Entropy solutions for linearly degenerate hyperbolic systems of rich type

Ta-Tsien Li\textsuperscript{a,*}, Yue-Jun Peng\textsuperscript{b}, Jérémy Ruiz\textsuperscript{b}

\textsuperscript{a} School of Mathematical Sciences, Fudan University, Shanghai 200433, PR China
\textsuperscript{b} Laboratoire de Mathématiques, CNRS UMR 6620, Université Blaise Pascal (Clermont-Ferrand 2), 63177 Aubière cedex, France

Received 2 October 2008
Available online 17 January 2009

Abstract

Consider a linearly degenerate hyperbolic system of rich type. Assuming that each eigenvalue of the system has a constant multiplicity, we construct a representation formula of entropy solutions in $L^\infty$ to the Cauchy problem. This formula depends on the solution of an autonomous system of ordinary differential equations taking $x$ as parameter. We prove that for smooth initial data, the Cauchy problem for such an autonomous system admits a unique global solution. By using this formula together with classical compactness arguments, we give a very simple proof on the global existence of entropy solutions. Moreover, in a particular case of the system, we obtain an another explicit expression and the uniqueness of the entropy solution. Applications include the one-dimensional Born–Infeld system and linear Lagrangian systems.

© 2009 Elsevier Masson SAS. All rights reserved.

Keywords: Explicit solution; Rich system; Linearly degenerate characteristic; Non-strict hyperbolicity; Existence of entropy solution

Résumé

On considère des systèmes hyperboliques linéairement dégénérés de type riche. On suppose que chaque valeur propre des systèmes considérés est de multiplicité constante, on construit une formule de représentation des solutions entropiques $L^\infty$ du problème de Cauchy. Cette formule dépend de la solution d’un système autonome d’équations différentielles ordinaires en prenant $x$ comme paramètre. Nous démontrons que pour des données initiales régulières, le problème de Cauchy pour un tel système autonome admet une unique solution globale. On utilise cette formule et des arguments classiques de compacité pour démontrer l’existence globale de solutions entropiques. De plus, pour des systèmes particuliers on obtient des formules explicites et l’unicité des solutions entropiques. En application on traite le cas du système unidimensionnel de Born–Infeld et des cas de systèmes lagrangiens linéaires.

© 2009 Elsevier Masson SAS. All rights reserved.

Keywords: Explicit solution; Rich system; Linearly degenerate characteristic; Non-strict hyperbolicity; Existence of entropy solution

\* Corresponding author.
E-mail addresses: dqli@fudan.edu.cn (T.-T. Li), peng@math.univ-bpclermont.fr (Y.-J. Peng), ruiz@math.univ-bpclermont.fr (J. Ruiz).

\textsuperscript{1} Supported by the Basic Research Program of China (No. 2007CB814800).

0021-7824/S – see front matter © 2009 Elsevier Masson SAS. All rights reserved.
doi:10.1016/j.matpur.2009.01.008
1. Introduction

For first order quasilinear hyperbolic systems, it is well known that, generically speaking, classical solutions exist only locally in time and singularities may appear in a finite time (see [10–12] and the references therein). In some special cases, however, the global existence of classical solutions can be obtained. In particular, these situations occur for linearly degenerate systems provided that the initial data are sufficiently small and decay at infinity (see [13–15]). In this paper we are interested in constructing explicit formulas of solutions to the Cauchy problem for linearly degenerate quasilinear hyperbolic systems of diagonal form:

\[
\partial_t w_i + \lambda_i(w) \partial_x w_i = 0 \quad (1 \leq i \leq n), \quad t > 0, \quad x \in \mathbb{R}
\]

with the initial data

\[
t = 0: \ w = w^0(x), \quad x \in \mathbb{R}.
\]

Here \(w = (w_1, \ldots, w_n)^t\) and \(w^0 = (w^0_1, \ldots, w^0_n)^t\). The eigenvalues \(\lambda_i(w) \ (1 \leq i \leq n)\) of the system are real valued smooth functions defined on an open domain of \(\mathbb{R}^n\). We first consider the case where system (1.1) is strictly hyperbolic, namely, on the domain under consideration we have:

\[
\lambda_1(w) < \lambda_2(w) < \cdots < \lambda_n(w).
\]

Then we explain why our formulas also hold for non-strictly hyperbolic systems with constant multiplicity eigenvalues. The motivation of the study on hyperbolic systems with constant multiplicity eigenvalues comes from two physical models. They are the Born–Infeld system and the augmented Born–Infeld system (see [2–4]). Each of them possesses the property of constant multiplicity eigenvalues. In what follows, we consider two cases that \(w^0 \in L^\infty(\mathbb{R})\) and \(w^0 \in C^1(\mathbb{R})\) with bounded \(C^0\) norm.

For system (1.1), the \(i\)th characteristic \(\lambda_i(w)\) is linearly degenerate if and only if \(\lambda_i(w)\) is independent of \(w_i\):

\[
\frac{\partial \lambda_i(w)}{\partial w_i} \equiv 0.
\]

If all characteristics \(\lambda_i(w) \ (1 \leq i \leq n)\) are linearly degenerate, system (1.1) is said to be linearly degenerate. Let \(\Lambda(w) = \text{diag}\{\lambda_1(w), \ldots, \lambda_n(w)\}\). A pair of functions \((E(w), F(w))\) is an entropy–entropy flux pair of (1.1) if \(F'(w) = E'(w)\Lambda(w)\), i.e.,

\[
\frac{\partial F(w)}{\partial w_j} = \lambda_j(w) \frac{\partial E(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}.
\]

It is well known that \((E(w), F(w))\) is an entropy–entropy flux pair if and only if it satisfies the conservation law (see [8,11,22]),

\[
\partial_t E(w) + \partial_x F(w) = 0.
\]

The \(i\)th characteristic is rich if, on the domain under consideration, for all \(j, k \in \{1, \ldots, n\}\) such that \(j \neq i\) and \(k \neq i\), we have:

\[
\frac{\partial}{\partial w_j} \left( \frac{\partial \lambda_i(w)}{\partial w_k} \right) = \frac{\partial}{\partial w_k} \left( \frac{\partial \lambda_i(w)}{\partial w_j} \right).
\]

This definition is equivalent to the existence of a smooth positive function \(N_i(w) > 0\) such that

\[
(\lambda_j(w) - \lambda_i(w)) \frac{\partial N_i(w)}{\partial w_j} = N_i(w) \frac{\partial \lambda_i(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}, \ j \neq i.
\]

When \(\lambda_i(w)\) is linearly degenerate, (1.8) is equivalent to

\[
(\lambda_j(w) - \lambda_i(w)) \frac{\partial N_i(w)}{\partial w_j} = N_i(w) \frac{\partial \lambda_i(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}.
\]

System (1.1) is said to be rich if all characteristics \(\lambda_i(w) \ (1 \leq i \leq n)\) are rich. This notion was introduced by Serre [20,22] for studying the existence of entropies and for applying the compensated compactness argument to hyperbolic systems of conservation laws. It is a natural extension of the reducible hyperbolic system composed of 2 equations. In particular, a system of conservation laws which can be put in diagonal form must be rich (see [22,23]).
Proposition 1. Assume that system (1.1) is strictly hyperbolic. For \( i \in \{1, \ldots, n\} \), if there is a function \( H(w) > 0 \) such that \( (H(w), H(w)\lambda_i(w)) \) is an entropy–entropy flux pair of (1.1), then the \( i \)th characteristic \( \lambda_i(w) \) is rich and linearly degenerate.

Proof. By (1.5) we have:

\[
\left( \lambda_j(w) - \lambda_i(w) \right) \frac{\partial H(w)}{\partial w_j} = H(w) \frac{\partial \lambda_i(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}.
\]

(1.10)

Taking \( j \neq i \) in (1.10), we get (1.8) for \( N_i(w) = H(w) \), then \( \lambda_i(w) \) is rich. Moreover, taking \( j = i \) in (1.10), we obtain (1.4), then \( \lambda_i(w) \) is linearly degenerate. \( \square \)

For any given smooth solution \( w = w(t,x) \) to the linearly degenerate strictly hyperbolic rich system (1.1), using (1.9), it is easy to check that for each \( i \in \{1, \ldots, n\} \),

\[
\partial_t N_i(w) + \partial_x \left( N_i(w)\lambda_i(w) \right) = 0.
\]

(1.11)

Therefore, \( (N_i(w), N_i(w)\lambda_i(w)) \) is an entropy–entropy flux pair of system (1.1). More generally, we have:

\[
\partial_t \left( N_i(w)g_i(w_i) \right) + \partial_x \left( N_i(w)\lambda_i(w)g_i(w_i) \right) = 0 \quad \text{for any smooth function} \ g_i.
\]

(1.12)

Thus, \( N_i(w)g_i(w_i) \) \( (1 \leq i \leq n) \) stand for \( n \) independent families of entropies and each entropy of the linearly degenerate strictly hyperbolic rich system (1.1) is a linear combination of \( N_i(w)g_i(w_i) \) \( (1 \leq i \leq n) \) (see [20,22]). In particular, taking \( g_i(s) = s \) in (1.12) gives:

\[
\partial_t \left( N_i(w)w_i \right) + \partial_x \left( N_i(w)\lambda_i(w)w_i \right) = 0.
\]

(1.13)

For getting explicit solutions to quasilinear hyperbolic systems, the first work is due to Lax [11] in which an explicit formula of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In [21] Serre also investigated explicit formulas of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In [21] Serre also investigated explicit formulas of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In particular, taking \( g_i(s) = s \) in (1.12) gives:

\[
\partial_t \left( N_i(w)w_i \right) + \partial_x \left( N_i(w)\lambda_i(w)w_i \right) = 0.
\]

(1.13)

For getting explicit solutions to quasilinear hyperbolic systems, the first work is due to Lax [11] in which an explicit formula of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In [21] Serre also investigated explicit formulas of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In [21] Serre also investigated explicit formulas of entropy solutions is constructed for the scalar conservation law with convex flux. Some extension works for Temple systems were given in [1,21] by characteristic methods. In particular, taking \( g_i(s) = s \) in (1.12) gives:

\[
\partial_t \left( N_i(w)w_i \right) + \partial_x \left( N_i(w)\lambda_i(w)w_i \right) = 0.
\]

(1.13)
is a consequence of the fact that for a linearly degenerate rich system, there exits a coordinate \((s, y_i)\) in which the \(i\)th eigenvalue is equal to zero.

This paper is organized as follows. In the next section, we derive the explicit expressions for entropy and smooth solutions, respectively, when the system is strictly hyperbolic. In Section 3, we give another expression for entropy solutions in the particular case that \(N_i = N\) for all \(i \in \{1, \ldots, n\}\). In this case, we have:

\[
Y_i(t, x) = Z(t, x) - \tilde{\lambda}_i t, \quad \forall i \in \{1, \ldots, n\},
\]

where \(\tilde{\lambda}_i\) (\(1 \leq i \leq n\)) are constants which are nothing but just \(n\) eigenvalues of the system in Lagrangian coordinates \((t, z)\) with \(z = Z(t, x)\). We show that the inverse function of \(x \mapsto Z(t, x)\) solves a linear wave equation if and only if the system contains the Chaplygin gas dynamics equations. Section 4 is devoted to the non-strictly hyperbolic case with constant multiplicity eigenvalues, for which we explain why our formulas still hold. Section 5 is concerned with the existence of entropy solutions in the non-strictly hyperbolic case. We prove this result in a very simple way by using the explicit formulas. Finally, in Section 6 we give two examples of our results.

2. Explicit formulas of solutions

2.1. Case of entropy solutions

We suppose that \(w^0 \in L^\infty(\mathbb{R})\). For a linearly degenerate hyperbolic system, the Rankine–Hugoniot conditions are independent of the choice of the conservation laws among all entropy–entropy flux pairs. As a consequence, in the definition of entropy solutions, entropy inequalities for all convex entropies should be replaced by entropy equalities in the sense of distribution for all \(i \in \{1, \ldots, n\}\). As a consequence of the fact that for a linearly degenerate rich system, there exits a coordinate \((s, y_i)\) in which the \(i\)th eigenvalue is equal to zero.

**Definition 1.** A function \(w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\) is an entropy solution of a linearly degenerate rich system of diagonal form (1.1) if the entropy equality (1.12) is satisfied in the sense of distribution for all \(i \in \{1, \ldots, n\}\). This definition is used in [6] to prove the existence of entropy solutions to the Cauchy problem (1.1)–(1.2) when the system is strictly hyperbolic. It is clear that (1.12) implies (1.11) and (1.13). Due to the explicit formulas below, we show that (1.12) is indeed equivalent to (1.11) and (1.13) (see Proposition 2 below). Hence we can give a simpler definition of an entropy solution of (1.1) as follows, in which the arbitrary functions \(g_i\) (\(1 \leq i \leq n\)) are not involved.

Let \(w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\) be an entropy solution. For each \(i \in \{1, \ldots, n\}\), Eq. (1.11) as a compatibility condition implies that there is a function \(y_i = Y_i(t, x)\) such that

\[
dy_i = N_i(w) dx - \left(N_i \lambda_i \right)(w) dt.
\]

Since

\[
\left.\frac{\partial Y_i}{\partial x}\right|_{t=0} = N_i\left(w(t, x)\right)|_{t=0} = N_i\left(w^0(x)\right),
\]

it is natural to define:

\[
Y_i(0, x) = Y_i^0(x) \overset{\text{def}}{=} \int_0^x N_i\left(w^0(\xi)\right) d\xi.
\]

For any given \(w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\), (2.1) and (2.3) determine a unique Lipschitz function \(y_i = Y_i(t, x)\) for all \(t \geq 0\). Since \(N_i > 0\), the function \(x \mapsto Y_i(t, x)\) is strictly increasing. Therefore, it is bijective for all \(t \geq 0\). Let \(X_i(t, \cdot) = Y_i^{-1}(t, \cdot)\) be the inverse function of \(Y_i(t, \cdot)\) and \(X_i^0 = X_i(0, \cdot) = (Y_i^0)^{-1}\). We verify easily that both \(X_i\) and \(X_i^0\) are Lipschitz functions on \(\mathbb{R}^+ \times \mathbb{R}\) and \(\mathbb{R}\), respectively. Then for each given \(i \in \{1, \ldots, n\}\), with \(s = t, (t, x) \mapsto (s, y_i)\) defines a change of coordinates of Euler–Lagrange type [24].

In what follows, we denote by \(u\) the variable \(u\) in Lagrangian coordinates. We have:

\[
\tilde{w}_i(t, y_i) = w_i\left(t, X_i(t, y_i)\right).
\]
Taking \( t = 0 \) gives:
\[
\tilde{w}_i(0, y_i) = w_i(0, X_i(0, y_i)) = w_i^0(0, X_i^0(y_i)).
\] (2.5)

More generally, in Lagrangian coordinates \((s, z)\) defined by:
\[
s = t, \quad dz = N \, dx - M \, dt
\] (2.6)
with \( N > 0 \), \( \tilde{N} \) and \( \tilde{M} \), etc. can be defined similarly to (2.4), then a straightforward computation shows that a conservative equation of the form,
\[
\partial_t U + \partial_x V = 0
\] (2.7)
can be equivalently expressed as (see also [18,24]):
\[
\partial_s \left( \frac{\tilde{U}}{\tilde{N}} \right) + \partial_z \left( \tilde{V} - \frac{\tilde{U} \tilde{M}}{\tilde{N}} \right) = 0.
\] (2.8)

Moreover, by (2.6), it is easy to see that the relation between the \( i \)th eigenvalue \( \lambda_i(w) \) and the eigenvalue \( \tilde{\lambda}_i(w) \) in Lagrangian coordinates is given by (see [23]):
\[
\tilde{\lambda}_i(w) = N \lambda_i(w) - M.
\] (2.9)

Taking \( N = N_i(w) \) and \( M = N_i(w) \lambda_i(w) \) and applying this general result of equivalence to the \( i \)th equation in (1.13) (namely, taking \( U = N_i(w)w_i \) and \( V = N_i(w)\lambda_i(w)w_i \), we get:
\[
\partial_s \tilde{w}_i = 0,
\] (2.10)
then
\[
\tilde{w}_i(s, y_i) = \tilde{w}_i(0, y_i) = w_i^0(X_i^0(y_i)).
\] (2.11)

Finally, we get:
\[
w_i(t, x) = \tilde{w}_i(t, Y_i(t, x)) = w_i^0(X_i^0(Y_i(t, x))) \quad (1 \leq i \leq n).
\] (2.12)

Thus, an entropy solution \( w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) to Cauchy problem (1.1)–(1.2) satisfies (2.12) together with (2.1) and (2.3), moreover, (2.12) gives the \( L^\infty \) boundedness of the entropy solution. Noting that (2.1) is equivalent to the conservation law (1.11) and by Proposition 2 below, (1.12) is equivalent to (1.11) and (1.13). We see the importance of variable \( N_i \) for defining an entropy solution to the non-conservative system (1.1).

**Proposition 2.** Assume \( w^0 \in L^\infty(\mathbb{R}) \) and the strictly hyperbolic system (1.1) is rich and linearly degenerate. Then each entropy solution \( w \in L^\infty(\mathbb{R}^+ \times \mathbb{R}) \) to Cauchy problem (1.1)–(1.2) satisfies (2.12) with (2.1) and (2.3), moreover, (2.12) gives the \( L^\infty \) boundedness of the entropy solution. Noting that (2.1) is equivalent to the conservation law (1.11) and by Proposition 2 below, (1.12) is equivalent to (1.11) and (1.13). We see the importance of variable \( N_i \) for defining an entropy solution to the non-conservative system (1.1).

**Proof.** It is clear that the function \( w \) defined by (2.12) satisfies the initial condition (1.2). It suffices to show that \( w \) satisfies (1.12) in the sense of distribution.

For any smooth function \( g_i \), since \( w_i^0 \in L^\infty(\mathbb{R}) \) and \( X_i^0 \) is a Lipschitz function, \( y_i \mapsto g_i(w_i^0(X_i^0(y_i))) \) is a bounded function in \( \mathbb{R} \), hence its primitive, denoted by \( \tilde{g}_i \), is Lipschitzian in \( \mathbb{R} \). Then, from (2.1) we have:
\[
N_i(w) g_i(w_i) = g_i(w_i) \partial_x Y_i = \partial_x G_i(Y_i)
\]
and
\[
N_i(w) \lambda_i(w) g_i(w_i) = -g_i(w_i) \partial_t Y_i = -\partial_t G_i(Y_i).
\]
This implies (1.12) in the sense of distributions. \( \Box \)

Now we give more details on function \( Y = (Y_1, \ldots, Y_n)^t \). For each \( i \in [1, \ldots, n] \), the differential relation in (2.1) together with (2.12) implies two partial differential equations for \( y_i = Y_i(t, x) \):
\[
\frac{\partial Y_i}{\partial t} = -(N_i \lambda_i(w_i^0(X_1^0(Y_1))), \ldots, w_n^0(X_n^0(Y_n))),
\] (2.13)
and

\[
\frac{\partial Y_i}{\partial t} = N_i\left(w_i^0(X_i(Y_i)), \ldots, w_n^0(X_n(Y_n))\right). \tag{2.14}
\]

If (2.12) defines an entropy solution, from Definition 1 we have (1.11). Then (2.14) is a consequence of (2.13). Indeed, from (2.13) we deduce, in the sense of distributions,

\[
\partial_t(\partial_x Y_i) = \partial_x(\partial_t Y_i) = -\partial_x\left(N_i \lambda_i\left(w^0(Y)\right)\right) = \partial_t N_i\left(\tilde{w}^0(Y)\right), \tag{2.15}
\]

namely,

\[
\partial_t(\partial_x Y_i - N_i) = 0. \tag{2.16}
\]

Hence, (2.14) follows from the definition (2.3) for \(Y_i(0, x)\). As a consequence of (2.13), taking \(x\) as parameter, it suffices to keep that \(Y\) is a solution of the following autonomous system of ordinary differential equations:

\[
\begin{align*}
\frac{dY_i}{dt} &= -(N_i \lambda_i)(w_i^0(X_i(Y_i)), \ldots, w_n^0(X_n(Y_n))), \quad t > 0, \\
 t = 0: &\quad Y_i = Y_i^0(x) \quad (1 \leq i \leq n). \tag{2.17}
\end{align*}
\]

Thus, by Proposition 2, it is easy to prove the following result:

**Corollary 1.** Under the assumptions of Proposition 2, each entropy solution \(w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\) to Cauchy problem (1.1)–(1.2) has the explicit expression (2.12) together with (2.17).

We point out that Proposition 2 and Corollary 1 do not concern the existence and uniqueness of entropy solutions. The existence of entropy solutions will be proved in Section 5 even for non-strictly hyperbolic systems with characteristics with constant multiplicity. Thus, there is at least one solution to Cauchy problem (1.1)–(1.2), which is given by (2.12) together with (2.17). The uniqueness of solutions \(w\) is equivalent to the uniqueness of \(Y\). In particular, the existence of an entropy solution \(w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})\) implies the existence of a Lipschitz solution \(Y\) to (2.17). This is not a trivial result since the differential equations in (2.17) have discontinuous flux functions.

### 2.2. Case of smooth solutions

Comparing with Proposition 2, Corollary 1 does not show that for a given Lipschitz solution \(Y\) of (2.17), expression (2.12) defines an entropy solution to Cauchy problem (1.1)–(1.2). In order to show this, it is equivalent to ask whether (2.12) and (2.17) imply (2.14). This question seems to be related to the uniqueness of entropy solutions to Cauchy problem (1.1)–(1.2). Unfortunately, for discontinuous initial data \(w^0 \in L^\infty(\mathbb{R})\), (2.17) is a system of differential equations with discontinuous flux functions, for which there is in general no uniqueness of solutions. However, this difficulty can be overcome for smooth solutions. To see this, we suppose that \(w^0 \in C^1(\mathbb{R})\) with bounded \(C^0\) norm. Then it is well known that Cauchy problem (1.1)–(1.2) admits a unique global smooth solution \(w\), then (2.17) has a unique global smooth solution \(Y \in C^2(\mathbb{R}^+ \times \mathbb{R})\).

**Theorem 1.** Assume \(w^0 \in C^1(\mathbb{R})\) with bounded \(C^0\) norm and the strictly hyperbolic system (1.1) is rich and linearly degenerate. Then Cauchy problem (1.1)–(1.2) admits a unique global \(C^1\) solution given by expression (2.12), where \(y_i = Y_i(t, x)\) (1 \(\leq i \leq n\) is the unique global \(C^2\) solution to the Cauchy problem for the autonomous system of ordinary differential equations (2.17).

**Proof.** We give a direct proof of Theorem 1. For each \(i \in \{1, \ldots, n\}\), there is a constant \(\alpha_i > 0\) such that \(N_i(w^0(x)) \geq \alpha_i > 0\). From (2.3) and noting that \(w^0 \in C^1(\mathbb{R})\) with bounded \(C^0\) norm and \(X_i^0 = (Y_i^0)^{-1}\), we see that both \(X_i^0\) and \(Y_i^0\) are \(C^2(\mathbb{R})\) functions, so \(\phi_i: Y \mapsto (N_i \lambda_i)(w_i^0(X_i(Y_i)), \ldots, w_n^0(X_n(Y_n)))\) is a \(C^1(\mathbb{R}^n)\) function with bounded \(C^0\) norm. It follows from the Cauchy–Lipschitz theorem that, for each \(x \in \mathbb{R}\), Cauchy problem (2.17) admits a unique \(C^1\) solution \(Y(\cdot, x)\) defined locally in time. Moreover, there is a constant \(C_1 > 0\) such that

\[
|N_i(w^0(x))| \leq C_1, \quad |\phi_i(y)| \leq C_1, \quad \forall x \in \mathbb{R}, \quad \forall y \in \mathbb{R}^n. \tag{2.18}
\]
On the other hand, for each \((t, x) \in \mathbb{R}^+ \times \mathbb{R}\), (2.17) implies that

\[
Y_i(t, x) = Y^0_i(x) - \int_0^t \phi_i(Y(\tau, x)) d\tau.
\] (2.19)

Then, for any \(T > 0\), from the definition of \(Y^0_i\), we have:

\[
|Y_i(t, x)| \leq C_1(|x| + T), \quad \forall (t, x) \in [0, T] \times \mathbb{R}.
\] (2.20)

This yields the global existence of \(Y(\cdot, x)\) for each \(x \in \mathbb{R}\). Since \(Y^0_i \in C^2(\mathbb{R})\), it is easy to check that \(Y(\cdot, x) \in C^2(\mathbb{R}^+)\).

It remains to show (2.14). Let \(\tilde{w}^0_i = w^0_i \circ X^0_i\) and \(\tilde{w}^0(Y) = (\tilde{w}^0_1(Y_1), \ldots, \tilde{w}^0_n(Y_n))^T\). From (2.17), we have:

\[
\partial_t N_i(\tilde{w}^0(Y)) = - \sum_{j=1}^n \frac{\partial N_i(\tilde{w}^0(Y))}{\partial w_j} (\tilde{w}^0_j)'(Y_j)(N_j \lambda_j)(\tilde{w}^0(Y))
\]
\[
= - \sum_{j=1}^n \left( \lambda_j \frac{\partial N_i}{\partial w_j} \right) (\tilde{w}^0(Y))(\tilde{w}^0_j)'(Y_j)N_j(\tilde{w}^0(Y)).
\] (2.21)

On the other hand, using the rich condition (1.9), we have:

\[
\partial_t (\partial_x Y_i) = - \partial_x (N_i \lambda_i(\tilde{w}^0(Y))) = - \sum_{j=1}^n \frac{\partial(N_i \lambda_i)(\tilde{w}^0(Y))}{\partial w_j} (\tilde{w}^0_j)'(Y_j)\partial_x Y_j
\]
\[
= - \sum_{j=1}^n \left( \lambda_j \frac{\partial N_i}{\partial w_j} \right) (\tilde{w}^0(Y))(\tilde{w}^0_j)'(Y_j)\partial_x Y_j.
\] (2.22)

Let \(q_i = \partial_x Y_i - N_j(\tilde{w}^0(Y))\). By (2.21)–(2.22) it is easy to see that \(q_i\) \((1 \leq i \leq n)\) satisfy the following linear system of ordinary differential equations with smooth coefficients:

\[
\partial_t q_i = - \sum_{j=1}^n \left( \lambda_j \frac{\partial N_i}{\partial w_j} \right) (\tilde{w}^0(Y))(\tilde{w}^0_j)'(Y_j)q_j \quad (1 \leq i \leq n).
\] (2.23)

Moreover, from (2.3), we have \(q_i(0, x) = 0\) \((1 \leq i \leq n)\). Then we deduce from the uniqueness of solution that \(q_i = 0\) \((1 \leq i \leq n)\). This shows (2.14).

Finally, noting \(Y^0_i \in C^2(\mathbb{R})\), a direct computation using (2.17) and (2.14) shows that \(Y \in C^2(\mathbb{R}^+ \times \mathbb{R})\). \(\Box\)

From (2.17) and (2.14), we further obtain:

\[
\begin{dcases}
\partial_t Y_i + \tilde{\lambda}_i(Y)\partial_x Y_i = 0, & t > 0, \\
\ell = 0: Y_i = Y^0_i(x) & (1 \leq i \leq n),
\end{dcases}
\] (2.24)

where

\[
\tilde{\lambda}_i(Y) = \lambda_i(w^0_1(X^0_1(Y_1)), \ldots, w^0_{i-1}(X^0_{i-1}(Y_{i-1})), w^0_{i+1}(X^0_{i+1}(Y_{i+1})), \ldots, w^0_n(X^0_n(Y_n))),
\] (2.25)

which is independent of \(Y_i\). Thus, the system in (2.24) is still linearly degenerate. Then Cauchy problem (2.24) admits a unique \(C^2\) solution \(Y\). Thus we get the following result:

**Corollary 2.** Under the assumptions of Theorem 1, the unique global \(C^1\) solution \(w\) of (1.1)–(1.2) can be expressed by (2.12) in which \(Y\) is the unique \(C^2\) solution of (2.24).

**Remark 1.** (1) The method used above depends strongly on the conservation law (1.11) corresponding to system (1.1), which gives a possibility to make the change of coordinates for each \(i \ (1 \leq i \leq n)\). From Proposition 1, (1.11) holds for all \(i \in \{1, \ldots, n\}\) if and only if the system is rich and linearly degenerate.
(2) This method employs \( n \) changes of coordinates. For each \((t,x) \mapsto (s,y_i)\), we obtain an equivalent system of (1.1) as follows:

\[
\partial_t \tilde{w}_j + N_i(\tilde{w})(\lambda_j(\tilde{w}) - \lambda_i(\tilde{w})) \partial_{y_i} \tilde{w}_j = 0 \quad (1 \leq j \leq n).
\]  

(2.26) However only the \( i \)th equation in (2.26) is used for deriving the explicit formula.

(3) By the characteristic method, for smooth solutions, for each \( i \) (\( 1 \leq i \leq n \)) we can also obtain the expression

\[
w_i(t,x) = w_0^i(\beta_i)
\]

where \( \beta_i \) is the intersection point of the \( i \)th characteristic curve passing through the point \((t,x)\) with the \( x \)-axis. However, the determination of \( \beta_i \) as function of \((t,x)\) is not obvious.

(4) For \( w_0 \in L^\infty(\mathbb{R}) \) and a given Lipschitz solution \( Y \) of (2.17), it is not sure that (2.12) defines an entropy solution to Cauchy problem (1.1)–(1.2). Indeed, without a uniqueness result, we do not know whether \( Y \) satisfies (2.14). Thus, it is interesting to give a supplementary condition on the initial data \( w_0 \) such that (2.12) together with (2.17) does define an entropy solution.

3. Entropy solutions in the case \( N_i = N \) for all \( i \)

In this section we consider the particular case that \( N_i = N \) for all \( i \), which is the common structure of the Born–Infeld system and the augmented Born–Infeld system (see Section 6). Our goal is to establish the existence and uniqueness of entropy solutions with an explicit expression. In this case, for each \( i \in \{1, \ldots, n\} \), Eq. (1.11) becomes:

\[
\partial_t N(w) + \partial_x (N(w)\lambda_i(w)) = 0,
\]

(3.1) or equivalently

\[
\partial_x (N(w)\lambda_i(w)) = -\partial_t N(w),
\]

(3.2) which implies that for each \( i \), there is a function \( \tilde{\lambda}_i \) of \( t \) such that

\[
N(w)\lambda_i(w) - \tilde{\lambda}_i = M,
\]

(3.3) where \( M \) is independent of \( i \), satisfying

\[
\partial_t N(w) + \partial_x M = 0.
\]

(3.4) Thus, we have:

\[
N(w)\lambda_i(w) - \tilde{\lambda}_i = N(w)\lambda_j(w) - \tilde{\lambda}_j,
\]

(3.5) then

\[
N = \frac{\tilde{\lambda}_i - \tilde{\lambda}_j}{\lambda_i(w) - \lambda_j(w)}, \quad \forall i, j \in \{1, \ldots, n\}, \ j \neq i.
\]

(3.6) This implies that \( \tilde{\lambda}_i - \tilde{\lambda}_j \) is independent of \( t \). Without loss of generality, we may suppose that each \( \tilde{\lambda}_i \) is independent of \( t \). Then \( M \) is a function of \( w \). From (3.4) we may make a change of coordinates \((t,x) \mapsto (s,z)\) with

\[
s = t, \quad dz = N \, dx - M \, dt.
\]

(3.7) Applying the result of equivalence between (2.7) and (2.8), in Lagrangian coordinates \((s,z)\), Eq. (1.13) becomes:

\[
\partial_s \tilde{w}_i + \tilde{\lambda}_i \partial_z \tilde{w}_i = 0, \quad s > 0, \ z \in \mathbb{R}.
\]

(3.8) This shows that \( \tilde{\lambda}_i \) is just the \( i \)th eigenvalue of system (1.1) in Lagrangian coordinates. Then we have:

\[
\tilde{w}_i(s,z) = \tilde{w}_i^0(z - \tilde{\lambda}_i s) \quad (1 \leq i \leq n)
\]

(3.9) with \( \tilde{w}_i^0 = \tilde{w}_i(0, \cdot) \).

For \( w^0 \in L^\infty(\mathbb{R}) \), let \( z = Z(t,x) \) be the unique Lipschitz function satisfying (3.7) and

\[
Z(0, x) = Z^0(x) \overset{\text{def}}{=} \int_0^x N(w^0(\xi)) \, d\xi.
\]

(3.10)
Then, as in Section 2, we have \( \tilde{w}_i^0 = u_i^0(X^0) \) with \( X^0 = (Z^0)^{-1} \). Since \( \tilde{w} = (\tilde{w}_1, \ldots, \tilde{w}_n)' \) is given explicitly by (3.9), it follows from (3.7) that

\[
\frac{dx}{dt} = \frac{1}{N(\tilde{w}(t,z))} \frac{dz}{dt} + \frac{M(\tilde{w}(t,z))}{N(\tilde{w}(t,z))} d\tau. \tag{3.11}
\]

Together with \( X(0,z) = X^0(z) \), (3.11) determines explicitly a unique Lipschitz function \( x = X(t,z) \) for all \( t \geq 0 \).

By Lemma 2.1 of [18], we know that \( Z(t,\cdot) = X^{-1}(t,\cdot) \). Then from (3.9), the unique entropy solution of Cauchy problem (1.1)–(1.2) is given by:

\[
Y_i(t,x) = Y_i^0(X^0(Z(t,x) - \lambda_i t)) \quad (1 \leq i \leq n). \tag{3.12}
\]

Comparing with expression (2.12) discussed in Section 2, for all \( i \in \{1, \ldots, n\} \), in the case that \( N_i = N \), we have:

\[
Y_i(t,x) = Z(t,x) - \lambda_i t. \tag{3.13}
\]

Now we try to give a further explicit expression of \( X \). Set:

\[
v = \frac{1}{N}, \quad u = \frac{M}{N}. \tag{3.14}
\]

From (3.7) we have:

\[
dx =vdz +uds. \tag{3.15}
\]

Then \( x = X(s,z) \) satisfies:

\[
\frac{\partial x}{\partial s} = \tilde{u}, \quad \frac{\partial x}{\partial z} = \tilde{v} \tag{3.16}
\]

and

\[
\frac{\partial^2 x}{\partial s^2} = \frac{\partial s}{\partial \lambda} \frac{\partial x}{\partial s} = \tilde{u}, \quad \frac{\partial^2 x}{\partial z^2} = \tilde{v}. \tag{3.17}
\]

Hence, under assumption (3.4), \( x = X(s,z) \) solves a linear wave equation:

\[
\frac{\partial x}{\partial s} X - a^2 \frac{\partial x}{\partial z} X = 0 \tag{3.18}
\]

if and only if

\[
\frac{\partial x}{\partial s} - a^2 \frac{\partial x}{\partial z} \tilde{v} = 0, \tag{3.19}
\]

where \( a > 0 \) is a constant. By (2.7)–(2.8) it is easy to see that in Eulerian coordinates \((t,x)\), (3.19) is equivalent to

\[
\frac{\partial X}{\partial t} (Nu) + a^2 \frac{\partial X}{\partial s} (Nu^2 - a^2 N^{-1}) = 0. \tag{3.20}
\]

On the other hand, since \( M = Nu \), Eq. (3.4) can be written as

\[
\frac{\partial x}{\partial t} N + \frac{\partial x}{\partial s} (Nu) = 0. \tag{3.21}
\]

Eqs. (3.20)–(3.21) form a system of Chaplygin gas dynamics, namely, an isentropic gas dynamic system with the state equation of Von Kármán–Tsien [7,2]:

\[
p(N) = p_0 - a^2 N^{-1} \quad \text{with a constant } p_0 > 0. \tag{3.22}
\]
Noting that Eq. (3.4) is equivalent to
\[ \partial_t \bar{u} - \partial_x \bar{u} = 0, \] (3.23)

system (3.20)–(3.21) is equivalent to the linear system of Eqs. (3.19) and (3.23). It is easy to see that ±\(a\) are just two eigenvalues of the last system. Since Eqs. (3.19) and (3.23) are included in system (3.8), we claim that ±\(a\) are two of the values \(\lambda_i\) \((i = 1, \ldots, n)\). Thus, we obtain the following result:

**Proposition 3.** Let \(z = Z(t, x)\) be the change of variables given by (3.7) and (3.10), and \(X(t, \cdot) = Z^{-1}(t, \cdot)\) for \(t \geq 0\). Then \(x = X(t, z)\) satisfies a linear wave Eq. (3.18) if and only if \((N, u)\) solves the system of Chaplygin gas dynamics (3.20)–(3.21).

Noting (3.14) and (3.16), the initial data of \(X\) are given explicitly by:
\[ X(0, z) = X^0(z), \quad \partial_t X(0, z) = \frac{M(u^0(z))}{N(u^0(z))} \] (3.24)

Then the d’Alembert formula provides an explicit expression of \(X\) in the case (3.18). This result can be applied to the Born–Infeld system (6.1), the augmented Born–Infeld system (6.16) and all \(2 \times 2\) linearly degenerate systems of conservation laws, respectively (see [18]).

4. The non-strictly hyperbolic case

In this section, we consider system (1.1) in the non-strictly hyperbolic case. We suppose that each eigenvalue \(\lambda_i(w)\) of (1.1) has a constant multiplicity. More precisely, on the domain of consideration, we suppose that
\[ \lambda_1(w) = \cdots = \lambda_r(w) < \lambda_{r+1}(w) = \cdots = \lambda_{s}(w) < \cdots < \lambda_{r+s-1}(w) = \cdots = \lambda_{r+s}(w), \] (4.1)

with \(r_1, r_2, \ldots, r_s\) being constants and
\[ 1 \leq r_1 < r_2 < \cdots < r_s = n. \] (4.2)

This implies that for any \(j \neq i\), we have either \(\lambda_j(w) = \lambda_i(w)\) or \(\lambda_j(w) \neq \lambda_i(w)\) for all \(w\). From the relation of eigenvalues (2.9) between two coordinate systems, it is easy to see that \(\lambda_i(w)\) and \(\tilde{\lambda}_i(w)\) have the same constant multiplicity. In this case, we prove that all the explicit formulas are still valid. For this propose, it suffices to check each condition in previous sections, where the strictly hyperbolic condition is used.

In the case of eigenvalues with constant multiplicity, the \(i\)th characteristic is rich if for all \(j, k \in \{1, \ldots, n\}\) such that \(\lambda_j(w) \neq \lambda_i(w)\) and \(\lambda_k(w) \neq \lambda_i(w)\), we have:
\[ \frac{\partial}{\partial w_j} \left( \frac{\partial \lambda_i(w)}{\partial w_k} - \lambda_i(w) \right) = \frac{\partial}{\partial w_k} \left( \frac{\partial \lambda_i(w)}{\partial w_j} - \lambda_i(w) \right). \] (4.3)

This definition is equivalent to the existence of a smooth positive function \(N_i(w) > 0\) such that
\[ (\lambda_j(w) - \lambda_i(w)) \frac{\partial N_i(w)}{\partial w_j} = N_i(w) \frac{\partial \lambda_i(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}, \lambda_j(w) \neq \lambda_i(w). \] (4.4)

When \(\lambda_i(w)\) is linearly degenerate, an equivalent condition of (4.4) is still (1.9), namely,
\[ (\lambda_j(w) - \lambda_i(w)) \frac{\partial N_i(w)}{\partial w_j} = N_i(w) \frac{\partial \lambda_i(w)}{\partial w_j}, \quad \forall j \in \{1, \ldots, n\}. \] (4.5)

Noting that if \(\lambda_i(w)\) is linearly degenerate and \(\lambda_j(w) \equiv \lambda_i(w)\) for \(j \neq i\), then \(\lambda_i(w)\) is independent of \(w_j\) too and we may take \(N_j(w) = N_i(w)\), hence, with the same definitions given by (2.1) and (2.3), we have \(Y_j = Y_i\) and then \(X_j = X_i\).

In the particular case that \(N_i = N\) for all \(i \in \{1, \ldots, n\}\) discussed in Section 3, (3.6) should be replaced by:
\[ N = \frac{\lambda_i - \tilde{\lambda}_j}{\lambda_i(w) - \lambda_j(w)}, \quad \forall i, j \in \{1, \ldots, n\}, \lambda_j(w) \neq \lambda_i(w). \] (4.6)

With these changes, we conclude that Propositions 1–3, Corollaries 1 and 2 and Theorems 1 and 2 are still valid when the strictly hyperbolic condition is replaced by the constant multiplicity condition of eigenvalues to system (1.1).
5. Existence of entropy solutions in the non-strictly hyperbolic case

In this section, we study the existence of entropy solutions in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$ to Cauchy problem (1.1)–(1.2) in the non-strictly hyperbolic case. To this end, we suppose that system (1.1) is rich and linearly degenerate with each eigenvalue $\lambda_i(w)$ being of constant multiplicity. We prove the existence of solutions by using the explicit expressions given by Proposition 2 and Theorem 1, together with classical compactness arguments. Thus, our proof is simpler than that of [6] in the strictly hyperbolic case, where the compensated compactness and Young measures are employed.

Note that for $w^0 \in BV(\mathbb{R})$, the existence of an entropy solution $w \in BV(\mathbb{R}^+ \times \mathbb{R})$ can be easily proved by the Glimm method [9]. Moreover, the total variation of each $w_i$ is decreasing in time. The goal of this section is to establish the global existence of entropy solutions in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$.

Theorem 3. Let $w^0 \in L^\infty(\mathbb{R})$. Under the assumptions above on system (1.1), Cauchy problem (1.1)–(1.2) admits a global entropy solution $w \in L^\infty(\mathbb{R}^+ \times \mathbb{R})$ in the sense of Definition 1.

Proof. Let $K = \prod_{i=1}^n [\inf_{x \in \mathbb{R}} w^0_i(x), \sup_{x \in \mathbb{R}} w^0_i(x)]$ and $(w^0_i)_{i>0}$ be the approximate sequence of $w^0$ defined by $w^0_i = \rho_i * w^0$, where $\rho_i$ is a standard mollifier (see [5]):

$$\rho_i \in C^\infty(\mathbb{R}), \quad \rho_i \geq 0, \quad \int_{\mathbb{R}} \rho_i(x) \, dx = 1, \quad \text{Supp} \, \rho_i \subset [-\varepsilon, \varepsilon].$$

Since $w^0 \in L^\infty(\mathbb{R}) \subset L^p_{\text{loc}}(\mathbb{R})$ for all $p \geq 1$, we have $w^0_i \in C^\infty(\mathbb{R})$ with values in $K$. Then, $w^0_i$ has a uniformly bounded $C^0$ norm. Moreover, as $\varepsilon \to 0$,

$$w^0_i \to w^0 \quad \text{strongly in} \quad L^p_{\text{loc}}(\mathbb{R}) \quad \text{and weakly-} \ast \quad \text{in} \quad L^\infty(\mathbb{R}),$$

which imply that, up to a subsequence,

$$w^0_i \to w^0 \quad \text{a.e. in} \quad \mathbb{R}.$$  

Let us consider an approximate Cauchy problem:

$$\begin{cases}
\partial_t w_i + \lambda_i(w_i) \partial_x w_i = 0 & (1 \leq i \leq n), \quad t > 0, \quad x \in \mathbb{R}, \\
t = 0: \quad w_i = w^0_i(x), \quad x \in \mathbb{R},
\end{cases} \quad (5.4)$$

with $w_e = (w_1, \ldots, w_n)'$. From Theorem 1, there is a unique global solution $w_e \in C^1(\mathbb{R}^+ \times \mathbb{R})$ given by

$$w_i(t, x) = w^0_i \left( X^0_{ie} (Y_e(t, x)) \right), \quad \forall (t, x) \in \mathbb{R}^+ \times \mathbb{R} \quad (1 \leq i \leq n),$$

where $Y_e = (Y_1, \ldots, Y_n)' \in C^2(\mathbb{R}^+ \times \mathbb{R})$ is the unique smooth solution to the following problem:

$$\begin{cases}
\frac{dY_{ie}}{dt} = -(N_i \lambda_i)(w^0_1(X^0_{1e}(Y_e(t, x))), \ldots, w^0_n(X^0_{ne}(Y_e(t, x)))), & t > 0, \\
t = 0: \quad Y_{ie} = Y^0_{ie}(x) & (1 \leq i \leq n)
\end{cases} \quad (5.5)$$

with $X^0_{ie} = (Y^0_{ie})^{-1}$ and

$$Y^0_{ie}(x) = \int_0^x N_i(w^0_e(\xi)) \, d\xi.$$  

This solution $Y_e$ also satisfies:

$$\frac{\partial Y_{ie}}{\partial x} = N_i(w^0_1(X^0_{1e}(Y_e(t)))), \ldots, w^0_n(X^0_{ne}(Y_e(t)))) \quad (1 \leq i \leq n), \quad \forall t > 0. \quad (5.8)$$

Thus, from the second part of Proposition 2, once we can prove the following proposition, we get Theorem 3 immediately. □
Proposition 4. With the notations above, up to a subsequence, as $\varepsilon \to 0$, we have, for all $1 \leq i \leq n$,
\[ w_{i\varepsilon} \to w_i^0(X_i^0(Y_i)) \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}), \tag{5.9} \]
where $X_i^0 = (Y_i^0)^{-1}$ with $Y_i^0$ given by (2.3) and $Y_i$ being a Lipschitz solution to (2.17) and (2.14).

Proof. From (5.1) and the definition of $Y_i^0$ and $Y_i^0$, it is clear that, as $\varepsilon \to 0$,
\[ Y_{i\varepsilon} \to Y_i^0 \quad \text{locally uniformly in } \mathbb{R}. \tag{5.10} \]
Noting that $X_i^0$ is a Lipschitz function, then
\[ X_i^0(Y_{i\varepsilon}^0) \to X_i^0(Y_i^0) = I, \quad \text{locally uniformly in } \mathbb{R}, \tag{5.11} \]
where $I$ denotes the unit operator from $\mathbb{R}$ to $\mathbb{R}$. For any given constant $L_1 > 0$, since $w_i^0$ is uniformly bounded, and $N_i \geq \alpha_i > 0$, there is a constant $L_2 > 0$ independent of $\varepsilon$, such that $z = Y_i^0(x) \in [-L_1, L_1]$ implies that $x \in [-L_2, L_2]$.

This yields:
\[ \sup \limits_{z \in [-L_1, L_1]} |X_{i\varepsilon}^0(z) - X_i^0(z)| = \sup \limits_{z = Y_{i\varepsilon}^0(x) \in [-L_1, L_1]} |x - X_{i\varepsilon}^0(Y_{i\varepsilon}^0(x))| \leq \sup \limits_{x \in [-L_2, L_2]} |x - X_i^0(Y_i^0(x))| \to 0. \]

Therefore,
\[ X_{i\varepsilon}^0 \to X_i^0 \quad \text{locally uniformly in } \mathbb{R}. \tag{5.12} \]

On the other hand, by the same argument as for proving (2.20), we deduce from (5.6) that there exists a constant $C_2 > 0$ independent of $\varepsilon$, such that for any $T > 0$ we have:
\[ |Y_{i\varepsilon}(t,x)| \leq C_2(|x| + T), \quad \forall (t,x) \in [0,T] \times \mathbb{R}. \tag{5.13} \]

Since $w_i^0$ is a uniformly bounded function in $\mathbb{R}$, we obtain from (5.6) and (5.8) that the sequence $(Y_{i\varepsilon})_{\varepsilon > 0}$ is bounded in $W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$. From the Ascoli theorem, there is a function $Y_i \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R})$ such that, up to a subsequence,
\[ Y_{i\varepsilon} \to Y_i \quad \text{locally uniformly in } \mathbb{R}^+ \times \mathbb{R} \tag{5.14} \]
and
\[ Y_{i\varepsilon} \to Y_i \quad \text{weakly-* in } W^{1,\infty}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{5.15} \]

It follows from (5.12) that
\[ X_{i\varepsilon}^0(Y_{i\varepsilon}) \to X_i^0(Y_i) \quad \text{locally uniformly in } \mathbb{R}^+ \times \mathbb{R}, \tag{5.16} \]
so, noting (5.3), we have:
\[ w_{i\varepsilon}^0(X_{i\varepsilon}^0(Y_{i\varepsilon})) \to w_i^0(X_i^0(Y_i)) \quad \text{a.e. in } \mathbb{R}^+ \times \mathbb{R}. \tag{5.17} \]

Thus, the Lebesgue dominated convergence theorem together with (5.5) implies that
\[ w_{i\varepsilon} \to w_i^0(X_i^0(Y_i)) \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{5.18} \]

It remains to show that $Y = (Y_1, \ldots, Y_n)'$ is a Lipschitz function and satisfies (2.1), i.e., (2.17) and (2.14). From (5.15), we obtain:
\[ \frac{\partial Y_{i\varepsilon}}{\partial t} \to \frac{\partial Y_i}{\partial t}, \quad \frac{\partial Y_{i\varepsilon}}{\partial x} \to \frac{\partial Y_i}{\partial x} \quad \text{weakly-* in } L^\infty_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{5.19} \]

Moreover, using (5.17) and again the Lebesgue dominated convergence theorem, up to a subsequence and for any smooth function $\psi$, we have:
\[ \psi(w_{i\varepsilon}^0(X_{i\varepsilon}^0(Y_{i\varepsilon}))), \ldots, w_{n\varepsilon}^0(X_{n\varepsilon}^0(Y_{n\varepsilon}))) \to \psi(w_i^0(X_i^0(Y_1)), \ldots, w_n^0(X_n^0(Y_n))) \quad \text{strongly in } L^p_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{5.20} \]

This allows to pass to the limit in (5.6) and (5.8) to obtain (2.17) and (2.14). Thus, $Y$ is a Lipschitz function in $\mathbb{R}^+ \times \mathbb{R}$. \( \square \)
6. Examples

We give two examples for the above results. Each example corresponds to the case $N_i = N$ for all $i$ $(1 \leq i \leq n)$ discussed in Sections 3 and 4. These two examples are the Born–Infeld system (of 4 equations) and the linear Lagrangian system. We mention that both Born–Infeld system and its augmented system possess constant multiplicity eigenvalues. In [17,18], the Born–Infeld system is solved through the resolution of the augmented Born–Infeld system and a fine analysis on its entropy–entropy flux pairs. For that purpose, we have to show that the Born–Infeld manifolds are time invariant for entropy solutions. Here, we treat these two systems in the same way due to their common structure: $N_i = N$ $(1 \leq i \leq n)$.

6.1. The Born–Infeld system

The one-dimensional Born–Infeld system reads as (see [4]):

\[
\begin{align*}
&\partial_t D_2 + \partial_x \left( \frac{B_3 + D_2 P_1 - D_1 P_2}{h} \right) = 0, \\
&\partial_t D_3 + \partial_x \left( \frac{-B_2 + D_3 P_1 - D_1 P_3}{h} \right) = 0, \\
&\partial_t B_2 + \partial_x \left( \frac{-D_3 + B_2 P_1 - B_1 P_2}{h} \right) = 0, \\
&\partial_t B_3 + \partial_x \left( \frac{D_2 + B_3 P_1 - B_1 P_3}{h} \right) = 0,
\end{align*}
\]

(6.1)

Here $u = (D_2, D_3, B_2, B_3)^t$ are the unknown variables, $B_1, D_1$ are real constants, and

\[ B = (B_1, B_2, B_3)^t, \quad D = (D_1, D_2, D_3)^t, \quad P = (P_1, P_2, P_3)^t. \]

From [17], the Riemann invariants and the eigenvalues of system (6.1) are:

\[ w_i = h^{-1}(u)l_i u \quad (1 \leq i \leq 4) \]

(6.2)

and

\[ \lambda_1(u) = \lambda_2(u) = \frac{P_1 - a}{h}, \quad \lambda_3(u) = \lambda_4(u) = \frac{P_1 + a}{h}, \]

(6.3)

respectively, where $l_i$ $(1 \leq i \leq 4)$ are linearly independent constant vectors given by:

\[
\begin{align*}
l_1 &= (a, \beta_1, 0, -\beta_3), \\
l_2 &= (-\beta_1, a, \beta_3, 0), \\
l_3 &= (0, -\beta_2, a, \beta_1), \\
l_4 &= (\beta_2, 0, -\beta_1, a),
\end{align*}
\]

(6.4)

in which

\[ \beta_1 = B_1 D_1, \quad \beta_2 = 1 + B_1^2, \quad \beta_3 = 1 + D_1^2 \quad \text{and} \quad a = \sqrt{1 + B_1^2 + D_1^2} > 0. \]

(6.5)

Hence, system (6.1) can be written in the form (1.1). It is not strictly hyperbolic but with constant multiplicity.

Since $a$ is a constant, from $h\lambda_i = P_1 \pm a$ and the following additional conservation law (see [4]):

\[ \partial_t h + \partial_x P_1 = 0, \]

we see that $(h, h\lambda_i)$ $(1 \leq i \leq 4)$ are entropy–entropy flux pairs. By Proposition 1, (6.1) is a linearly degenerate rich system with $N_i = h$ being independent of $i$. Furthermore, from the discussion in Section 4, the eigenvalues of system (6.1) in Lagrangian coordinates are:

\[ \tilde{\lambda}_1 = \tilde{\lambda}_2 = -a, \quad \tilde{\lambda}_3 = \tilde{\lambda}_4 = a. \]

(6.6)

Then Theorem 2 can be applied to the Born–Infeld system (6.1) with $M = P_1$. 

More precisely, given an initial condition for system (6.1):
\[ t = 0: u = u^0(x), \quad x \in \mathbb{R}, \] (6.7)
where \( u^0 \in L^\infty(\mathbb{R}) \). Let \( u^0_i = h^{-1}(u^0)l_iu^0 \) (1 \( \leq i \leq 4 \) ) and
\[ Z^0(x) \overset{\text{def}}{=} \int_0^x h(u^0(\xi)) d\xi \quad (1 \leq i \leq 4). \] (6.8)

Regarding \( M \) as a function of \( w = (w_1, w_2, w_3, w_4)^T \) and denoting it still by \( M \), by Theorem 2, the unique entropy solution of (6.1) and (6.7) is given by:
\[ w_i(t, x) = w_i^0(X^0(Z(t, x) - \tilde{\lambda}_i t)) \quad (1 \leq i \leq 4), \] (6.9)
where \( X^0 \) and \( Z(t, \cdot) \) are the inverse functions of \( Z^0 \) and \( X(t, \cdot) \), respectively, with \( x = X(t, z) \) being the unique solution of
\[ dx = \frac{1}{h(\tilde{w}(t, z))} dz + \frac{P_1(\tilde{w}(t, z))}{h(\tilde{w}(t, z))} dt, \quad X(0, z) = X^0(z), \] (6.10)
where \( \tilde{w}_i(t, z) = u_i^0(X^0(z - \tilde{\lambda}_i t)) \).

If the initial data \( u^0 \in C^1(\mathbb{R}) \) with bounded \( C^0 \) norm, we may also apply Theorem 1 to Cauchy problem (6.1) and (6.7), of which the unique smooth solution is given by:
\[ w_i(t, x) = w_i^0(X_i^0(Y_i^0(t, x))) \quad (1 \leq i \leq 4), \] (6.11)
where \( X^0 \) is the inverse function of \( Z^0 \) and \( y_i = Y_i^0(t, x) \) (1 \( \leq i \leq 4 \) ) are the unique global smooth solution of the following Cauchy problem:
\[
\begin{align*}
\frac{dy_1}{dt} &= -(P_1 - a)(w_1^0(X^0(y_1)), w_2^0(X^0(y_1)), w_3^0(X^0(y_2)), w_4^0(X^0(y_2))), \\
\frac{dy_2}{dt} &= -(P_1 + a)(w_1^0(X^0(y_1)), w_2^0(X^0(y_1)), w_3^0(X^0(y_2)), w_4^0(X^0(y_2))), \\
y_3 &= y_1, \quad y_4 = y_2.
\end{align*}
\] (6.12)

6.2. The linear Lagrangian system

A linear Lagrangian system takes the following conservative form [18,19]:
\[
\begin{align*}
\frac{\partial_x u_1 + \partial_t f_1(u)}{u_1} &= 0, \\
\frac{\partial_t u_i + \partial_x}{u_1} \left[ \sum_{j=2}^n \left( a_{ij} + u_i f_1(u) + \sum_{j=2}^n a_{ij} u_j \right) \right] &= 0 \quad (2 \leq i \leq n),
\end{align*}
\] (6.13)
where \( u_1 > 0, A = (a_{ij})_{1 \leq i, j \leq n} \) is a real constant matrix and
\[
u = (u_1, \ldots, u_n)^T, \quad f_1(u) = -a_{11} - \sum_{j=2}^n a_{1j} u_j.
\] (6.14)

Suppose that system (6.13) is hyperbolic, then matrix \( A \) is diagonalizable and its Riemann invariants are:
\[
w_i(u) = \frac{1}{u_1} l_i(1, u_2, \ldots, u_n)^T \quad (1 \leq i \leq n),
\] where \( l_i \) is the \( i \)th left eigenvector of \( A \). Let \( \tilde{\lambda}_i \) be the \( i \)th eigenvalue of \( A \). From (2.9), we have:
\[
f_1(u) = u_1 \tilde{\lambda}_i(u) - \tilde{\lambda}_i.
\] (6.15)
Hence, system (6.13) can be put in the form (1.1). Moreover, \((u_1, u_1\lambda_i(u))\) is an entropy–entropy flux pair for each \(i \in \{1, \ldots, n\}\). It follows from Proposition 1 that (6.13) is a linearly degenerate rich system with \(N_i = u_1\) and \(M = f_j(u)\). Thus, Theorem 2 can be applied to system (6.13). Here we omit the detailed expressions.

Finally, remark that the class of linear Lagrangian systems contains the augmented Born–Infeld system but does not contain the Born–Infeld system [18]. This difference comes from the fact that \(h\) is not a conservative variable of the Born–Infeld system but it is of the augmented Born–Infeld system. However, these two systems have a common structure: \(N_i = h\) \((1 \leq i \leq n)\) expressed by their Riemann invariants. Indeed, the augmented Born–Infeld system reads as

\[
\begin{align*}
\partial_t h + \partial_x P_1 &= 0, \\
\partial_t P_1 + \partial_x \left( \frac{P_1^2 - a^2}{h} \right) &= 0, \\
\partial_t D_2 + \partial_x \left( \frac{B_3 + D_2 P_1 - D_1 P_2}{h} \right) &= 0, \\
\partial_t D_3 + \partial_x \left( \frac{-B_2 + D_3 P_1 - D_1 P_3}{h} \right) &= 0, \\
\partial_t B_2 + \partial_x \left( \frac{-D_3 + B_2 P_1 - B_1 P_2}{h} \right) &= 0, \\
\partial_t B_3 + \partial_x \left( \frac{D_2 + B_3 P_1 - B_1 P_3}{h} \right) &= 0, \\
\partial_t P_2 + \partial_x \left( \frac{P_1 P_2 - D_1 D_2 - B_1 B_2}{h} \right) &= 0, \\
\partial_t P_3 + \partial_x \left( \frac{P_1 P_3 - D_1 D_3 - B_1 B_3}{h} \right) &= 0,
\end{align*}
\]

(6.16)

where \(h, P_1, D_2, D_3, B_2, B_3, P_2\) and \(P_3\) are unknown variables. The first two equations in (6.16) are just the system of Chaplygin gas dynamics. Moreover, comparing with (6.13), the corresponding matrix \(A\) is:

\[
A = \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -D_1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -D_1 \\
0 & 0 & 0 & -1 & 0 & 0 & -B_1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -B_1 & 0 \\
0 & 0 & -D_1 & 0 & -B_1 & 0 & 0 & 0 \\
0 & 0 & 0 & -D_1 & 0 & -B_1 & 0 & 0
\end{pmatrix},
\]

which is obviously diagonalizable on \(\mathbb{R}\). This implies that the augmented Born–Infeld system (6.16) is hyperbolic.

References


