

# On Some Universal Algebras Associated to the Category of Lie Bialgebras

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In our previous work (math/0008128), we studied the set  $\text{Quant}(\mathbb{K})$  of all universal quantization functors of Lie bialgebras over a field  $\mathbb{K}$  of characteristic zero

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universal group and  $\mathfrak{u}(\mathbb{K})$  is a quotient set of a set  $\mathcal{B}(\mathbb{K})$  of families of Lie polynomials by the action of a group  $\mathcal{G}(\mathbb{K})$ . We prove here that  $\mathcal{G}_0(\mathbb{K})$  is equal to the multiplicative group  $1 + \hbar\mathbb{K}[[\hbar]]$ . So  $\text{Quant}(\mathbb{K})$  is “as close as it can be” to  $\mathfrak{u}(\mathbb{K})$ . We also prove that the only universal derivations of Lie bialgebras are multiples of the composition of the bracket with the cobracket. Finally, we prove that the stabilizer of any element of  $\mathcal{B}(\mathbb{K})$  is reduced to the 1-parameter subgroup generated by the corresponding “square of the antipode.” © 2001 Elsevier Science

## 1. MAIN RESULTS

**1.1. Results on  $\text{Quant}(\mathbb{K})$ .** Let  $\mathbb{K}$  be a field of characteristic zero. In [2], we introduced a group  $\mathcal{G}_0(\mathbb{K})$ ; the elements of  $\mathcal{G}_0(\mathbb{K})$  are the universal automorphisms of the adjoint representations of  $\mathbb{K}[[\hbar]]$ -Lie bialgebras.

Let us recall the definition of  $\mathcal{G}_0(\mathbb{K})$  more explicitly. Let  $\hbar$  be a formal variable and let  $\text{LBA}_{\hbar}$  be the category of Lie bialgebras over  $\mathbb{K}[[\hbar]]$ , which are topologically free  $\mathbb{K}[[\hbar]]$ -modules. An element of  $\mathcal{G}_0(\mathbb{K})$  is a functorial assignment  $(\alpha, [, ], \delta_{\alpha}) \mapsto \rho_{\alpha}$ , where for each object  $(\alpha, [, ], \delta_{\alpha})$  of  $\text{LBA}_{\hbar}$ ,  $\rho_{\alpha}$  is an element of  $\text{End}_{\mathbb{K}[[\hbar]]}(\alpha)$ , given by a composition of tensor products of the bracket and cobracket of  $\alpha$ , this composition being the same for all Lie bialgebras (we express the latter condition by saying that  $\alpha \mapsto \rho_{\alpha}$  is universal), and such that the identities  $(\rho_{\alpha} \bmod \hbar) = \text{id}_{\alpha}$ ,  $\rho_{(\alpha^*)^{\text{cop}}} = (\rho_{\alpha})^t$  and  $\rho_{\alpha}([x, y]) = [\rho_{\alpha}(x), y]$  for  $x, y \in \alpha$  hold at the universal level.

In [2], we also introduced the group  $\text{Aut}(\text{LBA}_{\hbar})$  of transformations of the category of Lie bialgebras over  $\mathbb{K}[[\hbar]]$ . Elements of  $\text{Aut}(\text{LBA}_{\hbar})$  are assignments  $\alpha: (\mathfrak{a}, [\cdot, \cdot]_{\mathfrak{a}}, \delta_{\mathfrak{a}}) \mapsto (\text{br}_{\mathfrak{a}}^{\alpha}, \text{co}_{\mathfrak{a}}^{\alpha})$ , where  $\text{br}_{\mathfrak{a}}^{\alpha}$  and  $\text{co}_{\mathfrak{a}}^{\alpha}$  are universal elements of  $\text{Hom}(\wedge^2 \mathfrak{a}, \mathfrak{a})$  and  $\text{Hom}(\mathfrak{a}, \wedge^2 \mathfrak{a})$ , such that  $(\mathfrak{a}, \text{br}_{\mathfrak{a}}^{\alpha}, \text{co}_{\mathfrak{a}}^{\alpha})$  is a Lie bialgebra;  $\text{br}_{\mathfrak{a}}^{\alpha}$  and  $\text{co}_{\mathfrak{a}}^{\alpha}$  should be equal to  $[\cdot, \cdot]_{\mathfrak{a}}$  and  $\delta_{\mathfrak{a}}$  modulo  $\hbar$ , and also satisfy compatibility conditions with the operations of taking duals and doubles of Lie bialgebras. The composition law in  $\text{Aut}(\text{LBA}_{\hbar})$  is defined by  $\text{br}_{\mathfrak{a}}^{\alpha\alpha'} = \text{br}_{(\mathfrak{a}, \text{br}_{\mathfrak{a}}^{\alpha'}, \text{co}_{\mathfrak{a}}^{\alpha'})}^{\alpha}$  and  $\text{co}_{\mathfrak{a}}^{\alpha\alpha'} = \text{co}_{(\mathfrak{a}, \text{br}_{\mathfrak{a}}^{\alpha'}, \text{co}_{\mathfrak{a}}^{\alpha'})}^{\alpha}$ .

There is a unique map from  $\mathcal{G}_0(\mathbb{K})$  to  $\text{Aut}(\text{LBA}_{\hbar})$ , sending the assignment  $(\mathfrak{a} \mapsto \rho_{\mathfrak{a}})$  to the assignment  $\mathfrak{a} \mapsto (\mathfrak{a}, \rho_{\mathfrak{a}}^{-1/2} \circ [\cdot, \cdot]_{\mathfrak{a}} \circ (\rho_{\mathfrak{a}}^{1/2} \otimes \rho_{\mathfrak{a}}^{1/2}), (\rho_{\mathfrak{a}}^{1/2} \otimes \rho_{\mathfrak{a}}^{1/2}) \circ \delta_{\mathfrak{a}} \circ \rho_{\mathfrak{a}}^{-1/2})$ . This map is injective, and the composition law of  $\mathcal{G}_0(\mathbb{K})$  is uniquely defined by the condition that it is a group morphism.

View  $1 + \hbar \mathbb{K}[[\hbar]]$  as a multiplicative subgroup of  $\mathbb{K}[[\hbar]]^{\times}$ . There is a unique map  $\beta: 1 + \hbar \mathbb{K}[[\hbar]] \rightarrow \mathcal{G}_0(\mathbb{K})$ , such that for any Lie bialgebra  $\mathfrak{a}$ ,  $(\beta(\lambda))_{\mathfrak{a}} = \lambda \text{id}_{\mathfrak{a}}$ . This map makes  $1 + \hbar \mathbb{K}[[\hbar]]$  a subgroup of  $\mathcal{G}_0(\mathbb{K})$ .

We will show

**THEOREM 1.1.** (1) *Let  $(\mathfrak{a} \mapsto \rho_{\mathfrak{a}})$  be a universal assignment, where  $\mathfrak{a}$  is an object of  $\text{LBA}_{\hbar}$  and  $\rho_{\mathfrak{a}}$  is an element of  $\text{End}_{\mathbb{K}[[\hbar]]}(\mathfrak{a})$  such that for any  $x, y$  in  $\mathfrak{a}$ ,  $\rho_{\mathfrak{a}}([x, y]) = [\rho_{\mathfrak{a}}(x), y]$ . Then there is a scalar  $\lambda$  such that for any  $\mathfrak{a}$ ,  $\rho_{\mathfrak{a}} = \lambda \text{id}_{\mathfrak{a}}$ . The same statement holds if we replace  $\text{LBA}_{\hbar}$  by  $\text{LBA}$  and the base ring by  $\mathbb{K}$ .*

(2)  $\mathcal{G}_0(\mathbb{K})$  is equal to its subgroup  $1 + \hbar \mathbb{K}[[\hbar]]$ .

In [2], we defined  $\text{Quant}(\mathbb{K})$  as the set of all isomorphism classes of universal quantization functors of Lie bialgebras, compatible with duals and doubles. If we denote by  $\text{LBA}$  and  $\text{QUE}$  the categories of Lie bialgebras and of quantized universal enveloping algebras over  $\mathbb{K}$ , and by  $\text{class}: \text{QUE} \rightarrow \text{LBA}$  the semiclassical limit functor, then a universal quantization functor of Lie bialgebras, compatible with duals and doubles, is a functor  $Q: \text{LBA} \rightarrow \text{QUE}$ , such that

(1)  $\text{class} \circ Q$  is isomorphic to the identity;

(2) (universality) there exists an isomorphism of functors between  $\mathfrak{a} \mapsto Q(\mathfrak{a})$  and  $\mathfrak{a} \mapsto U(\mathfrak{a})[[\hbar]]$  (these are viewed as functors from  $\text{LBA}$  to the category of  $\mathbb{K}[[\hbar]]$ -modules) with the following properties: if we compose this isomorphism with the symmetrisation map  $U(\mathfrak{a})[[\hbar]] \rightarrow S(\mathfrak{a})[[\hbar]]$ , and if we transport the operations of  $Q(\mathfrak{a})$  on  $S(\mathfrak{a})[[\hbar]]$ , then the expansion in  $\hbar$  of these operations yields maps  $S^i(\mathfrak{a}) \otimes S^j(\mathfrak{a}) \rightarrow S^k(\mathfrak{a})$  and  $S^i(\mathfrak{a}) \rightarrow S^j(\mathfrak{a}) \otimes S^k(\mathfrak{a})$ ; we require that these maps be compositions of tensor products of the bracket and cobracket of  $\mathfrak{a}$ , these compositions being independent of  $\mathfrak{a}$  (see [1]);

(3) if  $Q^\vee$  (resp.,  $D(Q)$ ,  $Q^{\text{cop}}$ ) denotes the QUE-dual (resp., Drinfeld double, the QUE-algebra with opposite coproduct) of an object  $Q$  of QUE, and  $D(\mathfrak{a})$  denotes the double Lie bialgebra of an object  $\mathfrak{a}$  of LBA, then there are canonical isomorphisms  $Q((\mathfrak{a}^*)^{\text{cop}} \rightarrow Q(\mathfrak{a})^\vee)^{\text{cop}}$  and  $Q(D(\mathfrak{a})) \rightarrow D(Q(\mathfrak{a}))$  (see [2]).

In [2], we also introduced an explicit set  $\mathfrak{M}(\mathbb{K})$  of equivalence classes of families of Lie polynomials, satisfying associativity relations, and we constructed a canonical injection of  $\mathfrak{M}(\mathbb{K})$  in  $\text{Quant}(\mathbb{K})$ . Moreover, we constructed an action of  $\mathcal{G}_0(\mathbb{K})$  on  $\text{Quant}(\mathbb{K})$  and showed that the map

$$\mathcal{G}_0(\mathbb{K}) \times \mathfrak{M}(\mathbb{K}) \rightarrow \text{Quant}(\mathbb{K})$$

given by the composition  $\mathcal{G}_0(\mathbb{K}) \times \mathfrak{M}(\mathbb{K}) \subset \mathcal{G}_0(\mathbb{K}) \times \text{Quant}(\mathbb{K}) \rightarrow \text{Quant}(\mathbb{K})$  (in which the second map is the action map of  $\mathcal{G}_0(\mathbb{K})$  on  $\text{Quant}(\mathbb{K})$ ) is a bijection. Theorem 1.1 therefore implies

**COROLLARY 1.1.** *If  $\mathfrak{a} = (\mathfrak{a}, [\cdot, \cdot], \delta_{\mathfrak{a}})$  is an object of LBA and  $\lambda \in (1 + \hbar\mathbb{K}[[\hbar]])$ , let  $\mathfrak{a}_\lambda$  be the object of  $LBA_{\hbar}$  isomorphic to  $(\mathfrak{a}, [\cdot, \cdot], \lambda\delta)$ . The group  $1 + \hbar\mathbb{K}[[\hbar]]$  acts freely on  $\text{Quant}(\mathbb{K})$  by the rule  $(\lambda, Q) \mapsto Q_\lambda$ , where  $Q_\lambda$  is the functor  $\mathfrak{a} \mapsto \hat{Q}(\mathfrak{a}_\lambda)$  and  $\hat{Q}$  is the natural extension of  $Q$  to a functor from  $LBA_{\hbar}$  to QUE. Then the map*

$$(1 + \hbar\mathbb{K}[[\hbar]]) \times \mathfrak{M}(\mathbb{K}) \rightarrow \text{Quant}(\mathbb{K})$$

given by the composition  $(1 + \hbar\mathbb{K}[[\hbar]]) \times \mathfrak{M}(\mathbb{K}) \subset (1 + \hbar\mathbb{K}[[\hbar]]) \times \text{Quant}(\mathbb{K}) \rightarrow \text{Quant}(\mathbb{K})$  is a bijection.

Therefore  $\text{Quant}(\mathbb{K})$  is “as close as it can be” to  $\mathfrak{M}(\mathbb{K})$ .

**1.2. Universal (Co)Derivations of Lie Bialgebras.** Recall that a coderivation of a Lie coalgebra  $(\mathfrak{c}, \delta_{\mathfrak{c}})$  is an endomorphism  $d$  of  $\text{End}(\mathfrak{c})$ , such that  $(d \otimes \text{id}_{\mathfrak{c}} + \text{id}_{\mathfrak{c}} \otimes d) \circ \delta_{\mathfrak{c}} = \delta_{\mathfrak{c}} \circ d$ . By a derivation (resp., coderivation) of a Lie bialgebra we mean a derivation (resp., coderivation) of the underlying Lie algebra (resp., Lie coalgebra).

Let  $\mathcal{D}$  (resp.,  $\mathcal{C}$ ) be the space of all universal derivations (resp., coderivations) of Lie bialgebras. More explicitly,  $\mathcal{D}$  (resp.,  $\mathcal{C}$ ) is the linear space of all functorial assignments  $\mathfrak{a} \mapsto \lambda_{\mathfrak{a}}$ , where for each object  $\mathfrak{a}$  of LBA,  $\lambda_{\mathfrak{a}}$  belongs to  $\text{End}(\mathfrak{a})$ , is universal in the above sense, and is a derivation (resp., coderivation) of the Lie bialgebra structure of  $\mathfrak{a}$ . It is well-known that if  $[\cdot, \cdot]_{\mathfrak{a}}$  and  $\delta_{\mathfrak{a}}$  are the bracket and cobracket maps of  $\mathfrak{a}$ , then  $[\cdot, \cdot]_{\mathfrak{a}} \circ \delta_{\mathfrak{a}}$  is a derivation of  $\mathfrak{a}$ ; e.g., if  $\mathfrak{a}$  is finite-dimensional, if  $\sum_{i \in I} a_i \otimes b_i$  is the canonical element of  $\mathfrak{a} \otimes \mathfrak{a}^*$  and if we set  $u = \sum_{i \in I} [a_i, b_i]$ , then we have the identity  $([\cdot, \cdot]_{\mathfrak{a}} \circ \delta_{\mathfrak{a}})(x) = [u, x]$  in the double Lie algebra of  $\mathfrak{a}$ ; and since  $[\cdot, \cdot]_{\mathfrak{a}^*} \circ \delta_{\mathfrak{a}^*}$  is a derivation of  $\mathfrak{a}^*$ , its transpose  $[\cdot, \cdot]_{\mathfrak{a}} \circ \delta_{\mathfrak{a}}$  is a coderivation of  $\mathfrak{a}$ . Then

**THEOREM 1.2.**  $\mathcal{D}$  and  $\mathcal{C}$  both coincide with the one-dimensional vector space spanned by the assignment  $\alpha \mapsto [\cdot, \cdot]_{\alpha} \circ \delta_{\alpha}$ .

If  $V$  is a vector space, we denote by  $F(V)$  the free Lie algebra generated by  $V$ . Then the assignment  $c \mapsto F(c)$  is a functor from the category LCA of Lie coalgebras to LBA. The proof of Theorem 1.2 implies the following analogous statement for the subcategory of LBA of free Lie algebras of Lie coalgebras.

**PROPOSITION 1.1.** *Let  $(c \mapsto \lambda_c)$  be a functorial assignment, where for each object  $c$  of LCA,  $\lambda_c$  is both a derivation and a coderivation of the Lie bialgebra  $F(c)$ . Then there exists a scalar  $\lambda$ , such that for any object  $c$  of LBA,  $\lambda_c = \lambda[\cdot, \cdot]_{F(c)} \circ \delta_{F(c)}$ .*

The analogues of Theorem 1.2 and Proposition 1.1 also hold if we replace the categories LBA and LCA by the categories  $\text{LBA}_{\hbar}$  and  $\text{LCA}_{\hbar}$ , where  $\text{LCA}_{\hbar}$  is the category of Lie coalgebras, which are topologically free  $\mathbb{K}[[\hbar]]$ -modules.

**1.3. Isotropy of the Action of  $\mathcal{G}(\mathbb{K})$  on  $\mathcal{B}(\mathbb{K})$ .** If  $n$  is any integer  $\geq 0$ , we define  $\mathcal{F}\mathcal{A}_n$  as the free algebra with generators  $x_1, \dots, x_n$ . If we assign degree  $\delta_i$  to each  $x_i$ , then it is graded by  $\bigoplus_{i=1}^n \mathbb{N}\delta_i$ . We define  $\text{FA}_n$  as the degree  $\sum_{i=1}^n \delta_i$  part of  $\mathcal{F}\mathcal{A}_n$ . Let  $\mathcal{F}\mathcal{L}_n$  be the Lie subalgebra of  $\mathcal{F}\mathcal{A}_n$  generated by  $x_1, \dots, x_n$ . So  $\mathcal{F}\mathcal{L}_n$  is the free Lie algebra with generators  $x_1, \dots, x_n$ .  $\mathcal{F}\mathcal{L}_n$  is also graded by  $\bigoplus_{i=1}^n \mathbb{N}\delta_i$ . Define  $\text{FL}_n$  as the part of  $\mathcal{F}\mathcal{L}_n$  of degree  $\sum_{i=1}^n \delta_i$ .  $\text{FL}_n$  may be viewed as the space of all functorial assignments  $\alpha \mapsto \lambda_{\alpha}$ , where  $\alpha$  is a Lie algebra and  $\lambda_{\alpha} \in \text{Hom}(\alpha^{\otimes n}, \alpha)$ ;  $\text{FA}_n$  is the space of similar assignments for the category of associative algebras.

We record here the definition of  $\mathfrak{W}(\mathbb{K})$ . Let  $\mathcal{B}(\mathbb{K})$  be the set of families  $(B_{pq})_{p, q \geq 0}$ , such that for each  $p, q$ ,  $B_{pq}$  belongs to  $\text{FL}_{p+q}[[\hbar]]$ ,  $B_{10}(x) = B_{01}(x) = x$ ,  $B_{p0} = B_{0p}$  if  $p \neq 1$ ,  $B_{11}(x, y) = [x, y]$ , and for any integers  $p, q, r$ , the identity

$$\begin{aligned} & \sum_{\alpha > 0} \sum_{(p_{\beta})_{\beta=1, \dots, \alpha} \in \text{Part}_{\alpha}(p), (q_{\beta})_{\beta=1, \dots, \alpha} \in \text{Part}_{\alpha}(q)} B_{\alpha r} (B_{p_1 q_1}(x_1, \dots, x_{p_1} \mid y_1, \dots, y_{q_1}) \\ & \cdots B_{p_{\alpha} q_{\alpha}}(x_{\sum_{\beta=1}^{\alpha-1} p_{\beta} + 1}, \dots, x_p \mid y_{\sum_{\beta=1}^{\alpha-1} q_{\beta} + 1}, \dots, y_q) \mid z_1, \dots, z_r) \\ & = \sum_{\alpha > 0} \sum_{(q_{\beta})_{\beta=1, \dots, \alpha} \in \text{Part}_{\alpha}(q), (r_{\beta})_{\beta=1, \dots, \alpha} \in \text{Part}_{\alpha}(r)} B_{p\alpha}(x_1, \dots, x_p \mid \\ & B_{q_1 r_1}(y_1, \dots, y_{q_1} \mid z_1, \dots, z_{r_1}) \cdots B_{q_{\alpha} r_{\alpha}}(y_{\sum_{\beta=1}^{\alpha-1} q_{\beta} + 1}, \dots, y_q \mid z_{\sum_{\beta=1}^{\alpha-1} r_{\beta} + 1}, \dots, z_r)) \end{aligned}$$

holds; here  $\text{Part}_\alpha(n)$  is the set of  $\alpha$ -partitions of  $n$ , i.e., the set of families  $(n_1, \dots, n_\alpha)$  of positive integers such that  $n_1 + \dots + n_\alpha = n$ . Define  $\mathcal{G}(\mathbb{K})$  as the subset of  $\prod_{n \geq 1} \text{FL}_n[[\hbar]]$  of families  $(P_n)_{n \geq 1}$ , such that  $P_1(x) = x$ . Then we are going to define a group structure on  $\mathcal{G}(\mathbb{K})$ , and an action of  $\mathcal{G}(\mathbb{K})$  on  $\mathcal{B}(\mathbb{K})$ ;  $\mathfrak{M}(\mathbb{K})$  is the quotient set  $\mathcal{B}(\mathbb{K})/\mathcal{G}(\mathbb{K})$ .

Recall first that if  $(\mathfrak{c}, \delta_\mathfrak{c})$  is a Lie coalgebra and  $B \in \mathcal{B}(\mathbb{K})$ , then there is a unique Hopf algebra structure on the completed tensor algebra  $T(\mathfrak{c})[[\hbar]]$  with coproduct  $\Delta_B^\mathfrak{c}: T(\mathfrak{c})[[\hbar]] \rightarrow T(\mathfrak{c})^{\otimes 2}[[\hbar]]$  defined by

$$\Delta_B^\mathfrak{c}(x) = x \otimes 1 + 1 \otimes x + \sum_{p, q | p+q \geq 1} \hbar^{p+q-1} \alpha_{pq}(\delta_\mathfrak{c}^{(B_{pq})}(x))$$

for any  $x \in \mathfrak{c}$ , where for any  $P$  in  $\text{FL}_n[[\hbar]]$ ,  $\delta_\mathfrak{c}^{(P)}$  is the map from  $\mathfrak{c}$  to  $\mathfrak{c}^{\otimes n}$  dual to the map from  $(\mathfrak{c}^*)^{\otimes n}$  to  $\mathfrak{c}^*$  defined by  $P$  (when  $\mathfrak{c}$  is finite-dimensional), and  $\alpha_{pq}$  is the map from  $\mathfrak{c}^{\otimes p+q}[[\hbar]]$  to  $T(\mathfrak{c}) \otimes T(\mathfrak{c})[[\hbar]]$  sending  $x_1 \otimes \dots \otimes x_{p+q}$  to  $(x_1 \otimes \dots \otimes x_p) \otimes (x_{p+1} \otimes \dots \otimes x_{p+q})$ .

If  $P = (P_n)_{n \geq 1}$  belongs to  $\mathcal{G}(\mathbb{K})$ , and  $(\mathfrak{c}, \delta_\mathfrak{c})$  is a Lie coalgebra, define  $i_P^\mathfrak{c}$  as the unique automorphism of  $T(\mathfrak{c})[[\hbar]]$ , such that for any  $x$  in  $\mathfrak{c}$ , we have

$$i_P^\mathfrak{c}(x) = x + \sum_{n \geq 2} \hbar^{n-1} \delta_\mathfrak{c}^{(P_n)}(x).$$

Then the product  $*$ :  $\mathcal{G}(\mathbb{K}) \times \mathcal{G}(\mathbb{K}) \rightarrow \mathcal{G}(\mathbb{K})$  and the operation  $*$ :  $\mathcal{G}(\mathbb{K}) \times \mathcal{B}(\mathbb{K}) \rightarrow \mathcal{B}(\mathbb{K})$  are uniquely determined by the conditions that for any Lie coalgebra  $(\mathfrak{c}, \delta_\mathfrak{c})$ , and any  $P, Q$  in  $\mathcal{G}(\mathbb{K})$  and any  $B$  in  $\mathcal{B}(\mathbb{K})$ , we have

$$i_{P*Q}^\mathfrak{c} = i_P^\mathfrak{c} \circ i_Q^\mathfrak{c} \quad \text{and} \quad \Delta_{P*B}^\mathfrak{c} = (i_P^\mathfrak{c} \otimes i_P^\mathfrak{c}) \circ \Delta_B^\mathfrak{c} \circ (i_P^\mathfrak{c})^{-1}.$$

Then one checks that for any  $B$  in  $\mathcal{B}(\mathbb{K})$ , there exists a unique family  $(S_n)_{n \geq 2}$ , where  $S_n \in \text{FL}_n[[\hbar]]$ , such that for any Lie coalgebra  $(\mathfrak{c}, \delta_\mathfrak{c})$ , the antipode  $S_B^\mathfrak{c}$  of the bialgebra  $(T(\mathfrak{c})[[\hbar]], m_0, \Delta_B^\mathfrak{c})$  is such that for any  $x \in \mathfrak{c}$ ,

$$S_B^\mathfrak{c}(x) = -x + \sum_{n | n \geq 2} \hbar^{n-1} \delta_\mathfrak{c}^{(S_n)}(x)$$

( $m_0$  is the multiplication map in  $T(\mathfrak{c})[[\hbar]]$ ). It follows that there is a unique family  $(\tilde{S}_n)_{n \geq 2}$ , where  $\tilde{S}_n \in \text{FL}_n[[\hbar]]$ , such that

$$(S_B^\mathfrak{c})^2(x) = x + \sum_{n | n \geq 2} \hbar^{n-1} \delta_\mathfrak{c}^{(\tilde{S}_n)}(x)$$

for any  $x \in \mathfrak{c}$ . Let us set  $\tilde{S}_1(x) = x$  and set  $S_B^2 = (\tilde{S}_n)_{n \geq 1}$ . Then  $S_B^2$  is an element of  $\mathcal{G}(\mathbb{K})$ . Moreover,  $\mathcal{G}(\mathbb{K})$  is a pro-unipotent group. It follows that one can define the logarithm  $\log(S_B^2)$  of  $S_B^2$ , and the corresponding one-parameter subgroup  $\exp(\mathbb{K}[[\hbar]] \log(S_B^2))$  of  $\mathcal{G}(\mathbb{K})$ .

Since for any Lie coalgebra  $(\mathfrak{c}, \delta_{\mathfrak{c}})$ , we have  $((S_B^{\mathfrak{c}})^2 \otimes (S_B^{\mathfrak{c}})^2) \circ \Delta_B^{\mathfrak{c}} = \Delta_B^{\mathfrak{c}} \circ (S_B^{\mathfrak{c}})^2$ , we also have  $S_B^2 * B = B$ . It follows that for any element  $g$  of  $\exp(\mathbb{K}[[\hbar]] \log(S_B^2))$ , we have  $g * B = B$ , so  $\exp(\mathbb{K}[[\hbar]] \log(S_B^2))$  is contained in the isotropy group of  $B$ .

**PROPOSITION 1.2.** *For any  $B$  in  $\mathcal{B}(\mathbb{K})$ , the isotropy group of  $B$  for the action of  $\mathcal{G}(\mathbb{K})$  on  $\mathcal{B}(\mathbb{K})$  is equal to  $\exp(\mathbb{K}[[\hbar]] \log(S_B^2))$ .*

## 2. PROOF OF THEOREM 1.1

Let us define  $\mathcal{E}$  as the set of all universal  $\mathbb{K}[[\hbar]]$ -module endomorphisms of Lie bialgebras. More explicitly, an element  $\mathcal{E}$  is a functorial assignment  $(\mathfrak{a}, [., .], \delta_{\mathfrak{a}}) \mapsto \epsilon_{\mathfrak{a}} \in \text{End}_{\mathbb{K}[[\hbar]]}(\mathfrak{a})$ , where  $\mathfrak{a}$  is an object of  $\text{LBA}_{\hbar}$  and the universality requirement means that  $\epsilon_{\mathfrak{a}}$  is given by a composition of tensor products of the bracket and cobracket of  $\mathfrak{a}$ , this composition being the same for all Lie bialgebras. Then  $\mathcal{E}$  is a  $\mathbb{K}[[\hbar]]$ -module, and  $\mathcal{G}_0(\mathbb{K})$  is a subset of  $\mathcal{E}$ . We will first give a description of  $\mathcal{E}$  in terms of multilinear parts of free Lie algebras (Proposition 2.1). Then a computation in free algebras will prove Theorem 1.1.

**2.1. Description of  $\mathcal{E}$ .** Recall that  $\text{FL}_n$  is the multilinear part in each generator of the free Lie algebra over  $\mathbb{K}$  with  $n$  generators. Let  $\mathfrak{S}_n$  act diagonally on  $\text{FL}_n \otimes \text{FL}_n$  by simultaneous permutation of the generators  $x_1, \dots, x_n$  and  $y_1, \dots, y_n$  of each factor.

We define a linear map  $p \mapsto (i(p)_{\mathfrak{a}})$  from  $\hat{\bigoplus}_{n|n \geq 1} (\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} [[\hbar]]$  to  $\mathcal{E}$  as follows.

If  $Q$  is an element of  $\text{FL}_n$ , view  $Q$  as an element of the free algebra with generators  $x_1, \dots, x_n$  and write  $Q = \sum_{\sigma \in \mathfrak{S}_n} Q_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(n)}$ . Then set

$$\delta_{\mathfrak{a}}^{(Q)}(x) = \frac{1}{n} \sum_{\sigma \in \mathfrak{S}_n} Q_{\sigma} ((\text{id}_{\mathfrak{a}}^{\otimes n-2} \otimes \delta_{\mathfrak{a}}) \circ \cdots \circ \delta_{\mathfrak{a}}(x))^{\sigma(1) \dots \sigma(n)}$$

for any  $x \in \mathfrak{a}$ . The definition of  $\delta_{\mathfrak{a}}^{(Q)}$  is such that if  $\mathfrak{a}$  is a Lie coalgebra,  $\mathfrak{b}$  is a Lie algebra and we have a pairing  $\langle, \rangle: \mathfrak{a} \times \mathfrak{b} \rightarrow \mathbb{K}$ , then we have

$$\langle \delta_{\mathfrak{a}}^{(Q)}(x), y_1 \otimes \cdots \otimes y_n \rangle = \langle x, Q(y_1, \dots, y_n) \rangle$$

for any  $x \in \mathfrak{a}$  and  $y_1, \dots, y_n \in \mathfrak{b}$ .

If  $p = \sum_{\alpha} P_{\alpha} \otimes Q_{\alpha}$  is an element of  $\text{FL}_n \otimes \text{FL}_n$ , define  $i(p)_{\alpha}$  as the endomorphism of  $\mathfrak{a}$  such that

$$i(p)_{\alpha}(x) = \sum_{\alpha} P_{\alpha}(\delta^{(Q_{\alpha})}(x))$$

for any  $x \in \mathfrak{a}$ . This maps factors through a linear map  $p \mapsto i(p)_{\alpha}$  from  $(\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n}$  to  $\text{End}_{\mathbb{K}[[\hbar]]}(\mathfrak{a})$ , and induces a linear map  $p \mapsto \hat{i}(p)_{\alpha}$  from  $\hat{\bigoplus}_{n|n \geq 1} (\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} [[\hbar]]$  to  $\text{End}_{\mathbb{K}[[\hbar]]}(\mathfrak{a})$  (here  $\hat{\bigoplus}$  is the  $\hbar$ -adically completed direct sum).

Then if  $\mathfrak{a}$  is finite-dimensional over  $\mathbb{K}[[\hbar]]$ , and if we express the canonical element of  $\mathfrak{a} \otimes \mathfrak{a}^*$  as  $\sum_{i \in I} a_i \otimes b_i$ , then

$$i(p)_{\alpha}(x) = \sum_{\alpha} \sum_{i_1, \dots, i_n \in I} \langle x, Q_{\alpha}(b_{i_1}, \dots, b_{i_n}) \rangle P_{\alpha}(a_{i_1}, \dots, a_{i_n}).$$

**PROPOSITION 2.1.** *The linear map  $\hat{i}$  from  $\hat{\bigoplus}_{n|n \geq 1} (\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} [[\hbar]]$  to  $\mathcal{E}$  defined by  $p \mapsto (\mathfrak{a} \mapsto \hat{i}(p)_{\alpha})$  is a linear isomorphism.*

*Proof.* If  $n$  and  $m$  are  $\geq 1$ , define  $\mathcal{E}_{n,m}$  as the vector space of all universal linear homomorphisms from  $\mathfrak{a}^{\otimes n}$  to  $\mathfrak{a}^{\otimes m}$  (“universal” again means that these homomorphisms are compositions of tensor products of the bracket and cobracket map, this composition being the same for each  $\mathfrak{a}$ ). Then  $\mathcal{E}$  is just  $\mathcal{E}_{1,1}$ . The direct sum  $\hat{\bigoplus}_{n,m|n,m \geq 1} \mathcal{E}_{n,m}$  may be defined formally as the smallest  $\hbar$ -adically complete vector subspace of the space of all functorial assignments  $\mathfrak{a} \mapsto \rho_{\mathfrak{a}} \in \hat{\bigoplus}_{n,m|n,m \geq 1} \text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ , containing the assignments  $\mathfrak{a} \mapsto \text{id}_{\mathfrak{a}} \in \text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}, \mathfrak{a})$ , the bracket and the cobracket operations, stable under the external tensor products operations  $\text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}) \otimes \text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n'}, \mathfrak{a}^{\otimes m'}) \rightarrow \text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n+n'}, \mathfrak{a}^{\otimes m+m'})$ , under the natural actions of the symmetric groups  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$  on  $\text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ , and under the composition operation.

Let us define  $F^{(n,m)}$  as

$$F^{(n,m)} = \bigoplus_{(p_{ij}) \in \mathbb{N}^{\{1, \dots, n\} \times \{1, \dots, m\}}} \left( \bigotimes_{i=1}^n \text{FL}_{\sum_{j=1}^m p_{ij}} \otimes \bigotimes_{j=1}^m \text{FL}_{\sum_{i=1}^n p_{ij}} \right)_{\prod_{(i,j) \in \{1, \dots, n\} \times \{1, \dots, m\}} \mathfrak{S}_{p_{ij}}};$$

the generators of the  $i$ th factor of the first tensor product are  $x_{\alpha}^{(ij)}$ ,  $j = 1, \dots, m$ ,  $\alpha = 1, \dots, p_{ij}$ , and the generators of the  $j$ th factor of the second tensor product are  $y_{\alpha}^{(ij)}$ ,  $i = 1, \dots, n$ ,  $\alpha = 1, \dots, p_{ij}$ ; the group  $\mathfrak{S}_{p_{ij}}$  acts by simultaneously permuting the generators  $x_{\alpha}^{(ij)}$  and  $y_{\alpha}^{(ij)}$ ,  $\alpha \in \{1, \dots, p_{ij}\}$ .

Then there is a unique linear map  $i_{n,m}: F^{(n,m)} \rightarrow \mathcal{E}_{m,n}$ , such that if

$$p = \sum_{(p_{ij}) \in \mathbb{N}^{\{1,\dots,n\} \times \{1,\dots,m\}}} \sum_{\lambda} \bigotimes_{i=1}^n P_i^\lambda(x_\alpha^{(ij)}; j=1, \dots, m; \alpha=1, \dots, p_{ij}) \\ \otimes \bigotimes_{j=1}^m Q_j^\lambda(y_\alpha^{(ij)}; i=1, \dots, n; \alpha=1, \dots, p_{ij}),$$

and  $x_1, \dots, x_m$  belong to a Lie bialgebra  $\mathfrak{a}$ , then

$$(i_{n,m}(p))_{\mathfrak{a}}(x_1 \otimes \dots \otimes x_m) \\ = \sum_{(p_{ij}) \in \mathbb{N}^{\{1,\dots,n\} \times \{1,\dots,m\}}} \sum_{\lambda} \left( \bigotimes_{i=1}^n P_i^\lambda \right) \circ \alpha_{(p_{ij})} \left( \bigotimes_{j=1}^m \delta^{(Q_j^\lambda)}(x_j) \right). \quad (2)$$

Here  $\alpha_{(p_{ij})}$  is the linear endomorphism of  $\mathfrak{a}^{\otimes \sum_{i=1}^n \sum_{j=1}^m p_{ij}}$  given by the following permutation of factors

$$\alpha_{(p_{ij})} \left( \bigotimes_{j=1}^m \left( \bigotimes_{i=1}^n \left( \bigotimes_{\alpha=1}^{p_{ij}} x_\alpha^{(ij)} \right) \right) \right) = \bigotimes_{i=1}^n \left( \bigotimes_{j=1}^m \left( \bigotimes_{\alpha=1}^{p_{ij}} x_\alpha^{(ij)} \right) \right).$$

If  $\mathfrak{a}$  is a finite-dimensional Lie bialgebra, and if we write the canonical element of  $\mathfrak{a} \otimes \mathfrak{a}^*$  as  $\sum_{i \in I} a(i) \otimes b(i)$ , then the map  $(i_{n,m}(p))_{\mathfrak{a}}$  takes the following form

$$(i_{n,m}(p))_{\mathfrak{a}}(x_1 \otimes \dots \otimes x_m) \\ = \sum_{\lambda} \sum_{i_1^{(11)} \in I, \dots, i_{p_{nm}}^{(nm)} \in I} \prod_{j=1}^m \langle x_j, Q_j^\lambda(b(i_\alpha^{(ij)}); i=1, \dots, n; \alpha=1, \dots, p_{ij}) \rangle \\ \circ \bigotimes_{i=1}^n P_i^\lambda(a(i_\alpha^{(ij)}); j=1, \dots, m; \alpha=1, \dots, p_{ij}).$$

The map  $i_{n,m}$  induces linear maps  $\hat{i}_{n,m}: F^{(n,m)}[[\hbar]] \rightarrow \mathcal{E}_{n,m}$  and  $\hat{\bigoplus}_{n,m|n,m \geq 1} \hat{i}_{n,m}: \hat{\bigoplus}_{n,m|n,m \geq 1} F^{(n,m)}[[\hbar]] \rightarrow \hat{\bigoplus}_{n,m|n,m \geq 1} \mathcal{E}_{n,m}$ .

Let us show that  $\hat{\bigoplus}_{n,m|n,m \geq 1} \hat{i}_{n,m}$  is surjective. For this, let us study  $\hat{\bigoplus}_{n,m \geq 1} \text{Im}(\hat{i}_{n,m})$ . This is a subspace of the space of all functorial assignments

$$\mathfrak{a} \mapsto \rho_{\mathfrak{a}} \in \hat{\bigoplus}_{n,m|n,m \geq 1} \text{Hom}_{\mathbb{K}[[\hbar]]}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m}).$$



Let us show that it shares all the properties of  $\bigoplus_{n,m|n,m \geq 1} \mathcal{E}_{n,m}$ . The identity is the image of the element  $x_1^{(11)} \otimes y_1^{(11)}$  of  $F^{(1,1)}$ , the bracket is the image of the element  $[x_1^{(11)}, x_1^{(12)}] \otimes y_1^{(11)} \otimes y_1^{(12)}$  of  $F^{(1,2)}$ , and the cobracket is the image of the element  $x_1^{(11)} \otimes x_1^{(21)} \otimes [y_1^{(11)}, y_1^{(21)}]$  of  $F^{(2,1)}$ . The fact that  $\widehat{\bigoplus}_{n,m \geq 1} \text{Im}(\hat{t}_{n,m})$  is stable under the composition follows from the following lemma.

**LEMMA 2.1.** *Let  $P$  and  $Q$  be Lie polynomials in  $\text{FL}_n$  and  $\text{FL}_m$ , respectively. Then there exist an element  $p = \sum_{p \in \mathbb{N}^{\{1, \dots, n\} \times \{1, \dots, m\}}} \sum_{\lambda} \left( \bigotimes_{i=1}^n P_i^{p, \lambda} \right) \otimes \left( \bigotimes_{j=1}^m Q_j^{p, \lambda} \right)$  of  $F^{(n,m)}$ , such that if  $\mathfrak{a}$  is any Lie bialgebra over  $\mathbb{K}$ , we have*

$$(\delta^{(Q)} \circ P)_\alpha = i_{n,m}(p)_\alpha = \left( \sum_{p \in \mathbb{N}^{\{1, \dots, n\} \times \{1, \dots, m\}}} \left( \bigotimes_{i=1}^n P_i^{p, \lambda} \right) \circ \alpha_p \circ \left( \bigotimes_{j=1}^m \delta^{(Q_j^{p, \lambda})} \right) \right)_\alpha \quad (3)$$

in  $\text{End}(\mathfrak{a}^{\otimes n}, \mathfrak{a}^{\otimes m})$ .

*Proof.* Assume that  $\mathfrak{a}$  is finite-dimensional and  $\sum_{i \in I} a(i) \otimes b(i)$  is the canonical element of  $\mathfrak{a} \otimes \mathfrak{a}^*$ , then the statement is equivalent to the following formula

$$\begin{aligned} & \sum_{i_1, \dots, j_m \in I} \langle Q(a(j_1), \dots, a(j_m)), P(b(i_1), \dots, b(i_n)) \rangle \left( \bigotimes_{\alpha=1}^n a(i_\alpha) \right) \otimes \left( \bigotimes_{\beta=1}^m b(j_\beta) \right) \\ &= \sum_{p \in \mathbb{N}^{\{1, \dots, n\} \times \{1, \dots, m\}}} \sum_{\lambda} \sum_{i_1^{(11)} \in I, \dots, i_{p_{nm}}^{(nm)} \in I} \left( \bigotimes_{i=1}^n P_i^{p, \lambda} (a(i_\alpha^{(ij)}); j = 1, \dots, m; \alpha = 1, \dots, p_{ij}) \right) \\ & \quad \otimes \left( \bigotimes_{j=1}^m Q_j^{p, \lambda} (b(i_\alpha^{(ij)}); i = 1, \dots, n; \alpha = 1, \dots, p_{ij}) \right). \end{aligned} \quad (4)$$

To prove it, we may assume that  $P$  and  $Q$  have the form  $P(x_1, \dots, x_n) = [x_1, [x_2, \dots, x_n]]$  and  $Q(y_1, \dots, y_m) = [y_1, [y_2, \dots, y_m]]$ . Then the invariance of the canonical bilinear form in  $D(\mathfrak{a})$  and the fact that  $\sum_{i \in I} a(i) \otimes b(i)$  satisfies the classical Yang–Baxter identity in  $D(\mathfrak{a})$  imply the following formula. If  $\alpha$  is an integer and  $\underline{k} = (k_1, \dots, k_\alpha)$  is a sequence of integers such that  $1 \leq k_1 < \dots < k_\alpha < m$ , let  $\underline{k}' = (k'_1, \dots, k'_m)$  be the sequence such that  $k'_i = k_i$  for  $i = 1, \dots, \alpha$ ,  $(k'_i)_{i > \alpha}$  is decreasing and  $\{k'_1, \dots, k'_m\} = \{1, \dots, m\}$ . Then

$$\begin{aligned}
& \sum_{i_1, \dots, j_m \in I} \langle [a(j_1), [a(j_2), \dots, a(j_m)]], [b(i_1), [b(i_2), \dots, b(i_n)]] \rangle \\
& \quad \circ a(i_1) \otimes \dots \otimes a(i_n) \otimes b(j_1) \otimes \dots \otimes b(j_m) \\
& = \sum_{\alpha=1}^{m-1} \sum_{k_1, \dots, k_\alpha | 1 \leq k_1 < \dots < k_\alpha < m} \sum_{s=0}^{m-\alpha} \kappa(s) \\
& \quad \circ \sum_{i_1, \dots, j_m \in I} \langle [a(j_{k'_m}), \dots, [a(j_{k'_{\alpha+s+2}}), a(j_{k'_{\alpha+s+1}})]], [b(i_2), [b(i_3), \dots, b(i_n)]] \rangle \\
& \quad \circ [a(j_{k'_1}), \dots, [a(j_{k'_{\alpha+s}}), a(i_1)]] \otimes a(i_2) \otimes \dots \otimes a(i_m) \\
& \quad \otimes b(j_1) \otimes \dots \otimes [b(j_{k'_{\alpha+s+1}}), b(i_1)] \otimes \dots \otimes b(j_m),
\end{aligned}$$

where  $\kappa(0) = -1$  and  $\kappa(s) = (-1)^s$  if  $s \neq 0$ . Formula (4) then follows by induction on  $n$  and  $m$ . One then checks that the element  $p$  obtained from this computation also satisfies the identity (3) for any Lie bialgebra.  $\blacksquare$

*End of Proof of Proposition.* Let us explain why Lemma 3 implies that  $\hat{\bigoplus}_{n, m \geq 1} \text{Im}(\hat{\iota}_{n, m})$  is stable under composition. Tensoring  $r$  copies of identity (3), we find that if we are given any homogeneous Lie polynomials  $P_1, \dots, P_r, Q_1, \dots, Q_r$  (the sum of degree of  $P_i$  is  $n$ , the sum of degrees of  $Q_i$  is  $m$ ), then there exist families  $(P_i^{p, \lambda})_{i, \lambda}$  and  $(Q_i^{p, \lambda})_{i, \lambda}$ , such that the identity

$$\left( \left( \bigotimes_{i=1}^r \delta^{(Q_i)} \right) \circ \left( \bigotimes_{i=1}^r P_i \right) \right)_\alpha = \left( \sum_{i, \lambda} \left( \bigotimes_{i=1}^n P_i^{p, \lambda} \right) \circ \alpha_p \circ \left( \bigotimes_{j=1}^m \delta^{(Q_j^{p, \lambda})} \right) \right)_\alpha$$

holds for any  $\alpha$ . Then composing this identity from the left by a composition  $\sigma \circ (\bigotimes_{i'} P_{i'})$  and from the right by a composition  $(\bigotimes_{j'} \delta^{(Q_{j'})}) \circ \tau$ , where  $P_{i'}$  and  $Q_{j'}$  are homogeneous Lie polynomials and  $\sigma, \tau$  are permutations, we express the product of any pairs of elements of  $\hat{\bigoplus}_{n, m \geq 1} \text{Im}(\hat{\iota}_{n, m})$  as an element of the same space.

The other properties of  $\hat{\bigoplus}_{n, m | n, m \geq 1} \mathcal{E}_{n, m}$  are obviously shared by  $\hat{\bigoplus}_{n, m \geq 1} \text{Im}(\hat{\iota}_{n, m})$ . So  $\hat{\bigoplus}_{n, m \geq 1} \text{Im}(\hat{\iota}_{n, m})$  is contained in  $\hat{\bigoplus}_{n, m | n, m \geq 1} \mathcal{E}_{n, m}$  and shares all its properties; since  $\hat{\bigoplus}_{n, m | n, m \geq 1} \mathcal{E}_{n, m}$  is the smallest vector subspace of the space of functorial assignments  $\alpha \mapsto \rho_\alpha \in \hat{\bigoplus}_{n, m | n, m \geq 1} \text{Hom}_{\mathbb{k}[[\hbar]]}(\alpha^{\otimes n}, \alpha^{\otimes m})$  with these properties, we obtain  $\hat{\bigoplus}_{n, m | n, m \geq 1} \text{Im}(\hat{\iota}_{n, m}) = \hat{\bigoplus}_{n, m | n, m \geq 1} \mathcal{E}_{n, m}$ . This proves that  $\hat{\bigoplus}_{n, m | n, m \geq 1} \hat{\iota}_{n, m}$  is surjective. Since  $\hat{\iota} = \hat{\iota}_{1, 1}$ , this implies that  $\hat{\iota}$  is surjective.

Let us now show that the map  $\hat{i}$  is injective. Let  $(p_n)_{n \geq 1}$  be a family such that  $p_n \in (\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} [[\hbar]]$  and  $\hat{i}(\sum_{n|n \geq 1} p_n) = 0$ . Then if  $\mathfrak{a}$  is any Lie bialgebra, then  $\sum_{n|n \geq 1} \hat{i}(p_n)_{\mathfrak{a}} = 0$ .

If  $V$  is any vector space, let us denote by  $F(V)$  the free Lie algebra generated by  $V$ . If  $(\mathfrak{c}, \delta_{\mathfrak{c}})$  is a Lie coalgebra, then the map  $\mathfrak{c} \rightarrow \wedge^2 F(\mathfrak{c})$  defined as the composition of  $\mathfrak{c} \rightarrow \wedge^2 \mathfrak{c} \rightarrow \wedge^2 F(\mathfrak{c})$  of the cobracket map of  $\mathfrak{c}$  with the canonical inclusion extends to a unique cocycle map  $\delta_{F(\mathfrak{c})}: F(\mathfrak{c}) \rightarrow \wedge^2 F(\mathfrak{c})$ . Then  $(F(\mathfrak{c}), [, ], \delta_{F(\mathfrak{c})})$  is a Lie bialgebra. The assignment  $\mathfrak{c} \mapsto F(\mathfrak{c})$  is a functor from the category LCA of Lie coalgebras to LBA. Then if  $(\mathfrak{c}, \delta_{\mathfrak{c}})$  is any Lie coalgebra, we have

$$\sum_{n|n \geq 1} \hat{i}(p_n)_{F(\mathfrak{c})} = 0. \quad (5)$$

Recall that  $\text{FA}_n$  is the multilinear part of the free algebra with generators  $x_1, \dots, x_n$ . Then  $\mathfrak{S}_n$  acts on  $\text{FA}_n \otimes \text{FL}_n$  by simultaneously permuting the generators  $x_1, \dots, x_n$  of  $\text{FA}_n$  and  $y_1, \dots, y_n$  of  $\text{FL}_n$ . The injection  $\text{FL}_n \subset \text{FA}_n$  induces a linear map  $(\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} \rightarrow (\text{FA}_n \otimes \text{FL}_n)_{\mathfrak{S}_n}$ ; since  $\mathfrak{S}_n$  is finite, this linear map is an injection. Moreover, the map  $\text{FL}_n \rightarrow (\text{FA}_n \otimes \text{FL}_n)_{\mathfrak{S}_n}$ , sending  $P$  to the class of  $x_1 \cdots x_n \otimes P(y_1, \dots, y_n)$ , is a linear isomorphism.

For each  $n$ , define  $\bar{p}_n$  as the element of  $\text{FL}_n$  such that the equality

$$p_n = x_1 \cdots x_n \otimes \bar{p}_n(y_1, \dots, y_n)$$

holds in  $(\text{FA}_n \otimes \text{FL}_n)_{\mathfrak{S}_n} [[\hbar]]$ .

The restriction of  $\hat{i}(p_n)_{F(\mathfrak{c})}$  to  $\mathfrak{c} \subset F(\mathfrak{c})$  is a linear map  $\hat{i}(p_n)_{F(\mathfrak{c})|_{\mathfrak{c}}}$  from  $\mathfrak{c}$  to  $F(\mathfrak{c})[[\hbar]]$ . The image of this map is actually contained in the degree  $n$  part  $F(\mathfrak{c})_n [[\hbar]]$  of  $F(\mathfrak{c})[[\hbar]]$ . The space  $F(\mathfrak{c})_n$  is a vector subspace of  $\mathfrak{c}^{\otimes n}$ . Moreover, formula (1) shows that the composition of  $\hat{i}(p_n)_{F(\mathfrak{c})|_{\mathfrak{c}}}$  with the canonical inclusion  $F(\mathfrak{c})_n [[\hbar]] \subset \mathfrak{c}^{\otimes n} [[\hbar]]$  coincides with  $\delta^{(\bar{p}_n)}$ . So if  $(\mathfrak{c}, \delta_{\mathfrak{c}})$  is any Lie coalgebra, the map  $\sum_{n|n \geq 0} \delta^{(\bar{p}_n)}: \mathfrak{c} \rightarrow \bigoplus_{n|n \geq 0} \mathfrak{c}^{\otimes n} [[\hbar]]$  is zero. Now the linear map  $\text{FL}_n \rightarrow \{\text{functorial assignments } \mathfrak{c} \mapsto \tau_{\mathfrak{c}} \in \text{Hom}(\mathfrak{c}, \mathfrak{c}^{\otimes n}), \text{ where } \mathfrak{c} \text{ is an object of LCA}\}$  defined by  $P \mapsto \delta^{(P)}$ , is injective. This implies that each  $\bar{p}_n$  is zero. So  $\hat{i}$  is injective.

This ends the proof of Proposition 2.1.  $\blacksquare$

More generally, we have the following description of  $\mathcal{E}_{n,m}$  (we will not use it in the sequel). After it was written, I was informed by P. Etingof that this result and its proof were obtained in an e-mail message of Positselski [5].

**PROPOSITION 2.2.** *The map  $\hat{i}_{n,m}: F^{(n,m)} \rightarrow \mathcal{E}_{n,m} [[\hbar]]$  defined by (2) is a linear isomorphism.*

*Proof.* The surjectivity of  $\hat{h}_{n,m}$  has already been proved in Proposition 2.1, and we use the category of all Lie bialgebras of the form  $F(c)$ , where  $(c, \delta_c)$  is a Lie coalgebra, and the inclusion

$$\begin{aligned} F^{(n,m)} &\subset \bigoplus_{(p_{ij}) \in \mathbb{N}^{\{1,\dots,n\} \times \{1,\dots,p\}}} \left( \bigotimes_{i=1}^n \text{FA}_{\sum_{j=1}^m p_{ij}} \otimes \bigotimes_{j=1}^m \text{FL}_{\sum_{i=1}^n p_{ij}} \right)_{\prod_{(i,j) \in \{1,\dots,n\} \times \{1,\dots,m\}} \mathfrak{S}_{p_{ij}}}; \\ &= \bigoplus_{(p_{ij}) \in \mathbb{N}^{\{1,\dots,n\} \times \{1,\dots,p\}}} \bigoplus_{\sigma_1 \in \mathfrak{S}_{p_{11}, \dots, p_{1m}}, \dots, \sigma_n \in \mathfrak{S}_{p_{n1}, \dots, p_{nm}}} \bigotimes_{j=1}^m \text{FL}_{\sum_{i=1}^n p_{ij}} \end{aligned}$$

to prove the injectivity (here  $\mathfrak{S}_{n_1, \dots, n_k}$  is the set of permutations of  $\{1, \dots, n_1 + \dots + n_k\}$ , which are increasing on each subset  $\{n_1 + \dots + n_{i-1} + 1, \dots, n_1 + \dots + n_i\}$ ). ■

*Remark 1.* It would be interesting to understand (1) the algebra structure of  $F^{(1,1)}$  provided by the isomorphism of Proposition 2.1 and (2) the algebra structure of  $\bigoplus_{n,m | n, m \geq 1} F^{(n,m)}$  provided by Proposition 2.2. As we noted before, the latter algebra is also equipped with natural operations of the symmetric groups  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$  on each component  $F^{(n,m)}$ , and external product maps  $F^{(n,m)} \otimes F^{(n',m')} \rightarrow F^{(n+n', m+m')}$ .

*Remark 2.* Universal Gerstenhaber–Schack–Shnider–Sternberg complex. Let  $F_{n,m}^{\text{antisymm}}$  be the subspace of  $F_{n,m}$  corresponding to totally antisymmetric tensors with respect to the actions of the symmetric groups  $\mathfrak{S}_n$  and  $\mathfrak{S}_m$ . So  $F_{n,m}^{\text{antisymm}}$  is a universal version of the vector spaces  $\text{Hom}(\wedge^n \mathfrak{a}, \wedge^m \mathfrak{a})$ .

The bigraded vector space  $\bigoplus_{n,m} F_{n,m}^{\text{antisymm}}$  is then equipped with a double complex structure, which is a universal version of the Shnider–Sternberg variant of the Gerstenhaber–Schack complex for Lie bialgebra cohomology (see [4, 6]). Theorem 1.2 says that the first cohomology group of this complex is one-dimensional. On the other hand, universal deformations and obstructions to deformations of enveloping algebras of Lie bialgebras as QUE-algebras are parametrized by the second and third cohomologies of the associated complex. For example, the facts that the dimensions of these cohomology groups are respectively 1 and 0 would imply the unicity of universal quantization functors of Lie bialgebras up to rescalings of the deformation parameter, but we do not know any evidence for these facts.

*Remark 3.* Graph interpretation of the algebra  $\widehat{\bigoplus}_{n,m | n, m \geq 1} \mathcal{E}_{n,m}$ . In [3], Etingof and Kazhdan defined a universal category  $\mathcal{C}$  of Lie bialgebras as an example of “linear algebraic structures.”  $\mathcal{C}$  is a tensor category, whose objects are labelled by nonnegative integer numbers; if  $[n]$  is the object corresponding to the integer  $n$ , then we have  $[n] = [1]^{\otimes n}$ . Then Proposition 2.2 says that there are “enough” Lie bialgebras for the natural map  $\text{Hom}_{\mathcal{C}}([n], [m]) \rightarrow \mathcal{E}_{n,m}$  to be an isomorphism.

$\text{Hom}_{\mathcal{G}}([n], [m])$  can be described in the following way. Let  $\text{Graph}_{n,m}$  be the set of all graphs  $\gamma$ , oriented, without cycles, and with the following properties. Vertices of  $\gamma$  are of two types, ‘‘algebras’’ and ‘‘operations.’’ ‘‘Algebra’’ vertices are of two subtypes, ‘‘input’’ and ‘‘output’’; ‘‘operations’’ vertices are of two types, ‘‘bracket’’ and ‘‘cobracket.’’ Input (resp., output) vertices have exactly 1 outgoing (resp., 1 incoming) edge; bracket (resp., cobracket) vertices have exactly 2 incoming and 1 outgoing (resp., 1 ingoing and 2 outcoming) edges. There are exactly  $n$  (resp.,  $m$ ) input (resp., output) vertices, numbered from 1 to  $n$  (resp., from 1 to  $m$ ). Factor the vector space  $\hat{\bigoplus}_{\gamma \in \text{Graph}_{n,m}} \mathbb{K}[[\hbar]] \gamma$  by the relations arising from the Lie bialgebra axioms.

Then the resulting vector space is isomorphic to  $\text{Hom}_{\mathcal{G}}([n], [m]) = \mathcal{E}_{n,m}$ . The algebra structure of  $\mathcal{E} = \hat{\bigoplus}_{n,m \mid n,m \geq 1} \mathcal{E}_{n,m}$  corresponds to the composition of classes of graphs, the action of permutation groups to renumbering of the algebra vertices, and the external product corresponds to juxtaposition of graphs with renumbering of algebra vertices.

**2.2. Proof of Theorem 1.1.** Let  $(\alpha \mapsto \rho_\alpha)$  belong to  $\mathcal{G}_0(\mathbb{K})$ . Recall that this means that  $(\alpha \mapsto \rho_\alpha)$  belongs to  $\mathcal{E}$ , in particular,  $\rho_\alpha$  belongs to  $\text{End}_{\mathbb{K}[[\hbar]]}(\alpha)$  for any object  $\alpha$  of  $\text{LBA}_{\hbar}$ .

Let us prove 1). Let  $(\alpha \mapsto \rho_\alpha)$  be a universal assignment such that  $\rho_\alpha$  satisfies the identity  $\rho_\alpha([x, y]) = [\rho_\alpha(x), y]$  for any  $x, y$  in  $\alpha$ . Let  $p$  be the preimage of  $\rho_\alpha$  by the map  $\hat{i}$ . Then there is a unique sequence  $(p_n)_{n \geq 1}$ , where  $p_n$  belongs to  $(\text{FL}_n \otimes \text{FL}_n)_{\in_n} [[\hbar]]$ , such that  $p = \sum_{n \mid n \geq 1} p_n$ . Then if we set  $p_n = \sum_{\alpha} P_{\alpha}^{(n)} \otimes Q_{\alpha}^{(n)}$ , and if  $\alpha$  is finite-dimensional, and  $\sum_{i \in I} a(i) \otimes b(i)$  is the canonical element of  $\alpha \otimes \alpha^*$ , then we have

$$\rho_{\alpha}(x) = \sum_{n \mid n \geq 1} \sum_{i_1, \dots, i_n \in I} \sum_{\alpha} \langle x, Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)) \rangle P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)).$$

Then

$$\begin{aligned} \rho_{\alpha}([x, y]) &= \sum_{n \mid n \geq 1} \sum_{i_1, \dots, i_n \in I} \sum_{\alpha} \langle [x, y], Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)) \rangle \\ &\quad \circ P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)). \end{aligned}$$

**LEMMA 2.2.** *If  $\xi$  belongs to  $\alpha^*$ ,  $[\sum_{i \in I} a(i) \otimes b(i), \xi \otimes 1 + 1 \otimes \xi]$  belongs to  $\alpha^* \otimes \alpha^*$ , and we have*

$$\langle [x, y], \xi \rangle = \left\langle x \otimes y, \left[ \sum_{i \in I} a(i) \otimes b(i), \xi \otimes 1 + 1 \otimes \xi \right] \right\rangle.$$

*Proof of Lemma.* The first statement follows from the fact that  $\sum_i a(i) \otimes b(i) + b(i) \otimes a(i)$  is  $D(\mathfrak{a})$ -invariant.

Let us prove the second statement. The invariance of the bilinear form of  $D(\mathfrak{a})$  implies that  $\langle [x, y], \xi \rangle$  is equal to  $\langle x, [y, \xi] \rangle$ . This is equal to  $\langle \sum_{i \in I} [a(i), \xi] \otimes b(i), x \otimes y \rangle$ . Since  $\mathfrak{a}$  is an isotropic subspace of  $D(\mathfrak{a})$ , this is the same as

$$\left\langle \sum_{i \in I} [a(i), \xi] \otimes b(i) + a(i) \otimes [b(i), \xi], x \otimes y \right\rangle. \quad \blacksquare$$

So we get

$$\begin{aligned} \rho_{\mathfrak{a}}([x, y]) &= \sum_{n|n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n \in I} \left\langle x \otimes y, \sum_i [a(i), Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n))] \right. \\ &\quad \left. \otimes b(i) + a(i) \otimes [b(i), Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n))] \right\rangle \\ &\quad \circ P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)). \end{aligned} \quad (6)$$

On the other hand, we have

$$[\rho_{\mathfrak{a}}(x), y] = \sum_{n|n \geq 0} \sum_{i_1, \dots, i_n \in I} \langle x, Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)) \rangle [P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)), y]$$

so

$$\begin{aligned} [\rho_{\mathfrak{a}}(x), y] &= \sum_{n|n \geq 0} \sum_{\alpha} \sum_{i_1, \dots, i_n \in I} \sum_{i \in I} \langle x \otimes y, Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)) \otimes b(i) \rangle \\ &\quad \circ [P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)), a(i)]. \end{aligned} \quad (7)$$

Comparing (6) and (7), and using the first part of Lemma 2.2, we get

$$\begin{aligned} &\sum_{n|n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n, i \in I} [P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)), a(i)] \otimes Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)) \otimes b(i) \\ &= \sum_{n|n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n, i \in I} P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)) \otimes a(i) \\ &\quad \otimes [b(i), Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n))] \\ &\quad + P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)) \otimes [a(i), Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n))] \otimes b(i). \end{aligned} \quad (8)$$

In fact, it is easy to see that the identity

$$\begin{aligned}
 & \sum_{n \geq 1} \sum_{\alpha} [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes Q_{\alpha}^{(n)}(y_1, \dots, y_n) \otimes y \\
 &= \sum_{n | n \geq 1} \sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes x \otimes [y, Q_{\alpha}^{(n)}(y_1, \dots, y_n)] \\
 & \quad + P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes [x, Q_{\alpha}^{(n)}(y_1, \dots, y_n)] \otimes y
 \end{aligned} \tag{9}$$

holds in

$$F_n^{(abb)} = (\text{FL}_n \otimes (\mathcal{F}\mathcal{L}_n^{\otimes 2})_{\sum_{i=1}^n \delta_i}) \varepsilon_n,$$

where  $\mathfrak{S}_n$  acts by simultaneously permuting the generators of  $\text{FL}_n$  and  $(\mathcal{F}\mathcal{L}_n)^{\otimes 2}$ . Equation (8) is then a consequence of (9).

Applying the Lie bracket to the two last tensor factors of (8), we obtain

$$\begin{aligned}
 & \sum_{n | n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n, i \in I} [P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)), a(i)] \otimes [Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)), b(i)] \\
 &= \sum_{n | n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n, i \in I} P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)) \\
 & \quad \otimes [[a(i), b(i)], Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n))].
 \end{aligned} \tag{10}$$

Since  $\sum_{i \in I} a(i) \otimes b(i)$  satisfies CYBE, we have

$$\sum_{j \in I} a(j) \otimes \left[ \sum_{i \in I} [a(i), b(i)], b(j) \right] = \sum_{i, j \in I} [a(i), a(j)] \otimes [b(i), b(j)].$$

So identity (10) is rewritten as

$$\begin{aligned}
 & \sum_{n | n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n, i \in I} [P_{\alpha}^{(n)}(a(i_1), \dots, a(i_n)), a(i)] \otimes [Q_{\alpha}^{(n)}(b(i_1), \dots, b(i_n)), b(i)] \\
 &= \sum_{n | n \geq 1} \sum_{\alpha} \sum_{k=1}^n \sum_{i_1, \dots, i_n, i \in I} P_{\alpha}^{(n)}(a(i_1), \dots, [a(i), a(i_k)], \dots, a(i_n)) \\
 & \quad \otimes Q_{\alpha}^{(n)}(b(i_1), \dots, [b(i), b(i_k)], \dots, b(i_n)).
 \end{aligned} \tag{11}$$

On the other hand, one easily derives from (9) the identity

$$\begin{aligned} & \sum_{n|n \geq 1} \sum_{\alpha} [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes [Q_{\alpha}^{(n)}(y_1, \dots, y_n), y] \\ &= \sum_{n|n \geq 1} \sum_{k=1}^n \sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, [x, x_k], \dots, x_n) \otimes Q_{\alpha}^{(n)}(y_1, \dots, [y, y_k], \dots, y_n) \end{aligned} \quad (12)$$

valid in  $F^{(1,1)}$ ; this identity is the universal version of (11). Separating homogeneous components, we get for each  $n \geq 1$

$$\begin{aligned} & \sum_{\alpha} [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes [Q_{\alpha}^{(n)}(y_1, \dots, y_n), y] \\ &= \sum_{\alpha} \sum_{k=1}^n P_{\alpha}^{(n)}(x_1, \dots, [x, x_k], \dots, x_n) \otimes Q_{\alpha}^{(n)}(y_1, \dots, [y, y_k], \dots, y_n). \end{aligned} \quad (13)$$

For each  $n \geq 1$ , let  $R_n$  be the element of  $\text{FL}_n[[\hbar]]$  such that the identity  $R_n(x_1, \dots, x_n) \otimes y_1 \cdots y_n = \sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes Q_{\alpha}^{(n)}(y_1, \dots, y_n)$  holds in  $(\text{FL}_n \otimes \text{FA}_n)_{\mathfrak{E}_n}[[\hbar]]$ . Then (13) implies that  $R_n$  satisfies identity

$$\begin{aligned} & [R_n(x_1, \dots, x_n), x_{n+1}] - [R_n(x_2, \dots, x_{n+1}), x_1] \\ &= 2 \sum_{k=1}^n R_n(x_1, \dots, [x_k, x_{k+1}], \dots, x_{n+1}) \end{aligned} \quad (14)$$

in  $\text{FL}_{n+1}[[\hbar]]$ .

Recall that  $R_n$  belongs to  $\text{FA}_n[[\hbar]]$ . It follows that there are unique elements  $R_n^{(i)}$  of  $\text{FA}_{n-1}[[\hbar]]$  ( $i = 1, \dots, n$ ), such that

$$R_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i R_n^{(i)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Let us view (14) as an identity in  $\text{FA}_n[[\hbar]]$ , and let us project it on  $\bigoplus_{\sigma|\sigma \in \mathfrak{E}_n, \sigma(1)=1} \mathbb{K}[[\hbar]] x_{\sigma(1)} \cdots x_{\sigma(n)}$  parallel to  $\bigoplus_{\sigma|\sigma \in \mathfrak{E}_n, \sigma(1) \neq 1} \mathbb{K}[[\hbar]] x_{\sigma(1)} \cdots x_{\sigma(n)}$ . This means that we select in this identity the terms ‘‘starting with  $x_1$ .’’ This yields

$$\begin{aligned} & R_n^{(1)}(x_2, \dots, x_n) x_{n+1} + R_n(x_2, \dots, x_{n+1}) \\ &= 2x_2 R_n^{(1)}(x_3, \dots, x_{n+1}) + 2 \sum_{i=2}^n R_n^{(1)}(x_2, \dots, [x_i, x_{i+1}], \dots, x_{n+1}). \end{aligned} \quad (15)$$



This is an equality in  $\text{FA}_n[[\hbar]]$ . Let us denote by  $\mathcal{F}\mathcal{A}_n$  the free  $\mathbb{K}[[\hbar]]$ -algebra with generators  $x_2, \dots, x_{n+1}$ , and by  $\mathcal{F}\mathcal{L}_n$  the free Lie algebra with the same generators. Then  $\text{FA}_n[[\hbar]] \subset \mathcal{F}\mathcal{A}_n$ , and  $\mathcal{F}\mathcal{A}_n$  is the universal enveloping algebra  $U(\mathcal{F}\mathcal{L}_n)$  of  $\mathcal{F}\mathcal{L}_n$ . This structure of an enveloping algebra defines a filtration on  $\mathcal{F}\mathcal{A}_n$ . An element of  $\mathcal{F}\mathcal{A}_n$  has degree  $\leq \beta$  for this filtration iff it can be expressed as a polynomial of degree  $\leq \beta$  in elements of  $\mathcal{F}\mathcal{L}_n$ .

Let  $\alpha$  be the degree of  $R_n^{(1)}$  for the analogous filtration of  $\mathcal{F}\mathcal{A}_{n-1}$ . Let us assume that  $\alpha > 0$ . Then  $R_n^{(1)}(x_2, \dots, x_n) x_{n+1}$  and  $x_2 R_n^{(1)}(x_3, \dots, x_{n+1})$  both have degree  $\alpha + 1$  in  $\mathcal{F}\mathcal{A}_n$ ;  $R_n^{(1)}(x_2, \dots, [x_i, x_{i+1}], \dots, x_{n+1})$  has degree  $\alpha$  in  $\mathcal{F}\mathcal{A}_n$ ; and  $R_n(x_2, \dots, x_{n+1})$  has degree 1 (its degree is  $\leq 1$ , but if this degree is zero, then  $R_n$  vanishes identically).

Let  $\bar{R}_n^{(1)}$  be the image of  $R_n^{(1)}$  in the associated graded of  $\mathcal{F}\mathcal{A}_n$ , which is the symmetric algebra  $S(\text{FL}_n)[[\hbar]]$  of  $\text{FL}_n$ . Then (15) implies the identity

$$\bar{R}_n^{(1)}(x_2, \dots, x_n) x_{n+1} = 2x_2 \bar{R}_n^{(1)}(x_3, \dots, x_{n+1}) \quad (16)$$

in  $S(\mathcal{F}\mathcal{L}_n)[[\hbar]]$ . Recall that in this identity,  $\bar{R}_n^{(1)}(y_1, \dots, y_{n-1})$  is a polynomial in variables  $P_{k,\alpha}(y_{i_1}, \dots, y_{i_k})$ , where  $k$  runs over  $1, \dots, n-1$ ,  $i_1, \dots, i_k$  runs over all sequences of integers such that  $1 \leq i_1 < \dots < i_k \leq n-1$ , and  $P_{k,\alpha}$  runs over a basis of  $\text{FL}_k$ , so it is a polynomial in  $\sum_{k=1}^{n-1} \binom{n-1}{k} (k-1)!$  variables. Moreover, if  $(\delta_1, \dots, \delta_{n-1})$  is the canonical basis of  $\mathbb{N}^n$ , and we say that the variables  $P_{k,\alpha}(y_{i_1}, \dots, y_{i_k})$  have multidegree  $\delta_{i_1} + \dots + \delta_{i_k}$ , then  $\bar{R}_n^{(1)}$  is homogeneous of multidegree  $\delta_1 + \dots + \delta_{n-1}$ .

Equation (16) implies that  $x_2$  divides  $\bar{R}_n^{(1)}(x_2, \dots, x_n)$ ; if we set

$$\bar{R}_n^{(1)}(x_1, \dots, x_{n-1}) = x_1 R_n^{(1)'}(x_1, \dots, x_{n-1}),$$

where  $R_n^{(1)'}(x_1, \dots, x_{n-1})$  belongs to  $S(\mathcal{F}\mathcal{L}_n)[[\hbar]]$ , then  $R_n^{(1)'}(x_1, \dots, x_{n-1})$  is homogeneous of multidegree  $\delta_2 + \dots + \delta_{n-1}$ . So  $R_n^{(1)'}(x_1, \dots, x_{n-1})$  actually belongs to  $S(\mathcal{F}\mathcal{L}_{n-1})$  and may be written  $S_n^{(1)}(x_2, \dots, x_{n-1})$ . We have then

$$S_n^{(1)}(x_3, \dots, x_n) x_{n+1} = 2x_3 S_n^{(1)}(x_4, \dots, x_{n+1});$$

this equation is the same as (16), where the number of variables is decreased by 1. Repeating the reasoning above, we find  $\bar{R}_n^{(1)}(x_1, \dots, x_{n-1}) = \lambda x_1 \cdots x_{n-1}$ , where  $\lambda$  is scalar. Then Eq. (16) implies that  $\lambda = 0$ . This is a contradiction with  $\alpha > 0$ .

Therefore  $\alpha = 0$ , which means that  $R_n^{(1)}$  is scalar. The only cases when a scalar belongs to  $\text{FA}_{n-1}[[\hbar]]$  is  $n = 1$ , or this scalar is zero. We have therefore shown that if  $n > 1$ , then  $R_n^{(1)}$  is zero. Equation (15) then implies that  $R_n(x_1, \dots, x_n)$  also vanishes. On the other hand, when  $n = 1$ , all the solutions of (14) are  $R_n(x) = \lambda x$ , where  $\lambda \in \mathbb{K}[[\hbar]]$ .

Therefore the only solutions to Eq. (12) are such that if  $n > 1$ ,

$$\sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes Q_{\alpha}^{(n)}(y_1, \dots, y_n)$$

is zero, and  $\sum_{\alpha} P_{\alpha}^{(1)}(x_1) \otimes Q_{\alpha}^{(1)}(y_1)$  is of the form  $\lambda x_1 \otimes y_1$ , with  $\lambda \in \mathbb{K}[[\hbar]]$ . This solution corresponds to the assignment

$$\mathfrak{a} \mapsto \rho_{\mathfrak{a}} = \lambda \text{id}_{\mathfrak{a}}. \quad (17)$$

So all assignments of  $(\mathfrak{a} \mapsto \rho_{\mathfrak{a}})$  such that  $\rho_{\mathfrak{a}}([x, y]) = [\rho_{\mathfrak{a}}(x), y]$  have necessarily the form (17). This proves (1).

Let us now prove (2). It follows from (1) that any assignment of  $\mathcal{G}_0(\mathbb{K})$  is necessarily of the form (17). The necessary and sufficient condition for an assignment of the form (17) to actually belong to  $\mathcal{G}_0(\mathbb{K})$  is that  $\lambda \in (1 + \hbar \mathbb{K}[[\hbar]])$ . This ends the proof of Theorem 1.1.  $\blacksquare$

*Remark 4.* It is much simpler to prove Theorem 1.1 by noting that each homogeneous component of the right hand side of Eq. (9) is anti-symmetric in its two last tensor factors. However, the techniques of the above proof will again be used in the next proofs.

### 3. PROOFS OF THEOREM 1.2 AND PROPOSITION 1.1

3.1. *Proof of Theorem 1.2.* Let  $(\mathfrak{a} \mapsto \lambda_{\mathfrak{a}})$  be an element of  $\mathcal{D}$ . According to the “non- $\hbar$ -adically completed” version of Proposition 2.1,  $(\mathfrak{a} \mapsto \lambda_{\mathfrak{a}})$  is the image by  $i$  of an element  $q$  of  $\bigoplus_{n|n \geq 1} (\text{FL}_n \otimes \text{FL}_n)_{\varepsilon_n}$ . Let us write  $q$  as

$$q = \sum_{n|n \geq 1} \sum_{\alpha} P_n^{(\alpha)}(x_1, \dots, x_n) \otimes Q_n^{(\alpha)}(y_1, \dots, y_n);$$

then in the same way as identity (9), one shows that

$$\begin{aligned} & \sum_{n|n \geq 1} \sum_{\alpha} [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes Q_{\alpha}^{(n)}(y_1, \dots, y_n) \otimes y \\ & \quad - [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes y \otimes Q_{\alpha}^{(n)}(y_1, \dots, y_n) \\ & = \sum_{n|n \geq 1} \sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes x \otimes [y, Q_{\alpha}^{(n)}(y_1, \dots, y_n)] \\ & \quad + P_{\alpha}^{(n)}(x_1, \dots, x_n) \otimes [x, Q_{\alpha}^{(n)}(y_1, \dots, y_n)] \otimes y \end{aligned} \quad (18)$$

holds in  $F^{(abb)}$ . Applying the Lie bracket to the two last tensor factors of this identity, we get

$$\begin{aligned} & 2 \sum_{n|n \geq 1} \sum_{\alpha} [P_{\alpha}^{(n)}(x_1, \dots, x_n), x] \otimes [Q_{\alpha}^{(n)}(y_1, \dots, y_n), y] \\ &= \sum_{n|n \geq 1} \sum_{k=1}^n \sum_{\alpha} P_{\alpha}^{(n)}(x_1, \dots, [x, x_k], \dots, x_n) \otimes Q_{\alpha}^{(n)}(y_1, \dots, [y, y_k], \dots, y_n). \end{aligned} \quad (19)$$

Let us separate the homogeneous components of this equation, and let us denote by  $R'_n$  the element of  $\text{FL}_n$  such that the identity  $R'_n(x_1, \dots, x_n) \otimes y_1 \cdots y_n = \sum_{\alpha} P'_{\alpha}(x_1, \dots, x_n) \otimes Q'_{\alpha}(y_1, \dots, y_n)$  holds in  $(\text{FL}_n \otimes \text{FA}_n)_{\varepsilon_n}$ . Then  $R'_n$  satisfies the identity

$$\begin{aligned} & [R'_n(x_1, \dots, x_n), x_{n+1}] - [R'_n(x_2, \dots, x_{n+1}), x_1] \\ &= \sum_{k=1}^n R'_n(x_1, \dots, [x_k, x_{k+1}], \dots, x_{n+1}) \end{aligned} \quad (20)$$

in  $\text{FL}_{n+1}$ .

If  $n = 1$ , then (20) implies that  $R'_n = 0$ . If  $n = 2$ , then the solutions of (20) are of the form  $R'_n(x, y) = \lambda[x, y]$ , where  $\lambda$  is any scalar.

Let us assume that  $n > 2$ . We will show that the only solution to (20) is  $R'_n = 0$ .

Let us proceed as above and introduce the elements  $R_i^{(n)}$  of  $\text{FA}_{n-1}$  ( $i = 1, \dots, n$ ), such that

$$R'_n(x_1, \dots, x_n) = \sum_{i=1}^n x_i R_i^{(n)}(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n).$$

Let us select in (20) the terms “starting with  $x_1$ ”. We obtain

$$\begin{aligned} & R_n^{(1)}(x_2, \dots, x_n) x_{n+1} + R'_n(x_2, \dots, x_{n+1}) \\ &= x_2 R_n^{(1)}(x_3, \dots, x_{n+1}) + \sum_{i=2}^n R_n^{(1)}(x_2, \dots, [x_i, x_{i+1}], \dots, x_{n+1}). \end{aligned} \quad (21)$$

There exists unique scalars  $\lambda$  and  $(\lambda_{ij})_{2 \leq i < j \leq n}$ , such that

$$\begin{aligned} R'_n(x_2, \dots, x_n) &= \lambda x_2 \cdots x_n + \sum_{2 \leq i < j \leq n} x_2 \cdots x_{i-1} [x_i, x_j] x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_n \\ &\quad + \text{terms of degree } < n-2 \end{aligned}$$

(the degree is with respect to the enveloping algebra filtration of  $\mathcal{F}\mathcal{A}_{n-1} = U(\mathcal{F}\mathcal{L}_{n-1})$ ). We therefore obtain the equality

$$\begin{aligned}
& \sum_{i,j|2 \leq i < j \leq n} \lambda_{ij} x_2 \cdots x_{i-1} [x_i, x_j] x_{i+1} \cdots x_{j-1} x_{j+1} \cdots x_{n+1} \\
&= \sum_{i,j|2 \leq i < j \leq n} \lambda_{ij} x_2 \cdots x_i [x_{i+1}, x_{j+1}] x_{i+2} \cdots x_j x_{j+2} \cdots x_{n+1} \\
&+ \lambda \sum_{i=2}^n x_2 \cdots x_{i-1} [x_i, x_{i+1}] x_{i+2} \cdots x_{n+1} \tag{22}
\end{aligned}$$

modulo terms of degree  $\leq \max(1, n-2)$ , which is  $\leq n-2$  by assumption on  $n$ .

LEMMA 3.1. *If  $R'_n \neq 0$ , then  $\lambda \neq 0$ .*

*Proof.* Let  $d$  be the degree of  $R_n^{(1)}$  for the enveloping algebra filtration of  $\mathcal{F}\mathcal{A}_{n-1} = U(\mathcal{F}\mathcal{L}_{n-1})$ . If  $d = 0$ , then  $R_n^{(1)}$  is a scalar, so  $n = 1$ , which we ruled out. So  $d > 0$ . Let us denote by  $\overline{R_1^{(n)}}$  the image of  $R_1^{(n)}$  in the degree  $d$  part of the associated graded of  $\mathcal{F}\mathcal{A}_{n-1}$ . Then since  $R'_n(x_2, \dots, x_{n+1})$  has degree  $\leq 1 \leq d$ , and  $R_n^{(1)}(x_2, \dots, [x_i, x_{i+1}], \dots, x_{n+1})$  has degree  $d$ , the image of (21) in the degree  $d+1$  part of the associated graded of  $\mathcal{F}\mathcal{A}_{n-1}$  yields

$$\overline{R_1^{(n)}}(x_2, \dots, x_n) x_{n+1} = x_2 \overline{R_1^{(n)}}(x_3, \dots, x_{n+1}).$$

Therefore  $x_2$  divides  $\overline{R_1^{(n)}}(x_2, \dots, x_n)$ , and an induction as above shows that there exists a scalar  $\alpha$  such that

$$\overline{R_1^{(n)}}(x_2, \dots, x_n) = \alpha \times \text{class of } x_2 \cdots x_n.$$

Since  $\overline{R_1^{(n)}}$  cannot be zero,  $\alpha$  is not equal to zero. On the other hand, we have necessarily  $\alpha = \lambda$ , so  $\lambda \neq 0$ . ■

Let us assume that  $R'_n \neq 0$ . We have seen that then  $\lambda \neq 0$ . On the other hand, the image of (22) in the associated graded of  $\mathcal{F}\mathcal{A}_{n-1}$  implies the equalities

$$\lambda_{23} = \lambda, \lambda_{34} - \lambda_{23} = \lambda, \dots, \lambda_{n-1,n} - \lambda_{n-2,n-1} = \lambda, -\lambda_{n-1,n} = \lambda.$$

Summing up these equalities, we get  $(n-1)\lambda = 0$ , so  $\lambda = 0$ , a contradiction. Therefore  $R'_n = 0$ . It follows that  $q$  is homogeneous of degree 2, and is therefore proportional to  $[x_1, x_2] \otimes [y_1, y_2]$ . This ends the proof of the first part of Theorem 1.2 on universal derivations.

Let us prove the statement on universal coderivations. Assume that  $\alpha \mapsto \lambda_\alpha$  is a universal coderivation. Then the assignment  $\alpha \mapsto (\lambda_\alpha^*)^t$  is a universal derivation. We have shown that a universal derivation is necessarily proportional to  $u_\alpha = [\cdot, \cdot]_\alpha \circ \delta_\alpha$ . Since  $(u_\alpha^*)^t = u_\alpha$ , any universal coderivation is also proportional to  $\alpha \mapsto u_\alpha$ . This implies the second part of Theorem 1.2.

**3.2. Proof of Proposition 1.1.** We have shown that any functorial assignment  $(\mathfrak{c} \mapsto \lambda_\mathfrak{c})$ , where for each object  $\mathfrak{c}$  of LCA,  $\lambda_\mathfrak{c}$  is a derivation of  $F(\mathfrak{c})$ , is provided by an element  $q = (q_n)_{n \geq 1}$  of  $\bigoplus_{n \geq 1} (\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{E}_n}$ . More precisely,  $\lambda_\mathfrak{c}$  is uniquely determined by its restriction to  $\mathfrak{c}$ , which has the form (if  $\mathfrak{c}$  is finite-dimensional)

$$\lambda_\mathfrak{c}(x) = \sum_{n \geq 1} \sum_{\alpha} \sum_{i_1, \dots, i_n \in I} \langle x, S_\alpha^{(n)}(b_{i_1}, \dots, b_{i_n}) \rangle R_\alpha^{(n)}(a_{i_1}, \dots, a_{i_n}),$$

where  $q_n$  is the class of  $\sum_{\alpha} R_\alpha^{(n)} \otimes S_\alpha^{(n)}$  and we write the canonical element of  $\mathfrak{c} \otimes \mathfrak{c}^*$  as  $\sum_{i \in I} a_i \otimes b_i$ .

Since  $\lambda_\mathfrak{c}$  is a coderivation, it satisfies the identity

$$\delta_{F(\mathfrak{c})} \circ \lambda_\mathfrak{c}(x) = (\lambda_\mathfrak{c} \otimes \text{id}_{F(\mathfrak{c})} + \text{id}_{F(\mathfrak{c})} \otimes \lambda_\mathfrak{c}) \circ \delta_\mathfrak{c}(x) \quad (23)$$

for any  $x \in \mathfrak{c}$ .

Let us denote by  $\text{ad}^*$  the coadjoint action of  $\mathfrak{c}^*$  on  $\mathfrak{c}$ . For any  $x \in \mathfrak{c}$ , we have  $\delta_\mathfrak{c}(x) = \sum_{i \in I} a_i \otimes \text{ad}^*(b_i)(x)$ . Then

$$\begin{aligned} & (\lambda_\mathfrak{c} \otimes \text{id}_{F(\mathfrak{c})}) \circ \delta_\mathfrak{c}(x) \\ &= \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} \langle a_{i'}, S_\alpha^{(n)}(b_{i_1}, \dots, b_{i_n}) \rangle R_\alpha^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes \text{ad}^*(b_{i'})(x) \\ &= \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} R_\alpha^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes \text{ad}^*(S_\alpha^{(n)}(b_{i_1}, \dots, b_{i_n}))(a_{i'}) \langle x, b_{i'} \rangle. \end{aligned}$$

On the other hand, if we again denote by  $\text{ad}^*$  the action of  $\mathfrak{c}^*$  on  $F(\mathfrak{c})$  induced by the coadjoint action of  $\mathfrak{c}^*$  on  $\mathfrak{c}$ , we have

$$\begin{aligned} \delta_{F(\mathfrak{c})} \circ \lambda_\mathfrak{c}(x) &= \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} a_{i'} \otimes \text{ad}^*(b_{i'})(R_\alpha^{(n)}(a_{i_1}, \dots, a_{i_n})) \\ &\quad \times \langle x, S_\alpha^{(n)}(b_{i_1}, \dots, b_{i_n}) \rangle. \end{aligned}$$

Identity (23) therefore implies

$$\begin{aligned}
& \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} a_{i'} \otimes \text{ad}^*(b_{i'}) (R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n})) \otimes S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}) \\
&= \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes \text{ad}^*(S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}))(a_{i'}) \otimes b_{i'} \\
&\quad - \text{ad}^*(S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}))(a_{i'}) \otimes R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes b_{i'}.
\end{aligned}$$

Let us assume that  $\mathfrak{c}$  is a Lie bialgebra. Then there is a unique Lie algebra morphism  $\alpha_{\mathfrak{c}}$  from  $F(\mathfrak{c})$  to  $\mathfrak{c}$ , extending the identity on  $\mathfrak{c}$ . The image of this identity by  $\alpha_{\mathfrak{c}} \otimes \alpha_{\mathfrak{c}} \otimes \text{id}_{\mathfrak{c}^*}$  is the identity in  $D(\mathfrak{c}) \otimes D(\mathfrak{c}) \otimes \mathfrak{c}^*$  (where both sides belongs to  $\mathfrak{c} \otimes \mathfrak{c} \otimes \mathfrak{c}^*$ )

$$\begin{aligned}
& \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} a_{i'} \otimes [b_{i'}, R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n})] \otimes S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}) \\
&\quad + [a_{i'}, R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n})] \otimes b_{i'} \otimes S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}) \\
&= \sum_{n \geq 1} \sum_{\alpha} \sum_{i', i_1, \dots, i_n \in I} -R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes a_{i'} \otimes [S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}), b_{i'}] \\
&\quad + a_{i'} \otimes R_{\alpha}^{(n)}(a_{i_1}, \dots, a_{i_n}) \otimes [S_{\alpha}^{(n)}(b_{i_1}, \dots, b_{i_n}), b_{i'}];
\end{aligned}$$

this identity holds only due to the fact that  $\sum_{i \in I} a_i \otimes b_i$  is a solution of CYBE, so it holds at the universal level. It means that  $q^{(21)}$  satisfies (18). The proof of Theorem 1.2 then implies that  $q$  is homogeneous of degree 2, which is the conclusion of Proposition 1.1.

**3.3. Proof of Proposition 1.2.** Let  $P = (P_n)_{n \geq 1}$  be an element of  $\mathcal{G}(\mathbb{K})$ , such that  $P * B = B$ . Multiplying  $P$  by the suitable power of  $S_B^2$ , we may assume that  $P_2 = 0$ . The neutral element of  $\mathcal{G}(\mathbb{K})$  is the sequence  $e = (e_n)_{n \geq 1}$ , where  $e_i = 0$  for  $i \geq 2$ . Assume that  $P$  is not equal to  $e$  and let  $k$  be the smallest index such that  $P_k \neq 0$ . Then  $k \geq 3$ .

Let us denote by  $\Delta_0$  the usual (undeformed) coproduct of  $T(\mathfrak{c})[[\hbar]]$ , and by  $\Delta_1$  the first jet of its deformation: so  $\Delta_1$  is the unique map from  $T(\mathfrak{c})[[\hbar]]$  to  $T(\mathfrak{c})^{\otimes 2}[[\hbar]]$ , such that  $\Delta_1|_{\mathfrak{c}} = \delta_{\mathfrak{c}}$  and  $\Delta_1(xy) = \Delta_0(x) \Delta_1(y) + \Delta_1(x) \Delta_0(y)$  for any pair  $x, y$  of elements of  $T(\mathfrak{c})[[\hbar]]$ .

Then we have the identities

$$\Delta_0(\delta_{\mathfrak{c}}^{(P_k)}(x)) = \delta_{\mathfrak{c}}^{(P_k)}(x) \otimes 1 + 1 \otimes \delta_{\mathfrak{c}}^{(P_k)}(x),$$

and

$$\begin{aligned} & \Delta_0(\delta_c^{(P_{k+1})}(x)) + \Delta_1(\delta_c^{(P_k)}(x)) \\ &= \delta_c^{(P_{k+1})}(x) \otimes 1 + 1 \otimes \delta_c^{(P_{k+1})}(x) + (\delta_c^{(P_k)} \otimes \text{id}_{F(c)} + \text{id}_{F(c)} \otimes \delta_c^{(P_k)}) \circ \delta_c(x). \end{aligned}$$

The first identity means that  $\delta_c^{(P_k)}(x)$  is actually contained in  $F(c)$ . Let us expand  $P_k$  in the form  $P_k = \sum_{\sigma \in \mathfrak{S}_k} P_{k, \sigma} x_{\sigma(1)} \cdots x_{\sigma(k)}$ , then this means that  $P_k = \sum_{\sigma \in \mathfrak{S}_k} P_{k, \sigma^{-1}} x_{\sigma(1)} \cdots x_{\sigma(k)}$  also belongs to  $\text{FL}_k[[\hbar]]$ . Therefore the class of  $\sum_{\sigma \in \mathfrak{S}_k} P_{k, \sigma} x_{\sigma(1)} \cdots x_{\sigma(k)} \otimes y_1 \cdots y_k$  in  $(\text{FA}_n \otimes \text{FA}_n)_{\mathfrak{S}_n}$  actually belongs to  $(\text{FL}_n \otimes \text{FL}_n)_{\mathfrak{S}_n}$ .

In the second identity, the first and last terms are antisymmetric, while the others are symmetric. It follows that

$$\Delta_1(\delta_c^{(P_k)}(x)) = (\delta_c^{(P_k)} \otimes \text{id}_{F(c)} + \text{id}_{F(c)} \otimes \delta_c^{(P_k)}) \circ \delta_c(x).$$

Let us denote by  $\delta_{F(c)}$  the extension of  $\delta_c$  to a cocycle map from  $F(c)$  to  $\wedge^2 F(c)$ . Then the restriction of  $\Delta_1$  to  $F(c)$  coincides with  $\delta_{F(c)}$ , so

$$\delta_{F(c)} \circ \delta_c^{(P_k)}(x) = (\delta_c^{(P_k)} \otimes \text{id}_{F(c)} + \text{id}_{F(c)} \otimes \delta_c^{(P_k)}) \circ \delta_c(x),$$

for any  $x \in c$ . So  $\delta_c^{(P_k)}$  satisfies identity (23). Since  $k \geq 3$ , the proof of Proposition 1.1 implies that  $\delta_c^{(P_k)} = 0$ , so  $P_k = 0$ , a contradiction. This proves Proposition 1.2.

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## REFERENCES

1. V. Drinfeld, On some unsolved problems in quantum groups theory, in "Quantum Groups, Leningrad, 1990," Lecture Notes in Math., Vol. 1510, Springer-Verlag, New York/Berlin, 1992.
2. B. Enriquez, Quantization of Lie bialgebras and shuffle algebras of Lie algebras, preprint, math.QA/0008128.
3. P. Etingof and D. Kazhdan, Quantization of Lie bialgebras, II, *Selecta Math. (N.S.)* **4** (1998), 213–232.
4. M. Gerstenhaber and S. Schack, Bialgebra cohomology, deformations, and quantum groups, *Proc. Natl. Acad. Sci. U.S.A.* **87** (1990), 478–481.
5. L. Positselski, Letter to M. Finkelberg and R. Bezrukavnikov, 1995. [In Russian]
6. S. Shnider and S. Sternberg, "Quantum Groups: From Coalgebras to Drinfeld Algebras, A Guided Tour," Graduate Texts in Math., II, Intern. Press, Cambridge, MA, 1993.