perp = 0$, where u_α ranges over all the maximal uniform elements of L ; but each u_α^\perp is prime by (5.6). The last part follows from (3.7) and (2.9). ■

6. AN EXTENSION OF YOOD'S THEOREM TO PSEUDOCOMPLEMENTED LATTICES WITH A COMPATIBLE PRODUCT

As commented in the Introduction, the aim of this section is to extend to complete pseudocomplemented lattices a decomposition theorem for topological rings due to [Y, (2.6)]. Our main task will be to produce atoms starting from prime elements.

LEMMA 6.1. *Let L be a pseudocomplemented lattice with a compatible product xy . If c is a coatom of L such that $c^\perp \neq 0$, then*

- (1) c is a minimal prime, and
- (2) $(c^\perp)^2$ is an atom.

Proof. (1) Since $c^\perp \neq 0$, $c \leq c^{\perp\perp} \neq 1$, so $c = c^{\perp\perp}$ is a maximal annihilator and hence a minimal prime by (5.6).

(2) Let us see that $a = (c^\perp)^2$ is an atom. First note that $a \neq 0$ since the product is compatible (4.7). Now let $x \leq a \leq c^\perp$. Since c is a coatom, either $c \vee x = c$ or $c \vee x = 1$. In the first case, $x \leq c \wedge c^\perp = 0$. In the second, by (4.1), (4.2), and (4.3), $a = (c^\perp)^2 \leq c^\perp 1 = c^\perp (c \vee x) \leq x$. Therefore, a is an atom. ■

A careful analysis of the above proof allows us to see that the atom $(c^\perp)^2$ constructed in (2) is independent of the choice of the compatible product. Indeed, let a be any atom dominated by c^\perp . Since c is a coatom, we have as above that $1 = c \vee a$ and hence

$$(c^\perp)^2 \leq c^\perp 1 = c^\perp (c \vee a) \leq a = a^2 \leq (c^\perp)^2.$$

In general, a coatom c need not be prime. Consider the three-element chain $C_2 = \{0, x, 1\}$ with the compatible product defined by the condition $1^2 = x$ as in (4.8). Note that x is a coatom in C_2 which is not prime.

PROPOSITION 6.2. *Let L be an annihilator lattice with a compatible product xy . For $c \neq 1$ in L , the following conditions are equivalent:*

- (i) c is a minimal prime.
- (ii) c is prime.
- (iii) c is a coatom.
- (iv) $c = a^\perp$, where a is an atom.

Proof. (i) \Rightarrow (ii) is clear.

(ii) \Rightarrow (iii). Let $c \leq x$. Since c is prime $x \leq c$ or $x^\perp \leq c$. If the first holds, $x = c$; if the second, $x^\perp \leq c \leq x$, so $x^\perp = 0$ and hence $x = 1$ since L is annihilator.

(iii) \Rightarrow (iv). Since $c^\perp \neq 0$, it follows as in (6.1.1) that $c = c^{\perp\perp}$. Now, by (6.1.2), $a = (c^\perp)^2$ is an atom, and, by (4.17.1), $a^\perp = ((c^\perp)^2)^\perp = c^{\perp\perp} = c$.

(iv) \Rightarrow (i) follows from (5.6) and the fact that atoms are uniform elements.

THEOREM 6.3. *For a complete lattice L the following conditions are equivalent:*

(i) *L is pseudocomplemented and the greatest element 1 is the join of the atoms of L .*

(ii) *L is annihilator and atomic.*

(iii) *L is annihilator and there exists a product in L for which L is a strongly semiprime lattice.*

Proof. (i) \Rightarrow (ii) follows from (2.2).

(ii) \Rightarrow (iii). It follows from (5.3.2).

(iii) \Rightarrow (i). By (6.2) and (1.7.5), $0 = \bigwedge a_\alpha^\perp = (\bigvee A)^\perp$, where A is the set of the atoms of L . Hence $1 = \bigvee A$ since L is annihilator. ■

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REFERENCES

[B] G. Birkhoff, "Lattice Theory," American Mathematical Society Colloquium Publications, Vol. 25, 3rd ed., American Mathematical Society, New York, 1967.

[BoG] F. F. Bonsall and A. W. Goldie, Annihilator algebras, *Proc. London Math. Soc.* **4** (1954), 154–167.

[ChM] C. Chameni-Nembua and B. Monjardet, Les treillis pseudocomplémentés finis, *Europ. J. Combin.* **13** (1992), 89–107.

[CrD] P. Crawley and R. P. Dilworth, "Algebraic Theory of Lattices," Prentice–Hall, Englewood Cliffs, NJ, 1973.

[CuR] J. A. Cuenca Mira and A. Rodríguez Palacios, Structure theory for noncommutative Jordan H^* -algebras, *J. Algebra* **106** (1987), 1–14.

[D] R. P. Dilworth, "The Dilworth Theorems," Birkhauser Boston, Boston, MA, 1990.

[Di] J. Dixmier, "C*-Algebras," North-Holland Mathematics Library, Vol. 15, North-Holland, Amsterdam, 1982.

[FG1] A. Fernández López and E. García Rus, Nondegenerate Jordan algebras satisfying local Goldie conditions, *J. Algebra* **182** (1996), 52–59.

- [FG2] A. Fernández López and E. García Rus, Algebraic lattices and nonassociative structures, *Proc. Amer. Math. Soc.* **126** (1998), 3211–3221.
- [FR] A. Fernández López and A. Rodríguez Palacios, A Wedderburn theorem for non-associative complete normed algebras, *J. London Math. Soc.* **33** (1986), 328–338.
- [Fr1] O. Frink, Representations of Boolean algebras, *Bull. Amer. Math. Soc.* **47** (1941), 755–756.
- [Fr2] O. Frink, Pseudo-complements in semilattices, *Duke Math.* **29** (1962), 505–514.
- [GoW] K. R. Goodearl and F. Wehrung, Representations of distributive semilattices by dimension groups, regular rings, C^* -algebras, and complemented modular lattices, preprint.
- [Gr] G. Grätzer, “General Lattice Theory,” Birkhäuser, Basel, 1991.
- [GPu1] P. Grzeszczuk and E. R. Puczyłowski, On Goldie and dual Goldie dimension, *J. Pure Appl. Algebra* **31** (1984), 47–54.
- [GPu2] P. Grzeszczuk and E. R. Puczyłowski, On infinite Goldie dimension of modular lattices and modules, *J. Pure Appl. Algebra* **35** (1985), 151–155.
- [GPu3] P. Grzeszczuk and E. R. Puczyłowski, Relations between some dimensions of modular lattices, *Comm. Algebra* **17** (1989), 1723–1737.
- [H] H. Hanche-Olsen and E. Stormer, “Jordan Operators Algebras,” Pitman, 1984.
- [J] B. E. Johnson, A commutative semi-simple Banach algebra which is not dual, *Bull. Amer. Math. Soc.* **73** (1967), 407–409.
- [K] I. Kaplansky, “Selected Papers and Other Writings,” Springer-Verlag, New York, 1995.
- [Ke] K. Keimel, A unified theory of minimal prime ideals, *Acta Math. Acad. Sci.* **23** (1972), 51–69.
- [L] T. Y. Lam, “Lectures on Modules and Rings,” Springer-Verlag, New York, 1999.
- [La] J. Lambek, “Lectures on Rings and Modules,” Chelsea Publishing, New York, 1976.
- [Lb] M. S. Lambrou, Nontrivially pseudocomplemented lattices are complemented, *Proc. Amer. Math. Soc.* **77** (1979), 55–56.
- [Lo] O. Loos, “Jordan Pairs,” Lecture Notes in Mathematics, Vol. 460, Springer-Verlag, Berlin/New York, 1975.
- [Mc] K. McCrimmon, A characterization of the nondegenerate radical in quadratic Jordan triple systems, *Algebra, Groups and Geometries* **4** (1987), 145–146.
- [MoR] A. Moreno Galindo and A. Rodríguez Palacios, On the zelmanovian classification of prime JB^* -triples, *J. Algebra* **226** (2000), 577–613.
- [PC] D. Ponasse and J. C. Carrega, “Algèbre et topologie Boolèennes,” Masson, Paris, 1979.
- [R] A. Rodríguez Palacios, Jordan structures in analysis, in “Jordan Algebras, Oberwolfach, 1992,” pp. 97–186, de Gruyter, Berlin/New York, 1994.
- [Ro] J. Rosický, Multiplicative lattices and frames, *Acta Math. Hungar.* **49** (1987), 391–395.
- [Ru] B. Russo, Structure of JB^* -triples, in “Jordan Algebras, Oberwolfach, 1992,” pp. 209–280, de Gruyter, Berlin/New York, 1994.
- [S] E. Schenkman, The similarity between the properties of ideals in commutative rings and the properties of normal subgroups of groups, *Proc. Amer. Math. Soc.* **9** (1958), 375–3816.
- [Sm] M. F. Smiley, Right annihilator algebras, *Proc. Amer. Math. Soc.* **6** (1955), 698–701.
- [So1] D. W. B. Somerset, Minimal primal ideals in rings and Banach algebras, *J. Pure Appl. Algebra* **144** (1999), 67–89.
- [So2] D. W. B. Somerset, A Dauns–Hofmann theorem for TAF -algebras, *Proc. Amer. Math. Soc.* **127** (1999), 1379–1385.

- [Th] A. Thedy, Z -closed ideals of quadratic Jordan algebras, *Comm. Algebra* **13** (1985), 2537–2565.
- [To] B. J. Tomiuk, Structure theory of complemented Banach algebras, *Canad. J. Math.* **14** (1962), 651–659.
- [Y] B. Yood, Closed prime ideals in topological rings, *Proc. London Math. Soc.* **24** (1972), 307–323.
- [Z] K. A. Zhevlakov *et al.*, “Rings that Are Nearly Associative,” Academic Press, New York, 1982.