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Transference principles for semigroups and a theorem of Peller

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Abstract

A general approach to transference principles for discrete and continuous operator (semi)groups is described. This allows one to recover the classical transference results of Calderón, Coifman and Weiss and of Berkson, Gillespie and Muhly and the more recent one of the author. The method is applied to derive a new transference principle for (discrete and continuous) operator semigroups that need not be groups. As an application, functional calculus estimates for bounded operators with at most polynomially growing powers are derived, leading to a new proof of classical results by Peller from 1982. The method allows for a generalization of his results away from Hilbert spaces to L^p-spaces and—involving the concept of γ-boundedness—to general Banach spaces. Analogous results for strongly-continuous one-parameter (semi)groups are presented as well. Finally, an application is given to singular integrals for one-parameter semigroups.

Keywords: Transference; Operator semigroup; Functional calculus; Analytic Besov space; Peller; γ-boundedness; γ-radonifying; γ-summing; Power-bounded operator

1. Introduction and summary

The purpose of this article is twofold. The shorter part (Section 2) is devoted to a generalization of the classical transference principle of Calderón, Coifman and Weiss. In the major part (Sections 3–7) we give applications of this new abstract result to discrete and continuous operator (semi)groups; in particular we shall recover and generalize important results of Peller [26].

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In the classical transference principle(s) the objects under investigation are derived operators of the form

$$\mathrm{T}_\mu := \int_G T(s)\,\mu(\mathrm{d}s) \tag{1.1}$$

where G is a locally compact group and $\mathrm{T} = (T(s))_{s\in G} : G \to \mathcal{L}(X)$ is a bounded strongly continuous representation of G on a Banach space X. The integral (1.1) has to be understood in the strong sense, i.e.,

$$\mathrm{T}_\mu x = \int_G T(s)x\,\mu(\mathrm{d}s) \quad (x \in X),$$

and μ is a scalar measure that renders the expression meaningful. Since such operators occur in a variety of situations, the applications of transference principles are manifold, and the literature on this topic is vast. We therefore restrict ourselves to mentioning only a few 'landmarks' which we regard as most important for the understanding of the present paper.

Originally, Calderón [6] considered representations on L^p induced by a G-flow of measure-preserving transformations of the underlying measure space. His considerations were motivated by ergodic theory and his aim was to obtain maximal inequalities. Subsequently, Coifman and Weiss [7,8] shifted the focus to norm estimates and were able to drop Calderón's assumption of an underlying measure-preserving G-flow towards general G-representations on L^p-spaces. Some years later, Berkson, Gillespie and Muhly [2] were able to generalize the method towards general Banach spaces X. However, the representations considered in these works were still (uniformly) bounded. In the continuous one-parameter case (i.e., $G = \mathbb{R}$) Blower [3] showed that the original proof method could fruitfully be applied also to non-bounded representations. However, his result was in a sense 'local' and did not take into account the growth rate of the group $(T(s))_{s\in\mathbb{R}}$ at infinity. In [14] we re-discovered Blower's result and in [16] we could refine it towards a 'global' transference result for strongly continuous one-parameter groups, cf. also Section 3 below.

In the present paper, more precisely in Section 2, we develop a *method of generating transference results* and show in Section 3 that the known transference principles (the classical Berkson–Gillespie–Muhly result and the central results of [16]) are special instances of it. Our method has three important new features. Firstly, it allows to pass from *groups* (until now the standard assumption) to *semigroups*. More precisely, we consider closed sub-semigroups S of a locally compact group G together with a strongly continuous representation $\mathrm{T} : S \to \mathcal{L}(X)$ on a Banach space, and try to estimate the norms of operators of the form

$$\mathrm{T}_\mu = \int_S T(s)\,\mu(\mathrm{d}s) \tag{1.2}$$

by means of the transference method. The second feature is the role of *weights* in the transference procedure, somehow hidden in the classical version. Thirdly, our account brings to light the *formal structure* of the transference argument: in a first step one establishes a *factorization* of the operator (1.2) over a convolution (i.e., Fourier multiplier) operator on a space of X-valued functions on G; then, in a second step, one uses this factorization to estimate the norms; and finally, one may vary the parameters to optimize the obtained inequalities. So one can briefly subsume our method under the scheme

factorize—estimate—optimize,

where we use one particular way of constructing the initial factorization. One reason for the power of the method lies in choosing different weights in the factorization, allowing for the optimization in the last step. The second reason lies in the purely formal nature of the factorization; this allows to re-interpret the same factorization involving different function spaces.

The second part of the paper (Sections 4–7) is devoted to applications of the transference method. These applications deal exclusively with the cases $S = \mathbb{Z}, \mathbb{Z}_+$ and $S = \mathbb{R}, \mathbb{R}_+$, which we for short call the *discrete* and the *continuous* case, respectively. However, let us point out that the general transference method of Section 2 works even for sub-semigroups of *non-abelian* groups.

To clarify what kind of applications we have in mind, let us look at the discrete case first. Here the semigroup consists of the powers $(T^n)_{n\in\mathbb{N}_0}$ of one single bounded operator T, and the derived operators (1.2) take the form

$$\sum_{n\geqslant 0} \alpha_n T^n$$

for some (complex) scalar sequence $\alpha = (\alpha_n)_{n\geqslant 0}$. In order to avoid convergence questions, we suppose that α is a finite sequence, hence

$$\widehat{\alpha}(z) := \sum_{n\geqslant 0} \alpha_n z^n$$

is a complex polynomial. One usually writes

$$\widehat{\alpha}(T) := \sum_{n\geqslant 0} \alpha_n T^n$$

and is interested in continuity properties of the *functional calculus*

$$\mathbb{C}[z] \to \mathcal{L}(X), \quad f \mapsto f(T).$$

That is, one looks for a function algebra norm $\|\cdot\|_{\mathcal{A}}$ on $\mathbb{C}[z]$ that allows an estimate of the form

$$\big\|f(T)\big\| \lesssim \|f\|_{\mathcal{A}} \quad \big(f \in \mathbb{C}[z]\big). \tag{1.3}$$

(The symbol $\lesssim$ is short for $\leqslant C\cdot$ for some unspecified constant $C \geqslant 0$, see also the "Terminology"-paragraph at the end of this introduction.) A rather trivial instance of (1.3) is based on the estimate

$$\big\|f(T)\big\| = \bigg\|\sum_{n\geqslant 0} \alpha_n T^n\bigg\| \leqslant \sum_{n\geqslant 0} |\alpha_n| \big\|T^n\big\|.$$

Defining the positive sequence $\omega = (\omega_n)_n$ by $\omega_n := \|T^n\|$, we hence have

$$\|f(T)\| \leqslant \|f\|_\omega := \sum_{n\geqslant 0} |\alpha_n|\omega_n \tag{1.4}$$

and by the submultiplicativity $\omega_{n+m} \leqslant \omega_n\omega_m$ one sees that $\|\cdot\|_\omega$ is a function algebra (semi)norm on $\mathbb{C}[z]$.

The "functional calculus" given by (1.4) is tailored to the operator T and uses no other information than the growth of the powers of T. The central question now is: under which conditions can one obtain better estimates for $\|f(T)\|$, i.e., in terms of weaker function norms? The conditions we have in mind may involve T (or better: the semigroup $(T^n)_{n\geqslant 0}$) or the underlying Banach space. To recall a famous example: *von Neumann's inequality* [25] states that if $X = H$ is a Hilbert space and $\|T\| \leqslant 1$ (i.e., T is a contraction), then

$$\|f(T)\| \leqslant \|f\|_\infty \quad \text{for every } f \in \mathbb{C}[z], \tag{1.5}$$

where $\|f\|_\infty$ is the norm of f in the Banach algebra $\mathcal{A} = \mathrm{H}^\infty(\mathbb{D})$ of bounded analytic functions on the open unit disc $\mathbb{D}$.

Von Neumann's result is optimal in the trivial sense that the estimate (1.5) of course implies that T is a contraction, but also in the sense that one cannot improve the estimate without further conditions: If $H = \mathrm{L}^2(\mathbb{D})$ and $(Th)(z) = zh(z)$ is multiplication with the complex coordinate, then $\|f(T)\| = \|f\|_\infty$ for any $f \in \mathbb{C}[z]$. A natural question then is to ask which operators satisfy the slightly weaker estimate

$$\|f(T)\| \lesssim \|f\|_\infty \quad \big(f \in \mathbb{C}[z]\big)$$

(called "polynomial boundedness of T"). On a general Banach space this may fail even for a contraction: simply take $X = \ell^1(\mathbb{Z})$ and T the shift operator, given by $(Tx)_n = x_{n+1}$, $n \in \mathbb{Z}$, $x \in \ell^1(\mathbb{Z})$. On the other hand, Lebow [20] has shown that even on a Hilbert space polynomial boundedness of an operator T may fail if it is only assumed to be *power-bounded*, i.e., if one has merely $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$ instead of $\|T\| \leqslant 1$. The class of power-bounded operators on Hilbert spaces is notoriously enigmatic, and it can be considered one of the most important problems in operator theory to find good functional calculus estimates for this class.

Let us shortly comment on the continuous case. Here one is given a strongly continuous (in short: C_0-)semigroup $(T(s))_{s\geqslant 0}$ of operators on a Banach space X, and one considers integrals of the form

$$\int_{\mathbb{R}_+} T(s)\,\mu(\mathrm{d}s), \tag{1.6}$$

where we assume for simplicity that the support of the measure μ is bounded. We shall use only basic results from semigroup theory, and refer to [1,10] for further information. The *generator* of the semigroup $(T(s))_{s\geqslant 0}$ is a closed and in general unbounded operator $-A$ satisfying

$$(\lambda + A)^{-1} = \int_0^\infty e^{-\lambda s} T(s)\,\mathrm{d}s \tag{1.7}$$

for $\operatorname{Re}\lambda$ large enough. The generator is densely defined, i.e., its domain $\operatorname{dom}(A)$ is dense in X. In this paper we exclusively deal with semigroups satisfying a polynomial growth $\|T(s)\| \lesssim (1+s)^\alpha$ for some $\alpha \geqslant 0$, and hence (1.7) holds at least for all $\operatorname{Re}\lambda > 0$. One writes $T(s) = e^{-sA}$ for $s \geqslant 0$ and, more generally,

$$(\mathcal{L}\mu)(A) := \int_{\mathbb{R}_+} T(s)\,\mu(\mathrm{d}s)$$

where

$$(\mathcal{L}\mu)(z) := \int_{\mathbb{R}_+} e^{-zs}\,\mu(\mathrm{d}s)$$

is the *Laplace transform* of μ. So in the continuous case the Laplace transform takes the role of the Taylor series in the discrete case. Asking for good estimates for operators of the form (1.6) is as asking for *functional calculus estimates* for the operator A. The continuous version of von Neumann's inequality states that if $X = H$ is a Hilbert space and if $\|T(s)\| \leqslant 1$ for all $s \geqslant 0$ (i.e., if T is a *contraction semigroup*), then

$$\|f(A)\| \leqslant \|f\|_\infty \quad (f = \mathcal{L}\mu)$$

where $\|f\|_\infty$ is the norm of f in the Banach algebra $\mathrm{H}^\infty(\mathbb{C}_+)$ of bounded analytic functions on the open half-plane $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$, see [12, Theorem 7.1.7].

There are similarities in the discrete and in the continuous case, but also characteristic differences. The discrete case is usually a little more general, shows more irregularities, and often it is possible to transfer results from the discrete to the continuous case. (However, this may become quite technical, and we prefer direct proofs in the continuous case whenever possible.) In the continuous case, the role of power-bounded operators is played by *bounded semigroups*, and similar to the discrete case, the class of bounded semigroups on Hilbert spaces appears to be rather enigmatic. In particular, there is a continuous analogue of Lebow's result due to Le Merdy [21], cf. also [12, Section 9.1.3]. And there remain some notorious open questions involving the functional calculus, e.g., the power-boundedness of the Cayley transform of the generator, cf. [9] and the references therein.

The strongest results in the discrete case obtained so far can be found in the remarkable paper [26] by Peller from 1982. One of Peller's results is that if T is a power-bounded operator on a Hilbert space H, then

$$\|f(T)\| \lesssim \|f\|_{\mathrm{B}^0_{\infty,1}} \quad \bigl(f \in \mathbb{C}[z]\bigr)$$

where $\mathrm{B}^0_{\infty,1}(\mathbb{D})$ is the so-called *analytic Besov algebra* on the disc. (See Section 5 below for a precise definition.)

In 2005, Vitse [27] made a major advance in showing that Peller's Besov class estimate still holds true on general Banach spaces if the power-bounded operator T is actually of Tadmor–Ritt type, i.e., satisfies the “analyticity condition”

$$\sup_{n\geqslant 0}\left\|n\left(T^{n+1}-T^{n}\right)\right\|<\infty.$$

She moreover established in [28] an analogue for strongly continuous bounded *analytic* semigroups. Whereas Peller’s results rest on Grothendieck’s inequality (and hence are particular to Hilbert spaces) Vitse’s approach is based on repeated summation/integration by parts, possible because of the analyticity assumption.

In the present paper we shall complement Vitse’s result by devising an entirely new approach, using our transference methods (Sections 4 and 5). In doing so, we avoid Grothendieck’s inequality and reduce the problem to certain Fourier multipliers on vector-valued function spaces. By Plancherel’s identity, on Hilbert spaces these are convenient to estimate, but one can still obtain positive results on $\mathrm{L^p}$-spaces or on UMD spaces. Our approach works simultaneously in the discrete and in the continuous case, and hence we do not only recover Peller’s original result (Theorem 5.1) but also establish a complete continuous analogue (Theorem 5.3), conjectured in [28]. Moreover, we establish an analogue of the Besov-type estimates for $\mathrm{L^p}$-spaces and for UMD spaces (Theorem 5.7). These results, however, are less satisfactory since the algebras of Fourier multipliers on the spaces $\mathrm{L}^2(\mathbb{R};X)$ and $\mathrm{L}^2(\mathbb{Z};X)$ are not thoroughly understood if X is not a Hilbert space.

In Section 6 we show how our transference methods can also be used to obtain “γ-versions” of the Hilbert space results. The central notion here is the so-called γ-boundedness of an operator family, a strengthening of operator norm boundedness. It is related to the notion of R-boundedness and plays a major role in Kalton and Weis’ work [19] on the H^∞-calculus. The ‘philosophy’ behind this theory is that to each Hilbert space result based on Plancherel’s theorem there is a corresponding Banach space version, when operator norm boundedness is replaced by γ-boundedness.

We give evidence to this philosophy by showing how our transference results enable one to prove γ-versions of functional calculus estimates on Hilbert spaces. As examples, we recover the γ-version of a result of Boyadzhiev and deLaubenfels, first proved by Kalton and Weis in [19] (Theorem 6.5). Then we derive γ-versions of the Besov calculus theorems in both the discrete and the continuous forms. The simple idea consists of going back to the original factorization in the transference method, but exchanging the function spaces on which the Fourier multiplier operators act from an L^2-space into a γ-space. This idea is implicit in the original proof from [19] and has been employed independently of us recently by Le Merdy [22].

Finally, in Section 7 we discuss consequences of our estimates for full functional calculi and singular integrals for discrete and continuous semigroups. For instance, we prove in Theorem 7.1 that if $(T(s))_{s\geqslant 0}$ is any strongly continuous semigroup on a UMD space X, then for any $0<a<b$ the principal value integral

$$\lim_{\epsilon\searrow 0}\int\limits_{\epsilon<|s-b|<a}\frac{T(s)x}{s-b}$$

exists for all $x\in X$. For C_0-*groups* this is well known, cf. [14], but for semigroups which are not groups, this is entirely new.

Terminology. We use the common symbols $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ for the sets of natural, integer, real and complex numbers. In our understanding 0 is not a natural number, and we write

$$\mathbb{Z}_+ := \{n \in \mathbb{Z} \mid n \geqslant 0\} = \mathbb{N} \cup \{0\} \quad \text{and} \quad \mathbb{R}_+ := \{t \in \mathbb{R} \mid t \geqslant 0\}.$$

Moreover, $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ is the open unit disc, $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ is the torus, and $\mathbb{C}_+ := \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ is the open right half-plane.

We use X, Y, Z to denote (complex) Banach spaces, and A, B, C to denote closed possibly unbounded operators on them. By $\mathcal{L}(X)$ we denote the Banach algebra of all bounded linear operators on the Banach space X, endowed with the ordinary operator norm. The domain, kernel and range of an operator A are denoted by $\operatorname{dom}(A)$, $\ker(A)$ and $\operatorname{ran}(A)$, respectively.

The Bochner space of equivalence classes of p-integrable X-valued functions is denoted by $\mathrm{L}^p(\mathbb{R}; X)$. If Ω is a locally compact space, then $\mathrm{M}(\Omega)$ denotes the space of all bounded regular Borel measures on Ω. If $\mu \in \mathrm{M}(\Omega)$ then $\operatorname{supp}\mu$ denotes its topological support. If $\Omega \subseteq \mathbb{C}$ is an open subset of the complex plane, $\mathrm{H}^\infty(\Omega)$ denotes the Banach algebra of bounded holomorphic functions on Ω, endowed with the supremum norm $\|f\|_{\mathrm{H}^\infty(\Omega)} = \sup\{|f(z)| \mid z \in \Omega\}$.

We shall need notation and results from Fourier analysis as collected in [12, Appendix E]. In particular, we use the symbol $\mathcal{F}$ for the Fourier transform acting on the space of (possibly vector-valued) tempered distributions on $\mathbb{R}$, where we agree that

$$\mathcal{F}\mu(t) := \int_{\mathbb{R}} e^{-ist}\,\mu(\mathrm{d}s)$$

is the Fourier transform of a bounded measure $\mu \in \mathrm{M}(\mathbb{R})$. A function $m \in \mathrm{L}^\infty(\mathbb{R})$ is called a *bounded* Fourier multiplier on $\mathrm{L}^p(\mathbb{R}; X)$ if there is a constant $c \geqslant 0$ such that

$$\left\|\mathcal{F}^{-1}(m \cdot \mathcal{F}f)\right\|_p \leqslant c\,\|f\|_p \tag{1.8}$$

holds true for all $f \in \mathrm{L}^1(\mathbb{R}; X) \cap \mathcal{F}^{-1}(\mathrm{L}^1(\mathbb{R}; X))$. The smallest c that can be chosen in (1.8) is denoted by $\|\cdot\|_{\mathcal{M}_{p,X}}$. This turns the space $\mathcal{M}_{p,X}(\mathbb{R})$ of all bounded Fourier multipliers on $\mathrm{L}^p(\mathbb{R}; X)$ into a unital Banach algebra.

A Banach space X is a UMD space, if and only if the function $t \mapsto \operatorname{sgn} t$ is a bounded Fourier multiplier on $\mathrm{L}^2(\mathbb{R}; X)$. Such spaces are the right ones to study singular integrals for vector-valued functions. In particular, by results of Bourgain, McConnell and Zimmermann, a vector-valued version of the classical Mikhlin theorem holds, see [12, Appendix E.6] as well as Burkholder's article [5] and the literature cited there. Each Hilbert space is UMD, and if X is UMD, then $\mathrm{L}^p(\Omega, \Sigma, \mu; X)$ is also UMD whenever $1 < p < \infty$ and (Ω, Σ, μ) is a measure space.

The Fourier transform of $\mu \in \ell^1(\mathbb{Z})$ is

$$\widehat{\mu}(z) = \sum_{n \in \mathbb{Z}} \mu(n) z^n \quad (z \in \mathbb{T}).$$

Analogously to the continuous case, we form the algebra $\mathcal{M}_{p,X}(\mathbb{T})$ of functions $m \in \mathrm{L}^\infty(\mathbb{T})$ which induce bounded Fourier multiplier operators on $\ell^p(\mathbb{Z}; X)$, endowed with its natural norm.

Finally, given a set A and two real-valued functions $f, g : A \to \mathbb{R}$ we write

$$f(a) \lesssim g(a) \quad (a \in A)$$

to abbreviate the statement that there is $c \geqslant 0$ such that $f(a) \leqslant c\, g(a)$ for all $a \in A$.

2. Transference identities

We introduce the basic idea of transference. Let G be a locally compact group with left Haar measure $\mathrm{d}s$. Let $S \subseteq G$ be a closed sub-*semi*group of G and let

$$\mathrm{T} : S \to \mathcal{L}(X)$$

be a strongly continuous representation of S on a Banach space X. Let μ be a (scalar) Borel measure on S such that

$$\int_S \|T(s)\|\, |\mu|(\mathrm{d}s) < \infty,$$

and let the operator $\mathrm{T}_\mu \in \mathcal{L}(X)$ be defined by

$$\mathrm{T}_\mu x := \int_S T(s)x\, \mu(\mathrm{d}s) \quad (x \in X). \tag{2.1}$$

The aim of transference is an estimate of $\|\mathrm{T}_\mu\|$ in terms of a convolution operator involving μ. The idea to obtain such an estimate is, in a first step, purely formal. To illustrate it we shall need some preparation.

For a (measurable) function $\varphi : S \to \mathbb{C}$ we denote by $\varphi\mathrm{T}$ the pointwise product

$$(\varphi\mathrm{T}) : S \to \mathcal{L}(X), \quad s \mapsto \varphi(s)T(s)$$

and by $\varphi\mu$ the measure

$$(\varphi\mu)(\mathrm{d}s) = \varphi(s)\, \mu(\mathrm{d}s).$$

In the following we do not distinguish between a function/measure defined on S and its extension to G by 0 on $G \setminus S$. Also, for Banach spaces X, Y, Z and operator-valued functions $F : G \to \mathcal{L}(Z; Y)$ and $H : G \to \mathcal{L}(Y; X)$ we define the convolution $H * F : G \to \mathcal{L}(Z; X)$ formally by

$$(H * F)(t) := \int_G H(s)F\bigl(s^{-1}t\bigr)\, \mathrm{d}s \quad (t \in G) \tag{2.2}$$

in the strong sense, as long as this is well defined. (Actually, as we are to argue purely formally, at this stage we do not bother too much about whether all things are well defined. Instead, we shall establish formulae first and then explore conditions under which they are meaningful.)

Our first lemma expresses the fact that a semigroup representation induces representations of convolution algebras on S.

Lemma 2.1. *Let G, S, T, X be as above and let $\varphi, \psi : S \to \mathbb{C}$ be functions. Then, formally,*

$$(\varphi \mathrm{T}) * (\psi \mathrm{T}) = (\varphi * \psi)\, \mathrm{T}.$$

Proof. Fix $t \in G$. If $s \in G$ is such that $s \notin S \cap tS^{-1}$ then $\varphi(s) = 0$ (in case $s \notin S$) or $\psi(s^{-1}t) = 0$ (in case $s \notin tS^{-1}$). On the other hand, if $s \in S \cap tS^{-1}$ then $s, s^{-1}t \in S$ which implies that $t \in S$ and $T(s)T(s^{-1}t) = T(t)$. Hence, formally

$$\begin{aligned}
\bigl((\varphi \mathrm{T}) * (\psi \mathrm{T})\bigr)(t) &= \int_G (\varphi \mathrm{T})(s)(\psi \mathrm{T})\bigl(s^{-1}t\bigr)\, \mathrm{d}s \\
&= \int_G \varphi(s)\psi\bigl(s^{-1}t\bigr)T(s)T\bigl(s^{-1}t\bigr)\, \mathrm{d}s \\
&= \int_{S \cap tS^{-1}} \varphi(s)\psi\bigl(s^{-1}t\bigr)T(s)T\bigl(s^{-1}t\bigr)\, \mathrm{d}s \\
&= \int_{S \cap tS^{-1}} \varphi(s)\psi\bigl(s^{-1}t\bigr)\, \mathrm{d}s\, T(t) \\
&= \int_G \varphi(s)\psi\bigl(s^{-1}t\bigr)\, \mathrm{d}s\, T(t) = \bigl((\varphi * \psi)\, \mathrm{T}\bigr)(t). \qquad \square
\end{aligned}$$

For a function $F : G \to X$ and a measure μ on G let us abbreviate

$$\langle F, \mu \rangle := \int_G F(s)\, \mu(\mathrm{d}s)$$

defined in whatever weak sense. We shall stretch this notation to apply to all cases where it is reasonable. For example, μ could be a vector measure with values in X' or in $\mathcal{L}(X)$. The reflection $F^{\sim}$ of F is defined by

$$F^{\sim} : G \to X, \quad F^{\sim}(s) := F\bigl(s^{-1}\bigr).$$

If $H : G \to \mathcal{L}(X)$ is an operator-valued function, then the convolution $H * F$ is defined also by (2.2). Furthermore, we let

$$(\mu * F)(t) := \int_G F\bigl(s^{-1}t\bigr)\, \mu(\mathrm{d}s) \quad (t \in G),$$

which is in coherence with the definitions above if μ has a density and scalars are identified with their induced dilation operators.

The next lemma is almost a triviality.

Lemma 2.2. *Let $H : G \to \mathcal{L}(X)$, $F : G \to X$ and let μ be a measure on G. Then*

$$\langle H * F, \mu \rangle = \langle H, \mu * F^{\sim} \rangle$$

formally.

Proof. Writing out the brackets into integrals, it is just Fubini's theorem:

$$\begin{aligned}
\langle H * F, \mu \rangle &= \int_G \int_G H(s) F\bigl(s^{-1}t\bigr)\, \mathrm{d}s\, \mu(\mathrm{d}t) = \int_G \int_G H(s) F\bigl(s^{-1}t\bigr)\, \mu(\mathrm{d}t)\, \mathrm{d}s \\
&= \int_G H(s) \int_G F^{\sim}\bigl(t^{-1}s\bigr)\, \mu(\mathrm{d}t)\, \mathrm{d}s \\
&= \int_G H(s) \bigl(\mu * F^{\sim}\bigr)(s)\, \mathrm{d}s = \bigl\langle H, \mu * F^{\sim} \bigr\rangle. \qquad \square
\end{aligned}$$

If we combine Lemmas 2.1 and 2.2 we obtain the following.

Proposition 2.3. *Let S be a closed sub-semigroup of G and let $\mathrm{T} : S \to \mathcal{L}(X)$ be a strongly continuous representation. Let $\varphi, \psi : S \to \mathbb{C}$ and let μ be a measure on S. Then, writing $\eta := \varphi * \psi$,*

$$\mathrm{T}_{\eta\mu} = \bigl\langle \mathrm{T}, (\varphi * \psi)\mu \bigr\rangle = \bigl\langle \varphi \mathrm{T}, \mu * (\psi \mathrm{T})^{\sim} \bigr\rangle$$

formally.

This result can be interpreted as a *factorization* of the operator $\mathrm{T}_{\eta\mu}$ as

$$\begin{tikzcd}
\Phi(G;X) \arrow[r, "L_\mu"] & \Psi(G;X) \arrow[d, "P"] \\
X \arrow[u, "\iota"] \arrow[r, "\mathrm{T}_{\eta\mu}"] & X
\end{tikzcd} \tag{2.3}$$

i.e., $\mathrm{T}_{\eta\mu} = P \circ L_\mu \circ \iota$, where

- ι maps $x \in X$ to the weighted orbit

$$(\iota x)(s) := \psi\bigl(s^{-1}\bigr) T\bigl(s^{-1}\bigr) x \quad (s \in G);$$

- L_μ is the convolution operator with μ

$$L_\mu(F) := \mu * F;$$

- P maps an X-valued function on G back to an element of X by integrating against $\varphi\mathrm{T}$:

$$PF := \langle \varphi\mathrm{T}, F\rangle = \int_G \varphi(t)T(t)F(t)\,\mathrm{d}t,$$

- $\Phi(G;X), \Psi(G;X)$ are function spaces such that $\iota : X \to \Phi(G;X)$ and $P : \Psi(G;X) \to X$ are meaningful and bounded.

We call a factorization of the form (2.3) a *transference identity*. It induces a *transference estimate*

$$\|\mathrm{T}_{\eta\mu}\|_{\mathcal{L}(X)} \leqslant \|P\| \|L_\mu\|_{\mathcal{L}(\Phi(G;X);\Psi(G;X))} \|\iota\|. \tag{2.4}$$

3. Transference principles for groups

In the present section we shall explain that the classical transference principle of Berkson, Gillespie and Muhly [2] for uniformly bounded groups and the recent one for general C_0-groups [16] are instances of the explained technique.

3.1. Unbounded C_0-groups

We take $G = S = \mathbb{R}$ and let $\mathrm{U} = (U(s))_{s\in\mathbb{R}} : \mathbb{R} \to \mathcal{L}(X)$ be a strongly continuous representation on the Banach space X. Then U is exponentially bounded, i.e., its *exponential type*

$$\theta(\mathrm{U}) := \inf\bigl\{\omega \geqslant 0 \bigm| \exists M \geqslant 0\colon \ \bigl\|U(s)\bigr\| \leqslant M\mathrm{e}^{\omega|s|} \ (s \in \mathbb{R})\bigr\}$$

is finite. We choose $\alpha > \omega > \theta(\mathrm{U})$ and take a measure μ on $\mathbb{R}$ such that

$$\mu_\omega := \cosh(\omega\,\cdot)\mu \in \mathrm{M}(\mathbb{R})$$

is a finite measure. Then $\mathrm{U}_\mu = \int_{\mathbb{R}} U\,\mathrm{d}\mu$ is well defined. It turns out [16] that one can factorize

$$\eta := \frac{1}{\cosh(\omega\,\cdot)} = \varphi * \psi$$

where $\psi = 1/\cosh(\alpha\,\cdot)$ and $\cosh(\omega\,\cdot)\varphi = O(1)$. We obtain $\mu = \eta\mu_\omega$ and, writing μ_ω for μ in Proposition 2.3,

$$\mathrm{U}_\mu = \mathrm{U}_{\eta\mu_\omega} = \bigl\langle \varphi\mathrm{U}, \mu_\omega * (\psi\mathrm{U})^{\sim}\bigr\rangle = P \circ L_{\mu_\omega} \circ \iota. \tag{3.1}$$

If $-iA$ is the generator of U and $f = \mathcal{F}\mu$ is the Fourier transform of μ, one writes

$$f(A) := \mathrm{U}_\mu = \int_{\mathbb{R}} U(s)\,\mu(\mathrm{d}s),$$

which is well defined because the Fourier transform is injective. Applying the transference estimate (2.4) with $\Phi(\mathbb{R};X) = \Psi(\mathbb{R};X) := \mathrm{L}^{\mathrm{p}}(\mathbb{R};X)$ as the function spaces as in [16] leads to the estimate

$$\| f(A)\| \lesssim \frac{1}{2}\big(\| f(\cdot + i\omega)\|_{\mathcal{M}_{p,X}(\mathbb{R})} + \| f(\cdot - i\omega)\|_{\mathcal{M}_{p,X}(\mathbb{R})}\big),$$

where $\mathcal{M}_{p,X}(\mathbb{R})$ denotes the space of all (scalar-valued) bounded Fourier multipliers on $\mathrm{L^p}(\mathbb{R}; X)$. In the case that X is a UMD space one can now use the Mikhlin type result for Fourier multipliers on $\mathrm{L^p}(\mathbb{R}; X)$ to obtain a generalization of the Hieber–Prüss theorem [17] to unbounded groups, see [16, Theorem 3.6].

If $p = 2$ and $X = H$, this Fourier multiplier norm coincides with the sup-norm by Plancherel's theorem, and by the maximum principle one obtains the H^∞-estimate

$$\| f(A)\| \lesssim \| f\|_{\mathrm{H}^\infty(\mathrm{St}(\omega))}, \tag{3.2}$$

where

$$\mathrm{St}(\omega) := \{ z \in \mathbb{C} \mid |\mathrm{Im}\, z| < \omega \}$$

is the vertical strip of height 2ω, symmetric about the real axis. This result is originally due to Boyadzhiev and deLaubenfels [4] and is closely related to McIntosh's theorem on H^∞-calculus for sectorial operators with bounded imaginary powers from [23], see [16, Corollary 3.7] and [12, Chapter 7].

3.2. Bounded groups: the classical case

The classical transference principle, in the form put forward by Berkson, Gillespie and Muhly in [2] reads as follows: *Let G be a locally compact amenable group, let $\mathrm{U} = (U(s))_{s\in G}$ be a uniformly bounded, strongly continuous representation of G on a Banach space X, and let $p \in [1, \infty)$. Then*

$$\left\| \int_G U(s)\, \mu(\mathrm{d}s) \right\| \leqslant M^2 \| L_\mu\|_{\mathcal{L}(\mathrm{L^p}(G;X))}$$

for every bounded measure $\mu \in \mathrm{M}(G)$. (Here $M := \sup_{s\in G} \|U(s)\|$.)

We shall review its proof in the special case of $G = \mathbb{R}$ (but the general case is analogous using Følner's condition, see [7, p. 10]). First, fix $n, N > 0$ and suppose that $\mathrm{supp}(\mu) \subseteq [-N, N]$. Then

$$\eta := \varphi * \psi := \frac{1}{2n}\mathbf{1}_{[-n,n]} * \mathbf{1}_{[-N-n,N+n]} = 1 \quad \text{on } [-N, N].$$

So $\eta\mu = \mu$; applying the transference estimate (2.4) with the function space $\Phi(\mathbb{R}; X) = \Psi(\mathbb{R}; X) := \mathrm{L^p}(\mathbb{R}; X)$ together with Hölder's inequality yields

$$\begin{aligned}\|\mathrm{T}_\mu\| &\leqslant M^2 \|\varphi\|_{p'} \|\psi\|_p \|L_\mu\|_{\mathcal{L}(\mathrm{L^p}(\mathbb{R};X))}\\ &= M^2 (2n)^{\frac{1}{p'}-1} (2N + 2n)^{\frac{1}{p}} \|L_\mu\|_{\mathcal{L}(\mathrm{L^p}(\mathbb{R};X))}\\ &= M^2 \left(1 + \frac{N}{n}\right)^{1/p} \|L_\mu\|_{\mathcal{L}(\mathrm{L^p}(\mathbb{R};X))}.\end{aligned}$$

Finally, let $n \to \infty$ and approximate a general $\mu \in \mathrm{M}(\mathbb{R})$ by measures of finite support.

Remark 3.1. This proof shows a feature to which we pointed already in the introduction, but which was not present in the case of unbounded groups treated above. Here, an additional *optimization argument* appears which is based on some freedom in the choice of the auxiliary functions φ and ψ. Indeed, φ and ψ can vary as long as $\mu = (\varphi * \psi)\mu$, which amounts to $\varphi * \psi = 1$ on $\operatorname{supp}(\mu)$.

Remark 3.2. A transference principle for bounded *cosine functions* instead of groups was for the first time established and applied in [15].

4. A transference principle for discrete and continuous operator semigroups

In this section we shall apply the transference method from Section 2 to operator semigroups, i.e., strongly continuous representations of the semigroup $\mathbb{R}_+$ (continuous case) or $\mathbb{Z}_+$ (discrete case).

4.1. The continuous case

Let $\mathrm{T} = (T(s))_{s \geqslant 0}$ be a strongly continuous (i.e. C_0-) one-parameter semigroup on a (non-trivial) Banach space X. By standard semigroup theory [10], T is exponentially bounded, i.e., there exist $M, \omega \geqslant 0$ such that $\|T(s)\| \leqslant Me^{\omega s}$ for all $s \geqslant 0$. We consider complex measures μ on $\mathbb{R}_+ := [0, \infty)$ such that

$$\int_0^\infty \|T(s)\| \, |\mu|(\mathrm{d}s) < \infty.$$

If μ is Laplace transformable and if $f = \mathcal{L}\mu$ is its Laplace(–Stieltjes) transform

$$\mathcal{L}\mu(z) = \int_0^\infty e^{-zs} \, \mu(\mathrm{d}s),$$

then we use (similar to the group case) the abbreviation

$$f(A) := \mathrm{T}_\mu = \int_0^\infty T(s) \, \mu(\mathrm{d}s)$$

where $-A$ is the generator of the semigroup T. The mapping $f \mapsto f(A)$ is well defined since the Laplace transform is injective, and is called the *Hille–Phillips functional calculus* for A, see [12, Section 3.3] and [18, Chapter XV].

Theorem 4.1. *Let $p \in (1, \infty)$. Then there is a constant $c_p \geqslant 0$ such that*

$$\|\mathrm{T}_\mu\| \leqslant c_p \big(1 + \log(b/a)\big) M(b)^2 \, \|L_\mu\|_{\mathcal{L}(\mathrm{L}^p(\mathbb{R};X))} \tag{4.1}$$

whenever the following hypotheses are satisfied:

(1) $\mathrm{T} = (T(s))_{s \geqslant 0}$ *is a* C_0*-semigroup on the Banach space* X;
(2) $0 < a < b < \infty$;
(3) $M(b) := \sup_{0 \leqslant s \leqslant b} \|T(s)\|$;
(4) $\mu \in \mathrm{M}(\mathbb{R}_+)$ *such that* $\operatorname{supp}(\mu) \subseteq [a, b]$.

Proof. Take $\varphi \in \mathrm{L}^{\mathrm{p}'}(0, b)$, $\psi \in \mathrm{L}^{\mathrm{p}}(0, b)$ such that $\varphi * \psi = 1$ on $[a, b]$, and let $\eta := \varphi * \psi$. Then $\eta\mu = \mu$ and Proposition 2.3 yields

$$\mathrm{T}_\mu = \mathrm{T}_{\eta\mu} = \big\langle \varphi \mathrm{T}, \mu * (\psi \mathrm{T})^{\sim} \big\rangle.$$

Hölder's inequality leads to a norm estimate

$$\|\mathrm{T}_\mu\| \leqslant M(b)^2 \|\varphi\|_{p'} \|\psi\|_p \|L_\mu\|_{\mathcal{L}(\mathrm{L}^\mathrm{p}(\mathbb{R}; X))}.$$

Hence, to prove the theorem it suffices to show that

$$c(a, b) := \inf\big\{ \|\varphi\|_{p'} \|\psi\|_p \colon \varphi * \psi = 1 \text{ on } [a, b] \big\} \leqslant c_p \log\big(1 + (b/a)\big)$$

with c_p independent of a and b. This is done in Lemma A.1. □

Remarks 4.2.

(1) The conclusion of the theorem is also true in the case $p = 1$ or $p = \infty$, but in this case

$$\|L_\mu\|_{\mathcal{L}(\mathrm{L}^\mathrm{p}(\mathbb{R}; X))} = \|\mu\|_{\mathrm{M}(\mathbb{R}_+)}$$

is just the total variation norm of μ. And clearly $\|\mathrm{T}_\mu\| \leqslant M(b)\|\mu\|_{\mathrm{M}}$, which is stronger than (4.1).

(2) In functional calculus terms, (4.1) takes the form

$$\big\| f(A) \big\| \leqslant c_p \big(1 + \log(b/a)\big) M(b)^2 \|f\|_{\mathcal{AM}_{p,X}(\mathbb{C}_+)}$$

where $f = \mathcal{L}\mu$ and

$$\mathcal{AM}_{p,X}(\mathbb{C}_+) := \big\{ f \in \mathrm{H}^\infty(\mathbb{C}_+) \,\big|\, f(i\cdot) \in \mathcal{M}_{p,X}(\mathbb{R}) \big\}$$

is the (scalar) *analytic* $\mathrm{L}^\mathrm{p}(\mathbb{R}; X)$*-Fourier multiplier algebra*, endowed with the norm

$$\|f\|_{\mathcal{AM}_{p,X}(\mathbb{C}_+)} := \big\| f(i\cdot) \big\|_{\mathcal{M}_{p,X}(\mathbb{R})}.$$

Let us now state a corollary for semigroups with polynomial growth type.

Corollary 4.3. *Let* $p \in (1, \infty)$. *Then there is a constant* $c_p \geqslant 0$ *such that the following is true. If* $-A$ *generates a* C_0*-semigroup* $\mathrm{T} = (T(s))_{s \geqslant 0}$ *on a Banach space* X *such that there are* $M, \alpha \geqslant 0$

with

$$\|T(s)\| \leqslant M(1+s)^{\alpha} \quad (s \geqslant 0),$$

then

$$\|f(A)\| \leqslant c_p M^2 (1+b)^{2\alpha} \bigl(1+\log(b/a)\bigr) \|f\|_{\mathcal{AM}_{p,X}(\mathbb{C}_+)} \tag{4.2}$$

for $0 < a < b < \infty$, $f = \mathcal{L}\mu$ *and* $\mu \in \mathrm{M}[a,b]$.

The case that $\alpha = 0$, i.e., the case of a *bounded* semigroup, is particularly important, hence we state it separately.

Corollary 4.4. *Let* $p \in (1,\infty)$. *Then there is a constant* $c_p \geqslant 0$ *such that the following is true. If* $-A$ *generates a uniformly bounded* C_0*-semigroup* $\mathrm{T} = (T(s))_{s\geqslant 0}$ *on a Banach space* X *then, with* $M := \sup_{s\geqslant 0} \|T(s)\|$,

$$\|f(A)\| \leqslant c_p M^2 \bigl(1+\log(b/a)\bigr) \|f\|_{\mathcal{AM}_{p,X}(\mathbb{C}_+)} \tag{4.3}$$

for $0 < a < b < \infty$, $f = \mathcal{L}\mu$ *and* $\mu \in \mathrm{M}[a,b]$.

Remark 4.5. If $X = H$ is a Hilbert space and $p = 2$, by Plancherel's theorem and the maximum principle, Eq. (4.3) becomes

$$\|f(A)\| \lesssim M^2 \bigl(1+\log(b/a)\bigr) \|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \tag{4.4}$$

where $f = \mathcal{L}\mu$ is the Laplace–Stieltjes transform of μ. A similar estimate has been established by Vitse [28, Lemma 1.5] on a general Banach space X, but with the semigroup being holomorphic and bounded on a sector.

4.2. The discrete case

We now turn to the situation of a discrete operator semigroup, i.e., the powers of a bounded operator. Let $T \in \mathcal{L}(X)$ be a bounded operator and $\mathrm{T} = (T^n)_{n\in\mathbb{Z}_+}$ the corresponding semigroup representation. If $\mu \in \ell^1(\mathbb{Z}_+)$ is such that $\sum_{n=0}^{\infty} |\mu(n)| \|T^n\| < \infty$ then (2.1) takes the form

$$\mathrm{T}_\mu = \sum_{n=0}^{\infty} \mu(n) T^n.$$

Denoting $\widehat{\mu}(z) := \sum_{n=0}^{\infty} \mu(n) z^n$ for $|z| \leqslant 1$ we also write $\widehat{\mu}(T) := \mathrm{T}_\mu$.

Theorem 4.6. *Let* $p \in (1,\infty)$. *Then there is a constant* $c_p \geqslant 0$ *such that*

$$\|\widehat{\mu}(T)\| \leqslant c_p \bigl(1+\log(b/a)\bigr) M(b)^2 \|L_\mu\|_{\mathcal{L}(\ell^p(\mathbb{Z};X))} \tag{4.5}$$

whenever the following hypotheses are satisfied:

(1) T *is a bounded operator on a Banach space* X;
(2) $a, b \in \mathbb{Z}_+$ *with* $1 \leqslant a \leqslant b$;
(3) $M(b) := \sup_{0 \leqslant n \leqslant b} \|T^n\|$;
(4) $\mu \in \ell^1(\mathbb{Z}_+)$ *such that* $\operatorname{supp}(\mu) \subseteq [a, b]$.

Proof. This is completely analogous to the continuous situation. Take $\varphi \in \ell^{p'}(\mathbb{Z}_+)$, $\psi \in \ell^p(\mathbb{Z}_+)$ such that $\operatorname{supp}(\varphi), \operatorname{supp}(\psi) \subseteq [0, b]$ and $\eta := \varphi * \psi = 1$ on $[a, b]$. Then $\eta\mu = \mu$ and Proposition 2.3 yields

$$\widehat{\mu}(T) = \mathrm{T}_\mu = \mathrm{T}_{\eta\mu} = \big\langle \varphi\mathrm{T}, \mu * (\psi\mathrm{T})^\sim \big\rangle.$$

Hölder's inequality leads to a norm estimate

$$\|\mathrm{T}_\mu\| \leqslant M(b)^2 \|\varphi\|_{p'} \|\psi\|_p \|L_\mu\|_{\mathcal{L}(\ell^p(\mathbb{Z};X))}.$$

Minimizing with respect to φ, ψ yields $\|\mathrm{T}_\mu\| \leqslant c(a,b) M(b)^2 \|L_\mu\|_{\mathcal{L}(\ell^p(\mathbb{Z};X))}$, with

$$c(a,b) := \inf\big\{ \|\varphi\|_{p'} \|\psi\|_p \colon \varphi \in \ell^{p'}(\mathbb{Z}_+),\ \psi \in \ell^p(\mathbb{Z}_+),\ \varphi * \psi = 1 \text{ on } [a,b] \big\}.$$

(Note that we may allow infinite support for φ, ψ in this definition since the values of φ, ψ on $[b+1, \infty)$ don't affect the values of $\varphi * \psi$ on $[0, b]$.) Applying Lemma A.2 now concludes the proof. □

Remarks 4.7.

(1) As in the continuous case, the assertion remains true for $p = 1, \infty$, but is weaker than the obvious estimate $\|\widehat{\mu}\| \leqslant M(b)\|\mu\|_{\ell^1}$.
(2) If we write $f = \widehat{\mu}$, (4.5) takes the form

$$\big\|f(T)\big\| \leqslant c_p\big(1 + \log(b/a)\big) M(b)^2 \|f\|_{\mathcal{AM}_{p,X}(\mathbb{D})}.$$

Here

$$\mathcal{AM}_{p,X}(\mathbb{D}) := \big\{ f \in \mathrm{H}^\infty(\mathbb{D}) \ \big|\ f|_{\mathbb{T}} \in \mathcal{M}_{p,X}(\mathbb{T}) \big\}$$

is the (scalar) *analytic* $\ell^p(\mathbb{Z}; X)$*-Fourier multiplier algebra*, endowed with the norm

$$\|f\|_{\mathcal{AM}_{p,X}(\mathbb{D})} = \|f|_{\mathbb{T}}\|_{\mathcal{M}_{p,X}(\mathbb{T})}.$$

Similar to the continuous case we state a consequence for operators with polynomially growing powers.

Corollary 4.8. *Let* $p \in (1, \infty)$. *Then there is a constant* $c_p \geqslant 0$ *such that the following is true. If* T *is a bounded operator on a Banach space* X *such that there are* $M, \alpha \geqslant 0$ *with*

$$\big\|T^n\big\| \leqslant M(1+n)^\alpha \quad (n \geqslant 0),$$

then

$$\big\|f(T)\big\| \leqslant c_p\, M^2(1+b)^{2\alpha}\big(1+\log(b/a)\big)\|f\|_{\mathcal{AM}_{p,X}(\mathbb{D})} \tag{4.6}$$

for $f=\widehat{\mu}$, $\mu\in\ell^1([a,b]\cap\mathbb{Z})$, $a,b\in\mathbb{Z}_+$ *with* $1\leqslant a\leqslant b$.

Remark 4.9. For the applications to Peller's theorem in the next section the exact asymptotics of $c(a,b)$ is irrelevant, and one can obtain an effective estimate with much less effort. In the continuous case, the identity $c(a,b)=c(1,b/a)$ (cf. the proof of Lemma A.1) already shows that $c(a,b)$ only depends on b/a. For the special choice of functions

$$\varphi=\mathbf{1}_{[0,1]},\qquad \psi=\mathbf{1}_{[0,b]}$$

one has $\|\varphi\|_{p'}=1$ and $\|\psi\|_p=b^{1/p}$. Consequently $c(1,b)\leqslant b^{1/p}$ and symmetrizing yields

$$\big\|f(A)\big\|\leqslant M(b)^2(b/a)^{1/\max(p,p')}\|f\|_{\mathcal{AM}_{p,X}(\mathbb{C})}.$$

In the discrete case take η as in the proof of Lemma A.2 and factorize

$$\widehat{\eta}=\widehat{\varphi}\cdot\widehat{\psi}=\frac{1-z^a}{1-z}\cdot\frac{z}{a(1-z)}.$$

Then $\|\varphi\mathbf{1}_{[0,b]}\|_{p'}^{p'}=a$ and $\|\psi\mathbf{1}_{[0,b]}\|_p^p=b/a^p$, hence

$$c(a,b)\leqslant\|\varphi\mathbf{1}_{[0,b]}\|_{p'}\|\psi\mathbf{1}_{[0,b]}\|_p=a^{1/p'}b^{1/p}a^{-1}=(b/a)^{1/p}.$$

Symmetrizing yields the estimate

$$\big\|f(T)\big\|\leqslant M^2(b/a)^{1/\max(p,p')}\|f\|_{\mathcal{AM}_{p,X}(\mathbb{D})}$$

similar to the continuous case.

5. Peller's theorems

The results can be used to obtain a new proof of some classical results of Peller's about Besov class functional calculi for bounded Hilbert space operators with polynomially growing powers from [26]. In providing the necessary notions we essentially follow Peller, changing the notation slightly (cf. also [27]).

For an integer $n\geqslant 1$ let

$$\varphi_n(k):=\begin{cases}0, & k\leqslant 2^{n-1},\\ \frac{1}{2^{n-1}}\cdot(k-2^{n-1}), & 2^{n-1}\leqslant k\leqslant 2^n,\\ \frac{1}{2^n}\cdot(2^{n+1}-k), & 2^n\leqslant k\leqslant 2^{n+1},\\ 0, & 2^{n+1}\leqslant k.\end{cases}$$

That is, φ_n is supported in $[2^{n-1}, 2^{n+1}]$, zero at the endpoints, $\varphi_n(2^n) = 1$ and linear on each of the intervals $[2^{n-1}, 2^n]$ and $[2^n, 2^{n+1}]$. Let $\varphi_0 := (1, 1, 0, \ldots)$, then

$$\sum_{n=0}^{\infty} \varphi_n = \mathbf{1}_{\mathbb{Z}_+},$$

the sum being locally finite. For $s \geqslant 0$ the *Besov class* $\mathrm{B}^s_{\infty,1}(\mathbb{D})$ is defined as the class of analytic functions f on the unit disc $\mathbb{D}$ satisfying

$$\|f\|_{\mathrm{B}^s_{\infty,1}} := \sum_{n=0}^{\infty} 2^{ns} \|\widehat{\varphi_n} * f\|_{\mathrm{H}^\infty(\mathbb{D})} < \infty.$$

That is, if $f = \sum_{k \geqslant 0} \alpha_k z^k$, $\alpha := (\alpha_k)_{k \geqslant 0}$, then

$$\|f\|_{\mathrm{B}^s_{\infty,1}} = \sum_{n=0}^{\infty} 2^{ns} \|\widehat{\varphi_n \alpha}\|_{\mathrm{H}^\infty(\mathbb{D})} < \infty.$$

Following Peller [26, p. 347], one has

$$f \in \mathrm{B}^s_{\infty,1}(\mathbb{D}) \quad \Longleftrightarrow \quad \int_0^1 (1-r)^{m-s-1} \left\| f^{(m)} \right\|_{\mathrm{L}^\infty(r\mathbb{T})} \, \mathrm{d}r < \infty,$$

where m is an arbitrary integer such that $m > s$. Since we only consider $s \geqslant 0$, we have

$$\mathrm{B}^s_{\infty,1}(\mathbb{D}) \subseteq \mathrm{H}^\infty(\mathbb{D})$$

and it is known that $\mathrm{B}^s_{\infty,1}(\mathbb{D})$ is a Banach algebra in which the set of polynomials is dense. The following is essentially [26, p. 354, bottom line]; we give a new proof.

Theorem 5.1 *(Peller 1982). There exists a constant $c \geqslant 0$ such that the following holds: Let X be a Hilbert space, and let $T \in \mathcal{L}(X)$ such that*

$$\left\| T^n \right\| \leqslant M(1+n)^\alpha \quad (n \geqslant 0)$$

with $\alpha \geqslant 0$ and $M \geqslant 1$. Then

$$\left\| f(T) \right\| \leqslant c\, 9^\alpha M^2 \|f\|_{\mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D})}$$

for every polynomial f.

Proof. Let $f = \widehat{\nu} = \sum_{k \geqslant 0} \nu_n z^n$, and ν has finite support. If $n \geqslant 1$, then $\varphi_n \nu$ has support in $[2^{n-1}, 2^{n+1}]$, so we can apply Corollary 4.8 with $p = 2$ to obtain

$$\left\| \widehat{\varphi_n \nu}(T) \right\| \leqslant c_2 M^2 \left(1 + 2^{n+1}\right)^{2\alpha} (1 + \log 4) \|\widehat{\varphi_n \nu}\|_{\mathcal{AM}_{2,X}(\mathbb{D})}.$$

Since X is a Hilbert space, Plancherel's theorem (and standard Hardy space theory) yields that $\mathcal{AM}_{2,X}(\mathbb{D}) = \mathrm{H}^\infty(\mathbb{D})$ with equal norms. Moreover, $1 + 2^{n+1} \leqslant 3 \cdot 2^n$, and hence we obtain

$$\left\| \widehat{\varphi_n \nu}(T) \right\| \leqslant c_2\, 9^\alpha M^2 \cdot 2^{n(2\alpha)} \|\widehat{\varphi_n \nu}\|_{\mathrm{H}^\infty(\mathbb{D})}.$$

Summing up, we arrive at

$$\begin{aligned} \left\| f(T) \right\| &\leqslant \sum_{n\geqslant 0} \left\| \widehat{\varphi_n \nu}(T) \right\| \\ &\leqslant |\nu_0| + |\nu_1| M 2^\alpha + c_2\, 9^\alpha M^2 \sum_{n\geqslant 1} 2^{n(2\alpha)} \|\widehat{\varphi_n \nu}\|_\infty \\ &\leqslant c\, 9^\alpha M^2 \|f\|_{\mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D})} \end{aligned}$$

for some constant $c \geqslant 0$. $\square$

Remark 5.2. N. Nikolski has observed that Peller's Theorem 5.1 is only interesting if $\alpha \leqslant 1/2$. Indeed, define

$$\mathrm{A}_\alpha(\mathbb{D}) := \left\{ f = \sum_{k\geqslant 0} a_k z^k \;\middle|\; \|f\|_{\mathrm{A}_\alpha} := \sum_{k\geqslant 0} |a_k|(1+k)^\alpha < \infty \right\}.$$

Then $\mathrm{A}_\alpha(\mathbb{D})$ is a Banach algebra with respect to the norm $\|\cdot\|_{\mathrm{A}_\alpha}$, and one has the obvious estimate

$$\left\| f(T) \right\| \leqslant M \|f\|_{\mathrm{A}_\alpha} \quad \bigl(f \in \mathrm{A}_\alpha(\mathbb{D})\bigr)$$

if $\|T^k\| \leqslant M(1+k)^\alpha$, $k \in \mathbb{N}$. This is the 'trivial' functional calculus for T we mentioned in the introduction, see (1.4). For $f \in \mathrm{B}^{\alpha+1/2}_{\infty,1}(\mathbb{D})$ we have

$$\begin{aligned} \|f\|_{\mathrm{A}_\alpha} &= |a_0| + \sum_{k\geqslant 0} \sum_{2^k \leqslant n < 2^{k+1}} |a_n|(1+n)^\alpha \\ &\leqslant |a_0| + \sum_{k\geqslant 0} 2^{(k+1)\alpha} \sum_{2^k \leqslant n < 2^{k+1}} |a_n| \\ &\leqslant |a_0| + \sum_{k\geqslant 0} 2^{(k+1)\alpha} 2^{k/2} \biggl(\sum_{2^k \leqslant n < 2^{k+1}} |a_n|^2 \biggr)^{1/2} \\ &\leqslant |a_0| + \sum_{k\geqslant 0} 2^\alpha 2^{(\alpha+1/2)k} \bigl(\|\widehat{\varphi_{k-1}} * f\|_2 + \|\widehat{\varphi_k} * f\|_2 + \|\widehat{\varphi_{k+1}} * f\|_2 \bigr) \\ &\lesssim \sum_{k=0}^\infty 2^{(\alpha+1/2)k} \|\widehat{\varphi_k} * f\|_\infty = \|f\|_{\mathrm{B}^{\alpha+1/2}_{\infty,1}} \end{aligned}$$

by the Cauchy–Schwarz inequality, Plancherel's theorem and the fact that $\mathrm{H}^\infty(\mathbb{D}) \subseteq \mathrm{H}^2(\mathbb{D})$. This shows that $\mathrm{B}^{\alpha+1/2}_{\infty,1}(\mathbb{D}) \subseteq \mathrm{A}_\alpha(\mathbb{D})$. Hence, if $\alpha \geqslant 1/2$, then $2\alpha \geqslant \alpha + 1/2$, and therefore $\mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D}) \subseteq \mathrm{B}^{\alpha+1/2}_{\infty,1}(\mathbb{D}) \subseteq \mathrm{A}_\alpha(\mathbb{D})$, and the Besov calculus is weaker than the trivial A_α-calculus.

On the other hand, for $\alpha > 0$, the example

$$f(z) = \sum_{n=0}^{\infty} 2^{-2\alpha n} z^{2^n} \in \mathrm{A}_\alpha(\mathbb{D}) \setminus \mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D})$$

shows that $\mathrm{A}_\alpha(\mathbb{D})$ is *not* included into $\mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D})$, and so the Besov calculus does not cover the trivial calculus. (By a straightforward argument one obtains the embedding $\mathrm{A}_\alpha(\mathbb{D}) \subseteq \mathrm{B}^{\alpha}_{\infty,1}(\mathbb{D})$.)

5.1. An analogue in the continuous case

Peller's theorem has an analogue for continuous one-parameter semigroups. The role of the unit disc $\mathbb{D}$ is taken by the right half-plane $\mathbb{C}_+$, the power series representation of a function on $\mathbb{D}$ is replaced by a Laplace transform representation of a function on $\mathbb{C}_+$. However, a subtlety appears that is not present in the discrete case, namely the possibility (or even necessity) to consider also dyadic decompositions "at zero". This leads to so-called "homogeneous" Besov spaces, but due to the special form of the estimate (4.2) we have to treat the decomposition at 0 different from the decomposition at ∞.

To be more precise, consider the partition of unity

$$\varphi_n(s) := \begin{cases} 0, & 0 \leqslant s \leqslant 2^{n-1}, \\ \frac{1}{2^{n-1}} \cdot (s - 2^{n-1}), & 2^{n-1} \leqslant s \leqslant 2^n, \\ \frac{1}{2^n} \cdot (2^{n+1} - s), & 2^n \leqslant s \leqslant 2^{n+1}, \\ 0, & 2^{n+1} \leqslant s \end{cases}$$

for $n \in \mathbb{Z}$. Then $\sum_{n\in\mathbb{Z}} \varphi_n = \mathbf{1}_{(0,\infty)}$, the sum being locally finite in $(0, \infty)$. For $s \geqslant 0$, an analytic function $f : \mathbb{C}_+ \to \mathbb{C}$ is in the (mixed-order homogeneous) Besov space $\mathrm{B}^{0,s}_{\infty,1}(\mathbb{C}_+)$ if $f(\infty) := \lim_{t\to\infty} f(t)$ exists and

$$\|f\|_{\mathrm{B}^{0,s}_{\infty,1}} := \bigl|f(\infty)\bigr| + \sum_{n<0} \|\mathcal{L}\varphi_n * f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} + \sum_{n\geqslant 0} 2^{ns} \|\mathcal{L}\varphi_n * f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} < \infty.$$

Here $\mathcal{L}$ denotes (as before) the Laplace transform

$$\mathcal{L}\varphi(z) := \int_0^\infty e^{-sz} \varphi(s)\, \mathrm{d}s \quad (\operatorname{Re} z > 0).$$

Since we are dealing with $s \geqslant 0$ only, it is obvious that $\mathrm{B}^{0,s}_{\infty,1}(\mathbb{C}_+) \subseteq \mathrm{H}^\infty(\mathbb{C}_+)$. Clearly, our definition of $\mathrm{B}^{0,s}_{\infty,1}(\mathbb{C}_+)$ is a little sloppy, and to make it rigorous we would need to employ the theory of Laplace transforms of distributions. However, we shall not need that here, because we shall use only functions of the form $f = \mathcal{L}\mu$, where μ is a bounded measure *with compact support* in $[0, \infty]$. In this case

$$\mathcal{L}\varphi_n * f = \mathcal{L}\varphi_n * \mathcal{L}\mu = \mathcal{L}(\varphi_n \mu)$$

by a simple computation.

Theorem 5.3. *There is an absolute constant $c \geqslant 0$ such that the following holds: Let X be a Hilbert space, and let $-A$ be the generator of a strongly continuous semigroup $\mathrm{T} = (T(s))_{s\in\mathbb{R}_+}$ on X such that*

$$\|T(s)\| \leqslant M(1+s)^\alpha \quad (n \geqslant 0)$$

with $\alpha \geqslant 0$ and $M \geqslant 1$. Then

$$\|f(A)\| \leqslant c\, 9^\alpha M^2 \|f\|_{\mathrm{B}^{0,2\alpha}_{\infty,1}(\mathbb{C}_+)}$$

for every $f = \mathcal{L}\mu$, $\mu \in \mathrm{M}(\mathbb{R}_+)$ with compact support.

Proof. The proof is analogous to the proof of Theorem 5.1. One has

$$\mu = f(\infty)\delta_0 + \sum_{n<0} \varphi_n \mu + \sum_{n\geqslant 0} \varphi_n \mu$$

where the first series converges in $\mathrm{M}[0, 1]$ and the second is actually finite. Hence

$$\begin{aligned}
\|f(A)\| &\leqslant |f(\infty)| + \sum_{n\in\mathbb{Z}} \big\|\big[\mathcal{L}(\varphi_n\mu)\big](A)\big\| \\
&\lesssim |f(\infty)| + \sum_{n\in\mathbb{Z}} M^2\big(1+2^{n+1}\big)^{2\alpha} \|L_{\varphi_n\mu}\|_{\mathcal{L}(\mathrm{L}^2(\mathbb{R};X))} \\
&= |f(\infty)| + M^2 \sum_{n\in\mathbb{Z}} \big(1+2^{n+1}\big)^{2\alpha} \|\mathcal{L}\varphi_n * f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \\
&\lesssim |f(\infty)| + M^2 \sum_{n<0} 2^{2\alpha} \|\mathcal{L}\varphi_n * f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \\
&\quad + M^2 \sum_{n\geqslant 0} \big(3\cdot 2^n\big)^{2\alpha} \|\mathcal{L}\varphi_n * f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \\
&\leqslant M^2 9^\alpha \|f\|_{\mathrm{B}^{0,2\alpha}_{\infty,1}},
\end{aligned}$$

by Plancherel's theorem and Corollary 4.3. □

Remark 5.4. The space $\mathrm{B}^{0,0}_{\infty,1}(\mathbb{C}_+)$ has been considered by Vitse in [28] under the name $\mathrm{B}^{0}_{\infty,1}(\mathbb{C}_+)$, and we refer to that paper for more information. In particular, Vitse proves that

$f \in \mathrm{B}^{0,0}_{\infty,1}(\mathbb{C}_+)$ if and only if $f \in \mathrm{H}^\infty(\mathbb{C}_+)$ and

$$\int_0^\infty \sup_{s\in\mathbb{R}} \left| f'(t+is) \right| \mathrm{d}t < \infty.$$

Let us formulate the special case $\alpha = 0$ as a corollary, with a slight generalization.

Corollary 5.5. *There is a constant $c \geqslant 0$ such that the following is true. Whenever $-A$ generates a strongly continuous semigroup $(T(s))_{s\geqslant 0}$ on a Hilbert space such that $\|T(s)\| \leqslant M$ for all $s \geqslant 0$, then*

$$\left\| f(A) \right\| \leqslant c\, M^2 \|f\|_{\mathrm{B}^{0,0}_{\infty,1}(\mathbb{C}_+)} \tag{5.1}$$

for all $f = \mathcal{L}\mu$, $\mu \in \mathrm{M}(\mathbb{R}_+)$.

Proof. It is easy to see that the Laplace transform $\mathcal{L} : \mathrm{M}(\mathbb{R}_+) \to \mathrm{B}^{0,0}_{\infty,1}(\mathbb{C}_+)$ is bounded. Since (5.1) is true for measures with compact support and such measures are dense in $\mathrm{M}(\mathbb{R}_+)$, an approximation argument proves the claim. □

Remarks 5.6.

(1) Vitse [28, Introduction, p. 248] in a short note suggests to prove Corollary 5.5 by a discretization argument using Peller's Theorem 5.1 for $\alpha = 0$. This is quite plausible, but no details are given in [28] and it seems that further work is required to make this approach rigorous.
(2) (Cf. Remark 4.9.) To prove Theorems 5.1 and 5.3 we did not make full use of the logarithmic factor $\log(1 + b/a)$ but only of the fact that it is constant in n if $[a, b] = [2^{n-1}, 2^{n+1}]$. However, as Vitse notes in [28, Remark 4.2], the logarithmic factor appears a fortiori; indeed, if $\operatorname{supp}\mu \subseteq [a, b]$ then if we write

$$\mu = \sum_{n\in\mathbb{Z}} \varphi_n \mu$$

the number $N = \operatorname{card}\{n \in \mathbb{Z} \mid \varphi_n\mu \neq 0\}$ of non-zero terms in the sum is proportional to $\log(1 + b/a)$. Hence, for the purposes of functional calculus estimates neither Lemma A.1 nor A.2 is necessary.
(3) (Cf. Remark 5.2.) Different to the discrete case, the Besov estimates are not completely uninteresting in the case $\alpha \geqslant 1/2$, because α affects only the decomposition at ∞.

5.2. Generalizations for UMD spaces

Our proofs of Peller's theorems use essentially that the underlying space is a Hilbert space. Indeed, we have applied Plancherel's theorem in order to estimate the Fourier multiplier norm of a function by its L^∞-norm. Hence we do not expect Peller's theorem to be valid on other Banach spaces without modifications. In the next section below we shall show that replacing ordinary boundedness of an operator family by the so-called γ-boundedness, Peller's theorems carry over to arbitrary Banach spaces. Here we suggest a different path, namely to replace the algebra $\mathrm{H}^\infty(\mathbb{C}_+)$ in the construction of the Besov space $\mathrm{B}^{0,s}_{\infty,1}$ by the analytic multiplier algebra $\mathcal{AM}_{p,X}(\mathbb{C}_+)$, introduced in Remark 4.2(2). We restrict ourselves to the continuous case, leaving the discrete version to the reader.

To simplify notation, let us abbreviate $\mathcal{A}_p := \mathcal{AM}_{p,X}(\mathbb{C}_+)$. For $s \geqslant 0$ and $f : \mathbb{C}_+ \to \mathbb{C}$ we say $f \in \mathrm{B}^{0,s}_1[\mathcal{A}_p]$ if $f \in \mathrm{H}^\infty(\mathbb{C}_+)$, $f(\infty) := \lim_{t\to\infty} f(t)$ exists and

$$\|f\|_{\mathrm{B}^{0,s}_1[\mathcal{A}_p]} := \bigl|f(\infty)\bigr| + \sum_{n<0} \|\mathcal{L}\varphi_n * f\|_{\mathcal{A}_p} + \sum_{n\geqslant 0} 2^{ns} \|\mathcal{L}\varphi_n * f\|_{\mathcal{A}_p} < \infty.$$

Then the following analogue of Theorem 5.3 holds, with a similar proof.

Theorem 5.7. *Let $p \in (1,\infty)$. Then there is a constant $c_p \geqslant 0$ such that the following holds: Let $-A$ be the generator of a strongly continuous semigroup $\mathrm{T} = (T(s))_{s\in\mathbb{R}_+}$ on a Banach space X such that*

$$\bigl\|T(s)\bigr\| \leqslant M(1+s)^\alpha \quad (n \geqslant 0)$$

with $\alpha \geqslant 0$ and $M \geqslant 1$. Then

$$\bigl\|f(A)\bigr\| \leqslant c_p\, 9^\alpha M^2 \|f\|_{\mathrm{B}^{0,2\alpha}_1[\mathcal{M}_p]}$$

for every $f = \mathcal{L}\mu$, $\mu \in \mathrm{M}(\mathbb{R}_+)$ with compact support.

For $X = H$ is a Hilbert space and $p = 2$ one is back at Theorem 5.3. For special cases of X—typically if X is an L^1- or a $\mathrm{C}(K)$-space—one has $\mathrm{B}^{0,0}_1[\mathcal{M}_p] = \mathrm{M}(\mathbb{R}_+)$. But if X is a UMD space, one has more informative results. To formulate them let

$$\mathrm{H}^\infty_1(\mathbb{C}_+) := \bigl\{ f \in \mathrm{H}^\infty(\mathbb{C}_+) \bigm| zf'(z) \in \mathrm{H}^\infty(\mathbb{C}_+) \bigr\}$$

be the *analytic Mikhlin algebra*. This is a Banach algebra with respect to the norm

$$\|f\|_{\mathrm{H}^\infty_1} := \sup_{z\in\mathbb{C}_+} \bigl|f(z)\bigr| + \sup_{z\in\mathbb{C}_+} \bigl|zf'(z)\bigr|.$$

If X is a UMD space then the vector-valued version of the Mikhlin theorem [12, Theorem E.6.2] implies that one has a continuous inclusion

$$\mathrm{H}^\infty_1(\mathbb{C}_+) \subseteq \mathcal{AM}_{p,X}(\mathbb{C}_+)$$

where the embedding constant depends on p and (the UMD constant of) X. If one defines $\mathrm{B}^{0,s}_1[\mathrm{H}^\infty_1]$ analogously to $\mathrm{B}^{0,s}_1[\mathcal{M}_p]$ above, then we obtain the following.

Corollary 5.8. *If X is a* UMD *space, then Theorem 5.7 is still valid when $\mathcal{AM}_{p,X}(\mathbb{C}_+)$ is replaced by $\mathrm{H}^\infty_1(\mathbb{C}_+)$ and the constant c_p is allowed to depend on (the* UMD*-constant of) X.*

Now, fix $\theta \in (\pi/2, \pi)$ and consider the sector

$$\Sigma_\theta := \{ z \in \mathbb{C} \setminus \{0\} \mid |\arg z| < \theta \}.$$

Then $\mathrm{H}^\infty(\Sigma_\theta) \subseteq \mathrm{H}_1^\infty(\mathbb{C}_+)$, as follows from an application of the Cauchy integral formula, see [12, Lemma 8.2.6]. Hence, if we define $\mathrm{B}_{\infty,1}^{0,s}(\Sigma_\theta)$ by replacing the space $\mathrm{H}^\infty(\mathbb{C}_+)$ in the definition of $\mathrm{B}_{\infty,1}^{0,s}(\mathbb{C}_+)$ by $\mathrm{H}^\infty(\Sigma_\theta)$ we obtain the following UMD-version of Peller's theorem.

Corollary 5.9. *Let $\theta \in (\pi/2, \pi)$, let X be a* UMD *space, and let $p \in (1, \infty)$. Then there is a constant $c = c(\theta, X, p)$ such that the following holds. Let $-A$ be the generator of a strongly continuous semigroup $\mathrm{T} = (T(s))_{s \in \mathbb{R}_+}$ on X such that*

$$\|T(s)\| \leqslant M(1+s)^\alpha \quad (s \geqslant 0)$$

with $\alpha \geqslant 0$ and $M \geqslant 1$. Then

$$\|f(A)\| \leqslant c\, 9^\alpha M^2 \|f\|_{\mathrm{B}_{\infty,1}^{0,2\alpha}(\Sigma_\theta)}$$

for every $f = \mathcal{L}\mu$, $\mu \in \mathrm{M}(\mathbb{R}_+)$ with compact support.

Note that Theorem 5.3 above simply says that if X is a Hilbert space, one can choose $\theta = \pi/2$ in Corollary 5.9.

Remark 5.10. It is natural to ask whether $\mathrm{B}_1^{0,s}[\mathrm{H}_1^\infty]$ or $\mathrm{B}_{\infty,1}^{0,2\alpha}(\Sigma_\theta)$ are actually Banach algebras. This is probably not true, as the underlying Banach algebras $\mathrm{H}_1^\infty(\mathbb{C}_+)$ and $\mathrm{H}^\infty(\Sigma_\theta)$ are not invariant under shifting along the imaginary axis, and hence are not $\mathrm{L}^1(\mathbb{R})$-convolution modules. Consequently, Corollaries 5.8 and 5.9 are highly unsatisfactory from a functional calculus point of view.

6. Generalizations involving γ-boundedness

At the end of the previous section we discussed one possible generalization of Peller's theorems, involving still an assumption on the Banach space and a modification of the Besov algebra, but no additional assumption on the semigroup. Here we follow a different path, strengthening the requirements on the semigroups under consideration. Vitse has shown in [27,28] that the Peller-type results remain true without any restriction on the Banach space if the semigroup is bounded analytic (in the continuous case), or the operator is a Tadmor–Ritt operator (in the discrete case). (These two situations correspond to each other in a certain sense, see e.g. [12, Section 9.2.4].)

Our approach here is based on the ground-breaking work of Kalton and Weis, involving the concept of γ-boundedness. This is a stronger notion of boundedness of a set of operators between two Banach spaces. The "philosophy" of the Kalton–Weis approach is that every Hilbert space theorem which rests on Plancherel's theorem (and no other result specific for Hilbert spaces) can be transformed into a theorem on general Banach spaces, when operator norm boundedness (of operator families) is replaced by γ-boundedness.

The idea is readily sketched. In the proof of Theorem 5.3 we used the transference identity (2.3) with the function space $\mathrm{L}^2(\mathbb{R}; X)$ and factorized the operator T_μ over the Fourier multiplier L_μ. If X is a Hilbert space, the 2-Fourier multiplier norm of L_μ is just $\|\mathcal{L}\mu\|_\infty$ and this led to the Besov class estimate. We now replace the function space $\mathrm{L}^2(\mathbb{R}; X)$ by the space $\gamma(\mathbb{R}; X)$; in order to make sure that the transference identity (2.3) remains valid, we need that the embedding ι and the projection P from (2.3) are well defined. And this is where the concept of γ-boundedness comes in. Once we have established the transference *identity*, we can pass to the transference *estimate*; and since $\mathrm{L}^\infty(\mathbb{R})$ is also the Fourier multiplier algebra of $\gamma(\mathbb{R}; X)$, we recover the infinity norm as in the $\mathrm{L}^2(\mathbb{R}; H)$-case from above.

We shall now pass to more rigorous mathematics, starting with a (very brief) introduction to the theory of γ-spaces. For a deeper account we refer to [24].

6.1. γ-summing and γ-radonifying operators

Let H be a Hilbert space and X a Banach space. An operator $T : H \to X$ is called *γ-summing* if

$$\|T\|_\gamma := \sup_F \mathbb{E}\left(\left\| \sum_{e\in F} \gamma_e \otimes Te \right\|_X^2\right)^{1/2} < \infty,$$

where the supremum is taken over all finite orthonormal systems $F \subseteq H$ and $(\gamma_e)_{e\in F}$ is an independent collection of standard Gaussian random variables on some probability space. It can be shown that in this definition it suffices to consider only finite subsets F of some *fixed* orthonormal basis of H. We let

$$\gamma_\infty(H; X) := \{T : H \to X \mid T \text{ is } \gamma\text{-summing}\}$$

the space of γ-summing operators of H into X. This is a Banach space with respect to the norm $\|\cdot\|_\gamma$. The closure in $\gamma_\infty(H; X)$ of the space of finite rank operators is denoted by $\gamma(H; X)$, and its elements $T \in \gamma(H; X)$ are called *γ-radonifying*. By a theorem of Hoffman–Jørgensen and Kwapień, if X does not contain c_0 then $\gamma(H; X) = \gamma_\infty(H; X)$, see [24, Thm. 4.3].

From the definition of the γ-norm the following important *ideal property* of the γ-spaces is quite straightforward [24, Thm. 6.2].

Lemma 6.1 *(Ideal property). Let Y be another Banach space and K another Hilbert space, let $L : X \to Y$ and $R : K \to H$ be bounded linear operators, and let $T \in \gamma_\infty(H; X)$. Then*

$$LTR \in \gamma_\infty(K; Y) \quad \text{and} \quad \|LTR\|_\gamma \leqslant \|L\|_{\mathcal{L}(X;Y)} \|T\|_\gamma \|R\|_{\mathcal{L}(K;H)}.$$

If $T \in \gamma(H; X)$, then $LTR \in \gamma(K; Y)$.

If $g \in H$ we abbreviate $\overline{g} := \langle\cdot, g\rangle$, i.e., $g \mapsto \overline{g}$ is the canonical conjugate-linear bijection of H onto its dual $\overline{H}$. Every finite rank operator $T : H \to X$ has the form

$$T = \sum_{j=1}^n \overline{g_n} \otimes x_j,$$

and one can view $\gamma(H; X)$ as a completion of the algebraic tensor product $\overline{H} \otimes X$ with respect to the γ-norm. Since

$$\|\overline{g} \otimes x\|_\gamma = \|g\|_H \|x\|_X = \|\overline{g}\|_{\overline{H}} \|x\|_X$$

for every $g \in H$, $x \in X$, the γ-norm is a cross-norm. Hence every *nuclear* operator $T : H \to X$ is γ-radonifying and $\|T\|_\gamma \leqslant \|T\|_{\mathrm{nuc}}$. (Recall that T is a nuclear operator if $T = \sum_{n \geqslant 0} \overline{g_n} \otimes x_n$ for some $g_n \in H$, $x_n \in X$ with $\sum_{n \geqslant 0} \|g_n\|_H \|x_n\|_X < \infty$.) The following application turns out to be quite useful.

Lemma 6.2. *Let H, X be as before, and let (Ω, Σ, μ) be a measure space. Suppose that $f : \Omega \to H$ and $g : \Omega \to X$ are (strongly) μ-measurable and*

$$\int_\Omega \|f(t)\|_H \|g(t)\|_X \,\mu(\mathrm{d}t) < \infty.$$

Then $\overline{f} \otimes g \in \mathrm{L}^1(\Omega; \gamma(H; X))$, and $T := \int_\Omega \overline{f} \otimes g \,\mathrm{d}\mu \in \gamma(H; X)$ satisfies

$$Th = \int_\Omega \langle h, f(t) \rangle g(t) \,\mu(\mathrm{d}t) \quad (h \in H)$$

and

$$\|T\|_\gamma \leqslant \int_\Omega \|f(t)\|_H \|g(t)\|_X \,\mu(\mathrm{d}t).$$

Suppose that $H = \mathrm{L}^2(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ). Every function $u \in \mathrm{L}^2(\Omega; X)$ defines an operator $T_u : \mathrm{L}^2(\Omega) \to X$ by integration:

$$T_u : \mathrm{L}^2(\Omega) \to X, \quad T_u(h) = \int_\Omega h \cdot u \,\mathrm{d}\mu.$$

(Actually, one can do this under weaker hypotheses on u, but we shall have no occasion to use the more general version.) In this context we identify the operator T_u with the function u and write

$$u \in \gamma(\Omega; X) \quad \text{in place of } T_u \in \gamma\big(\mathrm{L}^2(\Omega); X\big)$$

(and likewise for $\gamma_\infty(\Omega; X)$). Extending an idea of [19, Remark 3.1] we can use Lemma 6.2 to conclude that certain vector-valued functions define γ-radonifying operators. Note that in the following $a = -\infty$ or $b = \infty$ are allowed; moreover we employ the convention that $\infty \cdot 0 = 0$.

Corollary 6.3. *Let $(a, b) \subseteq \mathbb{R}$, let $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}((a, b); X)$ and let $\varphi : (a, b) \to \mathbb{C}$. Suppose that one of the following two conditions is satisfied:*

(1) $\|\varphi\|_{\mathrm{L}^2(a,b)}\|u(a)\|_X < \infty$ *and* $\int_a^b \|\varphi\|_{\mathrm{L}^2(s,b)}\|u'(s)\|_X \,\mathrm{d}s < \infty$;
(2) $\|\varphi\|_{\mathrm{L}^2(a,b)}\|u(b)\|_X < \infty$ *and* $\int_a^b \|\varphi\|_{\mathrm{L}^2(a,s)}\|u'(s)\|_X \,\mathrm{d}s < \infty$.

Then $\varphi \cdot u \in \gamma((a,b); X)$ *with respective estimates for* $\|\varphi \cdot u\|_\gamma$.

Proof. In case (1) we use the representation $u(t) = u(a) + \int_a^t u'(s)\,\mathrm{d}s$, leading to

$$\varphi \cdot u = \varphi \otimes u(a) + \int_a^b \mathbf{1}_{(s,b)}\varphi \otimes u'(s)\,\mathrm{d}s.$$

Then we apply Lemma 6.2. In case (2) we start with $u(t) = u(b) - \int_t^b u'(s)\,\mathrm{d}s$ and proceed similarly. □

The space $\gamma(\mathrm{L}^2(\Omega); X)$ can be viewed as a space of generalized X-valued functions on Ω. Indeed, if $\Omega = \mathbb{R}$ with the Lebesgue measure, $\gamma_\infty(\mathrm{L}^2(\mathbb{R}); X)$ is a Banach space of X-valued tempered distributions. For such distributions their Fourier transform is coherently defined via its adjoint action: $\mathcal{F}T := T \circ \mathcal{F}$, and the ideal property mentioned above shows that $\mathcal{F}$ restricts to almost isometric isomorphisms of $\gamma_\infty(\mathrm{L}^2(\mathbb{R}); X)$ and $\gamma(\mathrm{L}^2(\mathbb{R}); X)$. Similarly, the multiplication with some function $m \in \mathrm{L}^\infty(\mathbb{R})$ extends via adjoint action coherently to $\mathcal{L}(\mathrm{L}^2(\mathbb{R}); X)$, and the ideal property above yields that $\gamma_\infty(\mathrm{L}^2(\mathbb{R}); X)$ and $\gamma(\mathrm{L}^2(\mathbb{R}); X)$ are invariant. Furthermore,

$$\|T \mapsto mT\|_{\gamma_\infty \to \gamma_\infty} = \|m\|_\infty$$

for every $m \in \mathrm{L}^\infty(\mathbb{R})$. Combining these two facts we obtain that for each $m \in \mathrm{L}^\infty(\mathbb{R})$ the *Fourier multiplier* operator with symbol m

$$F_m(T) := \mathcal{F}^{-1}(m\mathcal{F}T) \quad \bigl(T \in \mathcal{L}\bigl(\mathrm{L}^2(\mathbb{R}); X\bigr)\bigr)$$

is bounded on $\gamma_\infty(\mathrm{L}^2(\mathbb{R}); X)$ and $\gamma(\mathrm{L}^2(\mathbb{R}); X)$ with norm estimate

$$\bigl\|F_m(T)\bigr\|_\gamma \leqslant \|m\|_{\mathrm{L}^\infty(\mathbb{R})}\|T\|_\gamma.$$

Similar remarks apply in the discrete case $\Omega = \mathbb{Z}$.

An important result in the theory of γ-radonifying operators is the *multiplier theorem*. Here one considers a bounded operator-valued function $T : \Omega \to \mathcal{L}(X; Y)$ and asks under what conditions the multiplier operator

$$\mathcal{M}_T : \mathrm{L}^2(\Omega; X) \to \mathrm{L}^2(\Omega; Y), \quad \mathcal{M}_T f = T(\cdot)f(\cdot)$$

is bounded for the γ-norms. To formulate the result, one needs a new notion.

Let X, Y be Banach spaces. A collection $\mathcal{T} \subseteq \mathcal{L}(X; Y)$ is said to be γ*-bounded* if there is a constant $c \geqslant 0$ such that

$$\mathbb{E}\biggl(\biggl\|\sum_{T\in\mathcal{T}'}\gamma_T T x_T\biggr\|_X^2\biggr)^{1/2} \leqslant c\,\mathbb{E}\biggl(\biggl\|\sum_{T\in\mathcal{T}'}\gamma_T x_T\biggr\|_X^2\biggr)^{1/2} \tag{6.1}$$

for all finite subsets $\mathcal{T}' \subseteq \mathcal{T}$, $(x_T)_{T\in\mathcal{T}'} \subseteq X$. (Again, $(\gamma_T)_{T\in\mathcal{T}'}$ is an independent collection of standard Gaussian random variables on some probability space.) If $\mathcal{T}$ is γ-bounded, the smallest constant c such that (6.1) holds, is denoted by $\gamma(\mathcal{T})$ and is called the *γ-bound* of $\mathcal{T}$. We are now ready to state the result, established by Kalton and Weis in [19].

Theorem 6.4 *(Multiplier theorem). Let $H = \mathrm{L}^2(\Omega)$ for some measure space (Ω, Σ, μ), and let X, Y be Banach spaces. Let $T : \Omega \to \mathcal{L}(X; Y)$ be a strongly μ-measurable mapping such that*

$$\mathcal{T} := \{T(\omega) \mid \omega \in \Omega\}$$

is γ-bounded. Then the multiplication operator

$$\mathcal{M}_T : \mathrm{L}^2(\Omega) \otimes X \to \mathrm{L}^2(\Omega; Y), \quad f \otimes x \mapsto f(\cdot)T(\cdot)x$$

extends uniquely to a bounded operator

$$\mathcal{M}_T : \gamma\big(\mathrm{L}^2(\Omega); X\big) \to \gamma_\infty\big(\mathrm{L}^2(\Omega); Y\big)$$

with $\|\mathcal{M}_T S\|_\gamma \leqslant \gamma(\mathcal{T})\|S\|_\gamma$ $(S \in \gamma(\mathrm{L}^2(\Omega); X))$.

It is unknown up to now whether such a multiplier $\mathcal{M}_T$ always must have its range in the smaller class $\gamma(\mathrm{L}^2(\Omega); Y)$.

6.2. Unbounded C_0-groups

Let us return to the main theme of this paper. In Section 3.1 we have applied the transference identities to unbounded C_0-groups in Banach spaces. In the case of a Hilbert space this yielded a proof of the Boyadzhiev–deLaubenfels theorem, i.e., that every generator of a C_0-group on a Hilbert space has bounded H^∞-calculus on vertical strips, if the strip height exceeds the exponential type of the group. The analogue of this result for general Banach spaces but under γ-boundedness conditions is due to Kalton and Weis [19, Thm. 6.8]. We give a new proof using our transference techniques.

Recall that the *exponential type* of a C_0-group on a Banach space X is

$$\theta(U) := \inf\big\{\omega \geqslant 0 \,\big|\, \exists M \geqslant 0\colon\ \|U(s)\| \leqslant M\mathrm{e}^{\omega|s|}\ (s \in \mathbb{R})\big\}.$$

Let us call the number

$$\theta_\gamma(U) := \inf\big\{\omega \geqslant 0 \,\big|\, \{e^{-\omega|s|}U(s) \mid s \in \mathbb{R}\} \text{ is } \gamma\text{-bounded}\big\}$$

the *exponential γ-type* of the group U. If $\theta_\gamma(U) < \infty$ we call U *exponentially γ-bounded*. The following is the γ-analogue of the Boyadzhiev–deLaubenfels theorem, see Eq. (3.2).

Theorem 6.5 *(Kalton–Weis). Let $-iA$ be the generator of a C_0-group $(U(s))_{s\in\mathbb{R}}$ on a Banach space X. Suppose that U is exponentially γ-bounded. Then A has a bounded $\mathrm{H}^\infty(\mathrm{St}(\omega))$-calculus for every $\omega > \theta_\gamma(U)$.*

Proof. Choose $\theta_\gamma(U) < \omega < \alpha$. By usual approximation techniques [16, Proof of Theorem 3.6] it suffices to show an estimate

$$\|f(A)\| \lesssim \|f\|_{\mathrm{H}^\infty(\mathrm{St}(\omega))}$$

only for $f = \mathcal{F}\mu$ with μ a measure such that $\mu_\omega \in \mathrm{M}(\mathbb{R})$. (Recall from Section 3.1 that $\mu_\omega(\mathrm{d}t) = \cosh(\omega t)\,\mu(\mathrm{d}t)$, so that $f = \mathcal{F}\mu$ has a bounded holomorphic extension to $\mathrm{St}(\omega)$.) By the transference identity (3.1) the operator $f(A)$ factorizes as

$$f(A) = P \circ L_{\mu_\omega} \circ \iota.$$

Here L_{μ_ω} is convolution with μ_ω,

$$\iota x(s) = \frac{1}{\cosh \alpha s} U(-s)x \quad (x \in X,\ s \in \mathbb{R})$$

and

$$PF = \int_{\mathbb{R}} \psi(t) U(t) F(t)\,\mathrm{d}t.$$

In Section 3.1 this factorization was considered to go via the space $\mathrm{L}^2(\mathbb{R}; X)$, i.e.,

$$\iota : X \to \mathrm{L}^2(\mathbb{R}; X), \qquad P : \mathrm{L}^2(\mathbb{R}; X) \to X.$$

However, the exponential γ-boundedness of U will allow us to replace the space $\mathrm{L}^2(\mathbb{R}; X)$ by $\gamma(\mathrm{L}^2(\mathbb{R}); X)$. But on $\gamma(\mathrm{L}^2(\mathbb{R}); X)$ the convolution with μ_ω—which is the Fourier multiplier with symbol $\mathcal{F}\mu_\omega$—is bounded with a norm not exceeding $\|\mathcal{F}\mu_\omega\|_{\mathrm{L}^\infty(\mathbb{R})}$. That, in turn, by elementary computations and the maximum principle can be majorized by $\|\mathcal{F}\mu\|_{\mathrm{H}^\infty(\mathrm{St}(\omega))}$, cf. Section 3.1.

To see that indeed $\iota : X \to \gamma(\mathrm{L}^2(\mathbb{R}); X)$, we write

$$(\iota x)(s) = \frac{1}{\cosh \alpha s} U(-s)x = \left(e^{-\omega|s|} U(-s)\right)\left(\frac{e^{\omega|s|}}{\cosh \alpha s} x\right)$$

and use the multiplier Theorem 6.4 to conclude that $\iota : X \to \gamma_\infty(\mathbb{R}; X)$ boundedly. To see that $\operatorname{ran}(\iota) \subseteq \gamma(\mathbb{R}; X)$ we employ a density argument. If $x \in \operatorname{dom}(A)$, write $\iota x = \psi \cdot u$ with

$$\psi(s) = \cosh(\alpha s)^{-1} \quad \text{and} \quad u(s) = U(-s)x \quad (s \in \mathbb{R}).$$

Then $u \in \mathrm{C}^1(\mathbb{R}; X)$, $u'(s) = iU(-s)Ax$, $\psi \in \mathrm{L}^2(\mathbb{R})$, and

$$\int_0^\infty \|\psi\|_{\mathrm{L}^2(s,\infty)} \left\|u'(s)\right\|_X \mathrm{d}s, \qquad \int_{-\infty}^0 \|\psi\|_{\mathrm{L}^2(-\infty,s)} \left\|u'(s)\right\|_X \mathrm{d}s < \infty.$$

Hence,

$$\iota x = \psi \cdot u = \psi \otimes x + \int_0^\infty \mathbf{1}_{(s,\infty)} \psi \otimes u'(s)\,\mathrm{d}s - \int_{-\infty}^0 \mathbf{1}_{(-\infty,s)} \psi \otimes u'(s)\,\mathrm{d}s \in \gamma(\mathbb{R}; X)$$

by Corollary 6.3. (One has to apply (1) to the part of ψu on $\mathbb{R}_+$ and (2) to the part on $\mathbb{R}_-$.) Since $\mathrm{dom}(A)$ is dense in X, we conclude that $\mathrm{ran}(\iota) \subseteq \gamma(\mathrm{L}^2(\mathbb{R}); X)$ as claimed.

Finally, we show that $P : \gamma(\mathrm{L}^2(\mathbb{R}); X) \to X$ is well defined. Clearly

$$P = \big(\text{integrate against } e^{\theta|t|}\varphi(t)\big) \circ \big(\text{multiply with } e^{-\theta|t|}U(t)\big)$$

where $\theta_\gamma(U) < \theta < \omega$. We know that $\varphi(t) = O(e^{-\omega|t|})$, so by the multiplier Theorem 6.4, everything works out fine. Note that in order to be able to apply the multiplier theorem, we have to start already in $\gamma(\mathrm{L}^2(\mathbb{R}); X)$. And this is why we had to ensure that ι maps there in the first place. □

Remark 6.6. Independently of us, Le Merdy [22] has recently obtained a γ-version of the classical transference principle for bounded groups. The method is similar to ours, by re-reading the transference principle with the γ-space in place of a Bochner space.

6.3. Peller's theorem—γ-version, discrete case

We now turn to the extension of Peller's theorems (see Section 5) from Hilbert spaces to general Banach spaces. We begin with the discrete case.

Theorem 6.7. *There is an absolute constant $c \geqslant 0$ such that the following holds: Let X be a Banach space, and let $T \in \mathcal{L}(X)$ such that the set*

$$\mathcal{T} := \big\{(1+n)^{-\alpha} T^n \;\big|\; n \geqslant 0\big\}$$

is γ-bounded. Then

$$\big\| f(T) \big\| \leqslant c\, 9^\alpha \gamma(\mathcal{T})^2 \| f \|_{\mathrm{B}^{2\alpha}_{\infty,1}(\mathbb{D})}$$

for every polynomial f.

The theorem is a consequence of the following lemma, the arguments being completely analogous to the proof of Theorem 5.1.

Lemma 6.8. *There is a constant $c \geqslant 0$ such that*

$$\big\| \widehat{\mu}(T) \big\| \leqslant c\big(1 + \log(b/a)\big) M(b) \| \widehat{\mu} \|_{\mathrm{H}^\infty(\mathbb{D})}$$

whenever the following hypotheses are satisfied:

(1) T *is a bounded operator on a Banach space* X;
(2) $a, b \in \mathbb{Z}_+$ *with* $1 \leqslant a \leqslant b$;
(3) $M(b) := \gamma\{T^n \mid 0 \leqslant n \leqslant b\}$;
(4) $\mu \in \ell^1(\mathbb{Z}_+)$ *such that* $\operatorname{supp}(\mu) \subseteq [a, b]$.

Proof. This is analogous to Theorem 4.6. Take $\varphi, \psi \in \mathrm{L}^2(\mathbb{Z}_+)$ such that $\psi * \varphi = 1$ on $[a, b]$ and $\operatorname{supp}\varphi, \operatorname{supp}\psi \subseteq [0, b]$. Then

$$\widehat{\mu}(T) = \langle \varphi \mathrm{T}, \mu * (\psi \mathrm{T})^{\sim} \rangle = P \circ L_\mu \circ \iota,$$

see (2.3). Note that only functions of finite support are involved here, so $\operatorname{ran}(\iota) \subseteq \mathrm{L}^2(\mathbb{Z}) \otimes X$. Hence we can take γ-norms and estimate

$$\|\widehat{\mu}(T)\| \leqslant \|P\|_{\gamma(\mathrm{L}^2(\mathbb{Z});X)\to X} \|L_\mu\|_{\gamma\to\gamma} \|\iota\|_{X\to\gamma(\mathrm{L}^2(\mathbb{Z});X)}.$$

Note that

$$\iota x = \left(\mathrm{T}^{\sim}\mathbf{1}_{[-b,0]}\right)\left(\psi^{\sim} \otimes x\right)$$

so the multiplier theorem yields

$$\|\iota x\|_\gamma \leqslant M(b)\left\|\psi^{\sim} \otimes x\right\|_\gamma = M(b)\|\psi\|_2\|x\|.$$

Similarly, P can be decomposed as

$$P = (\text{integrate against } \varphi) \circ (\text{multiply with } \mathbf{1}_{[0,b]}\mathrm{T})$$

and hence the multiplier theorem yields

$$\|P\|_{\gamma\to X} \leqslant \|\varphi\|_2 M(b).$$

Finally note that

$$\|L_\mu\|_{\gamma\to\gamma} = \|\widehat{\mu}\|_{\mathrm{H}^\infty(\mathbb{D})}$$

since—similar to the continuous case—all bounded measurable functions on $\mathbb{T}$ define bounded Fourier multipliers on $\gamma(\mathrm{L}^2(\mathbb{Z}); X)$. Putting the pieces together we obtain

$$\|\widehat{\mu}(T)\| \leqslant M(b)^2 \|\varphi\|_2 \|\psi\|_2 \|\widehat{\mu}\|_{\mathrm{H}^\infty(\mathbb{D})}$$

and an application of Lemma A.2 concludes the proof. □

6.4. Peller's theorem—γ-version, continuous case

We turn to the continuous version(s) of Peller's theorem.

Theorem 6.9. *There is an absolute constant $c \geqslant 0$ such that the following holds: Let $-A$ be the generator of a strongly continuous semigroup $\mathrm{T} = (T(s))_{s\geqslant 0}$ on a Banach space X. Suppose that $\alpha \geqslant 0$ is such that the set*

$$\mathcal{T} := \bigl\{(1+s)^{-\alpha} T(s) \bigm| s \geqslant 0\bigr\}$$

is γ-bounded. Then

$$\bigl\|f(A)\bigr\| \leqslant c\, 9^{\alpha} \gamma(\mathcal{T})^2 \|f\|_{\mathrm{B}^{0,2\alpha}_{\infty,1}(\mathbb{C}_+)}$$

for every $f = \mathcal{L}\mu$, $\mu \in \mathrm{M}(\mathbb{R}_+)$ with compact support.

Let us formulate a minor generalization in the special case of γ-bounded semigroups, analogous to Corollary 5.5 above,

Corollary 6.10. *There is an absolute constant $c \geqslant 0$ such that the following holds: Let $-A$ be the generator of a strongly continuous semigroup $\mathrm{T} = (T(s))_{s\geqslant 0}$ on a Banach space X such that the set*

$$\mathcal{T} := \bigl\{T(s) \bigm| s \geqslant 0\bigr\}$$

is γ-bounded. Then

$$\bigl\|f(A)\bigr\| \leqslant c\, \gamma(\mathcal{T})^2 \|f\|_{\mathrm{B}^{0,0}_{\infty,1}(\mathbb{C}_+)}$$

for every $f = \mathcal{L}\mu$, μ a bounded measure on $\mathbb{R}_+$.

The proofs are analogous to the proofs in the Hilbert space case, based on the following lemma.

Lemma 6.11. *There is a constant $c \geqslant 0$ such that*

$$\bigl\|f(A)\bigr\| \leqslant c\bigl(1 + \log(b/a)\bigr) M(b)^2 \|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)} \tag{6.2}$$

whenever the following hypotheses are satisfied:

(1) $\mathrm{T} = (T(s))_{s\geqslant 0}$ *is a C_0-semigroup on the Banach space X;*
(2) $0 < a < b < \infty$;
(3) $M(b) := \gamma\{T(s) \mid 0 \leqslant s \leqslant b\}$;
(4) $f = \mathcal{L}\mu$, *where $\mu \in \mathrm{M}(\mathbb{R}_+)$ such that* $\operatorname{supp}(\mu) \subseteq [a, b]$.

Proof. We re-examine the proof of Theorem 4.1. Choose $\varphi, \psi \in \mathrm{L}^2(0,b)$ such that $\varphi * \psi = 1$ on $[a,b]$. Then

$$f(A) = \mathrm{T}_\mu = P \circ L_\mu \circ \iota,$$

where for $x \in X$ and $F : \mathbb{R} \to X$

$$\iota x = \psi^\sim \mathrm{T}^\sim x, \qquad PF = \int_0^b \varphi(t)T(t)F(t)\,\mathrm{d}t.$$

We claim that $\iota : X \to \gamma(\mathbb{R}; X)$ with

$$\|\iota\|_{X \to \gamma} \leqslant M(b)\|\psi\|_{\mathrm{L}^2(0,b)}.$$

As in the case of groups, the estimate follows from the multiplier theorem; and the fact that $\mathrm{ran}(\iota) \subseteq \gamma(\mathbb{R}; X)$ (and not just $\gamma_\infty(\mathbb{R}; X)$) comes from a density argument. Indeed, if $x \in \mathrm{dom}(A)$ then $\iota x = \psi^\sim \cdot u$ with $u(s) = T(-s)x$ for $s \leqslant 0$. Since $u \in \mathrm{C}^1[-b,0]$ and $\psi^\sim \in \mathrm{L}^2(-b,0)$, Corollary 6.3 and the ideal property yield that $\iota x = \psi^\sim \cdot u \in \gamma((-b,0); X) \subseteq \gamma(\mathbb{R}; X)$. Since $\mathrm{dom}(A)$ is dense in X, $\mathrm{ran}(\iota) \subseteq \gamma(\mathbb{R}; X)$, as claimed.

Note that P can be factorized as

$$P = (\text{integrate against } \varphi) \circ (\text{multiply with } \mathbf{1}_{(0,b)}\mathrm{T})$$

and so $\|P\|_{\gamma \to X} \leqslant M(b)\|\varphi\|_{\mathrm{L}^2(0,b)}$ by the multiplier theorem. We combine these results to obtain

$$\|f(A)\| \leqslant M(b)^2 \|\varphi\|_{\mathrm{L}^2(0,b)} \|\psi\|_{\mathrm{L}^2(0,b)} \cdot \|f\|_{\mathrm{H}^\infty(\mathbb{C}_+)}$$

and an application of Lemma A.1 concludes the proof. □

7. Singular integrals and functional calculus

7.1. Functional calculus

The results of Sections 5 and 6 provided estimates of the form

$$\|f(A)\| \lesssim \|f\|_{\mathrm{B}^{0,2\alpha}_{\infty,1}(\Sigma_\theta)}$$

under various conditions on the Banach space X, the semigroup T or the angle θ. However, to derive these estimates we required $f = \mathcal{L}\mu$, μ some bounded measure of compact support. It is certainly natural to ask whether one can extend the results to all $f \in \mathrm{B}^{0,2\alpha}_{\infty,1}(\Sigma_\theta)$, i.e., to a proper Besov class functional calculus.

The major problem here is not the norm estimate, but the *definition* of $f(A)$ in the first place. (If $f = \mathcal{L}\mu$ for a measure μ with compact support, this problem does not occur.) Of course one could pass to a closure with respect to the Besov norm, but this yields a too small function class in general. And it does not show how this definition of $f(A)$ relates with all the others in the literature, especially, with the functional calculus for sectorial operators [12] and the one for half-plane type operators [13].

Unfortunately, although these questions appear to have quite satisfying answers, a diligent treatment of them would extend this already long paper beyond a reasonable size, so we postpone it to a future publication.

7.2. Singular integrals for semigroups

A usual consequence of transference estimates is the convergence of certain singular integrals. It has been known for a long time that if $(U(s))_{s\in\mathbb{R}}$ is a C_0-group on a UMD space X then the principal value integral

$$\int_{-1}^{1} U(s)x\,\frac{\mathrm{d}s}{s}$$

exists for every $x \in X$. This was the decisive ingredient in the Dore–Venni theorem and in Fattorini's theorem, as discussed in [14]. For semigroups, these proofs fail and this is not surprising as one has to profit from cancellation effects around 0 in order to have a principal value integral converging. Our results from Sections 4 and 5 now imply that if one shifts the singularity away from 0 then the associated singular integral for a semigroup will converge, under suitable assumptions on the Banach space or the semigroup. For groups we gave a fairly general statement in [16, Theorem 4.4].

Theorem 7.1. *Let $(T(s))_{s\geqslant 0}$ be a C_0-semigroup on a* UMD *Banach space X, let $0 < a < b$, and let $g \in \mathrm{BV}[b-a, b+a]$ be such that $g(\cdot + b)$ is even. Then the principal value integral*

$$\lim_{\epsilon \searrow 0} \int_{\epsilon a \leqslant |s-b| \leqslant a} g(s)T(s)x\,\frac{\mathrm{d}s}{s-b} \tag{7.1}$$

converges for every $x \in X$.

Proof. Define

$$f_\epsilon(z) := \int_{\epsilon a \leqslant |s-b| \leqslant a} g(s)e^{-sz}\,\frac{\mathrm{d}s}{s-b} \quad (z \in \mathbb{C}).$$

Then, since g is even about the singularity b,

$$f_\epsilon(A)x = \int_{\epsilon a \leqslant |s-b| \leqslant a} g(s)T(s)\frac{\mathrm{d}s}{s-b} = \int_{\epsilon a \leqslant |s-b| \leqslant a} g(s)\bigl(T(s)x - T(b)x\bigr)\frac{\mathrm{d}s}{s-b}.$$

If $x \in \operatorname{dom}(A)$ then $T(\cdot)x$ is continuously differentiable, and therefore $\lim_{\epsilon \searrow 0} f_\epsilon(A)x$, which is (7.1), exists. Hence, by density, one only has to show that $\sup_{0<\epsilon<1} \| f_\epsilon(A)\|$ is finite. In order to establish this, we use Theorem 4.1 to estimate

$$\| f_\epsilon(A)\| \lesssim \left(1 + \log\left(\frac{b+a}{b-a}\right)\right) \| f_\epsilon(i\cdot)\|_{\mathcal{M}_{p,X}}.$$

Writing $h(x) = g(ax+b)$ we obtain

$$f_\epsilon(it) = e^{-itb} \int\limits_{\epsilon \leqslant |s| \leqslant 1} e^{-iats} \frac{h(s)}{s}\,\mathrm{d}s = e^{-ibt}\mathcal{F}\left(\mathrm{PV}-\frac{h_\epsilon}{s}\right)(at),$$

where $h_\epsilon = h\mathbf{1}_{\{\epsilon \leqslant |s| \leqslant 1\}}$. It is a standard fact from Fourier multiplier theory that the exponential factor in front and the dilation by a in the argument do not change Fourier multiplier norms. So one is reduced to estimate the $\mathcal{M}_{p,X}$-norms of the functions

$$m_\epsilon := \mathcal{F}\left(\mathrm{PV}-\frac{h_\epsilon}{s}\right) \quad (0<\epsilon<1).$$

By the UMD-version of Mikhlin's theorem, $\|m_\epsilon\|_{\mathcal{M}_p}$ can be estimated by its Mikhlin norm, and by [16, Lemma 4.3] this in turn can be estimated by the BV-norm of m_ϵ. But since $\mathrm{BV}[-1,1]$ is a Banach algebra, and the characteristic functions $\mathbf{1}_{\{\epsilon \leqslant |s| \leqslant 1\}}$ have uniformly bounded BV-norms for $\epsilon \in (0,1)$, we are done. □

Remark 7.2. The result is also true on a general Banach space if $\{T(s) \mid 0 \leqslant s \leqslant 1\}$ is γ-bounded. The proof is analogous, but in place of Theorem 4.1 one has to employ Lemma 6.11.

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Appendix A. Two lemmata

We provide two lemmata concerning an optimization problem for convolutions on the halfline or the positive integers.

Lemma A.1 *(Haase–Hytönen). Let $p \in (1,\infty)$. For $0<a<b$ let*

$$c(a,b) := \inf\bigl\{\|\varphi\|_{p'}\|\psi\|_p\colon \varphi \in \mathrm{L}^{\mathrm{p}'}(0,b),\ \psi \in \mathrm{L}^{\mathrm{p}}(0,b),\ \varphi * \psi = 1 \text{ on } [a,b]\bigr\}.$$

Then there are constants $D_p, C_p > 0$ *such that*

$$D_p\big(1+\log(b/a)\big) \leqslant c(a,b) \leqslant C_p\big(1+\log(b/a)\big)$$

for all $0 < a < b$.

Proof. We fix $p \in (1,\infty)$. Suppose that $\varphi \in \mathrm{L}^{\mathrm{p}'}(\mathbb{R}_+)$ and $\psi \in \mathrm{L}^{\mathrm{p}}(\mathbb{R}_+)$ with $\varphi * \psi = 1$ on $[a,b]$. Then, by Hölder's inequality,

$$1 = \big|(\varphi * \psi)(a)\big| \leqslant \|\varphi\|_{p'} \|\psi\|_p,$$

which implies $c(a,b) \geqslant 1$. Secondly,

$$\begin{aligned}\log(b/a) &= \left|\int_a^b (\varphi * \psi)(t)\frac{\mathrm{d}t}{t}\right| \leqslant \int_a^b \int_0^t \big|\varphi(t-s)\big|\big|\psi(s)\big|\,\mathrm{d}s\,\frac{\mathrm{d}t}{t} \\ &\leqslant \int_0^\infty \int_s^\infty \frac{|\varphi(t-s)|}{t}\,\mathrm{d}t\,\big|\psi(s)\big|\,\mathrm{d}s = \int_0^\infty \int_0^\infty \frac{|\varphi(t)||\psi(s)|}{t+s}\,\mathrm{d}t\,\mathrm{d}s \\ &\leqslant \frac{\pi}{\sin(\pi/p)} \|\varphi\|_{p'}\|\psi\|_p.\end{aligned}$$

(This is "Hilbert's absolute inequality", see [11, Chapter 5.10].) This yields

$$c(a,b) \geqslant \frac{\sin(\pi/p)}{\pi} \log\left(\frac{b}{a}\right).$$

Taking both we arrive at

$$1 \vee \frac{\sin(\pi/p)}{\pi} \log\left(\frac{b}{a}\right) \leqslant c(a,b).$$

Since $\sin(\pi/p) \neq 0$, one can find $D_p > 0$ such that

$$D_p\big(1+\log(b/a)\big) \leqslant 1 \vee \frac{\sin(\pi/p)}{\pi} \log(b/a)$$

and the lower estimate is established.

To prove the upper estimate we note first that without loss of generality we may assume that $a = 1$. Indeed, passing from (φ, ψ) to $(a^{1/p'}\varphi(a\cdot), a^{1/p}\psi(a\cdot))$ reduces the (a,b)-case to the $(1, b/a)$-case and shows that $c(a,b) = c(1, b/a)$. The idea is now to choose φ, ψ in such a way that

$$(\varphi * \psi)(t) = \begin{cases} t, & t \in [0,1], \\ 1, & t \geqslant 1, \end{cases}$$

and cut them after b. Taking Laplace transforms, this means

$$\big[(\mathcal{L}\varphi)\cdot(\mathcal{L}\psi)\big](z) = \frac{1-e^{-z}}{z^2}$$

for $\operatorname{Re} z > 0$. Fix $\theta \in (0,1)$ and write

$$\frac{1-e^{-z}}{z^2} = \frac{(1-e^{-z})^{(1-\theta)}}{z} \cdot \frac{(1-e^{-z})^{\theta}}{z}.$$

Now, by the binomial series,

$$\frac{(1-e^{-z})^{\theta}}{z} = \sum_{k=0}^{\infty} \alpha_k^{(\theta)} \frac{e^{-kz}}{z} = \sum_{k=0}^{\infty} \alpha_k^{(\theta)} \mathcal{L}(\mathbf{1}_{(k,\infty)})(z),$$

and writing $\mathbf{1}_{(k,\infty)} = \sum_{j=k}^{\infty} \mathbf{1}_{(j,j+1)}$ we see that we can take

$$\psi = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \alpha_k^{(\theta)} \mathbf{1}_{(j,j+1)} = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} \alpha_k^{(\theta)}\right)\mathbf{1}_{(j,j+1)}$$

and likewise

$$\varphi = \sum_{j=0}^{\infty}\left(\sum_{k=0}^{j} \alpha_k^{(1-\theta)}\right)\mathbf{1}_{(j,j+1)}.$$

Let $\beta_j^{(\theta)} = \sum_{k=0}^{j} \alpha_k^{(\theta)}$. By standard asymptotic analysis

$$\alpha_k^{(\theta)} = O\left(\frac{1}{k^{1+\theta}}\right) \quad \text{and} \quad \beta_j^{(\theta)} = O\left(\frac{1}{(1+j)^{\theta}}\right).$$

It is clear that

$$c(1,b) \leqslant \|\varphi \mathbf{1}_{(0,b)}\|_{p'} \|\psi \mathbf{1}_{(0,b)}\|_{p}.$$

Now,

$$\|\psi \mathbf{1}_{(0,b)}\|_p^p = \int_0^b \big|\psi(t)\big|^p \,\mathrm{d}t = \sum_{j=0}^{\infty} \big(\beta_j^{\theta}\big)^p \int_0^b \mathbf{1}_{(j,j+1)}(t)\,\mathrm{d}t \lesssim \sum_{j=0}^{\infty} (1+j)^{-\theta p} \gamma_{j,b}$$

with

$$\gamma_{j,b} = \begin{cases} 1, & j \leqslant b-1, \\ b-j, & j \leqslant b \leqslant j+1, \\ 0, & b \leqslant j. \end{cases}$$

With $\theta := 1/p$ this yields

$$\|\psi \mathbf{1}_{(0,b)}\|_p^p \leqslant 1 + \sum_{j=1}^{\lfloor b\rfloor -1} \int_j^{j+1} \frac{dx}{x} + \frac{b-\lfloor b\rfloor}{1+\lfloor b\rfloor} \leqslant 2 + \log\bigl(\lfloor b\rfloor\bigr) \leqslant 2(1+\log b).$$

Analogously, noting that $1-\theta = 1-(1/p) = 1/p'$,

$$\|\varphi \mathbf{1}_{(0,b)}\|_{p'}^{p'} \lesssim 2(1+\log b)$$

which combines to

$$c(1,b) \lesssim (1+\log b)$$

as was to be proved. □

Now we state and prove an analogue in the discrete case.

Lemma A.2. *Let* $p \in (1,\infty)$. *For* $a, b \in \mathbb{N}$ *with* $a \leqslant b$ *let*

$$c(a,b) := \inf\bigl\{\|\varphi\|_{p'}\|\psi\|_p \colon \varphi \in \ell^{p'}(\mathbb{Z}_+),\ \psi \in \ell^p(\mathbb{Z}_+),\ \varphi * \psi = 1 \text{ on } [a,b]\bigr\}.$$

Then there are constants $C_p, D_p > 0$ *such that*

$$D_p\bigl(1+\log(b/a)\bigr) \leqslant c(a,b) \leqslant C_p\bigl(1+\log(b/a)\bigr)$$

for all $0 < a < b$.

Proof. The proof is similar to the proof of Lemma A.1. The lower estimate is obtained in a totally analogous fashion, making use of the discrete version of Hilbert's absolute inequality [11, Thm. 5.10.2] and the estimate

$$\sum_{n=a}^{b} \frac{1}{n+1} \geqslant \frac{1}{2}\log(b/a).$$

For the upper estimate we let

$$\eta(j) := \begin{cases} j/a, & j = 0,1,\ldots,a, \\ 1, & j \geqslant a+1, \end{cases}$$

and look for a factorization $\varphi * \psi = \eta$. Considering the Fourier transform we find

$$\widehat{\eta}(z) = \frac{z}{a}\frac{1-z^a}{(1-z)^2}$$

and so we try (as in the continuous case) the "Ansatz"

$$\psi = \frac{z}{a^\theta} \frac{(1-z^a)^\theta}{1-z} \quad \text{and} \quad \varphi = \frac{1}{a^{1-\theta}} \frac{(1-z^a)^{1-\theta}}{1-z}$$

for $\theta := 1/p$. Note that

$$\psi(z) = \frac{z}{a^\theta(1-z)} \sum_{j=0}^{\infty} \alpha_j^{(\theta)} z^{aj} = \frac{z}{a^\theta(1-z)} \sum_{k=0}^{\infty} \gamma_k z^k,$$

where

$$\gamma_k = \gamma_k(a,\theta) = \begin{cases} \alpha_{k/a}^{(\theta)} & \text{if } a \mid k, \\ 0 & \text{else.} \end{cases}$$

Consequently,

$$\psi(z) = \frac{z}{a^\theta} \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \gamma_k \right) z^n = \frac{z}{a^\theta} \sum_{n=0}^{\infty} \beta_{\lfloor n/a \rfloor}^{(\theta)} z^n$$

and, likewise,

$$\varphi(z) = \frac{1}{a^{1-\theta}} \sum_{n=0}^{\infty} \beta_{\lfloor n/a \rfloor}^{(1-\theta)} z^n.$$

As in the continuous case, it suffices to cut off φ and ψ after b, so

$$c(a,b) \leqslant \|\varphi \mathbf{1}_{[0,b]}\|_{p'} \|\psi \mathbf{1}_{[0,b]}\|_p.$$

Now write $b = ka + r$ with $0 \leqslant r < a$ and $k := \lfloor b/a \rfloor$; then

$$\begin{aligned} \|\psi \mathbf{1}_{[0,b]}\|_p^p &\leqslant \frac{1}{a} \sum_{n=0}^{b} \left|\beta_{\lfloor n/a \rfloor}^{(\theta)}\right|^p \lesssim \frac{1}{a} \sum_{n=0}^{b} \big(1 + \lfloor n/a \rfloor\big)^{-1} \\ &= \frac{1}{a} \left(\frac{a}{1} + \frac{a}{2} + \cdots + \frac{a}{k} + \frac{r}{k+1} \right) \leqslant \sum_{j=1}^{k+1} \frac{1}{j} \\ &\leqslant 1 + \int_1^{k+1} \frac{dx}{x} = 1 + \log(k+1) \leqslant 2\big(1 + \log(b/a)\big). \end{aligned}$$

A similar estimate holds for $\|\varphi \mathbf{1}_{[0,b]}\|_p^p$. $\square$

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