Improvement of Some Ostrowski-Grüss Type Inequalities

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Abstract—In this paper, we improve and further generalize some Ostrowski-Grüss type inequalities involving bounded once and twice differentiable mappings. © 2001 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In 1997, Dragomir and Wang [1] proved the following Ostrowski-Grüss type integral inequality.

Theorem 1.1. Let \( f : I \rightarrow \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval, be a mapping differentiable in the interior \( \overset{\circ}{I} \) of \( I \), and let \( a, b \in \overset{\circ}{I} \) with \( a < b \). If \( \gamma_1 \leq f'(x) \leq \Gamma_1 \), \( x \in [a, b] \) for some constants \( \gamma_1 \), \( \Gamma_1 \in \mathbb{R} \), then

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4} (b-a)(\Gamma_1 - \gamma_1),
\]

(1.1)

for all \( x \in [a, b] \).

This inequality is a connection between the Ostrowski inequality [2] and the Grüss inequality [3]. It can be applied to bound some special mean and some numerical quadrature rules, see [4–6]. The right-hand side of inequality (1.1) contains maximum \( \Gamma_1 \) and minimum \( \gamma_1 \) of the function \( f' \). It is usually more precise than an inequality used the maximum of function \( |f'| \). Cerone et al. [7] generalized inequality (1.1) to twice differentiable mappings.

Theorem 1.2. Let \( f : I \rightarrow \mathbb{R} \), where \( I \subseteq \mathbb{R} \) is an interval. Suppose that \( f \) is twice differentiable in the interior \( \overset{\circ}{I} \) of \( I \), and let \( a, b \in \overset{\circ}{I} \) with \( a < b \). If \( \gamma_2 \leq f''(x) \leq \Gamma_2 \), \( x \in [a, b] \) for some
Let the assumptions of Theorem 1.1 hold. Then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{4\sqrt{3}} (b-a)(\Gamma_1 - \gamma_1). \tag{1.3}
\]

**Theorem 1.4.** Let the assumptions of Theorem 1.2 hold. Then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \left( x - \frac{a+b}{2} \right) f'(x) + \frac{1}{24} (b-a)^2 + \frac{1}{2} \left( x - \frac{a+b}{2} \right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\
- \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq (\Gamma_2 - \gamma_2) F(a, b, x),
\]

where

\[
F(a, b, x) = \frac{b-a}{12\sqrt{5}} \left( \frac{1}{4} (b-a)^2 + 15 \left( x - \frac{a+b}{2} \right)^2 \right)^{1/2}. \tag{1.5}
\]

In this paper, we further improve above Ostrowski-Griiss type inequalities and get some sharp inequalities. We will prove the following inequalities.

**Theorem 1.5.** Let the assumptions of Theorem 1.1 hold. Then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{f(b) - f(a)}{b-a} \left( x - \frac{a+b}{2} \right) \right| \leq \frac{1}{8} (b-a)(\Gamma_1 - \Gamma_2). \tag{1.6}
\]

The constant \( 1/8 \) is sharp in the sense that it cannot be replaced by a smaller one.

**Theorem 1.6.** Let the assumptions of Theorem 1.2 hold. Then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \left( x - \frac{a+b}{2} \right) f'(x) + \frac{1}{24} (b-a)^2 + \frac{1}{2} \left( x - \frac{a+b}{2} \right)^2 \right) \frac{f'(b) - f'(a)}{b-a} \right. \\
- \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq (\Gamma_2 - \gamma_2) G(a, b, x),
\]

where

\[
G(a, b, x) = \begin{cases} 
\frac{1}{3(b-a)} \left| (x-a) \left( x - \frac{a+b}{2} \right) \right| (b-x) \\
+ \left( \frac{1}{12} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right)^{3/2}, & a \leq x \leq \frac{1}{3}(2a+b), \\
\frac{2}{3(b-a)} \left( \frac{1}{12} (b-a)^2 + \left( x - \frac{a+b}{2} \right)^2 \right)^{3/2}, & \frac{1}{3}(2a+b) \leq x \leq \frac{1}{3}(a+2b). 
\end{cases} \tag{1.8}
\]
We can prove that
\[ G(a, b, x) \leq \frac{2\sqrt{15}}{9} F(a, b, x). \] (1.9)
So we improve Theorem 1.4 by a factor about 0.85.

The organization of the paper is as follows. In the next section, we give the proof of Theorems 1.5 and 1.6. In Section 3, we give some generalizations of Theorems 1.5 and 1.6.

2. PROOF OF THEOREMS

PROOF OF THEOREM 1.5. First we know that
\[ f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \left( x - \frac{a+b}{2} \right) \frac{f(b) - f(a)}{b-a} = \frac{1}{b-a} \int_a^b G_1(x, t)f'(t) \, dt, \] (2.1)
where
\[ G_1(x, t) = \begin{cases} (t-a) - \left( x - \frac{a+b}{2} \right), & a \leq t < x, \\ (t-b) - \left( x - \frac{a+b}{2} \right), & x \leq t \leq b. \end{cases} \] (2.2)

By symmetry, we can suppose \( a \leq x \leq \frac{1}{2}(a+b) \). Then
\[ G_1(x, t) \geq 0, \quad t \in [a, x) \cup (t^*, b), \]
\[ G_1(x, t) \leq 0, \quad t \in (x, t^*], \]
where
\[ t^* = x + \frac{1}{2}(b-a). \] (2.4)

As
\[ \int_a^b G_1(x, t)f'(t) \, dt = \left( \int_a^x + \int_{t^*}^b \right) G_1(x, t)f'(t) \, dt + \int_{t^*}^x G_1(x, t)f'(t) \, dt \]
\[ \leq \frac{1}{8} (b-a)^2 \] (2.5)
and
\[ \left( \int_a^x + \int_{t^*}^b \right) G_1(x, t) \, dt = \frac{1}{8} (b-a)^2, \quad \int_{t^*}^x G_1(x, t) \, dt = -\frac{1}{8} (b-a)^2, \] (2.6)
so we get
\[ \int_a^b G_1(x, t)f'(t) \, dt \leq \frac{1}{8} (b-a)^2 (\Gamma_1 - \gamma_1). \] (2.7)
Similarly, we can obtain
\[ -\int_a^b G_1(x, t)f'(t) \, dt \leq \frac{1}{8} (b-a)^2 (\Gamma_1 - \gamma_1). \] (2.8)
Then (2.1), (2.7), and (2.8) imply the result of (1.6). It completes the proof of Theorem 1.5.

From the proof, we can construct the function
\[ f(t) = \begin{cases} \Gamma_1(t-a), & a \leq t < x, \\ \Gamma_1(x-a) + \gamma_1(t-x), & x \leq t < t^*, \\ \Gamma_1(x-a) + \gamma_1(t^*-x) + \Gamma_1(t-t^*), & t^* \leq t \leq b, \end{cases} \] (2.9)
where \( t^* \) is defined as (2.4). Then (1.6) holds equality. Thus, the constant 1/8 is sharp in the sense that it cannot be replaced by a smaller one.
For \( x = (a + b)/2 \), we obtain a sharper bound than that stated in [1,5]

\[
\left| f\left(\frac{a + b}{2}\right) - \frac{1}{b - a} \int_a^b f(t) \, dt \right| \leq \frac{1}{8}(b - a)(\Gamma_1 - \gamma_1).
\] (2.10)

**Proof of Theorem 1.6.** From [5], we can derive that

\[
\frac{1}{b - a} \int_a^b f(t) \, dt - f(x) + \left( x - \frac{a + b}{2} \right) f'(x)
- \left( \frac{1}{24} (b - a)^2 + \frac{1}{2} \left( x - \frac{a + b}{2} \right)^2 \right) \frac{f'(b) - f'(a)}{b - a}
= \frac{1}{b - a} \int_a^b f''(t) G_2(x, t) \, dt,
\] (2.11)

where

\[
G_2(x, t) = \begin{cases} 
\frac{1}{2} (t - a)^2 - \left( \frac{1}{24} (b - a)^2 + \frac{1}{2} \left( x - \frac{a + b}{2} \right)^2 \right), & a \leq t < x, \\
\frac{1}{2} (t - b)^2 - \left( \frac{1}{24} (b - a)^2 + \frac{1}{2} \left( x - \frac{a + b}{2} \right)^2 \right), & x \leq t \leq b.
\end{cases}
\] (2.12)

We assume that \( a \leq x \leq (1/2)(a + b) \). First we consider the case \( a \leq x \leq (2a + b)/3 \). By (2.12), we know that

\[
G_2(x, t) \leq 0, \quad t \in [a, x) \cup [t^{**}, b],
\]

\[
G_2(x, t) \geq 0, \quad t \in [x, t^{**}),
\] (2.13)

where

\[
t^{**} = b - \left( \frac{1}{12} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \right)^{1/2}.
\] (2.14)

As \( \gamma_2 \leq f''(x) \leq \Gamma_2 \) for all \( x \in [a, b] \), then

\[
\int_a^b G_2(x, t) f''(t) \, dt \leq \gamma_2 \left( \int_a^x + \int_{t^{**}}^b \right) G_2(x, t) \, dt + \Gamma_2 \int_x^{t^{**}} G_2(x, t) \, dt,
\] (2.15)

\[
- \int_a^b G_2(x, t) f''(t) \, dt \leq \Gamma_2 \left( \int_a^x + \int_{t^{**}}^b \right) - G_2(x, t) \, dt + \gamma_2 \int_x^{t^{**}} -G_2(x, t) \, dt.
\] (2.16)

By simple computation, we can derive

\[
\int_a^x G_2(x, t) \, dt = \frac{1}{3} (x - a) \left( x - \frac{a + b}{2} \right) (b - x),
\] (2.17)

\[
\int_{t^{**}}^b G_2(x, t) \, dt = - \frac{1}{3} \left( \frac{1}{12} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \right)^{3/2}.
\] (2.18)

Since

\[
\int_a^b G_2(x, t) \, dt = 0,
\] (2.19)

thus

\[
\int_x^{t^{**}} G_2(x, t) \, dt = - \left( \int_a^x + \int_{t^{**}}^b \right) G_2(x, t) \, dt.
\] (2.20)

By (2.15)-(2.20) and (2.11), we prove (1.7) easily for \( a \leq x \leq (1/3)(2a + b) \).
For \((1/3)(2a + b) \leq x \leq (1/2)(a + b)\), then

\[
G_2(x, t) \leq 0, \quad t \in [a, t_1^{**}] \cup [t_2^{**}, b],
\]
\[
G_2(x, t) \geq 0, \quad t \in [t_1^{**}, t_2^{**}),
\]
where

\[
t_1^{**} = a + \left(\frac{1}{12}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right)^{1/2},
\]

and \(t_2^{**} = t^{**}\) is defined as in (2.14). Using the same argument of (2.15)-(2.20), we get

\[
\int_a^{t_1^{**}} G_2(x, t) \, dt = -\frac{1}{2} \left(\frac{1}{12}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right)^{3/2},
\]
\[
\int_{t_2^{**}}^b G_2(x, t) \, dt = -\frac{1}{2} \left(\frac{1}{12}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2\right)^{3/2}.
\]

We can get the inequality (1.7) easily. It completes the proof of Theorem 1.6.

It is not difficult to see that

\[
r(x) = \frac{G(a, b, x)}{F(a, b, x)}, \quad \tilde{r}(\xi) = r(a + \xi(b-a)) = \frac{G(0, 1, \xi)}{F(0, 1, \xi)}, \quad \xi \in [0, 1],
\]

attains the maximum value at \(x = a\) and \(x = (1/2)(a+b)\) by numerical computations in computer. So we obtain the inequality (1.9).

### 3. FURTHER GENERALIZATION

In [5], the authors discuss the Ostrowski-Grüss type inequalities involving \(n\)-time differentiable mappings. We can also improve their results by the trick provided in the proofs of Theorems 1.5 and 1.6. In this section, we do not go towards this direction. We will generalize the Ostrowski-Grüss type inequalities as follows.

**Theorem 3.1.** Let the assumptions of Theorem 1.1 hold. Then for all \(x \in [a, b]\), we have

\[
\left|\frac{1}{2}f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)}\right| \leq \frac{1}{8(b-a)} \left((x-a)^2 + (x-b)^2\right) \left(\Gamma_1 - \gamma_1\right).
\]

**Proof.** We can write

\[
\frac{1}{2}f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{(x-b)f(b) - (x-a)f(a)}{2(b-a)} = \frac{1}{b-a} \int_a^b G_3(x, t) f'(t) \, dt,
\]

where

\[
G_3(x, t) = \begin{cases} 
(t-a) - \frac{1}{2} (x-a), & a \leq t < x, \\
(t-b) - \frac{1}{2} (x-b), & x \leq t \leq b.
\end{cases}
\]

Thus, using the trick of Section 2, we can derive the inequality (3.1) easily.
For \( x = a \) or \( x = b \), we get the trapezoid inequality

\[
\left| \frac{1}{b-a} \int_a^b f(t) \, dt - \frac{1}{2} \left( f(a) + f(b) \right) \right| \leq \frac{b-a}{8} (\Gamma_1 - \gamma_1). \tag{3.4}
\]

For \( x = \frac{1}{2} (a+b) \), we get a composite trapezoid quadrature rule (two equal subintervals).

**Theorem 3.2.** Let the assumptions of Theorem 1.2 hold. Then for all \( x \in [a, b] \), we have

\[
\left| f(x) - \frac{2}{3} \left( x - \frac{a+b}{2} \right) f'(x) + \frac{(x-b)^2 f'(b) - (x-a)^2 f'(a)}{6(b-a)} - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{18 \sqrt{3} (b-a)} \left( (x-a)^3 + (b-x)^3 \right) (\Gamma_2 - \gamma_2). \tag{3.5}
\]

**Proof.** We can write

\[
f(x) - \frac{2}{3} \left( x - \frac{a+b}{2} \right) f'(x) + \frac{(x-b)^2 f'(b) - (x-a)^2 f'(a)}{6(b-a)} - \frac{1}{b-a} \int_a^b f(t) \, dt = \frac{1}{b-a} \int_a^b G_4(x, t) f''(t) \, dt,
\]

where

\[
G_4(x, t) = \begin{cases} 
\frac{1}{2} (t-a)^2 - \frac{1}{6} (x-a)^2, & a \leq t < x, \\
\frac{1}{2} (t-b)^2 - \frac{1}{6} (x-b)^2, & x \leq t \leq b.
\end{cases} \tag{3.7}
\]

Similar to Section 2, we can get the result easily.

For \( x = \frac{1}{2} (a+b) \) and \( x = a \) or \( x = b \), we obtain the same result of Theorem 1.6 and improve the perturbed midpoint inequality and perturbed trapezoid inequality [5, p. 173], respectively:

\[
\left| f \left( \frac{a+b}{2} \right) + \frac{1}{24} (b-a) (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{72 \sqrt{3}} (\Gamma_2 - \gamma_2) (b-a)^2, \tag{3.8}
\]

\[
\left| f(a) + f(b) - \frac{b-a}{12} (f'(b) - f'(a)) - \frac{1}{b-a} \int_a^b f(t) \, dt \right| \leq \frac{1}{18 \sqrt{3}} (\Gamma_2 - \gamma_2) (b-a)^2. \tag{3.9}
\]

**References**