Quadratic Liénard Equations with Quadratic Damping

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In this paper we study the generalized Liénard equations with quadratic polynomials $f$ and $g$. We prove that these kind of equations can have at most one limit cycle, and we give the complete bifurcation diagram and classification of the phase portraits. The paper also contains a shorter proof for the result in A. Lins, W. de Melo, and C. C. Pugh, 1977, Lecture Notes in Math. 597, 335–357 on the unicity of the limit cycle for (standard) Liénard equations with quadratic damping. © 1997 Academic Press

1. INTRODUCTION

In 1977, Lins, de Melo and Pugh studied the Liénard equations

$$\dot{x} = y - F(x), \quad \dot{y} = -x, \quad (1)$$

where $F$ is a polynomial of degree $n + 1$, or equivalently,

$$\ddot{x} + f(x) \dot{x} + x = 0, \quad (2)$$

with $f(x) = F'(x)$. They proposed the following

Conjecture 1.1 ([LMP]). If $f(x)$ has degree $n$, then (1) has at most $\lfloor n/2 \rfloor$ limit cycles ($\lfloor n/2 \rfloor$ is the integer part of $n/2$, $n \geq 2$).

They also proved the conjecture for $n = 2$. The problem for $n \geq 3$ is still open. In 1988, Lloyd and Lynch [LL] considered the similar problem for generalized Liénard equations

$$\dot{x} = y - F(x), \quad \dot{y} = -g(x), \quad (3)$$

or equivalently

$$\ddot{x} + f(x) \dot{x} + g(x) = 0, \quad (4)$$

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TABLE 1
A Conjecture for $N(m, n)$, in Case $n + m \leq 5$,
(the Cases with an Asterisk Were Proved)

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where $f(x) = F'(x)$ is a polynomial of degree $n$, and $g(x)$ is a polynomial of degree $m$. In most cases, they gave an upper bound for the number of small amplitude limit cycles that can bifurcate out of a single non-degenerate singularity. If we denote by $N(m, n)$ the uniform upper bound for the number of limit cycles (admitting a priori that $N(m, n)$ could be infinite), then the results in [LL] give a lower bound for $N(m, n)$.

The result in [LMP] shows that $N(1, 2) = 1$. In 1988 Coppel [C] proved that $N(2, 1) = 1$ (see also the appendix of [DR]. We remark here that the theorem in [C] about the uniqueness of limit cycles for Liénard equations can be obtained from a result of Xian-wu Zeng [Ze], see [ZDHD], pp. 244-245, Theorem 4.11 and Remark 2.) In [DR] and [DL] it was proved that $N(3, 1) = 1$. Up to now, as far as we know, only these three cases, marked with asterisks in Table 1, have been completely investigated.

The main purpose of this paper is to prove that $N(2, 2) = 1$, and to investigate the bifurcation diagram and classification of the phase portraits for $f$ and $g$ quadratic.

Before studying the $(2, 2)$-case, in Section 2 we first introduce our basic argument, and use it to give a shorter proof for the cases $N(1, 2) = 1$ and $N(2, 1) = 1$.

2. A SHORTER PROOF FOR $N(1, 2) = 1$ AND $N(2, 1) = 1$

We essentially rely on two theorems. The first of them was obtained by Zhi-fen Zhang in 1958, and had many applications since then (the hyperbolicity of the limit cycle was obtained in this work, although it was not stated explicitly); the second one is usually called the “Dulac Criterion,” its proof is very simple and as such we prefer to recall it.
Theorem 2.1 ([Zh1, 2]). Consider Liénard equations

\[ \dot{x} = y - F(x), \quad \dot{y} = -g(x), \]  

(5)

where \( x \in (\alpha, \beta) \) with \( \alpha < 0 < \beta \) and where \( \alpha \) (resp. \( \beta \)) could be \( -\infty \) (resp. \( +\infty \)). Suppose that

(i) \( xg(x) > 0 \) for \( x \neq 0 \), and \( g(x) \) is continuous, satisfying a Lipschitz condition in any bounded interval of \( (\alpha, \beta) \).

(ii) \( f(x) = F'(x) \) is continuous, and \( f(x)/g(x) \) is nondecreasing (resp. nonincreasing) for \( x \in (\alpha, 0) \cup (0, \beta) \); \( (f(x)/g(x)) \neq 0 \) in a neighborhood of \( x = 0 \).

Then (5) has at most one limit cycle in the strip \( \alpha < x < \beta \); the limit cycle, if it exists, is hyperbolic and attracting (resp. repelling).

Theorem 2.2. Consider the differential equation

\[ \dot{x} = X(x, y), \quad \dot{y} = Y(x, y), \]  

(6)

where \( X, Y \in C^1(D) \), and \( D \) is a simply connected domain in \( \mathbb{R}^2 \). Suppose that there exists a function \( B(x, y) \in C^1(D) \), with \( B(x, y) > 0 \) (or \( < 0 \)), such that

\[ \frac{\partial (XB)}{\partial x} + \frac{\partial (YB)}{\partial y} \geq 0 \quad \text{(or } \leq 0) \]  

(7)

in \( D \), and the equality can not hold identically in any subregion of \( D \) with positive area. Then (6) has no periodic orbits in \( D \).

Proof. If on the contrary, there is a periodic orbit \( \Gamma \) in \( D \), then

\[ 0 \equiv \int_{\Gamma} (XB) \, dy - (YB) \, dx = \pm \int_{\Omega(\Gamma)} \left( \frac{\partial (XB)}{\partial x} + \frac{\partial (YB)}{\partial y} \right) \, dx \, dy, \]

where \( \Omega(\Gamma) \) is the region surrounded by \( \Gamma \), and “\( \pm \)” depends on the sign of (7) and the orientation of \( \Gamma \). In any case, this gives a contradiction.

The function \( B(x, y) \) in Theorem 2.2 is usually called a Dulac function. Based on the above theorems, we have the following new proofs.

Application 2.3. \( N(1, 2) = 1 \) (see [LMP]).

We consider (4) with \( f(x) = ax^2 + bx + c (a \neq 0) \) and \( g(x) = x \). Then

\[ \frac{(f(x)')}{(g(x))} = \frac{ax^2 - c}{x^2}. \]
If \( ac < 0 \), then by Theorem 2.1 system (4) has at most one limit cycle; if \( ac \geq 0 \), we take \( B(y) = e^{-by} \) and see that
\[
\frac{\partial}{\partial x} \left[ (y - F(x))B(y) \right] + \frac{\partial}{\partial y} \left[ -xB(y) \right] = -(ax^2 + c) e^{-by}
\]
has a fixed sign for \( x \neq 0 \), hence by Theorem 2.2 system (4) has no limit cycles.

Application 2.4. \( N(2, 1) = 1 \) (see [C]).

Without loss of generality, we take \( f(x) = ax + b \), \( g(x) = x^2 + x \) in (4), with \( a \neq 0 \), (see the next section for an explanation). Then
\[
\begin{align*}
\left( \frac{f(x)}{g(x)} \right)' &= \frac{-ax^2 - 2bx + b}{(x^2 + x)^2}.
\end{align*}
\]
Since \((x, y) = (-1, 0)\) is a saddle point of (3), the limit cycles can occur only in the strip \(-1 < x < +\infty\).

If \( b(b - a) < 0 \), then \((f(x)g(x))'\) has the same sign as \(-b\) for \( x \in (-1, 0) \cup (0, +\infty) \), hence by Theorem 2.1 system (3) has at most one limit cycle; if \( b(b - a) \geq 0 \), we choose (as a Dulac function) \( B(y) = e^{2b - ay} \), then
\[
\begin{align*}
\frac{\partial}{\partial x} \left[ (y - F(x))B(y) \right] + \frac{\partial}{\partial y} \left[ -g(x)B(y) \right] &= [(a - 2b)x^2 - 2bx - b] e^{2b - ay}.
\end{align*}
\]
has the same sign as \((-b)\) if \( b \neq 0 \), or has the same sign as \(a\) if \( b = 0 \) and \( x \neq 0 \), hence by Theorem 2.2 system (3) has no limit cycles.

3. QUADRATIC LIÉNARD EQUATIONS WITH QUADRATIC DAMPING

In general, we need to consider (3) with \( F(x) = ax^3 + bx^2 + cx \) and \( g(x) = ax^2 + bx + c \). To keep \( f(x) = F'(x) \) and \( g(x) \) quadratic, we need the condition \( a \neq 0 \). Let us hence consider
\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -(ax^2 + bx + c) y - (3ax^2 + 2bx + c) y,
\end{align*}
\]
keeping \( a \neq 0 \).
3.1. Study near Infinity

We first consider (8) near infinity by introducing

\[ \begin{align*}
    x &= \frac{\bar{x}}{r}, \\
    y &= \frac{\bar{y}}{r^3}
\end{align*} \tag{9}\]

with \( x^2 + y^2 = 1 \), multiplying with \( r^2 \) and taking \( r \to 0 \).

This is a version of the traditional Poincaré-compactification, which we call a Poincaré-Lyapunov compactification.

As a result we can extend the polynomial vector field (8) to an analytic vector field on the “Poincaré-Lyapunov disc.” See e.g., [DR] for a use of this technique. The circle \( \{ r = 0 \} \) is called the “circle at infinity” or “equator (at infinity).” In the present case it is better to use, as charts, the mapping

\[ \begin{align*}
    x &= \frac{1}{r}, \\
    y &= \frac{u}{r^3}
\end{align*} \tag{10}\]

and

\[ \begin{align*}
    x &= \frac{v}{r}, \\
    y &= \frac{1}{r^3}.
\end{align*} \tag{11}\]

After performing (10) and multiplying with \( r^2 \), (8) changes into:

\[ \begin{align*}
    \dot{r} &= -ur, \\
    \dot{u} &= -(ax^3 + br^4 + yr^5) - u(3a + 2br + cr^2) - 3u^2. \tag{12}\]

For \( r = 0 \) we find singularities at \( u = 0 \) and at \( u = -a \).

At \( (0, -a) \) we clearly have a hyperbolic node, with eigenvalues \( a \) and \( 3a \); it is repelling for \( a > 0 \) and attracting for \( a < 0 \). At \( (0, 0) \) the vector field is semi-hyperbolic with eigenvalues 0 and \( -3a \). A center manifold can be expressed as \( (r, w(r)) \) and standard center manifold calculations show that

\[ w(r) = -\frac{\alpha}{3a} r^3 + O(r^4), \tag{13}\]

while the behaviour on the center manifold is

\[ \dot{r} = \frac{\alpha}{3a} r^4 + O(r^5). \tag{14}\]

As such \( (0, 0) \) is a saddle-node, with the saddle-type behaviour in \( \{ r \geq 0 \} \).
As \( x = 1/r \) in (9), the power of \( r \) being odd, we recover the behaviour in the negative \( x \)-direction by restricting to \( \{ r \leq 0 \} \) in the previous calculations.

To complete the picture we have to use (11), but it will not provide new singularities. The results are now summarized in Fig. 1. In this figure we also draw the relative position of the vector field with respect to the \( x \)-axis.

Later on we will also need the behaviour near infinity for the cases \( a = 0, b \neq 0 \). The best way to study it consists in working on a Poincaré–Lyapunov disc of degree \((1, 2)\). The calculations are similar to the ones done before; we hence do not repeat them, but summarize the results in Fig. 2.

### 3.2. Study of the Limit Cycles

If we now want to look at the possible number of limit cycles for systems (8), we may take \( \gamma = 0 \), by putting a singularity at \((0, 0)\). If moreover \( \beta = 0 \),
then \( \dot{y} \) has fixed sign for \( y = 0 \) and \( x \neq 0 \), and the system has no periodic orbits. We can hence suppose that \( \beta \neq 0 \), and use the rescaling

\[(x, y, t) \to (\xi x, \eta y, \zeta t),\]

where \( \zeta = (\text{sign } \beta) \sqrt{|1/|\beta||}, \eta = (1/\zeta^4), \xi = \eta \zeta \). System (8), with \( \gamma = 0 \), keeps the same form but \( g(x) = x^2 - x \). If \( g(x) = x^2 - x \), we consider \( \tilde{x} = x - 1, \tilde{y} = y - y_0 \), for a suitable choice of \( y_0 \). Writing \((\tilde{x}, \tilde{y})\) as \((x, y)\) and bringing (8) to the standard Liénard form, we get expression (3) with \( g(x) = x^2 + x \).

As such, without loss of generality, we can suppose that \( g(x) = x^2 + x \).

We hence consider

\[
\dot{x} = y - (ax^3 + bx^2 + cx), \quad \dot{y} = -(x^2 + x) \tag{15}
\]

with \( a \neq 0 \).

The situation near infinity for system (15), can be found in figure 1, under the condition \( x > 0 \). Moreover we know that (15) has two singularities in the finite plane, a hyperbolic saddle \( s \) at \((-1, 0)\) and an antisaddle \( s' \) (point of index +1) at \((0, 0)\). In Fig. 3 we summarize the known information on the phase portraits, taking into account the behaviour near infinity and the knowledge at the singularities.

In classifying all phase portraits and describing the related bifurcation diagrams, the main problem often deals with the limit cycles. Next theorem is hence of crucial importance.

**Theorem 3.1** \( N(2, 2) = 1 \). More precisely, if \( c(3a - 2b + c) < 0 \), then (15) has at most one limit cycle, which is hyperbolic if it exists; if \( c(3a - 2b + c) \geq 0 \), then (15) has no limit cycles.

**FIG. 2.** Behaviour near infinity on a Poincaré–Lyapunov disc of degree (1, 2).
**Proof.** As explained in Application 2.4, we need only consider the uniqueness of limit cycles in the strip $-1 < x < +\infty$. Since

$$\frac{f(x)}{g(x)} = \frac{(3a - 2b)x^2 - 2cx - c}{(x^2 + x)^2}$$

and $4c^2 + 4c(3a - 2b) = 4c(3a - 2b + c)$, the conclusion for $c(3a - 2b + c) < 0$ is straightforward by Theorem 2.1. If $c(3a - 2b + c) \geq 0$, we take $B(y) = e^{2(b-c)y}$ as a Dulac function, then the non-existence of limit cycles can be obtained by using Theorem 2.2.

3.3. Bifurcation Diagram and Phase Portraits

In order to study the bifurcation diagram and phase portraits, we rewrite (15) in the equivalent form:

$$\begin{cases} \dot{x} = v, \\ \dot{v} = -(x + x^2) - (c + 2bx + 3ax^2) v. \end{cases}$$

(16)

Note that under the change $(v, t) \mapsto (-v, -t)$, (16) keeps its form, but $(a, b, c) \mapsto (-a, -b, -c)$. Hence the bifurcation diagram in $(a, b, c)$-space must be symmetric with respect to the origin. In further treatment we could restrict to $a > 0$, but as we will see sometimes it is better not to do this. In fact, depending on the arguments, it is sometimes better to write $c + 2bx + 3ax^2 = c + ex + 3a(x + x^2) = c + 2b(x + x^2) - ex^3$ with $e = 2b - 3a$ and to describe the bifurcation diagram in $(a, c, e)$-space or in $(b, c, e)$-space.
Lemma 3.2. If $c = 2b - 3a = 0$, then system (16) is integrable. The periodic orbits surround the center $(0, 0)$ and are bounded by a loop, homoclinic to the saddle point $(-1, 0)$, as in Fig. 5.

Proof. Take $c = 0$ and $e = 2b = 3a = 0$, then (16) has the form
\[
\begin{aligned}
\dot{x} &= v, \\
\dot{v} &= -(x + x^2)(1 + 2bv).
\end{aligned}
\]
For $b \neq 0$, $v = -(1/2b)$ is an invariant line. For $1 + 2bv \neq 0$, the above system is obviously integrable, and the other conclusions are easy to obtain. See also Lemma 3.5.

Lemma 3.3. Along the two half planes $H^+ = \{(a, b, c) \mid c = 0, e = 2b - 3a > 0\}$ and $H^- = \{(a, b, c) \mid c = 0, e = 2b - 3a < 0\}$ a Hopf bifurcation of order 1 takes place. More precisely, if $e > 0$ (resp. $< 0$) and $0 < c \ll 1$ (resp. $c < 0$, $|c| \ll 1$), then system (16) has a repelling (resp. attracting) limit cycle in a small neighborhood of the origin; and if $e > 0$ (resp. $< 0$) and $c < 0$, $|c| \ll 1$ (resp. $0 < c \ll 1$), then system (16) has no limit cycles in a small neighborhood of the origin.

Proof. It is obvious that $c = 0$ is the necessary condition for Hopf bifurcation. If $c = 0$ then, by using the formula as e.g., given in [GH], the first focal value is \( W_1 = \pm e \). The conclusion is straightforward.

Lemma 3.4. There are two surfaces $HL^+ = \{(a, b, c) \mid c = \varphi(a, b), e > 0\}$ and $HL^- = \{(a, b, c) \mid c = \varphi(a, b), e < 0\}$, defined by an analytic function $\varphi(a, b)$, defined on $(a, b) \in \mathbb{R}^2$ satisfying $0 < \varphi(a, b) < e$ for $0 < e$ and $e < \varphi(a, b) < 0$, for $e < 0$. Along $HL^+$ system (16) has a homoclinic loop bifurcation of order 1. If $(a, b, c)$ belongs to one of the two regions between $H^+$ and $HL^+$, and $H^-$ and $HL^-$ (as shaded in Fig. 4), then the corresponding system has exactly one limit cycle which is hyperbolic; if $(a, b, c)$ is located outside these two regions, then the corresponding system has no limit cycles.

Proof. Let us fix $a \in \mathbb{R}$. As we remarked before Lemma 3.2, the bifurcation diagram is symmetric with respect to $(c, e) = (0, 0)$, we hence only consider the case $e \geq 0$.

We first prove the existence of the function $\varphi(a, b)$, and discuss the existence and unicity of the limit cycle.

It is easy to verify that family (16) is a family of rotated vector fields (mod $v = 0$) with respect to the parameter $e$ (more precisely, called in [P] a semi-complete family (mod $v = 0$) of rotated vector fields, see also [D] for an account). Hence for $e = 0$, no periodic orbits are possible except at $c = 0$. Let us now take $e^* > 0$ fixed, and let us vary $c$ in a monotone way,
starting at $c = 0$. If $c$ decreases from 0, then by the rotational property it is easy to see that system (16) has no limit cycles (this agrees with the conclusion in Theorem 3.1). If $c$ increases and $0 < c \leq 1$, then by Lemma 3.3 system (16) has locally a hyperbolic attracting limit cycle. By Theorem 3.1, this is the only limit cycle, even globally. As $c$ increases continuously and monotonically, by the rotational property again, the limit cycle is also increasing monotonically. But by Theorem 3.1, a limit cycle cannot exist for $c \geq e^*$ (i.e., $c - 2b + 3a \geq 0$, $c > 0$). Hence, taking into account the situation near infinity (see Fig. 1 for $a \neq 0$ and Fig. 2 for $a = 0$, $b \neq 0$) the only possibility is that for some unique $c^* = \varphi(a, b)$, $0 < c^* \leq e^*$, the limit cycle tends to a homoclinic loop, disappearing for $c > e^*$ (see also Fig. 3 in case $a > 0$).

For $e = 2b - 3a = 0$, we define $\varphi(a, b) = 0$. The surface of homoclinic loop-bifurcations (including $(c, e) = (0, 0)$) is analytic, as can be seen as follows. Both the stable and unstable manifold of the saddle are analytic 1-manifolds, depending in an analytic way on $(a, b, c)$. Because of the rotational property with respect to $c$, we can use the implicit function theorem to obtain the result.

There remains to prove that $\varphi(a, b) < e = 2b - 3a$ for $e > 0$. This is equivalent to proving that along $L \setminus \{(c, e) = (0, 0)\}$ the corresponding system (16) has no saddle loops, where $L = \{(a, b, c) \mid c = e\}$. Let $c = e$, i.e., $c = 2b - 3a$, then system (16) becomes

$$
\begin{align*}
\dot{x} = v &= P, \\
\dot{v} = -(x + 1)[x + (e + 3ax)v] &= Q.
\end{align*}
$$

(17)
Since
\[
\begin{vmatrix} P & Q \\ P_e & Q_e \end{vmatrix} = -(x + 1) v^2 < 0 \quad \text{for} \quad x + 1 > 0, \quad v \neq 0,
\]
family (17) is a family of rotated vector fields (mod \( v = 0 \)) with respect to the parameter \( e \) and this in the half plane \( x + 1 > 0 \), in which the homoclinic loop or limit cycle could appear. At \( e = 0 \), by Lemma 3.2, (17) is integrable with a phase portrait as in Fig. 5. Hence, by the rotational property, system (17) has no saddle loop for \( e \neq 0 \). This means that along \( L \setminus \{(c, e) = (0, 0)\} \) system (16) has no saddle loop.

Finally, we need to prove that, for \( e > 0 \), the homoclinic loop bifurcation is of order 1, with respect to \( c \), by using \( 0 < \varphi(a, b) < e \). Actually the divergence of (16) at the saddle point \((-1, 0)\) with \( e = \varphi(a, b) \) is
\[-(c - 2b + 3a)|_{\varphi(a, b)} = e - c|_{\varphi(a, b)} = e - \varphi(a, b) > 0 \]
for \( e > 0 \). This, together with the rotational property, finishes the proof of Lemma 3.4.

As an extra information on \( \varphi \) we prove the following result.

**Lemma 3.5.** At \( e = 2b - 3a = 0 \) we have \(-3 < (\partial \varphi / \partial a)(a, b) < 0 \), where \( c = \varphi(a, b) \) is the function defined in Lemma 3.4. This means that the surface \( \{ c = \varphi(a, b) \} \) is transverse to both \( \{ c = 0 \} \) and \( L = \{(a, b, c) \mid c = e\} \) at \((c, e) = (0, 0)\).

**Proof.** Let \( 3a = 2b - e \), system (16) becomes
\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -(x + x^2)(1 + 2bv) + (ex^2 - c)v.
\end{align*}
\]
Let us fix \( b \in \mathbb{R} \) and consider \( e \) as an independent variable. We define
\[
\bar{\varphi}_e(b) = \bar{\varphi}(e, b) = \varphi \left( \frac{2b - e}{3}, b \right).
\]
As such
\[
\frac{\partial \varphi}{\partial a}(a, b) \bigg|_{2b - 3a = 0} = -3 \frac{\partial \bar{\varphi}}{\partial e}(0, b) = -3 \cdot \bar{\varphi}'_e(0)
\]
and it suffices to show that
\[
0 < \bar{\varphi}'_e(0) < 1.
\]
We only consider \((c, e)\) near \((0, 0)\), and as we already know that \(0 \leq \varphi_0'(0) \leq 1\), we take

\[
    c = \delta \tilde{c}, \quad e = \delta
\]

(22)

with \(\delta > 0\) small; (18) becomes

\[
\begin{align*}
    \dot{x} &= v, \\
    \dot{v} &= -(x + x^2)(1 + 2bv) + \delta(x^2 - \tilde{c}) v.
\end{align*}
\]

(23)

When \(\delta = 0\), (23) is integrable with a first integral

\[
    H(x, v) = h,
\]

where \(H = (v^2/2) + (x^2/2) + (x^3/3)\) if \(b = 0\), and \(H = (v/2b) - (1/4b^2) \ln |1 + 2bv| + (x^2/2) + (x^3/3)\) if \(b \neq 0\). The phase portraits of (23) with \(\delta = 0\) are shown in Fig. 5.

In any case, the closed level curves are

\[
    \gamma_h = \{(x, v) \mid H(x, v) = h, 0 < h < \frac{1}{2}\}
\]

and the abscis of the intersection point of the saddle loop with the positive \(x\)-axis, \(x^*\), satisfies \(0 < x^* < 1\).

Since, for small \(\delta > 0\), any limit cycle or homoclinic loop, if it exists, never meets the straight line \(1 + 2bv = 0\) for \(b \neq 0\), to study the limit cycle or the homoclinic bifurcation we can change (23) to the form

\[
\begin{align*}
    \dot{x} &= \frac{v}{1 + 2bv}, \\
    \dot{y} &= -(x + x^2) + \frac{\delta(x^2 - \tilde{c}) v}{1 + 2bv}.
\end{align*}
\]

(24)
The homoclinic bifurcation curve in $(\hat{c}, \delta)$-plane is given by
\[ I_1(\frac{1}{6}) - \partial I_2(\frac{1}{6}) + O(\delta) = 0, \] (25)
where
\[ I_1(h) = \int_{f_2}^{x^*} \frac{x^2}{1 + 2bv} \, dx, \quad I_2(h) = \int_{f_2}^{x^*} \frac{v}{1 + 2bv} \, dx. \]

We note that along $I_1$, we have that $(v/1 + 2bv) \, dx > 0$ and $-1 < x < x^* < 1$ except for the saddle point $(x, v) = (-1, 0)$. Hence
\[ 0 < I_1(1/6) < I_2(1/6) < 1. \] (26)

As such, from (25) and (22) we obtain that the homoclinic bifurcation is given, for small $\epsilon$, by
\[ c = \varphi(\epsilon) = \frac{I_1(1/6)}{I_2(1/6)} \epsilon + O(\epsilon^2). \] (27)

As $\varphi(\epsilon, b)$ is analytic, (26) and (27) induce the required result.

The bifurcations described in Lemma 3.4 (see Fig. 4) are not the only ones that one encounters in system (16). For symmetry reasons let us restrict to the case $a > 0$ (cfr. the first paragraph of 3.3). As is clear from Fig. 3, the right unstable separatrix of $s$ either can tend to $s'$, to a limit cycle, to $s$ in case of a loop or to infinity, where it has to tend to $p_1$. However for the right stable separatrix of $s$ there are more possibilities. It can come from $s'$, from a limit cycle, from $s$ or from infinity, but in the latter case it can come from $p_2$, $p_3$, or $p_4$; having $p_3$ as an $\alpha$-limit set occurs at a bifurcation value, representing a heteroclinic connection between $s$ and $p_3$.

**Lemma 3.6.** There is a surface $HC^+=\{(a, b, c)|c=\psi(a, b), a > 0\}$ for some $C^\infty$ function $\psi$ defined on $\mathbb{R}_+^0 \times \mathbb{R}$ and satisfying $\psi(a, b) > \varphi(a, b)$, with $\varphi$ as in Lemma 3.4. At $HC^+$ the analytic extension of (16) on the Poincaré-Lyapunov disc of degree $(1, 3)$ has a heteroclinic connection between $s$ and $p_3$ (see Fig. 3). $HC^+$ contains all possible heteroclinic connections for this analytic extension, in case $a > 0$.

**Proof.** As main ingredient in the proof let us recall that (16) is a family of rotated vector fields (mod $v=0$) with respect to the parameter $c$ (cfr. [P]). The same is true for the expression (12) representing (16) near.
infinity, by taking $x = 1$, $\beta = 1$, $\gamma = 0$. The implicit function theorem will provide the necessary arguments for the existence and the regularity of $\psi$, taking into account the limit situations when $c \to \pm \infty$. As we deal with a center manifold at $p_3$, the essential arguments are of class $C^r$, with $r$ arbitrary. The unicity of $\psi$, based on the rotational property, hence implies that it is $C^\infty$.

That no other heteroclinic connections can occur besides the ones from $s$ to $p_3$ is clear from the information summarized in Fig. 3. The fact that $\psi(a, b) > \phi(a, b)$ is also clear from Fig. 3.

This bifurcation $HC^+$ represents the fact that for $c > \psi(a, b)$ the right stable separatrix of $s$ consists of a solution whose $y$-variable stays negative, while for $c \leq \psi(a, b)$ this is not the case.

$HC^+$ has an empty intersection both with $HL$ and $H^+$, reflected by the property $\psi(a, b) > \phi(a, b)$. Concerning the relative position of $HC^+$ and $H^-$ we have the following result.

**Lemma 3.7.** There exists a $C^\infty$ function $a = r(b)$, with $r(b) > \max(0, (2b/3))$, such that $HC^+$ and \{c = 0\} cut transversally along \{(a, b, c) = (r(b), b, 0)\}. For $0 < a < r(b)$ we have $\psi(a, b) > 0$, while for $a > r(b)$ we have $\psi(a, b) < 0$.

**Proof.** If we take $c = 0$ and keep $b$ constant we can write (16) as:

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -(x + x^2)(1 + 2be) + ex^2v,
\end{align*}
\]  

(28)

where $e = 2b - 3a$. The map $a \to 2b - 3a$ is an affine coordinate change, so it suffices to check what happens to (28) when varying $e$ from 0 to $-\infty$. Again, for $x \neq 0$, (28) is a family of rotated vector fields (mod $v = 0$) with respect to $e$ and as such the heteroclinic connection between $s$ and $p_3$ (cfr. Lemma 3.6) will occur in a generic way at a unique point $e = \overline{e}(b) < 0$. This already implies most of the statement in the lemma. To see that $r(b) > 0$ for $b < 0$ it suffices to remark that $r(b)$ is $C^\infty$, and hence continuous, and that $r(b) = 0$ is impossible. Indeed, for $a = c = 0$ and $b < 0$, we know to be in $H^-$, meaning that $s'$ is attracting, and the behaviour near infinity, as represented in Fig. 2, does not permit heteroclinic connections.

In the next theorem we will now summarize the impact of the previous lemmas on the classification of the phase portraits.

**Theorem 3.8.** Quadratic Liénard equations with quadratic damping, having a general expression as given in (8) with $\alpha - a \neq 0$, can be extended to
an analytic vector field on the Poincaré–Lyapunov disc of degree $(1, 3)$. On this disc, the possible equivalence classes of phase portraits, for topological equivalence respecting the halfplane \( \{ y \geq 0 \} \), are those presented in Fig. 6, up to changes of the form \((x, y) \mapsto (-x, -y), \ (x, t) \mapsto (-x, -t), \ (y, t) \mapsto (-y, -t)\).

**Remark.** The pictures in Fig. 6 correspond to the case \( a > 0, \ x > 0 \). The behaviour at infinity is as described in Sect. 3.1; we refer to Sect. 3.1 for an account of the Poincaré–Lyapunov compactification of degree $(1, 3)$.

**Proof.** By means of changes of the form \((x, y, t) \mapsto (Ax, By, Ct)\) with \( A, B, C \in \mathbb{R} \setminus \{0\} \), and using a translation \( x \mapsto x + D(x, \beta, \gamma) \) for some well chosen \( A, B, C \) and \( D(x, \beta, \gamma) \) we can reduce system (8) to

\[
\begin{align*}
\dot{x} &= y, \\
\dot{y} &= -x^2 + \mu - (x^2 + v_1 x + v_0) y.
\end{align*}
\]  

(29)

Of course, a translation in \( x \) does not change the equivalence classes of phase portraits in system (8), neither does a change \((x, y, t) \mapsto (Ax, By, Ct)\) with \( A, B, C \in \mathbb{R} \setminus \{0\} \). Knowing that we want to respect the first component \("x = y"\) the only changes which change the equivalence classes of phase portraits and that we have to consider are \((x, y) \mapsto (-x, -y), \ (x, t) \mapsto (-x, -t)\) and \((y, t) \mapsto (-y, -t)\). The last two are orientation reversing and change the sense of the orbits, the first one respects orientation as well as the orientation of the orbits but it interchanges \( \{y \geq 0\} \) and \( \{y \leq 0\} \).

For \( \mu < 0 \), system (29) has no singularities and its phase portrait clearly has to be like in Fig. 6.1. For \( \mu = 0 \), system (29) has one singularity. At the unique singularity it is quite easy to calculate the eigenvalues, eigenspaces, invariant manifolds and determining jets. Let us not give these data but only present the resulting phase portraits. For \( v_0 \neq 0 \) the singularity is semi-hyperbolic, while it is a nilpotent cusp for \( v_0 = 0 \). Important again is that (29) is a family of rotated vector fields (mod \( y = 0 \)) with respect to \( v_0 \). When \( v_0 < 0 \) (resp. \( v_0 = 0 \)) the phase portrait is like in 6.2 (resp. 6.3). For \( v_0 > 0 \) we have the possibilities shown in 6.4, 6.5 and 6.6.

For \( \mu > 0 \) system (29) has two singularities \( s \) and \( s' \), of which the left one is a hyperbolic saddle, while \( s' \) is of index +1. Using changes of the form \((x, y, t) \mapsto (Ax, By, Ct)\) with \( A, B, C \in \mathbb{R} \setminus \{0\} \) and a parameter-dependent translation in \( x \) we can change (29) into expression (16). We can now rely on the results obtained in the Lemmas 3.2, 3.3, 3.4, 3.6 and 3.7, where all phase portraits were described together with the related bifurcation diagram. From Lemma 3.2 we know that \( s' \) can be a center, that we represent in Fig. 6.7; \( s' \) can also be hyperbolically attracting or hyperbolically repelling and it can both be a focus (case \( c^2 < 4 \)) or a node (case \( c^2 \geq 4 \)).
Fig. 6. Classification of phase portraits of (8) with $a > 0$, $\pi > 0$. 
Fig. 6—Continued
In the Figs. 6.8 to 6.17 we represent the cases where $s'$ is a focus. In the Figs. 6.8, 6.11, 6.12 and 6.13, $s'$ could be a weak focus as well.

To finish we can look at the possibilities when $s'$ is a hyperbolic node, nondegenerate or degenerate (in case $c = \pm 2$); the last distinction does not provide different (equivalence classes of) phase portraits. For sure we encounter the Figs. 6.18 and 6.19 for $|c|$ near infinity.

Also Fig. 6.20 has to occur, since both 6.18 and 6.20 can be encountered in the unfolding of the cusp singularity that (29) exhibits at $\mu = v_0 = 0$, for $v_1 \neq 0$ (cf. Fig. 6.3). At such values, for small $(\mu, v_0)$, (29) represents a Bogdanov-Takens bifurcation (see e.g., [D]).

Of course, if we let $c$ increase, starting from Fig. 6.20, we will encounter 6.21 before reaching 6.19.

Finally we have to check whether one can have a node (nondegenerate or degenerate) together with a limit cycle or a homoclinic loop. If this happens, then because of Lemma 3.4 (describing the bifurcation diagram of limit cycles), it has to happen at least for a degenerate node, hence for $c = \pm 2$. Let us therefore consider (16) with $c = \pm 2$.

\[
\begin{align*}
\dot{x} &= v, \\
\dot{v} &= -(x + x^2) - (c + 2bx + 3ax^2) v.
\end{align*}
\]  

(30)

In case $c = 2$, the node at the origin is attracting.

For $a = 0$, (30) is quadratic and as such no limit cycle nor loop can occur together with the node. In fact this is easy to see by looking along the eigendirection $\{v = x\}$. The right unstable separatrix of $s$ is hence attracted by $s'$, meaning that in the finite plane the phase portrait has to be like in 6.19–6.21. Now, with respect to $a$, for $x \neq 0$, (30) is a family of rotated vector fields (mod $v = 0$), and for increasing $a$, the right unstable separatrix of $s$ continues having $s'$ as its $-\infty$-limit, not permitting a limit cycle nor a loop.

In case $c = -2$, the node at the origin is repelling. Again for $a = 0$, there can be no limit cycle nor loop, and the phase portrait in the finite plane has to be like in Fig. 6.18. If we now increase $a$ in system (30) we will tend to phase portraits like in 6.23, passing by 6.22, using the fact that (30) is a family of rotated vector fields (mod $v = 0$), and relying on the Poincaré–Bendixson theorem as well as on the unicity of the limit cycle.

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