Algebraically closed MV-algebras and their sheaf representation

A. Di Nola, A.R. Ferraioli, G. Lenzi *

Università di Salerno, via Ponte Don Melillo, 84084 Fisciano (SA), Italy

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ABSTRACT

In this paper we first provide a new axiomatization of algebraically closed MV-algebras based on McNaughton’s Theorem. Then we turn to sheaves, and we represent algebraically closed MV-algebras as algebras of global sections of sheaves, where the stalks are divisible MV-chains and the base space is Stonean.

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1. Introduction

In this paper we are interested in MV-algebras. These algebras are the many-valued equivalents of Boolean algebras, which are the algebraic semantics of Classical 2-valued Logic. It is well known that free MV-algebras over any set of generators are MV-algebras of McNaughton’s functions, which are piecewise linear functions, so, it is natural to associate to free MV-algebras the geometry of simplexes [4]. However, simplexes could be no more sufficient when one passes to more general MV-algebras. So we would like to understand the algebraic geometry of general MV-algebras, in analogy with classical algebraic geometry over rings.

A basic tool of algebraic geometry consists of algebraically closed fields which provide a natural environment for algebraic varieties (that are sets of zeros of systems of polynomial equations). Another modern approach due to Grothendieck and Serre is based on the notions of sheaf and scheme. We would like to combine both tools in order to try to understand some algebraic geometry for MV-algebras. This paper is a first step in this direction.

Algebraically closed MV-algebras (a.c. MV-algebras) are studied by Lacava in [11] and [12]. In the latter paper he characterizes a.c. MV-algebras as regular and divisible MV-algebras. The main results of this paper are:

• an axiomatization of algebraically closed MV-algebras different from the one of [11];
• a sheaf representation of algebraically closed MV-algebras via divisible MV-chains and Stone spaces.

We would like to stress that the sheaf representation of a.c. MV-algebras is more concrete than the purely algebraic characterization by Lacava since it gives a decomposition of an a.c. MV-algebra into simpler and better known MV-algebras.
(remember that divisible MV-chains correspond via Mundici’s functor \( \Gamma \) (see [4]) to totally ordered divisible abelian groups).

2. Preliminaries

The main tool we use in this paper is sheaf representation of algebras. We borrow from [5] the standard definition of a sheaf of sets or a sheaf of algebras belonging to a given variety. In particular, we have the notion of a sheaf of MV-algebras.

In order to prove the completeness theorem of Łukasiewicz infinite-valued logic, Chang introduced MV-algebras in [3]. In order to make this preliminary section not too long we give only a quick review of MV-algebras, referring to [4] for further details.

An MV-algebra is a structure \((A, \oplus, \ast, 0)\), where \(\oplus\) is a binary operation, \(\ast\) is a unary operation and \(0\) is a constant such that the following axioms are satisfied for any \(a, b \in A\):

1. \((A, \oplus, 0)\) is an abelian monoid,
2. \((a^\ast)^\ast = a\),
3. \(0^\ast \oplus a = 0^\ast\),
4. \((a^\ast \oplus b)^\ast \oplus b = (b^\ast \oplus a)^\ast \oplus a\).

On an MV-algebra \(A\) we define the constant \(1\) and the auxiliary operation \(\odot\) as follows:

1. \(1 := 0^\ast\),
2. \(a \odot b := (a^\ast \oplus b^\ast)^\ast\)

for any \(a, b \in A\).

An example of an MV-algebra is given by the real interval \([0, 1]\) where \(x \oplus y = \min\{x + y, 1\}\) and \(x^\ast = 1 - x\). This MV-algebra is important because it generates the variety of all MV-algebras.

For the partial order on MV-algebras \(x \leq y\) and the infimum and supremum \(x \land y, x \lor y\) we refer to [4]. Also, for boolean (i.e. idempotent) elements we refer to [4]. We denote by \(B(A)\) the set of all idempotents of \(A\); \(B(A)\) is the largest boolean subalgebra of \(A\).

Since MV-algebras form a variety, the notions of MV-homomorphism, quotient, ideal are just the particular cases of the corresponding universal algebraic notions. For prime ideals, maximal ideals, and the radical ideal we refer to [4]. The set of all prime ideals of an MV-algebra \(A\) is denoted by \(\text{Spec}(A)\), while \(\text{Max}(A)\) and \(\text{Min}(A)\) denote the sets of the maximal and minimal prime ideals of \(A\), respectively. \(\text{Rad}(A)\) denotes the radical ideal of \(A\).

We can equip \(\text{Spec}(A)\) with the Zariski topology. With such topology, \(\text{Spec}(A)\) is a compact topological space [1]. With the induced topology, \(\text{Max}(A)\) is a compact Hausdorff topological space.

For every \(\emptyset \neq X \subseteq A\), we denote \(X^\perp = \{y \in A \mid y \land x = 0, \text{ for every } x \in X\}\). Then \(X^\perp\) is an ideal of \(A\) [1]. If \(X = \{x\}\), then we write \(x^\perp\) for \(X^\perp\).

For local and simple MV-algebras we refer to [4].

An MV-algebra \(A\) is said to be divisible if and only if for any \(x \in A\) and for any natural number \(n \geq 1\) there is \(y \in A\) such that \(ny = x\) and \(y \land (n - 1) y = 0\). Note that the second condition intuitively means \(y \leq 1/n\), but not quite, since \(1/n\) is not necessarily present in an MV-algebra. In every divisible MV-algebra, for each given \(x \in A\) there is a unique \(y\) as above, and \(y\) will be denoted by \(\delta_n(x)\).

It is worth to stress that divisible MV-algebras hold an important role in the proof of Chang’s Completeness Theorem for MV-algebras. Moreover, the class of divisible MV-algebras is closed under quotient.

An ideal \(H\) of an MV-algebra \(A\) is called primary if it is contained in a unique maximal ideal. Note that for every ideal \(H\), the quotient MV-algebra \(A/H\) is local if and only if \(H\) is primary.

For every \(P \in \text{Spec}(A)\), we define \(O(P) = \bigcap\{m \in \text{Min}(A) \mid m \subseteq P\}\). It follows that \(O(P)\) is an ideal of \(A\) such that \(O(P) \subseteq P\). Moreover in [9] the following characterization of \(O(P)\) is given: \(O(P) = \bigcup\{a^{\perp} \mid a \notin P\}\). Moreover it results that \(O(P)\) is a primary ideal for each \(P \in \text{Spec}(A)\) and the intersection \(\bigcap\{O(M) \mid M \in \text{Max}(A)\} = \{0\}\).

An MV-algebra \(A\) is said to be regular if and only if each minimal prime ideal is Stonean in the lattice reduct of \(A\), i.e. it is generated by idempotent elements. In [12], Lacava proved that an MV-algebra is regular if and only if it is quasi-completely boolean dominated. Recall that an MV-algebra \(A\) is said to be quasi-completely boolean dominated if and only if, for each \(x, y \in A\) such that \(x \land y = 0\), there exist \(b_1, b_2 \in B(A)\) such that \(b_1 \supseteq x\), \(b_2 \supseteq y\) and \(b_1 \land b_2 = 0\). Moreover, the maximal spectrum of a regular MV-algebra is a Stone space with the Zariski topology [2].

Proposition 1. Let \(A\) be a regular MV-algebra. For each \(M \in \text{Max}(A)\), \(O(M)\) is a minimal prime ideal.

Proof. Remember that \(O(M) = \bigcap\{m \in \text{Min}(A) \mid m \subseteq M\}\), for each \(M \in \text{Max}(A)\). From Proposition 27 in [2], there exists a unique minimal prime ideal \(m \subseteq M\), and so \(O(M) = m\). \(\square\)
An MV-algebra $A$ is called algebraically closed if every polynomial with coefficients in $A$ having a root in some extension of $A$ has already a root in $A$.

In this paper we rely on the sheaf representation of MV-algebras provided by Filipoiu and Georgescu in [8].

Remark 2. Consider the sheaf of MV-algebras $(\text{Max}(A), \pi_A, E_A)$ where $\text{Max}(A)$ is endowed with the Zariski topology, $E_A$ is the disjoint union of the quotients $A/O(M)$ with $M \in \text{Max}(A)$ and $\pi_A : E_A \to \text{Max}(A)$ is defined by $\pi_A(a/O(M)) = M$, for every $a \in A$.

Let $\varphi : A \to E_A(\text{Max}(A))$ defined as $\varphi(a) = \hat{a}$, $\hat{a}(M) = a/a(O(M))$ for each $M \in \text{Max}(A)$. In [8], the authors proved that $\varphi$ is an MV-isomorphism. In the same paper, the authors provide the following definition.

Definition 3. Let $X$ be a Hausdorff space and $F$ a sheaf of MV-algebras over $X$. The pair $(X, F)$ will be called an MV-algebraic space.

An MV-algebraic space $(X, F)$ will be called compact if the topological space $X$ is compact and, for any $x \in X$ and any open neighbourhood $U$, $F(x) = K_x + \text{opKer}(U)$, where $K_x = \{\sigma \in F(x) \mid \sigma(x) = 0\}$ and $\text{opKer}(U) = \bigcap\{K_x \mid x \notin U\}$.

An MV-algebraic space $(X, F)$ will be called local if every stalk $F_x$ is a local MV-algebra.

A morphism between MV-algebraic spaces, $\lambda : (X, F) \to (Y, G)$, consists of a continuous map $g : Y \to X$ and a collection of MV-algebra morphisms $\lambda_{U, V} : F(V) \to G(U)$ for all $U \subseteq Y$ and $V \subseteq X$ open such that $U \subseteq g^{-1}(V)$, and satisfying commutativity of diagrams with respect to restrictions of sections.

Theorem 4. (See [8].) The category $\mathcal{MV}$ of MV-algebras is equivalent to the category $\mathcal{CLMV}$ of compact and local MV-algebraic spaces.

In the sequel, we refer to compact and local MV-algebraic spaces as FG MV-algebraic spaces.

We conclude this preliminary section with a theorem which in many cases reduces the study of MV-algebras to the study of ultrapowers of $[0, 1]$ or their powers.

Theorem 5 (Di Nola Embedding Theorem). (See [6] and [7].)

1. Every linearly ordered MV-algebra admits an embedding in some ultrapower $[0, 1]^*$ of $[0, 1]$.
2. More generally, every MV-algebra admits an embedding in a power of an ultrapower of $[0, 1]$, that is, in an MV-algebra of the form $([0, 1]^*)^1$.

3. An alternative axiomatization of algebraically closed MV-algebras

In [12], Lacava proved the following characterization of algebraically closed MV-algebras.

Theorem 6. An MV-algebra $A$ is algebraically closed if and only if $A$ is regular and divisible.

Here we propose an axiomatization of algebraically closed MV-algebras different from the above. Our axiomatization is based on an essentially well-known characterization of MV-algebraic sets by means of finite unions of convex polyhedra, taken from the theory of [4].

3.1. MV-polynomials and DMV-polynomials

As usual, we call MV-polynomials the ordinary polynomials in the language of MV-algebras, built from variables and function symbols of the language. And as usual, the value of a polynomial is calculated inductively from the value of its variables.

Moreover, following [10], we generalize MV-polynomials to DMV-polynomials, which are built from MV-algebra symbols plus a unary function symbol $\delta_n$ for every positive integer $n$. DMV-polynomials have a value in every divisible MV-algebra (not in every MV-algebra, however).

If $A \subseteq B$ and $A, B$ are divisible MV-algebras, the value of any DMV-polynomial with coefficients in $A$ lies still in $A$.

We conclude this subsection with a bit of terminology on rational combinations, which will be useful in the next subsections.

If $x$ is a finite vector of variables, let us denote by $\text{Lcomb}(x)$ the set of all rational linear combinations of elements of $x$, and by $\text{Acomb}(x)$ the set of all rational affine combinations of elements of $x$ (that is, sums of elements of $\text{Lcomb}(x)$ and rational numbers).

The following lemma (quite easy to prove) allows us to switch from rational combinations to DMV-polynomials.

Lemma 7. Let $y$ be a vector of variables ranging in $[0, 1]$. Let $qy \in \text{Acomb}(y)$. Then there is a DMV-polynomial $py$ such that $py = qy$ for every value of $y$ such that $qy \in [0, 1]$. 

3.2. Recalls on MV-algebraic sets

In this subsection we set up our study of MV-polynomials by recalling some facts about the particular MV-algebra \([0, 1]\).

MV-polynomials over \([0, 1]\) are well studied in the literature, for instance it is known that they are equivalent to certain “simple” functions called McNaughton’s functions (this is McNaughton’s Theorem, see [4]).

In analogy with classical algebraic sets, we give the following:

Definition 8. Let \(A\) be an MV-algebra. A set \(X \subseteq A^n\) is called MV-algebraic over \(A\) if

\[
X = \{ x \in A^n \mid f(x, a) = 0 \}
\]

for some MV-polynomial \(f(x, a)\) with parameters \(a \in A^m\).

A set \(X \subseteq [0, 1]^n\) is simply called MV-algebraic if it is algebraic over \([0, 1]\).

By applying well-known transformations, one sees that, in the previous definition, it would be equivalent to replace one polynomial equation with any finite system of equations and inequalities (but not, unlike in algebraic geometry, with infinite systems).

Now we recall a normal form lemma for MV-polynomials in \([0, 1]\), given in terms of linear programming.

Lemma 9. A subset of \([0, 1]^n\) is MV-algebraic if and only if it is a finite union of convex polyhedra \(Px \leq Qy\), where \(Px \in Lcomb(x)^N\) and \(Qy \in Acomb(y)^N\) for some \(N\).

Proof. The assertion follows from McNaughton’s Theorem, because zero sets of McNaughton’s functions coincide with finite unions of polyhedra as above, see e.g. [13].

Note that Lemma 9 generalizes to divisible MV-chains. In fact, let us recall another result which follows from the theory of [4]:

Theorem 10. Every divisible MV-chain is elementarily equivalent to \([0, 1]\).

Now, if we fix an MV-polynomial \(f\) and a finite union \(\Pi\) of polyhedra, the equality of the zero set of \(f\) with \(\Pi\) over \([0, 1]\) can be expressed in first order logic, so it is invariant under elementary equivalence, and by the previous theorem, it “transfers” to divisible MV-chains.

3.3. Eliminating quantifiers

In this section we arrive at a certain extent of “elimination of quantifiers”, which will allow us to axiomatize algebraically closed MV-algebras in the next subsection.

We begin with a standard piece of linear programming:

Lemma 11. Let \(Px \leq Qy\) be a polyhedron where \(Px \in Lcomb(x)^N\) and \(Qy \in Acomb(y)^N\) for some \(N\). Then there is a finite subset \(F(y)\) of \(Acomb(y)^N\), such that for every value \(a\) of \(y\), the coordinates of all the vertexes of the polyhedron \(Px \leq Qa\) range in \(F(a)\).

Now it follows:

Corollary 12. Let \(Px \leq Qy\) be a polyhedron where \(Px \in Lcomb(x)^N\) and \(Qy \in Acomb(y)^N\) for some \(N\). For every value \(a\) of \(y\), if the polyhedron contains a real point \(x\), then it contains a point belonging to \(F(a)\).

Proof. If a bounded convex polyhedron \(Px \leq Qa\) is nonempty, then it has some vertexes, and by the previous lemma, all of them belong to \(F(a)\). Now take any vertex.

The following “elimination of quantifiers-like” result will prove useful:

Corollary 13. If \(f(x, y)\) is an MV-polynomial, then there are vectors of DMV-polynomials \(t_1(y), \ldots, t_N(y)\) such that

\[ [0, 1] \models \forall y. (\exists x. f(x, y) = 0 \iff f(t_1(y), y) = 0 \lor \cdots \lor f(t_N(y), y) = 0) \]

where \(\lor\) denotes logical disjunction.
Proof. Take the equation \( f(x, y) = 0 \). Let \( \Pi_f \) be a finite set of polyhedra as in Lemma 9 whose union gives the solutions. For each member \( \pi \) of \( \Pi_f \), let \( F_\pi(y) \) be like in Lemma 11. Let \( s_1(y), \ldots, s_n(y) \) be an enumeration of the union of \( F_\pi(y) \) over the polyhedra \( \pi \) of \( \Pi_f \). By Lemma 7, each \( s_i(y) \) is equal to a vector of DMV-polynomials \( t_i(y) \) for every \( y \) such that \( s_i(y) \) has all coordinates in \([0, 1]\). In particular, \( s_i(y) = t_i(y) \) for every \( y \) such that \( s_i(y) \) is a vertex of a polyhedron of \( \Pi_f \) \( \Box \).

**Corollary 14.** For every MV-polynomial \( f(x, y) \) there is a single DMV-polynomial \( g_f(y) \) such that \([0, 1]\) verifies the following formula:

\[
\phi_f : \forall y.(\exists x.f(x, y) = 0 \iff g_f(y) = 0).
\]

**Proof.** Take

\[
g_f(y) = f(t_1(y), y) \land \cdots \land f(t_n(y), y). \Box
\]

From Theorem 10 and Corollary 14 we obtain:

**Corollary 15.** If \( f(x, y) \) is an MV-polynomial, then every divisible MV-chain \( A \) verifies \( \phi_f \).

**Proof.** This is true because \( \phi_f \) is a formula of first order logic, so it is invariant under elementarily equivalence and "transfers" from \([0, 1]\) to divisible MV-chains. \( \Box \)

In the same vein, we can say that an MV-algebra admits partial elimination if it verifies the left to right implication of \( \phi_f \). It is useful to notice:

**Lemma 16.** Every divisible MV-algebra admits partial elimination.

**Proof.** Let \( A \) be divisible and suppose \( f(x, a) = 0 \) for some \( x \in A^n \). Let \( B = ([0, 1]^*)^I \) be an extension of \( A \) as in Theorem 5. Then \( f(x, a_i) = 0 \) in \([0, 1]^*\) for every \( i \in I \). By Corollary 15, we have \( g_f(a_i) = 0 \) for every \( i \in I \), and the DMV-polynomial \( g_f \) makes sense in \( A \) because \( A \) is divisible; hence \( g_f(a) = 0 \). \( \Box \)

3.4. The axiom system

In the previous subsections we carried out some preliminary work. In this section, by building on this work, we are led to the following characterization:

**Theorem 17.** An MV-algebra \( A \) is algebraically closed if and only if:

1. \( A \) is divisible;
2. for every MV-polynomial \( f \), \( A \) verifies the formula \( \phi_f \).

**Proof.** Suppose \( A \) is algebraically closed. We know \( A \) is divisible by [11]. Now let \( f \) be an MV-polynomial. The left-to-right implication of \( \phi_f \) holds because \( A \) is divisible. To check the right-to-left implication of \( \phi_f \), suppose \( a \) is a vector of elements of \( A \) and \( g_f(a) = 0 \). By Theorem 5, \( A \) is a subalgebra of some power of an ultrapower of \([0, 1]\), say \( B = ([0, 1]^*)^I \). Then for every \( i \in I \), we have \( g_f(a_i) = 0 \) in \([0, 1]^*\). By Corollary 15, we have \( f(x, a_i) = 0 \) for some vector \( x_i \) valued in \([0, 1]^*\). Hence \( f(x, a) = 0 \) for some vector \( x \) valued in \( B \), so \( f \) has a root in some extension of \( A \); and since \( A \) is supposed to be algebraically closed, we must have \( f(x, a) = 0 \) for some vector \( x \) valued in \( A \). This proves \( \phi_f \) in \( A \).

Conversely, suppose that \( A \) is divisible and \( A \) models the formula \( \phi_f \). Suppose that \( f(b, a) = 0 \) for some vector \( a \) of elements of \( A \) and some vector \( b \) of elements of an extension \( B \) of \( A \). By Theorem 5 there is an embedding \( j \) of \( B \) in some power of ultrapower of \([0, 1]\), say \( D = ([0, 1]^*)^I \). Then \( f(j(b), j(a)) = 0 \).

For every \( i \in I \) we have \( f(j_i(b), j_i(a)) = 0 \) in \([0, 1]^*\). Again by Corollary 15, we have \( g_f(j_i(a)) = 0 \), so (since \( A \) is divisible and \( g_f \) makes sense) \( j_i(g_f(a)) = 0 \) and, since \( i \in I \) is arbitrary we have \( j(g_f(a)) = 0 \) and \( g_f(a) = 0 \). By the formula \( \phi_f \), \( f(x, a) \) has a root in \( A \). This proves that \( A \) is algebraically closed. \( \Box \)

Strictly speaking, the DMV-function symbols \( \delta_n \) cannot appear in the axioms because they are not in the language of MV-algebras; however, for every \( n \) the binary relation \( y = \delta_n(x) \) is definable in the language of MV-algebras, so we can define away the \( \delta_n \) symbols by iterating the replacement of \( \phi(\delta_n(x)) \) with \( \exists y. \phi(y) \land y = \delta_n(x) \).
4. A sheaf representation of algebraically closed MV-algebras

In this section we prove that the category of algebraically closed MV-algebras is equivalent to the full subcategory of FG MV-algebraic spaces which stalks are divisible MV-chains and whose base space is Stonean. To prove this we will show that

- the category of divisible MV-algebras is equivalent to the subcategory of FG MV-algebraic spaces which stalks are divisible;
- the category of regular MV-algebras is equivalent to the subcategory of FG MV-algebraic spaces which stalks are linearly ordered and which base space is Stonean.

The representation of algebraically closed MV-algebras follows from these results joined with Theorem 6.

4.1. Divisible MV-algebras

Since the class of all divisible MV-algebras is closed under quotient, from Remark 2 it follows

Theorem 18. Each divisible MV-algebra is isomorphic to the MV-algebra of all global sections of a sheaf of local and divisible MV-algebras on a compact Hausdorff space.

Theorem 18 has a complete translation in terms of MV-algebraic spaces.

Definition 19. An MV-algebraic space \((X, F)\) will be called divisible if every stalk \(F_x\) is a divisible MV-algebra.

Remark 20. Let \(A\) be a divisible MV-algebra and consider its associated FG MV-algebraic space \((\text{Max}(A), F_A)\). Since divisible algebras are stable under quotient, the stalks of \(F_A\) are divisible too and so \((\text{Max}(A), E_A)\) is a compact local and divisible MV-algebraic space.

In the sequel, we will indicate by \(\text{DMV} \) the full subcategory of all divisible MV-algebras and by \(\text{DCLMV}\) the full subcategory of compact local and divisible MV-algebraic spaces.

Consider \(\text{Sc}_D: \text{DCLMV} \rightarrow \text{DMV}\) to be the restriction of the equivalence \(\text{Sc}\) defined in [8]. Since \(\text{Sc}\) is an equivalence of categories, \(\text{Sc}\) is full and faithful. Being \(\text{DCLMV}\) and \(\text{DMV}\) full subcategories, \(\text{Sc}_D\) is full and faithful too. Moreover, \(\text{Sc}_D\) is essentially surjective. Indeed, let \(A\) be a divisible MV-algebra and consider the compact local and divisible MV-algebraic space \((\text{Max}(A), F_A)\), it follows that the MV-algebra \(\text{Sc}((\text{Max}(A), F_A)) = F_A(\text{Max}(A))\) of all global sections is divisible too and from Remark 2 \(F_A(\text{Max}(A)) \cong A\). In this way, we obtain

Corollary 21. The full subcategory of all divisible MV-algebras is equivalent to the full subcategory of divisible FG MV-algebraic spaces.

4.2. Regular MV-algebras

From Proposition 1 and Remark 2, it follows

Theorem 22. Each regular MV-algebra is isomorphic to the MV-algebra of all global sections of a sheaf of MV-chains on a Stone space.

This isomorphism has a complete translation in terms of MV-algebraic spaces for regular MV-algebras, too.

Lemma 23. Let \((E, \pi, X)\) be a sheaf of MV-algebras such that for each \(x \in X\) the stalk \(E_x = \pi^{-1}(\{x\})\) is an MV-chain and \(X\) is a Stone space. The MV-algebra \(E(X)\) of all global sections is regular.

Proof. Using Proposition 17 in [12], we shall prove that for each \(\sigma, \tau \in E(X)\) such that \(\sigma \wedge \tau = 0\), there exist \(\alpha, \beta \in B(E(X))\) such that \(\sigma \leq \alpha, \tau \leq \beta\) and \(\alpha \wedge \beta = 0\).

Since \(\sigma \wedge \tau = 0\), for each \(x \in X\), \(\sigma(x) \wedge \tau(x) = 0\), with \(\sigma(x), \tau(x) \in E_x\) which an MV-chain. So \(\sigma(x) = 0\) or \(\tau(x) = 0\). Moreover for Lemma 1.1(ii) [5], there exist a clopen neighbourhood \(V_1\) of \(x\) and a section \(\sigma_x \in E(V_1)\) such that \(\sigma(x) = \sigma_x(x)\). Analogously for \(\tau\), there exist a clopen neighbourhood \(V_2\) of \(x\) and a section \(\tau_x \in E(V_2)\) such that \(\tau(x) = \tau_x(x)\). Now if \(\sigma(x) = 0\), consider an open set \(V_3\) of \(X\), with \(x \in V_3\) and a section \(\alpha_x = \sigma_x\). If \(\sigma(x) > 0\), consider the section \(\alpha_x \in E(V_4)\), with \(V_4\) open set of \(X\), such that \(\alpha_x(x) = 1_x\). In summary, we have

\[
\alpha_x(x) := \begin{cases} 
0_x, & \text{if } \sigma_x(x) = 0, \\
1_x, & \text{if } \sigma_x(x) \neq 0.
\end{cases}
\]

Analogously, we define...
\[
\beta_x(x) := \begin{cases} 
0_x, & \text{if } \tau_x(x) = 0, \\
1_x, & \text{if } \tau_x(x) \neq 0.
\end{cases}
\]

where \( \beta_x \in E(V_4) \), being \( V_4 \) an open neighbourhood of \( x \).

Set \( W_x = V_1 \cap V_2 \cap V_3 \cap V_4 \). We have obtained that \( \sigma_x(y) \wedge \tau_x(y) = 0, \sigma_x(y) \leq \alpha_x(y), \tau_x(y) \leq \beta_x(y), \alpha_x(y) \wedge \beta_x(y) = 0, \alpha_x(y), \beta_x(y) \in \{0_x, 1_x\} \).

Now \( \{W_x \mid x \in X\} \) is an open covering of \( X \). Without loss of generality, we can assume these sets disjoint. Being \( (E, \pi, X) \) is a sheaf, there exist \( \alpha, \beta \in B(E(X)) \) such that \( \alpha_{|W_x} = \alpha_x, \beta_{|W_x} = \beta_x \), for each \( x \in X \) and \( \sigma \leq \alpha, \tau \leq \beta \) and \( \alpha \wedge \beta = 0 \). Hence \( E(X) \) is regular. \( \square \)

**Definition 24.** An MV-algebraic space \( (X, F) \) will be called Stonean if and only if \( X \) is a Stone space.

**Remark 25.** Let \( A \) be a regular MV-algebra and let \( (\text{Max}(A), E_A) \) be the associated FG MV-algebraic space. From Theorem 22 it follows that \( (\text{Max}(A), E_A) \) is a linearly ordered Stonean FG MV-algebraic space.

In the sequel, we'll indicate by \( \text{ROMV} \) the full subcategory of all divisible MV-algebras and by \( \text{ROMVS} \) the full subcategory of linearly ordered, Stonean FG MV-algebraic spaces.

Consider \( \text{Sc}_R : \text{ROMVS} \to \text{ROMV} \) to be the restriction of the equivalence \( \text{Sc} \) defined in \([8]\). Since \( \text{Sc} \) is an equivalence of categories, \( \text{Sc}_R \) is full and faithful. Being \( \text{ROMVS} \) and \( \text{ROMV} \) full subcategories, \( \text{Sc}_R \) is essentially surjective. Indeed, let \( A \) be a regular MV-algebra and consider the associated FG MV-algebraic space \( (\text{Max}(A), E_A) \). From Lemma 23 it follows that the MV-algebra \( \text{Sc}((\text{Max}(A), E_A)) = E_A(\text{Max}(A)) \) of all global sections is regular too and from Remark 2, \( E_A(\text{Max}(A)) \) is a Stonean FG MV-algebraic space. In this way, we obtain

**Corollary 26.** The full subcategory of regular MV-algebras is equivalent to the full subcategory of linearly ordered, Stonean FG MV-algebraic spaces.

4.3. Representation of algebraically closed MV-algebras

By Theorem 6, Theorem 18 and Theorem 22, it follows

**Theorem 27.** An algebraically closed MV-algebra is isomorphic to the MV-algebras of global sections in a sheaf over a Stone space with stalks which are divisible MV-chains.

In terms of MV-algebraic spaces, from Corollary 21 and Corollary 26 we obtain

**Corollary 28.** The full subcategory of algebraically closed MV-algebras is equivalent to the full subcategory of linearly ordered, divisible and Stonean FG MV-algebraic spaces.

**References**


