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Lifting fixed points of completely positive mappings

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ABSTRACT

Let $1 \leqslant n \leqslant \infty$, and let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a row contraction on some Hilbert space $\mathcal{H}$. Let $\mathcal{F}(T)$ be the space of all $X \in \mathcal{B}(\mathcal{H})$ such that $\sum_{j=1}^n T_j X T_j^* = X$. We show that, if non-zero, this space is completely isometric to the commutant of the Cuntz part of the minimal isometric dilation of $\{T_j\}_{j=1}^n$.

1. Introduction

Suppose $1 \leqslant n \leqslant \infty$ and let $\{T_j\}_{j=1}^n$ be a row contraction on some complex Hilbert space $\mathcal{H}$. This means that $\{T_j\}_{j=1}^n$ is a sequence of bounded operators on $\mathcal{H}$ such that $\sum_{j=1}^n T_j T_j^* \leqslant I_{\mathcal{H}}$. We define

$$\mathcal{F}(T) = \left\{ X \in \mathcal{B}(\mathcal{H}) \colon \sum_{j=1}^n T_j X T_j^* = X \right\}$$

and call it the space of T-Toeplitz operators. It is obvious that $\mathcal{F}(T)$ is a selfadjoint weak* closed subspace of $\mathcal{B}(\mathcal{H})$. This space has been studied in several papers, see for instance [6,16] and the references therein. In [6] it was shown that, if $\sum_{j=1}^n T_j T_j^* = I_{\mathcal{H}}$, then $\mathcal{F}(T)$ is isometric and order isomorphic with the commutant of the minimal isometric dilation of $\{T_j\}_{j=1}^n$. A very detailed study of this space in connection with the ergodic properties of completely positive mappings was given in [16]. In particular, for an arbitrary row contraction $\{T_j\}_{j=1}^n$ it was proved that every positive element in $\mathcal{F}(T)$ can be lifted to a positive element in the corresponding fixed point space of the minimal isometric dilation of $\{T_j\}_{j=1}^n$ and moreover this lifting is norm preserving.

In this paper it will be shown that $\mathcal{F}(T)$ is completely isometric to the commutant $\{\{R_j\}_{j=1}^n\}'$ of the Cuntz part of the minimal isometric dilation of $\{T_j\}_{j=1}^n$. We also show that this complete isometry can be extended to a *-homomorphism

$$\pi : C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\} \to \{\{R_j\}_{j=1}^n\}'$$

and provide an asymptotic formula for the lifting.

The main ingredients of the proof are the lifting theorem for positive elements in $\mathcal{F}(T)$ from [16] mentioned above together with a well-known theorem of M.D. Choi and E.G. Effros [8] on the structure of completely positive and completely contractive idempotents on $\mathcal{B}(\mathcal{H})$. The Choi–Effros theorem says that the image of any such idempotent mapping is completely isometric to a C^*-algebra and provides a formula for the multiplication. In our case, a completely positive idempotent is constructed using a standard averaging process such that its image is precisely the space $\mathcal{F}(T)$. This construction

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has been frequently used in the past for studying fixed point spaces of completely positive mappings. We will give more details and references in the next section.

2. Preliminaries

Let $1 \leqslant n \leqslant \infty$. Let us denote $\Lambda_n = \{1, \dots, n\}$ if $1 \leqslant n < \infty$ and $\Lambda_\infty = \{1, 2, \dots\}$. For each $1 \leqslant m < \infty$ let $\Lambda_n^{(m)}$ be the set of all m-tuples $\alpha = (j_1, \dots, j_m)$ with $j_k \in \Lambda_n$ for $1 \leqslant k \leqslant m$. If $\{A_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ and $\alpha = (j_1, \dots, j_m) \in \Lambda_n^{(m)}$ for some $m \geqslant 1$, one denotes $A_\alpha = A_{j_1} A_{j_2} \cdots A_{j_m}$.

The following result is well known, see the proof of Lemma 2 on p. 286 in [2]. A similar result for amenable semigroups of completely positive maps on dual operator spaces is given in [5].

Lemma 2.1. *Let $\mathcal{M}$ be a von Neumann algebra and let $\Psi : \mathcal{M} \to \mathcal{M}$ be a completely positive, contractive and weak* continuous linear mapping. Let $\Psi^{(m)}$ be its m-power as an operator on $\mathcal{M}$. Let*

$$\mathcal{F}(\Psi) = \{X \in \mathcal{M}:\ \Psi(X) = X\}.$$

Then there exists a completely positive and contractive linear mapping $\Phi : \mathcal{M} \to \mathcal{M}$ such that $\Phi \circ \Phi = \Phi$ and whose range coincides with the space $\mathcal{F}(\Psi)$. Moreover, if $A, B \in \mathcal{M}$ satisfy $\Psi(AXB) = A\Psi(X)B$ for all $X \in \mathcal{M}$ then the same holds true for Φ. In addition,

$$\Phi(I) = \text{s.o.-}\lim_{m\to\infty} \Psi^{(m)}(I)$$

where I is the identity of $\mathcal{M}$.

An immediate consequence is that there exists a non-zero operator $X \in \mathcal{M}$ such that $\Psi(X) = X$ if and only if $\text{s.o.-}\lim_{m\to\infty} \Psi^{(m)}(I) \neq 0$. The following very useful result is essentially proved in [8], see also Theorem 6.1.3 in [10].

Theorem 2.2. *(See [8].) Let $\mathcal{A}$ be a unital C^*-algebra with unit I and let $\Phi : \mathcal{A} \to \mathcal{A}$ be a non-zero completely positive and contractive mapping such that $\Phi \circ \Phi = \Phi$. Let $\mathcal{E}$ denote its range and let $C^*\{I, \mathcal{E}\}$ be the unital C^*-subalgebra of $\mathcal{A}$ generated by $\mathcal{E}$. Then there exist a unital C^*-algebra $\mathcal{A}_\Phi$ and a surjective, order preserving, complete isometry $\theta : \mathcal{A}_\Phi \to \mathcal{E}$ such that $\theta^{-1} \circ \Phi$ is a unit preserving *-homomorphism when restricted to $C^*\{I, \mathcal{E}\}$. If moreover, $\mathcal{A}$ is a von Neumann algebra and if $\mathcal{E}$ is weak* closed, then $\mathcal{A}_\Phi$ is a von Neumann algebra and θ is a weak* homeomorphism.*

If $\Psi : \mathcal{M} \to \mathcal{M}$ is a unital completely positive mapping on a von Neumann algebra, then its fixed point space $\mathcal{F}(\Psi)$ is called a non-commutative harmonic space. The von Neumann algebra appearing in Theorem 2.2 which is completely isometric to $\mathcal{F}(\Psi)$ (and which is uniquely determined) is called the non-commutative Poisson boundary of $\mathcal{F}(\Psi)$, see [12,13]. A short survey of these topics is also given in [3]. The non-commutative Poisson boundary has been recently used in [4] in the study of asymptotic lifts of completely positive maps on von Neumann algebras.

Let $\mathcal{K}$ be a complex Hilbert space. A sequence $\{V_j\}_{j=1}^n$ of bounded operators on $\mathcal{K}$ is called a row isometry if each V_j is an isometry and $V_i^* V_j = 0$ whenever $i \neq j$. If $\sum_{j=1}^n V_j V_j^* = I_{\mathcal{K}}$ then $\{V_j\}_{j=1}^n$ is called a Cuntz isometry. On the other hand, if

$$w^*\text{-}\lim_{m\to\infty} \sum_{\alpha\in\Lambda_n^{(m)}} V_\alpha V_\alpha^* = 0$$

then $\{V_j\}_{j=1}^n$ is called a pure isometry. It was proved in [15] that every row isometry can be expressed uniquely as the orthogonal sum of a pure isometry and a Cuntz isometry, either of which may be absent.

It was proved in [19] (for $n = 1$), in [11] (for $n = 2$), in [7] (for n finite) and in [15] (for $n = \infty$), that for each row contraction $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ there exist a larger Hilbert space $\mathcal{K}$ and a row isometry $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ such that $\mathcal{H}$ is invariant for all V_j^*'s and the corresponding restrictions agree with T_j^* on $\mathcal{H}$. If moreover $\mathcal{K}$ is the smallest reducing subspace for $\{V_j\}_{j=1}^n$ containing $\mathcal{H}$ then $\{V_j\}_{j=1}^n$ is unique (up to a unitary equivalence), and it is called the minimal isometric dilation of $\{T_j\}_{j=1}^n$. The following important result proved in [16] will be very useful in the sequel.

Theorem 2.3. *(See [16].) Let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a row contraction, let $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ be its minimal isometric dilation, and let $\mathcal{F}(V) \subset \mathcal{B}(\mathcal{K})$ be the space of V-Toeplitz operators. Then for every $X \in \mathcal{F}(T)$ with $X \geqslant 0$ there exists $\tilde{X} \in \mathcal{F}(V)$ with $\tilde{X} \geqslant 0$ such that $X = P_{\mathcal{H}} \tilde{X}|_{\mathcal{H}}$.*

The proof of the following lemma is rather straightforward and it will be omitted.

Lemma 2.4. *Let $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ be a row isometry and let*

$$V_j = W_j \oplus R_j \in \mathcal{B}(\mathcal{S} \oplus \mathcal{R}), \quad j \in \Lambda_n,$$

be its Wold decomposition, where $\{W_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{S})$ *is a pure isometry and* $\{R_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{R})$ *is a Cuntz isometry. Then an operator* $Y \in \mathcal{B}(\mathcal{K})$ *belongs to* $\mathcal{F}(V)$ *if and only* $Y = 0 \oplus Z$ *where* $Z \in \{C^*\{\{R_j\}_{j=1}^n\}\}'$.

Let $2 \leqslant n < \infty$. The Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by n isometries $\{S_1, \ldots, S_n\}$ acting on some infinite dimensional Hilbert space $\mathcal{H}$ such that $\sum_{j=1}^n S_j S_j^* = I_{\mathcal{H}}$. Moreover $\mathcal{O}_\infty$ is the C^*-algebra generated by an infinite countable sequence of isometries $\{S_j\}_{j=1}^\infty \subset \mathcal{B}(\mathcal{H})$ with $\sum_{j=1}^\infty S_j S_j^* \leqslant I_{\mathcal{H}}$. As shown in [9], the isomorphism class of $\mathcal{O}_n$ (for either $n \geqslant 2$ or $n = \infty$) does not depend on the particular choice of the underlying Hilbert space. We shall also denote by $\mathcal{O}_1$ the commutative C^*-algebra $C(\mathbb{T})$ of all continuous complex valued functions on the unit circle.

3. A representation theorem

Let $1 \leqslant n \leqslant \infty$ and let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a row contraction on some Hilbert space $\mathcal{H}$. Let

$$Q = s.o.\text{-}\lim_{m\to\infty} \sum_{\alpha \in \Lambda_n^{(m)}} T_\alpha T_\alpha^*$$

and assume that $Q \neq 0$. Let $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ be its minimal isometric dilation. Then its Cuntz part $\{R_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{R})$ is non-zero [15]. Consider the operator $\Gamma : \mathcal{R} \to \mathcal{H}$ defined by

$$\Gamma(\xi) = P_{\mathcal{H}}(0_{\mathcal{S}} \oplus \xi), \quad \xi \in \mathcal{R},$$

where $\mathcal{S} \subset \mathcal{K}$ is the space of the pure isometric part of $\{V_j\}_{j=1}^n$. Then it follows from the dilation theorem that

(a) $Q = \Gamma\Gamma^*$;
(b) $T_j\Gamma = \Gamma R_j$ for $j \in \Lambda_n$;
(c) $\mathcal{R} = \overline{span}\{R_\alpha \Gamma^* h:\ \alpha \in \bigcup_{m\geqslant 1} \Lambda_n^{(m)},\ h \in \mathcal{H}\}$.

The main result of this paper is the following theorem. It may be viewed as a non-commutative version of similar results for commuting row contractions proved in [17] and [18]. The difference is that the role played in [17] and [18] by a sequence $\{U_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{R})$ of commuting normal operators with $\sum_{j=1}^n U_j U_j^* = I_{\mathcal{R}}$ is now played by a Cuntz isometry. In the case when $\sum_{j=1}^n T_j T_j^* = I_{\mathcal{H}}$, the fact that the mapping ρ appearing below is isometric was proved in [6].

Theorem 3.1. *Let* $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ *be a row contraction, let*

$$\mathcal{F}(T) = \left\{ X \in \mathcal{B}(\mathcal{H}):\ \sum_{j=1}^n T_j X T_j^* = X \right\},$$

and assume that $\mathcal{F}(T) \neq \{0\}$*, equivalently, that the operator*

$$Q = s.o.\text{-}\lim_{m\to\infty} \sum_{\alpha \in \Lambda_n^{(m)}} T_\alpha T_\alpha^*$$

is non-zero. Let $\{R_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{R})$ *be the Cuntz part of its minimal isometric dilation* $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$*. Let* $\Gamma : \mathcal{R} \to \mathcal{H}$ *be defined by*

$$\Gamma(\xi) = P_{\mathcal{H}}(0 \oplus \xi), \quad \xi \in \mathcal{R},$$

and let $\lambda : \mathcal{O}_n \to \mathcal{B}(\mathcal{R})$ *be the unique* **-representation such that* $\lambda(S_j) = R_j$ *for* $j \in \Lambda_n$*. Then the following hold true:*

(1) *The mapping* $\rho : \{\lambda(\mathcal{O}_n)\}' \to \mathcal{B}(\mathcal{H})$ *defined by*

$$\rho(Y) = \Gamma Y \Gamma^*, \quad Y \in \{\lambda(\mathcal{O}_n)\}',$$

implements a complete isometry and an order isomorphism from $\{\lambda(\mathcal{O}_n)\}'$ *onto* $\mathcal{F}(T)$*, where* $\{\lambda(\mathcal{O}_n)\}'$ *holds for the commutant of* $\lambda(\mathcal{O}_n)$ *in* $\mathcal{B}(\mathcal{R})$*.*

(2) *There exists a unique* **-representation*

$$\pi : C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\} \to \{\lambda(\mathcal{O}_n)\}'$$

such that

$$\pi(\rho(Y)) = Y, \quad Y \in \{\lambda(\mathcal{O}_n)\}'.$$

Proof. Let $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ be the minimal isometric dilation of $\{T_j\}_{j=1}^n$ and let

$$V_j = W_j \oplus R_j \in \mathcal{B}(\mathcal{S} \oplus \mathcal{R}), \quad j \in \Lambda_n,$$

be its Wold decomposition, where $\{W_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{S})$ is a pure isometry and $\{R_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{R})$ is a Cuntz isometry. In order to prove (1) we begin by observing that for every $Z \in \mathcal{B}(\mathcal{R})$ we have

$$\Gamma Z \Gamma^* = P_{\mathcal{H}}(0 \oplus Z)|_{\mathcal{H}}.$$

This, together with Lemma 2.4 shows that the mapping ρ defined at (1) takes values in $\mathcal{F}(T)$. Moreover, it easily follows from assertions (a), (b) and (c) above that ρ is one-to-one. Let us show that ρ is onto and completely isometric. For this purpose, we shall make use of Theorem 2.3 as follows. We define a mapping $\Psi_T : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ by

$$\Psi_T(X) = \sum_{j=1}^n T_j X T_j^*, \quad X \in \mathcal{B}(\mathcal{H}).$$

This mapping is completely positive, contractive and weak* continuous. Then there exists, by Lemma 2.1, a completely positive and contractive mapping $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ such that $\Phi \circ \Phi = \Phi$ and $Ran\,\Phi = \mathcal{F}(T)$. It then follows from Theorem 2.2 that there exist a unital C^*-algebra $\mathcal{A}_\Phi$ and a completely positive and completely isometric surjective mapping $\theta : \mathcal{A}_\Phi \to \mathcal{F}(T)$ such that $\theta^{-1} \circ \Phi$ is a unital *-homomorphism on $C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$. In particular it follows that $\theta(I) = Q$. Now, according to Theorem 2.3 and using again Lemma 2.4 it follows that for each $X \in \mathcal{F}(T)$ with $X \geqslant 0$ there exists $Y \in \{\lambda(\mathcal{O}_n)\}'$ such that $\rho(Y) = X$. Since $\mathcal{F}(T)$ is linearly generated by its positive cone this shows that ρ is onto. Obviously ρ is completely positive and contractive. It now follows from Theorem 2.2 that the mapping

$$\rho^{-1} \circ \theta : \mathcal{A}_\Phi \to \{\lambda(\mathcal{O}_n)\}'$$

is positive and unital. It then follows that it is also contractive. To show that ρ^{-1} is completely positive, it suffices to apply, for each $m \geqslant 2$, the previous arguments to the row contraction $\{T_j^{(m)}\}_{j=1}^n$ where $T_j^{(m)}$ is the m-fold ampliation of T_j. Therefore, the mapping $\rho^{-1} \circ \theta$ is completely positive. Since it is also unital it is completely contractive as well. This shows that ρ is indeed completely isometric and completely order preserving. The proof of (2) becomes now an easy consequence of Theorem 2.2 together with a well-known result from [14] that a unit preserving complete isometry between two C^*-algebras is a *-isomorphism. □

If Φ is a completely positive idempotent as in the proof of this theorem, then one can easily see that $\rho \circ \pi = \Phi$ on $C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$. This shows that π is a Stinespring dilation of the restriction of Φ to $C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$. However, it is not necessarily minimal. For instance, it was proved in [18] that if $\{T_j\}_{j=1}^n$ is a commuting row contraction, then the commutant of the range of the minimal Stinespring dilation of this restriction is an abelian von Neumann algebra.

Under the conditions of Theorem 3.1, it can be proved that the mapping $\Theta : \{\{T_j\}_{j=1}^n\}' \to \{\lambda(\mathcal{O}_n)\}'$ defined on the commutant of $\{T_j\}_{j=1}^n$ in $\mathcal{B}(\mathcal{H})$ by $\Theta(X) = \pi(XQ)$ for $X \in \{\{T_j\}_{j=1}^n\}'$ is completely contractive and multiplicative and, moreover $X\Gamma = \Gamma\Theta(X)$ for every $X \in \{\{T_j\}_{j=1}^n\}'$. A similar result was proved in [18] in the context of commuting row contractions, and that proof can easily be adapted to the present settings.

Corollary 3.2. *Let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a commuting row contraction such that $\mathcal{F}(T) \neq \{0\}$. Then the Cuntz part $\{R_j\}_{j=1}^n$ of its minimal isometric dilation generates a type I von Neumann algebra.*

Proof. It was proved in [18] that, given a commuting row contraction $\{T_j\}_{j=1}^n$, there exist a commuting family of normal operators $\{U_j\}_{j=1}^n$ on some Hilbert space $\mathcal{G}$ with $\sum_{j=1}^n U_j U_j^* = I_{\mathcal{G}}$ and a surjective, complete isometry

$$\beta : \{\{U_j\}_{j=1}^n\}' \to \mathcal{F}(T)$$

such that $\beta(I_{\mathcal{G}}) = Q$ where Q is as defined in Theorem 3.1. The conclusion now follows from Theorem 3.1 and the result of R.V. Kadison quoted above. □

Corollary 3.3. *Let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a row contraction, and assume there exists an operator $X \in \mathcal{F}(T)$ such that $\|X\|_e < \|X\|$, where $\|\cdot\|_e$ holds for the essential (Calkin) norm. Then either $\{T_j\}_{j=1}^n \subset \mathbb{C}I_{\mathcal{H}}$ or there exists a closed nontrivial subspace of $\mathcal{H}$ invariant for all operators in the commutant of $\{T_j\}_{j=1}^n$.*

Proof. Let $\mathcal{K}(\mathcal{H})$ be the ideal of compact operators on $\mathcal{H}$. Suppose first that the C^*-algebra $C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$ is irreducible. It then follows that it contains $\mathcal{K}(\mathcal{H})$. Let

$$\pi : C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\} \to \{\lambda(\mathcal{O}_n)\}'$$

be the *-homomorphism appearing in Theorem 3.1, and let $J = Ker(\pi)$. If $J \neq \{0\}$ then $J \supset \mathcal{K}(\mathcal{H})$. Since $\{\lambda(\mathcal{O}_n)\}'$ is isometric with $\mathcal{F}(T)$ this implies that the Calkin map is isometric on $\mathcal{F}(T)$ which contradicts the hypothesis. It follows that $J = \{0\}$ that is π is faithful. This shows that $\mathcal{F}(T)$ contains the identity and is closed under multiplication hence it contains $\mathcal{K}(\mathcal{H})$ which means that $\mathcal{F}(T) = \mathcal{B}(\mathcal{H})$. By Lemma 3.3 in [6], this means that every operator T_j is a scalar multiple of the identity. Assume now that $C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$ is not irreducible, and let $\mathcal{H}_0 \subset \mathcal{H}$ be a nontrivial reducing subspace. We may assume that there exists a non-zero operator $X_0 \in \mathcal{F}(T)$ which is not identically zero on $\mathcal{H}_0$. It then follows that the closed linear span of the set $\{AX_0h\colon A \in \{\{T_j\}_{j=1}^n\}',\ h \in \mathcal{H}_0\}$ is a closed nontrivial hyperinvariant subspace for $\{T_j\}_{j=1}^n$. □

The first part of the proof of this corollary is largely inspired from the proof of Theorem 2.1 in [2], see also Remark 2 on p. 288 in the same paper.

4. An asymptotic formula for the lifting

Lemma 4.1. *Let $\mathcal{M}$ be a von Neumann algebra, let $\Psi : \mathcal{M} \to \mathcal{M}$ be a completely positive, contractive and weak* continuous linear mapping, and let $\mathcal{F}(\Psi)$ be its fixed point space. Let $\Phi : \mathcal{M} \to \mathcal{M}$ be an idempotent whose range is $\mathcal{F}(\Psi)$ as in Lemma* 2.1. *Then for every $X \in C^*\{I, \mathcal{F}(\Psi)\}$ we have*

$$\Phi(X) = w^*\text{-}\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \Psi^{(k)}(X).$$

Proof. Let $\mathcal{N} = C^*\{I, \mathcal{F}(\Psi)\}$ and let us consider the sequence of mappings $\Theta_m : \mathcal{N} \to \mathcal{M}$ defined by

$$\Theta_m(X) = \frac{1}{m} \sum_{k=0}^{m-1} \Psi^{(k)}(X), \quad X \in \mathcal{N},\ m \geqslant 1.$$

Consider now the Banach space $\mathcal{B}(\mathcal{N}, \mathcal{M})$ of all bounded linear mappings $L : \mathcal{N} \to \mathcal{M}$. This is also a dual space where the weak* topology (BW-topology) is generated by the seminorms

$$\gamma_{X,\phi}(L) = \big(L(X), \phi\big), \quad L \in \mathcal{B}(\mathcal{N}, \mathcal{M}),\ X \in \mathcal{N},\ \phi \in \mathcal{M}_*$$

(see [1]). It is now easy to see that every BW-limit point $\tilde{\Phi}$ of the sequence $\{\Theta_m\}_{m=1}^\infty$ is a completely positive and contractive mapping with the property that $\Psi(\tilde{\Phi}(X)) = \tilde{\Phi}(X)$ for any $X \in \mathcal{N}$. This shows that $Ran(\tilde{\Phi}) = \mathcal{E}$ and that $\tilde{\Phi} \circ \tilde{\Phi} = \tilde{\Phi}$. Moreover, $\tilde{\Phi}(I) = \Phi(I)$. It then follows from Theorem 2.2 that actually Φ and $\tilde{\Phi}$ coincide on $\mathcal{N}$. □

Corollary 4.2. *Let $\mathcal{M}$, Ψ and Φ be as in Lemma* 4.1. *If $X \circ Y$ denotes the multiplication of $X, Y \in \mathcal{F}(\Psi)$ induced by the complete isometry with a C^*-algebra, then*

$$X \circ Y = w^*\text{-}\lim_{m\to\infty} \Psi^{(m)}(XY).$$

Proof. Let $X \in \mathcal{F}(\Psi)$. Then

$$X^*X = \Psi(X)^*\Psi(X) \leqslant \Psi(X^*X)$$

and, by iterating

$$X^*X \leqslant \Psi(X^*X) \leqslant \cdots \leqslant \Psi^{(m)}(X^*X)$$

for every $m \geqslant 1$. It now follows from Lemma 4.1 that

$$\Phi(X^*X) = w^*\text{-}\lim_{m\to\infty} \Psi^{(m)}(X^*X)$$

and, by polarization, we get the desired conclusion. □

The main result in this section is the following theorem, which extends a similar one for commuting row contractions, proved in [18]. Its proof is very close to the one given in [18]. We include, for completeness, all details.

Theorem 4.3. *Let $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ be a row contraction. Then, in the settings of Theorem* 3.1, *the *-homomorphism*

$$\pi : C^*\big\{I_{\mathcal{H}}, \mathcal{F}(T)\big\} \to \big\{\lambda(\mathcal{O}_n)\big\}'$$

has the following expression:

$$\pi(X) = w^*\text{-}\lim_{m\to\infty} \frac{1}{m} \sum_{j=0}^{m-1} \Psi_R^{(j)}(\Gamma^* X \Gamma), \quad X \in C^*\big\{I_{\mathcal{H}}, \mathcal{F}(T)\big\},$$

where

$$\Psi_R(Y) = \sum_{j=1}^{n} R_j Y R_j^*, \quad Y \in \mathcal{B}(\mathcal{R}),$$

and $R_j = \lambda(S_j)$ *for* $j \in \Lambda_n$.

Proof. Let us denote

$$\Theta_m(X) = \frac{1}{m} \sum_{j=0}^{m-1} \Psi_T^{(j)}(X), \quad X \in C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\},$$

and similarly

$$\Pi_m(Y) = \frac{1}{m} \sum_{j=0}^{m-1} \Psi_R^{(j)}(Y), \quad Y \in \mathcal{B}(\mathcal{R}).$$

Since $T_j \Gamma = \Gamma R_j$ for all $j \in \Lambda_n$ we have

$$\Gamma \Pi_m(Y) \Gamma^* = \Theta_m(\Gamma Y \Gamma^*), \quad Y \in \mathcal{B}(\mathcal{R}).$$

Also recall from Lemma 4.1 that

$$\Phi(X) = w^*\text{-}\lim_{m\to\infty} \Theta_m(X)$$

for every $X \in C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$. Let $X \in C^*\{I_{\mathcal{H}}, \mathcal{F}(T)\}$ and let $Z = \Gamma^* X \Gamma$. Also, let $Y \in \mathcal{B}(\mathcal{R})$ be any weak* limit point of the sequence $\{\Pi_m(Z)\}_{m\geqslant 1}$. It is easy to see that $\Psi_R(Y) = Y$ therefore $Y \in \{\lambda(\mathcal{O}_n)\}'$. Let $\xi, \eta \in \mathcal{H}$. Then, for some subsequence $1 < m_1 < m_2 < \cdots$ we have

$$\begin{aligned}
(\Gamma Y \Gamma^* \xi, \eta) &= \lim_{k\to\infty} \big(\Gamma \Pi_{m_k}(Z) \Gamma^* \xi, \eta\big) \\
&= \lim_{k\to\infty} \big(\Theta_{m_k}(QXQ)\xi, \eta\big) \\
&= \big(\Phi(QXQ)\xi, \eta\big) \\
&= \big(\Gamma \pi(QXQ) \Gamma^* \xi, \eta\big) \\
&= \big(\Gamma \pi(X) \Gamma^* \xi, \eta\big) \\
&= \big(\Phi(X)\xi, \eta\big).
\end{aligned}$$

This shows that $\rho(Y) = \Gamma Y \Gamma^* = \Phi(X)$. Since ρ is one-to-one on $\{\lambda(\mathcal{O}_n)\}'$ and since $\rho(\pi(X)) = \Phi(X)$ (see the remark following Theorem 3.1), it follows that $Y = \pi(X)$. This completes the proof. □

The following corollary follows easily from the previous theorem. An even simpler proof can be given using a compactness argument together with the uniqueness of the lifting. We leave the details to the reader.

Corollary 4.4. *Let* $\{T_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{H})$ *be a row contraction and let* $\{V_j\}_{j=1}^n \subset \mathcal{B}(\mathcal{K})$ *be its minimal isometric dilation. Suppose* $X \in \mathcal{F}(T)$ *and let* $Y \in \mathcal{F}(V)$ *such that* $X = P_{\mathcal{H}} Y|_{\mathcal{H}}$. *Then*

$$Y = w^*\text{-}\lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \Psi_V^{(k)}(J_{\mathcal{H}} X P_{\mathcal{H}})$$

where $J_{\mathcal{H}} : \mathcal{H} \to \mathcal{K}$ *is the canonical inclusion.*

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