Journal of Computational and Applied Mathematics 28 (1989) 105–117
North-Holland

Approximate solution of a flow over a ramp for large Froude number

Mina B. ABD-EL-MALEK and Sarwat N. HANNA
Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria, Egypt

Received 25 July 1988
Revised 14 September 1988

Abstract: An approximate method is presented to solve the problem of steady free-surface flow of an ideal fluid over a semi-infinite ramp in the bottom. Schwartz–Christoffel transformation is used to map the region of flow, in the complex potential-plane, onto the upper half-plane. The Hilbert transformation as well as the perturbation technique are used as a basis for the approximate solution of the problem for large Froude number and small inclination angle of the ramp. General equations, in integral form, for any order of approximation are obtained. Solution up to first-order approximation is discussed and illustrated.

Elevation of the free-surface for different ramp heights, different inclination angles of the ramp and different Froude numbers are plotted. An approximate formula of maximum elevation of the free-surface in terms of the ramp height and its inclination angle is found.

Keywords: Inviscid flow, free-surface flow problems, Schwartz–Christoffel transformation, Hilbert transformation, perturbation technique, nonlinear integral equations.

1. Introduction

This paper considers the problem of steady free-surface of a running two-dimensional, irrotational, inviscid and incompressible flow in an open channel with a nonuniform bottom in a shape of a ramp. This problem has many applications in the fields of coastal engineering and hydraulics. That's why this type of problems has received considerable attention throughout the history of fluid mechanics, during the past 30 years, and the literature on the topic is rich.

One will find an attractive discussion of the subject in the four papers of Thomson (Lord Kelvin) [22] in 1886, Wien [26] in 1900, Lamb [13] in 1945, Wehausen and Laitone [25] in 1960, and Kochin et al. [12] in 1964. Analytical methods fail to find out exact solutions for the problem of steady flow with a free-surface over irregular bottom, therefore numerical methods entered the field intensively to solve this problem within a certain range of accuracy. The first attempts on numerical solution of free-surface flows were made by Southwell and Vaisey [21] in 1946, using relaxation method. Steady flow over a step was successfully treated using the finite-difference method by Bellevaux and Fruman [5] in 1969. More recently the use of the finite-element method has been widespread among the field of fluid mechanics, in particular Aitchison [4] in 1979 was able to solve the difficult problem of a critical flow over a weir.

An alternative to the direct numerical solution of the equations of motion is to map the flow region onto an auxiliary plane with a simpler geometry where the solution is known, as given by Watters and Street [24] in 1964. In this case the problem reduces to a set of simultaneous equations for unknowns in the transformation. Bloor [6] in 1978 and then McIver [16] in 1980 applied that technique in solving problems of large amplitude surface waves and stationary waves in open channels, respectively. Recently, King and Bloor [11], in 1987, modified the transformation applied by Bloor [6] to cope with obstructions to a free-surface flow. In 1981, Forbes [9] investigated the flow over a submerged semi-elliptical body, by allowing the velocity potential and stream function to be independent variables and then the solution is obtained by a boundary integral technique. In 1981, Abd-el-Malek [1] considered the nonlinear problem of a flow over a ramp by applying the Hilbert transformation. In 1982, Forbes and Schwartz [10] considered the case of a flow over a semi-circular obstruction. Recently, Boutros, Abd-el-Malek and Masoud [7] considered the linear problem of a flow over a crump weir, applying the Fourier double integral and in [8] they solved the same problem, but in the nonlinear case using the Hilbert transformation. In 1988, Abd-el-Malek and Masoud [2] applied the Fourier double integral to solve the linear problem of a flow over a ramp.

In this paper, the Hilbert method as well as the perturbation technique are used as a basis for the approximate solution of a two-dimensional problem. First, as customary, the physical and complex potential-planes are mapped onto an auxiliary upper half-plane. Then nonlinear integral equations are found giving the solution of a mixed boundary-value problem. Lastly, several approximations are done based upon large Froude number and the perturbation technique is applied based upon small inclination angles of the ramp.

2. Formulation of the problem

The study is concerned with two-dimensional, irrotational motion, i.e., possessing a velocity potential $\phi(x, y)$ satisfying the equation

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0, \tag{2.1}$$

of a nonviscous and incompressible fluid, subject to the action of gravitational force and having a free upper-surface.

Consider a uniform flow with velocity U_1 in the positive x-direction at depth h_1 flowing over a bottom which consists of a horizontal plane AB, inclined plane BC with an inclination angle α and length L, and a horizontal plane CD, where the bottom extends from $-\infty$ (point A) to $+\infty$ (point D), as shown in Fig. 1.

For convenience, we choose point B to be the origin in the z-plane, the x-axis from left to right and the y-axis upwards. The complex potential

$$W(z) = \phi(x, y) + \mathrm{i}\,\psi(x, y),$$

is an analytic function of z within the region of flow, with complex conjugate velocity

$$\frac{\mathrm{d}W(z)}{\mathrm{d}z} = u(x, y) - \mathrm{i}\,v(x, y) = q\,\mathrm{e}^{-\mathrm{i}\theta}. \tag{2.2}$$

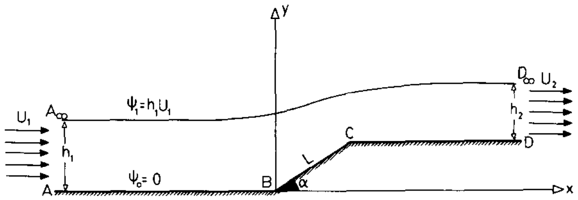

Fig. 1. Physical plane of a flow over a ramp of length L and inclination angle α.

Let the dimensionless variables z', q', and W' be

$$z' = \frac{z}{h_1}, \qquad q' = \frac{q}{U_1}, \qquad W' = \frac{W}{\psi_1}, \tag{2.3}$$

where $\psi_1 = U_1 h_1$.

The condition on the free-surface, where the pressure is uniform, is obtained from Bernoulli's equation. The Bernoulli condition on the free-surface, in dimensionless form, can be written

$$q'^2 + \frac{2}{F^2}(y' - 1) = 1, \tag{2.4}$$

where F is the Froude number defined by

$$F = \frac{U_1}{\sqrt{g h_1}}. \tag{2.5}$$

Denote the dimensionless ramp height by ϵ, where

$$\epsilon = \frac{L \sin \alpha}{h_1} = L' \sin \alpha. \tag{2.6}$$

In dimensionless form, (2.2) becomes

$$\xi = \frac{\mathrm{d}W'}{\mathrm{d}z'} = q' \,\mathrm{e}^{-\mathrm{i}\theta}. \tag{2.7}$$

Let

$$\omega = \ln \xi = \ln q' - \mathrm{i}\theta, \tag{2.8}$$

where ω is called the logarithmic hodograph variable.

Then, from (2.7) and (2.8) we get

$$z' = \int \mathrm{e}^{-\omega} \,\mathrm{d}W'. \tag{2.9}$$

Using the Schwartz–Christoffel transformation, we map the region of flow in the W'-plane onto the upper half of an auxiliary t-plane, so that the following points correspond (see Figs. 2 and 3):

$$\begin{aligned} B \ &: W' = 0, && t = 0, \\ D,\ D_\infty \ &: W' \to +\infty, && t = 1, \\ A,\ A_\infty \ &: W' \to -\infty, && t \to +\infty. \end{aligned} \tag{2.10}$$

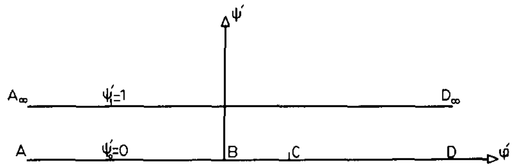

Fig. 2. Normalized complex potential (or W'-)plane.

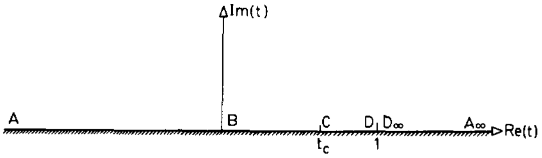

Fig. 3. The auxiliary upper half (or t-)plane.

The mapping function is

$$W'(t) = -\frac{1}{\pi} \ln(1-t), \quad 0 \leqslant \arg(1-t) \leqslant \pi. \tag{2.11}$$

For the second half of the problem, to express ω as a function of the single variable t, we introduce the Hilbert method for a mixed boundary-value problem in the upper half-plane, the general solution of which for an analytic function $Q(t)$ in the upper half-plane, is well known (see [1,3,14,15,17–19]).

Let $Q(t)$ be an analytic function defined in the upper half-plane, the t-plane, and suppose $\operatorname{Im}(Q(t))$ satisfies the Hölder condition on the boundary, $\operatorname{Im}(t) = 0$, of the t-plane.

If $\operatorname{Im}(Q(t))$ is known on the boundary, then $Q(t)$ is given by the Poisson integral formula

$$Q(t) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\operatorname{Im}(Q(\nu))}{\nu - t}\, d\nu + \sum_{j=0}^{\infty} A_j t^j, \tag{2.12}$$

where A_j are real constants, and $⨍$ denotes a singular integral in the sense of Cauchy.

Now, we try to relate the function $\omega(t)$ to function $Q(t)$. From (2.12), we find that $Q(t)$ is expressed in terms of its imaginary parts along the real axis, the boundary of the t-plane.

Thus, we have to examine the value of $\omega(t)$ along the boundary of the t-plane, and we find that

$$\begin{aligned} &\operatorname{Im}(\omega(t)) = -\theta(t), && t<1, \\ &\operatorname{Re}(\omega(t)) = \frac{1}{2} \ln\left(1 - \frac{2}{F^2}\eta'(t)\right), && t>1, \end{aligned} \tag{2.13}$$

where

$$\theta(t) = \begin{cases} 0, & t<0, \\ \alpha, & 0<t<t_C, \\ 0, & t_C<t<1, \end{cases} \tag{2.14}$$

and

$$\eta'(t) = y'(t) - 1. \tag{2.15}$$

This means that we know either the real or the imaginary part of $\omega(t)$ along the boundary of the t-plane. Then we must construct an auxiliary function $H(t)$ which makes the imaginary part of the quotient $Q(t) = \omega(t)/H(t)$ known on the boundary of the t-plane. The general form of $H(t)$ is

$$H(t) = a \prod_j (t - b_j)^{\pm\frac{1}{2}}, \tag{2.16}$$

where b_j are real constants, and $a = \pm\sqrt{\pm 1}$.

Song [20] has shown that the final solution is independent of the particular choice of $H(t)$. We choose

$$H(t) = -\mathrm{i}\sqrt{t-1}, \quad 0 \leqslant \arg(t-1) \leqslant \pi. \tag{2.17}$$

Using (2.13) and (2.17), we obtain

$$\operatorname{Im}(Q(t)) = \begin{cases} \dfrac{\theta(t)}{\sqrt{1-t}}, & t<1, \\ \dfrac{\ln\left(1 - \dfrac{2}{F^2}\eta'(t)\right)}{2\sqrt{t-1}}, & t>1. \end{cases} \tag{2.18}$$

Next, we examine the upstream condition. As we approach point A_∞, i.e., as $t \to +\infty$, $H(t) \simeq -\mathrm{i}\sqrt{t}$ and $\omega(t) \simeq \ln q'(t) = 0$. Therefore, $Q(t) \simeq \omega(t)/(-\mathrm{i}\sqrt{t}) = 0$, and from (2.12), $A_j = 0$ for $j = 0, 1, 2, \ldots, n$. Thus, (2.12) takes the form

$$Q(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Im}(Q(\nu))}{\nu - t}\, \mathrm{d}\nu. \tag{2.19}$$

Using (2.7), we obtain

$$Q(t) = \frac{\omega(t)}{H(t)} = \frac{\ln q'(t) + \mathrm{i}(-\theta(t))}{H(t)} = R(t) + \mathrm{i}S(t). \tag{2.20}$$

An equivalent form of (2.19) is

$$R(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{S(\nu)}{\nu - t}\, \mathrm{d}\nu, \qquad S(t) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{R(\nu)}{\nu - t}\, \mathrm{d}\nu; \tag{2.21}$$

for more details about the Hilbert transformation, see [17,23].

Using (2.21) we obtain a set of integral equations, and for our work we need only the following equations

$$\theta(t) = \frac{\sqrt{t-1}}{\pi} \int_{1}^{\infty} \frac{\ln q'(\nu)}{(\nu - t)\sqrt{\nu - 1}}\, \mathrm{d}\nu + \frac{2\alpha}{\pi} \tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad t>1, \tag{2.22}$$

$$\ln q_B' = -\frac{\sqrt{1-t}}{\pi} \int_{1}^{\infty} \frac{\ln q'(\nu)}{(\nu - t)\sqrt{\nu - 1}}\, \mathrm{d}\nu - \frac{2\alpha}{\pi} \tanh^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad 0<t<t_C, \tag{2.23}$$

where

$$r = 1 - \sqrt{1 - t_C}, \tag{2.24}$$

and $q_B'(t)$ is the flow speed over the ramp.

The coordinates (x', y') of a point on the free-surface can be obtained using (2.9) and (2.11) as follows:

$$z'(t) = (x_\infty' + \mathrm{i}) + \frac{1}{\pi} \int_{\infty}^{t} \frac{\mathrm{e}^{\mathrm{i}\theta(\nu)}}{(1-\nu)q'(\nu)}\, \mathrm{d}\nu, \quad t>1.$$

Separating real and imaginary part, we get

$$x'(t) = x'_\infty + \frac{1}{\pi}\int_\infty^t \frac{\cos\theta(\nu)}{(1-\nu)q'(\nu)}\,\mathrm{d}\nu, \quad t > 1, \tag{2.25}$$

$$y'(t) = 1 + \frac{1}{\pi}\int_\infty^t \frac{\sin\theta(\nu)}{(1-\nu)q'(\nu)}\,\mathrm{d}\nu, \quad t > 1. \tag{2.26}$$

From (2.26) we may find the length of the ramp L', and it has the form (see [1]):

$$L' = \frac{1}{\pi}\int_0^{t_C} \frac{1}{(1-\nu)q'_B(\nu)}\,\mathrm{d}\nu, \tag{2.27}$$

where $q'_B(t)$ is given by (2.23).

Hence, the system of equations (2.4), (2.22), (2.23), (2.25), (2.26), and (2.27) describes completely our problem.

3. The approximate equations

For large Froude number F and for small inclination angle α, in which case the change in $\theta(t)$ will be very small, we can approximate sin $\theta(t)$ by $\theta(t)$ and cos $\theta(t)$ by one. Having done so, we proceed to drop the primes and the system of equations takes the form:

$$q(t) \simeq 1 - \frac{1}{F^2}\eta(t), \quad t > 1, \tag{3.1}$$

$$\eta(t) \simeq -\frac{1}{\pi}\int_t^\infty \frac{\theta(\nu)}{1-\nu}\left(1 + \frac{1}{F^2}\eta(\nu)\right)\mathrm{d}\nu, \quad t > 1, \tag{3.2}$$

$$\theta(t) \simeq -\frac{\sqrt{t-1}}{\pi F^2} \int_1^\infty \frac{\eta(\nu)}{(\nu - t)\sqrt{\nu - 1}}\,\mathrm{d}\nu + \frac{2\alpha}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad t > 1, \tag{3.3}$$

$$x(t) \simeq x_\infty - \frac{1}{\pi}\int_t^\infty \frac{1}{1-\nu}\left(1 + \frac{1}{F^2}\eta(\nu)\right)\mathrm{d}\nu, \quad t > 1, \tag{3.4}$$

$$\ln q_B(t) \simeq \frac{\sqrt{1-t}}{\pi F^2}\int_1^\infty \frac{\eta(\nu)}{(\nu - t)\sqrt{\nu - 1}}\,\mathrm{d}\nu - \frac{2\alpha}{\pi}\tanh^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad 0 < t < t_C, \tag{3.5}$$

$$L \simeq \frac{1}{\pi}\int_0^{t_C} \frac{1}{(1-\nu)q_B(\nu)}\,\mathrm{d}\nu. \tag{3.6}$$

4. Perturbation technique

Expand $q(t)$, $\eta(t)$, $\theta(t)$ and $x(t)$ in terms of the small parameter α,

$$q(t) = q_0(t) + \alpha q_1(t) + \alpha^2 q_2(t) + \cdots, \tag{4.1}$$

$$\eta(t) = \eta_0(t) + \alpha\eta_1(t) + \alpha^2\eta_2(t) + \cdots, \tag{4.2}$$

$$\theta(t) = \theta_0(t) + \alpha\theta_1(t) + \alpha^2\theta_2(t) + \cdots, \tag{4.3}$$

$$x(t) = x_0(t) + \alpha x_1(t) + \alpha^2 x_2(t) + \cdots. \tag{4.4}$$

Upon employing the expansions (4.1)–(4.4) into the equations (3.1)–(3.4) and equating terms of similar power of α, we get the:

(i) Zero-order approximation

$$q_0(t) = 1 - \frac{1}{F^2}\eta_0(t), \tag{4.5}$$

$$\eta_0(t) = -\frac{1}{\pi}\int_t^\infty \frac{\theta_0(\nu)}{1-\nu}\left(1 + \frac{1}{F^2}\eta_0(\nu)\right) d\nu, \quad t > 1, \tag{4.6}$$

$$\theta_0(t) = -\frac{\sqrt{t-1}}{\pi F^2}\int_1^\infty \frac{\eta_0(\nu)}{(\nu - t)\sqrt{\nu - 1}} d\nu, \quad t > 1, \tag{4.7}$$

$$x_0(t) = x_\infty - \frac{1}{\pi}\int_t^\infty \frac{1}{1-\nu}\left(1 + \frac{1}{F^2}\eta_0(\nu)\right) d\nu, \quad t > 1. \tag{4.8}$$

This zero-order approximation corresponds to the case of flow over a flat bottom. Solution of the zero-order approximation is

$$q_0(t) = 1, \tag{4.9}$$

$$\eta_0(t) = 0, \tag{4.10}$$

$$\theta_0(t) = 0, \tag{4.11}$$

$$x_0(t) = x_\infty - \frac{1}{\pi}\int_t^\infty \frac{d\nu}{1-\nu}.$$

Writing

$$x_\infty = \frac{1}{\pi}\int_0^\infty \frac{d\nu}{1-\nu},$$

hence

$$x_0 = -\frac{1}{\pi}\ln(t-1). \tag{4.12}$$

(ii) First-order approximation

$$q_1(t) = -\frac{1}{F^2}\eta_1(t), \tag{4.13}$$

$$\eta_1(t) = -\frac{1}{\pi}\int_t^\infty \frac{\theta_1(\nu)}{1-\nu} d\nu, \tag{4.14}$$

$$\theta_1(t) = -\frac{\sqrt{t-1}}{\pi F^2}\int_1^\infty \frac{\eta_1(\nu)}{(\nu - t)\sqrt{\nu-1}} d\nu + \frac{2}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \tag{4.15}$$

$$x_1(t) = -\frac{1}{\pi F^2}\int_t^\infty \frac{\eta_1(\nu)}{1-\nu} d\nu. \tag{4.16}$$

(iii) In general, nth-order approximation

$$\begin{aligned} q_n(t) &= -\frac{1}{F^2}\eta_n(t), \\ \eta_n(t) &= -\frac{1}{\pi}\int_t^\infty \frac{1}{1-\nu}\left(\theta_n(\nu) + \frac{1}{F^2}\sum_{j=1}^{n-1}\theta_j(\nu)\eta_{n-j}(\nu)\right)\mathrm{d}\nu, \\ \theta_n(t) &= -\frac{\sqrt{t-1}}{\pi F^2}\int_1^\infty \frac{\eta_n(\nu)}{(\nu-t)\sqrt{\nu-1}}\,\mathrm{d}\nu, \\ x_n(t) &= -\frac{1}{\pi F^2}\int_t^\infty \frac{\eta_n(\nu)}{1-\nu}\,\mathrm{d}\nu, \end{aligned} \tag{4.17}$$

where $n = 2, 3, 4, \ldots$.

5. Solution of the problem

Solution of the first-order approximation: from (4.15), for very large Froude number F, we may neglect the first term with respect to the second one and one gets

$$\theta_1(t) \simeq \frac{2}{\pi}\tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad t > 1. \tag{5.1}$$

Upon substituting in (4.14) and carrying out integration, one finds

$$\eta_1(t) \simeq \frac{4r}{\pi^2\sqrt{1-r}}\tan^{-1}\sqrt{\frac{1-r}{t-1}}, \quad t > 1. \tag{5.2}$$

Hence $q_1(t)$ takes the form, from (4.13)

$$q_1(t) \simeq -\frac{4r}{\pi^2\sqrt{1-r}\,F^2}\tan^{-1}\sqrt{\frac{1-r}{t-1}}, \quad t > 1, \tag{5.3}$$

and

$$x_1(t) \simeq \frac{8r}{\pi^3F^2\sqrt{t-1}}, \quad t > 1. \tag{5.4}$$

For very large Froude number F, the first term in (3.5) may be neglected with respect to the second one and we get

$$\ln q_B(t) \simeq -\frac{2\alpha}{\pi}\tanh^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right), \quad 0 < t_C < 1, \tag{5.5}$$

from which we can find an approximate expression for the length of the ramp, using (3.6),

$$L \simeq -\frac{1}{\pi}\ln(1-t_C), \quad 0 < t_C < 1. \tag{5.6}$$

For small inclination angle α, we will consider only an approximate solution to an order of α, and we have

$$q(t) \simeq 1 - \alpha \frac{4r}{\pi^2 F^2 \sqrt{1-r}} \tan^{-1}\sqrt{\frac{1-r}{t-1}} + o(\alpha), \quad t > 1, \tag{5.7}$$

$$y(t) \simeq 1 + \alpha \frac{4r}{\pi^2 \sqrt{1-r}} \tan^{-1}\sqrt{\frac{1-r}{t-1}} + o(\alpha), \quad t > 1, \tag{5.8}$$

$$\theta(t) \simeq \frac{2\alpha}{\pi} \tan^{-1}\left(\frac{r\sqrt{t-1}}{t-r}\right) + o(\alpha), \quad t > 1, \tag{5.9}$$

$$x(t) \simeq -\frac{1}{\pi} \ln(t-1) + \alpha \frac{8r}{\pi^3 F^2 \sqrt{t-1}} + o(\alpha), \quad t > 1. \tag{5.10}$$

One quantity of physical interest is the drag force on the ramp caused by the fluid flow, which is the horizontal component of the force acting on unit width of the ramp. In fact this drag force represents the change in momentum flux due to the change in the stream produced by the ramp. The drag force D made dimensionless by reference to the quantity $\rho g h_1^2$, where ρ is the fluid density, is given by

$$D = \frac{1}{2} F^2 \int_{y=0}^{\epsilon} (1 - q^2)\, \mathrm{d}y, \tag{5.11}$$

where ϵ is given by (2.6). Equation (5.11) can be integrated most conveniently by using (5.7) and (5.8), we get

$$D = \frac{8\alpha \tan\alpha}{\pi^3} (\mathrm{e}^{\pi\epsilon \cot\frac{1}{2}\alpha} - 1)(1 - \mathrm{e}^{-\pi\epsilon/2 \sin\alpha}). \tag{5.12}$$

6. Numerical results and discussion

Approximate solutions for flows with large Froude number are found throughout a wide range of ramp heights. Results showing the free-surface elevation over a range of ramp heights ϵ, 0.002567 to 0.035513, for fixed $\alpha = \frac{1}{6}\pi$, $F^2 = 10.0$ and over a range of inclination angles α, $\frac{1}{12}\pi$ to $\frac{1}{6}\pi$, for fixed $\epsilon = 0.005012$, $F^2 = 10.0$ are given in Figs. 4 and 5.

Effect of the Froude number F on the free-surface elevation for fixed $\alpha = \frac{1}{6}\pi$ and $\epsilon = 0.005012$ is given in Fig. 6.

These results demonstrate the monotonic rise in free-surface elevation with increase of the ramp height, the inclination angle, and the Froude number.

From Figs. 4 and 5 we are able to give an approximate formula for maximum elevation in terms of the ramp height and the inclination angle, namely

$$\frac{y_{\max}}{h_1} \simeq 1.0 + 1.045807\,\epsilon, \quad \text{for } F^2 = 10.0 \text{ and } \alpha = \frac{\pi}{6}, \tag{6.1}$$

and

$$\frac{y_{\max}}{h_1} \simeq 1.0 + 0.049636\,\alpha, \quad \text{for } F^2 = 10.0 \text{ and } \epsilon = 0.005012; \tag{6.2}$$

they are plotted in Figs. 7 and 8, respectively.

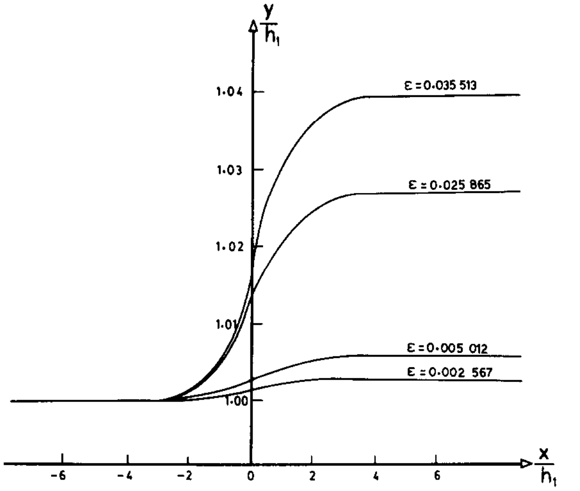

Fig. 4. Effect of the ramp height ϵ on the free-surface profile for $\alpha = \frac{1}{6}\pi$ and $F^2 = 10.0$.

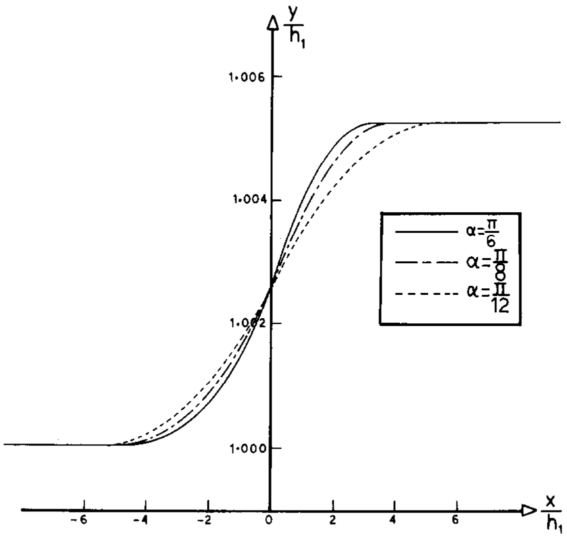

Fig. 5. Effect of the inclination angle α of the ramp on the free-surface profile for $\epsilon = 0.005012$ and $F^2 = 10.0$.

The drag force D, as shown in Fig. 9 and given in Table 1, increases monotonically with ramp height for different values of the inclination angle.

Forbes and Schwartz [10] considering the case of supercritical flows over a semi-circular obstruction with an otherwise flat bottom, concluded that the limiting form of the free-surface profile would have a sharp 120°-corner at its peak. A similar conclusion could be made with regard to the flow over a ramp. Behaviour of curves, in Fig. 4, for different ϵ, support this idea.

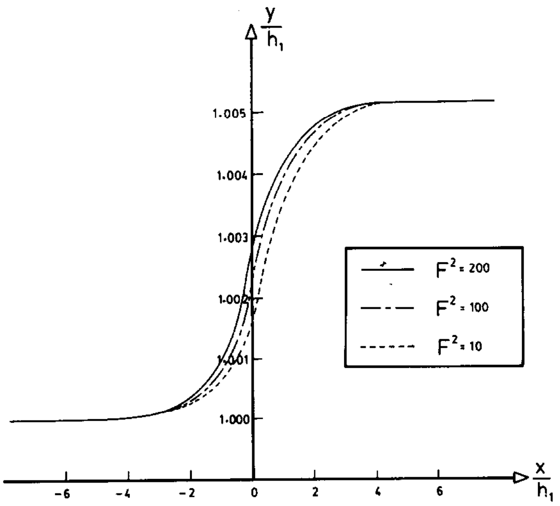

Fig. 6. Effect of the Froude number on the free-surface profile for $\alpha = \frac{1}{6}\pi$ and $\epsilon = 0.005012$.

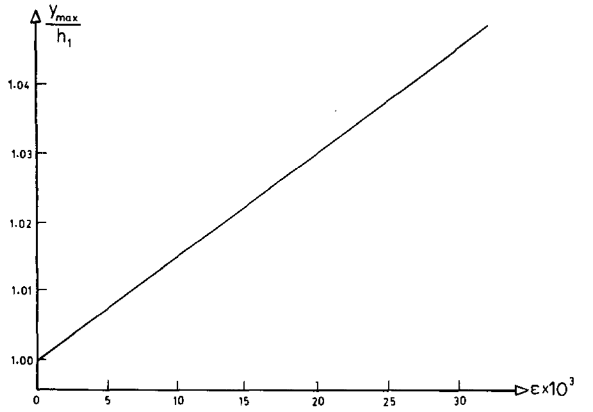

Fig. 7. Normalized maximum free-surface height against the ramp height ϵ for $\alpha = \frac{1}{6}\pi$ and $F^2 = 10.0$.

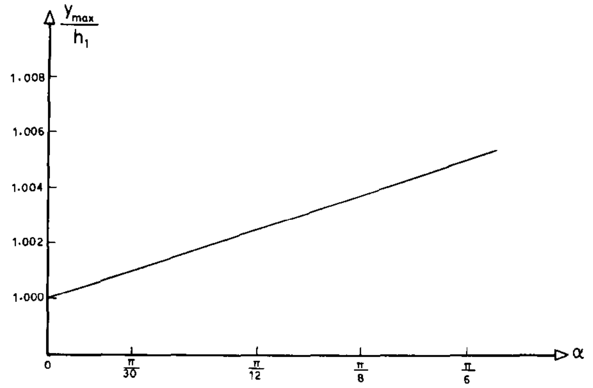

Fig. 8. Normalized maximum free-surface height against the inclination angle α of the ramp for $\epsilon = 0.005012$ and $F^2 = 10.0$.

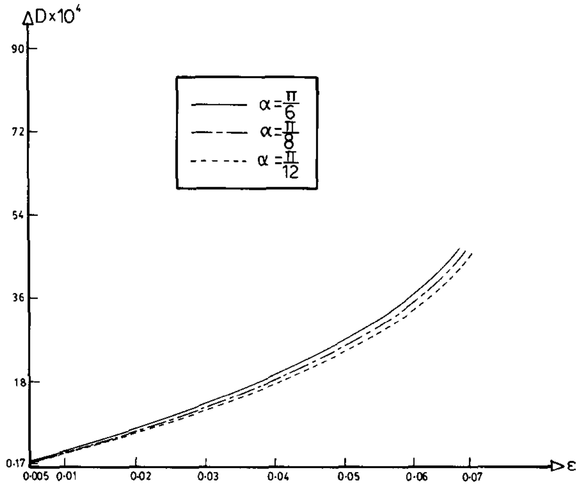

Fig. 9. The drag force D on the ramp as a function of the ramp height ϵ for different values of the inclination angle α.

Table 1
Effect of the ramp height on the drag force D for different inclination angles α

ϵ	$\alpha = \frac{1}{6}\pi$	$\alpha = \frac{1}{8}\pi$	$\alpha = \frac{1}{9}\pi$
0.005000	0.000017	0.000016	0.000016
0.025000	0.000415	0.000407	0.000405
0.050000	0.001652	0.001625	0.001619
0.060000	0.002375	0.002340	0.002333
0.070000	0.003229	0.003185	0.003177
0.080000	0.004213	0.004162	0.004155
0.090000	0.005328	0.005271	0.005266
0.100000	0.006572	0.006513	0.006512

Acknowledgements

One of the authors, namely Dr. Mina B. Abd-el-Malek, would like to express his gratitude to Dr. A.C. Smith, Professor of Applied Mathematics, University of Windsor, Ontario, Canada, for suggesting the problem, in the nonlinear solution, while supervising the author's doctoral study. Also sincere thanks to Dr. D.MacDonald, Department of Applied Mathematics and Theoretical Physics, University of Liverpool, England, for the valuable discussions, while visiting the University.

References

[1] M.B. Abd-el-Malek, Boundary integral methods and free surface problems, Ph.D. dissertation, University of Windsor, Windsor, Ontario, Canada, 1981.

[2] M.B. Abd-el-Malek and S.Z. Masoud, Linearized solution of a flow over a ramp, *Appl. Math. Modelling* **12** (1988) 406–410.

[3] K.M. Agrawal, Non-linear theories in two-dimensional fully and partially cavitating flow, Ph.D. dissertation, University of Windsor, Windsor, Ontario, Canada, 1969.

[4] J.M. Aitchison, A variable finite element method for the calculation of flow over a weir, Rutherford Laboratory Report, RL-79-069, 1979.

[5] C. Bellevaux and D. Fruman, Surfaces libers non-lineaires dans les canaux decouverts, *La Houille Blanche* **24** (1969) 119–122.

[6] M.I.G. Bloor, Large amplitude surface waves, *J. Fluid Mech.* **84** (1) (1978) 167–179.

[7] Y.Z. Boutros, M.B. Abd-el-Malek and S.Z. Masoud, Linearized solution of a flow over a nonuniform bottom, *J. Comput. Appl. Math.* **16** (1986) 105–116.

[8] Y.Z. Boutros, M.B. Abd-el-Malek and S.Z. Masoud, Hilbert's method for numerical solution of flow from a uniform channel over irregular bottom topographies, *Comput. Methods Appl. Mech. Engrg.* **65** (1987) 215–228.

[9] L.K. Forbes, On the wave resistance of a submerged semi-elliptical body, *J. Engrg. Math.* **15** (1981) 287–198.

[10] L.K. Forbes and L.M. Schwartz, Free-surface flow over a semi-circular obstruction, *Fluid Mech.* **114** (1982) 299–314.

[11] A.C. King and M.I.G. Bloor, Free-surface flow over a step, *J. Fluid Mech.* **184** (1987) 193–208.

[12] N.E. Kochin, I.A. Kibel and N.V. Roze, *Theoretical Hydrodynamics* (Wiley/Interscience, New York, 1964).

[13] H. Lamb, *Hydrodynamics* (Dover, New York, 6th ed., 1945).

[14] B.E. Larock and R.L. Street, A Riemann–Hilbert problem for non-linear fully cavitating flow, *J. Ship Research* **9** (3) (1965) 170–178.

[15] T.H. Lim, Integral equation methods for the numerical solution of free surface problems in inviscid flow, Ph.D. dissertation, University of Windsor, Windsor, Ontario, Canada, 1978.

[16] P. McIver, Stationary waves in open channels, Ph.D. thesis, University of Liverpool, Liverpool, United Kingdom, 1980.

[17] N.I. Muskhelishvili, *Singular Integral Equations* (Noordhoff, Groningen, The Netherlands, 1953).

[18] A.C. Smith and M.B. Abd-el-Malek, Hilbert's methods for numerical solution of flow from a uniform channel over a shelf, *J. Engrg. Math.* **17** (1983) 27–39.

[19] A.C. Smith and T.H. Lim, The steady water wave: A numerical solution using the Riemann–Hilbert method, *Internat. J. Engrg. Sci.* **18** (1980) 139–152.

[20] S.C. Song, A quasi-linear and linear for non-separated and separated two-dimensional irrotational flow about lifting bodies, Univ. Minnesota, Min. SAF Lab. Tech. Paper, B-43, 1963.

[21] R.V. Southwell and G. Vaisey, Relaxation methods applied to engineering problems. XII: Fluid motions characterised by free streamlines, *Philos. Trans. Roy. Soc. London Ser. A* **240** (1946) 117–161.

[22] W. Thomson, On stationary waves in flowing water, *Phil. Mag.* **5** (xxii) (1886) pp. 353, 454, 516 and (xxiii) (1887) p. 52.

[23] F.G. Tricomi, *Integral Equations* (Interscience, New York, 1957).
[24] G.Z. Watters and R.L. Street, Two-dimensional flow over sills in open channels, *Proc. ASCE* **90** (HY4) (1964) 107–140.
[25] J.V. Wehausen and E.V. Laitone, Surface waves, *Handbuch der Physik, Vol. 9* (Springer, Berlin, 1960).
[26] W. Wien, *Lehrbuch der Hydrodynamik* (Hirzel, Leipzig, 1900).