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3-branes and uniqueness of the Salam–Sezgin vacuum

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Abstract

We prove the uniqueness of the supersymmetric Salam–Sezgin $(\text{Minkowski})_4 \times S^2$ ground state among all non-singular solutions with a four-dimensional Poincaré, de Sitter or anti-de Sitter symmetry. We construct the most general solutions with an axial symmetry in the two-dimensional internal space, and show that included amongst these is a family that is non-singular away from a conical defect at one pole of a distorted 2-sphere. These solutions admit the interpretation of 3-branes with negative tension.

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1. Introduction

There has recently been a revival of interest in the six-dimensional gauged supergravity model of Salam and Sezgin, which has long been known to admit a $(\text{Minksowki})_4 \times S^2$ supersymmetric vacuum [1], and to have potentially interesting applications in cosmology [2–7]. On the theoretical side, it was recently found that this is one of the very few supergravity models that admits a fully consistent Pauli-type reduction on a coset space. Specifically, it was shown that it admits such a consistent reduction on S^2, yielding a chiral four-dimensional $N = 1$ supergravity coupled to an $SU(2)$ Yang–Mills multiplet and a scalar multiplet [8]. It was also shown that there exists an extended family of supersymmetric $AdS_3 \times S^3$ vacua, with a parameter characterising the degree of squashing of the S^3, which in an appropriate limit reduce (locally) to the $(\text{Minkowski})_4 \times S^2$ vacuum [9]. On the phenomenological side, the current interest in large extra dimensions favours six-dimensional models, and the Salam–Sezgin model has featured in recent studies (see [6,7], and references therein).

The Salam–Sezgin model as it stands, being chiral, is anomalous. These anomalies can be cancelled by the inclusion of additional matter multiplets [10–12]. A surprising feature of the six-dimensional model is that it has a positive scalar potential and this fact has hindered attempts to obtain it from higher-dimensional models such as eleven-dimensional supergravity or

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ten-dimensional string theory. Recently, in [13], it has been shown that the bosonic sector of the model can be obtained via a generalised dimensional reduction from $D = 7$ and in [14] an M/string-theory origin for the Salam–Sezgin theory has been found.

In this Letter, we shall show that the remarkable supersymmetric background found by Salam and Sezgin is in fact unique among all non-singular backgrounds with four-dimensional Poincaré, de Sitter or anti-de Sitter invariance. Thus any four-dimensional model based on the Salam–Sezgin theory must necessarily be supersymmetric unless 3-branes are included, as, for example, introduced in [7] by inserting conical defects at the north and south poles of the 2-sphere. By contrast with many compactifications, such as those of Calabi–Yau type, which have many moduli corresponding to flat supersymmetry-preserving directions in the relevant effective potential, the Salam–Sezgin vacuum has just one free parameter, which may be taken to be the expectation value of the dilaton field.

Although the full $SO(3)$ rotational symmetry of the 2-sphere is broken by the presence of the conical defects in the 3-branes introduced in [7], the solutions are still axisymmetric. We construct the most general Poincaré-invariant axisymmetric solution, and find that within this class there exist additional 3-brane solutions (first constructed, in a general framework, in [15]) with conical defects in which the local geometry of the 2-sphere is modified from the usual round S^2 geometry, and the dilaton field is no longer constant. The Einstein equations in these solutions force the existence of conical defects, without the necessity of introducing additional delta-function sources in the equations. By contrast with the 3-branes introduced in [7], which retain supersymmetry in the bulk, in our new solutions supersymmetry is broken in the bulk.

Unfortunately, the Dirac quantisation condition forces these branes to have *negative* tension. Following earlier suggestions [7,16], one may incorporate additional six-dimensional gauge fields in the solutions. These modify the Dirac quantisation condition in a way which is similar to the modification required for the conical defects introduced in [7] but do not alter the sign of the tension.

The new 3-brane solutions have a non-constant dilaton field, but are nevertheless apparently consistent with the suggestion of [7] that the 3-brane dilaton coupling should vanish.

2. Proof of uniqueness

In this section we shall show that any non-singular solution with a compact internal 2-space and with a four-dimensional spacetime of maximal symmetry must be the Salam–Sezgin (Minkowski)$_4 \times S^2$ ground state. We shall do so by first showing that any smooth solution with compact internal 2-space must be axisymmetric. All axisymmetric solutions, whether they be singular or not, are then obtained explicitly. We then show that the only non-singular solution with compact internal 2-space in this class is that of Salam and Sezgin. It follows therefore that any smooth ground state with compact internal space must be the Salam–Sezgin solution. Note that we do not assume axisymmetry; we prove it for all non-singular solutions. Of course, singular solutions need not be axisymmetric. However, the explicit axisymmetric (but singular) solutions which we obtain in this section provide explicit 3-brane solutions whose properties will be explored in the next section.

The bosonic sector of the six-dimensional $N = (1,0)$ gauged Einstein–Maxwell supergravity is described by the Lagrangian [1,17]

$$\begin{aligned}\mathcal{L} = {}& \hat{R}\,\hat{*}\,\mathbb{1} - \tfrac{1}{4}\,\hat{*}\,d\phi\wedge d\phi - \tfrac{1}{2}e^{\phi}\,\hat{*}\,H_{(3)}\wedge H_{(3)} \\ & - \tfrac{1}{2}e^{\frac{1}{2}\phi}\,\hat{*}\,F_{(2)}\wedge F_{(2)} - 8g^2 e^{-\frac{1}{2}\phi}\,\hat{*}\,\mathbb{1},\end{aligned} \tag{2.1}$$

where $F_{(2)} = dA_{(1)}$, $H_{(3)} = dB_{(2)} + \frac{1}{2}F_{(2)}\wedge A_{(1)}$, and we place a hat on the six-dimensional metric. (We use conventions where $\hat{*}\omega\wedge\omega = \frac{1}{p!}\omega^{M_1\cdots M_p}\omega_{M_1\cdots M_p}\,\hat{*}\,\mathbb{1}$ for any p-form ω.) Here g is the gauge-coupling constant, and the fermions all carry charge g in their minimal coupling to the $U(1)$ gauge field $A_{(1)}$. The bosonic equations of motion following from (2.1) are

$$\begin{aligned}\hat{R}_{MN} = {}& \tfrac{1}{4}\partial_M\phi\partial_N\phi + \tfrac{1}{2}e^{\frac{1}{2}\phi}\big(F^2_{MN} - \tfrac{1}{8}F^2\hat{g}_{MN}\big) \\ & + \tfrac{1}{4}e^{\phi}\big(H^2_{MN} - \tfrac{1}{6}H^2\hat{g}_{MN}\big) + 2g^2 e^{-\frac{1}{2}\phi}\hat{g}_{MN}, \\ \hat{\nabla}^2\phi = {}& \tfrac{1}{4}e^{\frac{1}{2}\phi}F^2 + \tfrac{1}{6}e^{\phi}H^2 - 8g^2 e^{-\frac{1}{2}\phi}, \\ d\big(e^{\frac{1}{2}\phi} * F_{(2)}\big) = {}& e^{\phi} * H_{(3)}\wedge F_{(2)}, \\ d\big(e^{\phi} * H_{(3)}\big) = {}& 0.\end{aligned} \tag{2.2}$$

Note that the dimensionful coupling constant g can be rescaled at will by adding a constant to ϕ, together with compensating rescalings of the other fields [8].

It has long been known that this theory admits a solution of the form $(\text{Minkowski})_4 \times S^2$, and furthermore, that this solution has $N = 1$ supersymmetry in the four-dimensional spacetime [1]. In what follows, we shall demonstrate that the supersymmetric Salam–Sezgin solution is in fact the only one with four-dimensional Poincaré, de Sitter or anti-de Sitter invariance and a smooth, non-singular, two-dimensional, compact internal space Y. We shall do so by first showing that the cosmological constant for the four-dimensional maximally-symmetric spacetime vanishes. Then, we shall show that every solution must admit a rotational Killing vector acting on the internal space, and then we exhibit explicitly all such axisymmetric solutions. The only non-singular one is that of Salam and Sezgin, but there are also non-supersymmetric solutions with conical singularities, which may be interpreted as containing 3-branes. Thus in this case, non-singularity together with Poincaré, de Sitter or anti-de Sitter invariance implies Poincaré supersymmetry, and in order to break supersymmetry one *must* introduce 3-branes.

The most general ansatz for a configuration with four-dimensional maximal symmetry is

$$\begin{aligned} &d\hat{s}_6^2 = W(y)^2\, ds_4^2 + ds_2^2, \\ &H_{(3)} = 0, \qquad F_{\mu\nu} = 0, \\ &F_{\mu a} = 0, \qquad F_{ab} = f(y)\epsilon_{ab}, \end{aligned} \tag{2.3}$$

where $ds_2^2 = g_{mn}\, dy^m\, dy^n$ is the metric on the internal space Y, $W(y)$ is a warp factor, and ds_4^2 is a four-dimensional Minkowski, de Sitter or anti-de Sitter metric. In the obvious tangent frame, the components of the six-dimensional Ricci tensor are given by

$$\begin{aligned} &\hat{R}_{\mu\nu} = \frac{1}{W^2} R_{\mu\nu} - \frac{1}{4W^4} \nabla^2 W^4 \eta_{\mu\nu}, \\ &\hat{R}_{ab} = R_{ab} - \frac{4}{W} \nabla_a \nabla_b W, \qquad \hat{R}_{\mu a} = 0, \end{aligned} \tag{2.4}$$

where $R_{\mu\nu}$ and R_{ab} are the tangent-frame components of the Ricci tensor for the four-dimensional spacetime and the internal space, and ∇_a is the covariant derivative on Y. Our assumption of maximal four-dimensional symmetry implies that we shall have $R_{\mu\nu} = \Lambda \eta_{\mu\nu}$.

The $\hat{R}_{\mu\nu}$ and ϕ equations become, from (2.2),

$$\frac{1}{4} F_{(2)}^2 e^{\frac{1}{2}\phi} - 8g^2 e^{-\frac{1}{2}\phi} = \frac{1}{W^4} \nabla^2 W^4 - \frac{4\Lambda}{W^2},$$

$$\frac{1}{4} F_{(2)}^2 e^{\frac{1}{2}\phi} - 8g^2 e^{-\frac{1}{2}\phi} = \frac{1}{W^4} \nabla^a \big(W^4 \nabla_a \phi\big). \tag{2.5}$$

It follows that

$$\nabla^a \big(W^4 \nabla_a (\phi - 4 \log W)\big) + 4\Lambda W^2 = 0. \tag{2.6}$$

Integrating over the compact internal manifold Y, we immediately see that $\Lambda \int_Y W^2 = 0$ and hence the cosmological constant must vanish.

Having established that the four-dimensional metric is flat, we now have

$$\nabla^a \big(W^4 \nabla_a (\phi - 4 \log W)\big) = 0. \tag{2.7}$$

Assuming as before that the internal space Y is complete and non-singular, and that ϕ and W are everywhere smooth functions on Y, with W everywhere positive, we may multiply Eq. (2.7) by $(\phi - 4 \log W)$ and integrate by parts, to get

$$\int_Y \sqrt{g}\, d^2 y\, W^4 \big|\nabla(\phi - 4 \log W)\big|^2 = 0, \tag{2.8}$$

and hence

$$\phi = 4 \log W. \tag{2.9}$$

(There is no loss of generality in omitting the addititive constant.) The equation of motion for $F_{(2)}$ now gives

$$F_{(2)} = \frac{1}{2} q W^{-6} \epsilon_{mn}\, dy^m \wedge dy^n, \tag{2.10}$$

where q is a magnetic charge.

Because Y is two-dimensional, we have $R_{mn} = K g_{mn}$, where $K = K(y)$ is the Gauss curvature. The R_{mn} equation becomes

$$\begin{aligned} &K g_{mn} - \frac{2}{W^2} \nabla_m \nabla_n W^2 \\ &\quad = \frac{3}{8} q^2 W^{-10} g_{mn} + 2g^2 W^{-2} g_{mn}. \end{aligned} \tag{2.11}$$

The tracefree part gives

$$\nabla_m \nabla_n W^2 = \frac{1}{2} \nabla^2 W^2 g_{mn}, \tag{2.12}$$

which shows that $\nabla^m W^2$ is a conformal Killing vector on Y. It then follows that

$$K^m \equiv \epsilon^{mn} \nabla_n W^2 \tag{2.13}$$

is a Killing vector on Y, which is orthogonal to the level sets of W (and hence ϕ).

By integrating the trace of (2.11) over Y, one finds that

$$\chi = \frac{1}{2\pi}\int_Y \sqrt{g}K\,d^2y = \frac{1}{2\pi}\int_Y \sqrt{g}\,d^2y \times\left(\frac{4(\nabla W)^2}{W^2} + \tfrac{3}{8}q^2W^{-10} + 2g^2W^{-2}\right), \tag{2.14}$$

and hence the Euler number must be positive. Since we are assuming that Y is complete, orientable and non-singular, it follows that we must have $\chi = 2$ and Y must be topologically S^2. Moreover, the Killing vector field K^m must have circular orbits with two fixed points, that is, K^m is a rotational Killing vector and Y has axial symmetry. The most general metric can therefore be written in the form

$$d\hat{s}_6^2 = W^2\,dx^\mu\,dx_\mu + d\rho^2 + a^2\,d\psi^2, \tag{2.15}$$

where W and a are functions only of ρ. The equations of motion then take the form

$$\begin{aligned}
&\frac{\ddot{W}}{W} + \frac{3\dot{W}^2}{W^2} + \frac{\dot{W}\dot{a}}{Wa} = \tfrac{1}{4}e^{-\frac{1}{2}\phi}\left(\tfrac{1}{2}q^2W^{-8} - 8g^2\right),\\
&\frac{4\ddot{W}}{W} + \frac{\ddot{a}}{a} + \tfrac{1}{4}\dot{\phi}^2 = -e^{-\frac{1}{2}\phi}\left(\tfrac{3}{8}q^2W^{-8} + 2g^2\right),\\
&\frac{4\dot{W}\dot{a}}{Wa} + \frac{\ddot{a}}{a} = -e^{-\frac{1}{2}\phi}\left(\tfrac{3}{8}q^2W^{-8} + 2g^2\right),\\
&\frac{1}{aW^4}\frac{d(aW^4\dot{\phi})}{d\rho} = e^{-\frac{1}{2}\phi}\left(\tfrac{1}{2}q^2W^{-8} - 8g^2\right),
\end{aligned} \tag{2.16}$$

where the dot signifies a derivative with respect to ρ. These equations can be derived from the Lagrangian

$$L = -8W^3\dot{W}\dot{a} - 12aW^2\dot{W}^2 + \tfrac{1}{4}aW^4\dot{\phi}^2 - ae^{-\frac{1}{2}\phi}\left(\tfrac{1}{2}q^2W^{-4} + 8g^2W^4\right), \tag{2.17}$$

subject to the constraint that the associated Hamiltonian vanishes.

It follows from (2.16) that there is a constant of the motion given by

$$a\left(W^4\dot{\phi} - 4W^3\dot{W}\right) = k. \tag{2.18}$$

As shown above, there are two fixed points of the axial Killing vector K^m on the smooth S^2 manifold, at which the Killing vector field vanishes. At these points, therefore, $a^2 = g_{mn}K^mK^n = 0$. If we take one of these points, without loss of generality, to be at $\rho = 0$, then if W and ϕ are smooth functions, bounded at $\rho = 0$, then it is evident that the integration constant k must vanish. In Appendix A, we construct the most general solutions with non-vanishing k. Here, we restrict attention to the cases with $k = 0$ because, as explained above, only these can give smooth compact internal spaces.

The local solutions with $k = 0$ were written down in [15]. They have $\phi = 4\log W$, with

$$\begin{aligned}
&ds_2^2 = e^{\frac{1}{2}\phi}\left(\frac{dr^2}{f_0^2} + \frac{r^2}{f_1^2}\,d\psi^2\right),\\
&F_{(2)} = \frac{qr}{W^4 f_0 f_1}\,dr\wedge d\psi,\\
&e^{-\phi} = \frac{f_0}{f_1},\qquad f_0 \equiv 1 + \frac{r^2}{r_0^2},\qquad f_1 \equiv 1 + \frac{r^2}{r_1^2}.
\end{aligned} \tag{2.19}$$

The constants r_0 and r_1 are given by

$$r_0^2 = \frac{1}{2g^2},\qquad r_1^2 = \frac{8}{q^2}. \tag{2.20}$$

If $r_1 = r_0$, then setting $r = r_0\tan\frac{1}{2}\theta$ one obtains $W = 1$, $\phi = 0$ and

$$ds_2^2 = \tfrac{1}{4}r_0^2\left(d\theta^2 + \sin^2\theta\,d\psi^2\right), \tag{2.21}$$

which is the round S^2 metric of the Salam–Sezgin solution. As we shall see in detail in the next section, this is the only completely regular solution. Our proof of the uniqueness is thus complete.

3. 3-brane solutions

When $r_0 \neq r_1$, one finds that if $\psi \in [0, 2\pi)$, then Y is smooth at $r = 0$ but has a conical singularity at $r = \infty$, with deficit angle δ given by

$$\frac{\delta}{2\pi} = 1 - \frac{r_1^2}{r_0^2}. \tag{3.1}$$

This conical singularity represents a 3-brane with positive tension if $r_0 > r_1$, and negative tension if $r_0 < r_1$. The field $F_{(2)}$ can be written locally in terms of the 1-form potential

$$A_{(1)} = -\frac{4}{q f_1}\, d\psi. \tag{3.2}$$

This is well-behaved as r goes to infinity, but not at the origin. Performing the gauge transformation $A_1 \to A_1 + d(4\psi/q)$ gives a potential which is regular near the origin, and so single-valuedness of the fermionic fields requires that the Dirac quantisation condition

$$\frac{4g}{q} = N \tag{3.3}$$

must be satisfied, where N is an integer. Equivalently, the flux

$$\frac{1}{4\pi}\int_Y F_{(2)} = \frac{2}{q} \tag{3.4}$$

is quantised in units of $1/(2g)$.

From (2.20) it follows that the deficit angle at $r = \infty$ is given by

$$\frac{\delta}{2\pi} = 1 - N^2, \tag{3.5}$$

and that the ratio r_1/r_0 is quantised

$$\frac{r_1}{r_0} = |N|. \tag{3.6}$$

Unfortunately, this implies for $|N| > 1$ that the 3-brane tension is necessarily negative.

More generally, one may identify ψ with period $2\pi\alpha$, where $\alpha > 0$. The deficit angle is given by

$$\delta = 2\pi - \lim_{\rho\to 0} \frac{C(\rho)}{\rho}, \tag{3.7}$$

where $C(\rho)$ is the circumference of a small circle of radius ρ. Thus at $r = 0$ and $r = \infty$ the deficits are

$$\delta_0 = 2\pi(1-\alpha), \qquad \delta_\infty = 2\pi\left(1 - \frac{N^2}{\alpha}\right). \tag{3.8}$$

The tension is given in terms of the deficit angle by

$$T = \frac{\delta}{8\pi G}, \tag{3.9}$$

which implies

$$T_0 = \frac{1}{4G}(1-\alpha), \qquad T_\infty = \frac{1}{4G}\left(1 - \frac{N^2}{\alpha}\right). \tag{3.10}$$

Thus both T_0 and T_∞ are less than $\frac{1}{4G}$, and

$$(1 - 4GT_0)(1 - 4GT_\infty) = N^2. \tag{3.11}$$

If the integer N exceeds 1, then it follows that both tensions, T_0 and T_∞ cannot be positive.

4. Solutions with additional gauge fields

In [7], following earlier work of [16], the 2-form supporting the solution was taken to be a linear combination of the supergravity 2-form $F_{(2)}$ that we have been using thus far, and a $U(1)$ subgroup of an additional Yang–Mills gauge sector $F^I_{(2)}$ in the six-dimensional theory. Thus now

$$\begin{aligned} F_{(2)} &= \frac{qr\cos\beta}{W^4 f_0 f_1}\, dr \wedge d\psi, \\ T_I F^I_{(2)} &= T_0 \frac{qr\sin\beta}{W^4 f_0 f_1}\, dr \wedge d\psi, \end{aligned} \tag{4.1}$$

where β is the mixing angle, and T_0 denotes the $U(1)$ generator within the Yang–Mills sector. There are now two Dirac quantisation conditions, associated with the requirement of single-valuedness for the supergravity and gauge-sector fermions, respectively:

$$\frac{4g\cos\beta}{q} = N, \qquad \frac{4g'\sin\beta}{q} = N', \tag{4.2}$$

where g' is the relevant gauge coupling constant in the Yang–Mills sector, and N and N' are integers.

Using (2.20), we can re-express these conditions as

$$\frac{r_1}{r_0} = \frac{N}{\cos\beta}, \qquad \frac{g'}{g} = \frac{N'}{N}\cot\beta. \tag{4.3}$$

The first equation can always be solved, provided that $r_1 > r_0$, which implies as before that the 3-brane will not have a positive tension. The second equation may then be regarded as determining g'. Note that these Dirac quantisation conditions are similar to those obtained in [8], where, following [7], conical deficits $2\pi\epsilon$ were introduced at the north and south poles of a round S^2. In that case, the analogous quantisation conditions were [8]

$$\cos\beta = \frac{N}{1-\epsilon}, \qquad \frac{g'\sin\beta}{g} = \frac{N'}{1-\epsilon}. \tag{4.4}$$

The special cases $\beta = 0$ and $\beta = \frac{1}{2}\pi$ were obtained earlier in [7]. It was noted in [8] that the first equation in (4.4) could not be satisfied for any integer N when $|\cos\beta| \neq 1$ or 0, unless ϵ was taken to be negative; in other words the 3-brane tension had to be negative.

5. 3-brane/dilaton coupling

In [7], 3-branes were introduced into the Salam–Sezgin model by inserting conical deficits at the north and south poles of the 2-sphere, with the dilaton being independent of the coordinates on S^2. The 3-brane action was taken to be

$$S_b = -T \int d^4x\, e^{-\frac{1}{2}\lambda\phi} (-\det \gamma_{\mu\nu})^{1/2}, \tag{5.1}$$

where $\gamma_{\mu\nu} = \hat{g}_{MN}\partial_\mu X^M \partial_\nu X^N$ is the induced metric on the 3-brane.[3] In the detailed calculations in [7], the 3-brane/dilaton coupling λ was taken to be zero.

In the more general solutions (2.19) obtained in this Letter, 3-branes arise naturally when $r_1 \neq r_0$. In these solutions the dilaton is not constant, and this allows us to make qualitative statements about the 3-brane/dilaton coupling. For negative-tension 3-branes, i.e., $r_1 > r_0$, the dilaton decreases from its value at the origin as one aproaches the 3-brane at $r = \infty$. Conversely, if the tension is positive, i.e., $r_1 < r_0$, the dilaton increases as the 3-brane at $r = \infty$ is approached. The fact that in our solutions ϕ is a smooth function without singularities is consistent with the idea that the 3-brane/dilaton coupling λ is in fact zero, as proposed in [7], because otherwise one would expect singular behaviour near the 3-brane from the delta-function in the dilaton equation arising from the contribution (5.1) to the action.

6. Modulus and breathing mode

Our proof of uniqueness shows that the Salam–Sezgin ground state has just one modulus, namely the value of ϕ_0. One can consider solutions in which the radius of the 2-sphere varies in space and time, with the six-dimensional fields taking the forms

$$d\hat{s}_6^2 = e^{\frac{1}{2}(\phi_1+\phi_2)}\, ds_4^2 + e^{-\frac{1}{2}(\phi_1+\phi_2)} g_{mn}\, dy^m\, dy^n,$$
$$F_{(2)} = 4g\epsilon_{(2)}, \qquad \phi = \phi_2 - \phi_1, \qquad H_{(3)} = 0, \tag{6.1}$$

where $\epsilon_{(2)}$ is the volume-form of metric $g_{mn}\, dy^m\, dy^n$ on S^2, which we normalise to $R_{mn} = 8g^2 g_{mn}$. Substituting into the higher-dimensional action, which is a valid procedure since this dimensional reduction is trivially consistent, yields the four-dimensional action

$$\mathcal{L} = R - \tfrac{1}{2}(\partial\phi_1)^2 - \tfrac{1}{2}(\partial\phi_2)^2 - 8g^2 e^{\phi_1}\big(1 - e^{\phi_2}\big)^2. \tag{6.2}$$

The potential in (6.2) was first derived, in the purely time-dependent case, in [2], and some cosmological applications were given in [2–5].

The field ϕ_2 plays the role of a breathing mode (or "radion"). Its mass $M_{\rm KK}$ is given by

$$M_{\rm KK} = 4g e^{\frac{1}{2}\phi_0}, \tag{6.3}$$

where ϕ_0 denotes the expectation value of the massless "modulus scalar" ϕ_1. As pointed out in [8], all Kaluza–Klein modes have masses set by the mass of this radion field.

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Appendix A. General axisymmetric solutions

Here we construct the most general solution to Eq. (2.16) for axially-symmetric configurations. It is advantageous first to introduce the "lapse function" $\mathcal{N}$ in the Lagragian (2.17), which enforces the vanishing of the associated Hamiltonian:

$$L = \big(-8W^3\dot{W}\dot{a} - 12aW^2\dot{W}^2 + \tfrac{1}{4}aW^4\dot{\phi}^2\big)\mathcal{N}$$
$$\qquad - ae^{-\frac{1}{2}\phi}\mathcal{N}^{-1}\big(\tfrac{1}{2}q^2W^{-4} + 8g^2W^4\big). \tag{A.1}$$

We next send $\mathcal{N} \to \mathcal{N}/(aW^4)$, make the coordinate gauge transformation $d\rho = aW^4\, d\eta$, and then suppress the lapse function. After introducing new independent variables by defining

$$W = e^{\frac{1}{4}(y-x)}, \qquad a = e^{\frac{1}{4}(3x+y+2z)},$$
$$\phi = y - x + 2z, \tag{A.2}$$

[3] Our ϕ is (-2) times the ϕ in [7], and so λ is the same as that used in [7].

we obtain the Lagrangian

$$x'^2 - y'^2 + z'^2 - \tfrac{1}{2}q^2 e^{2x} + 8g^2 e^{2y}, \tag{A.3}$$

together with the Hamiltonian constraint

$$x'^2 - y'^2 + z'^2 + \tfrac{1}{2}q^2 e^{2x} - 8g^2 e^{2y} = 0, \tag{A.4}$$

where a prime denotes a derivative with respect to η.

In terms of the new variables, the general system of equations of motion is decoupled, reducing to two Liouville equations for x and y, and a free-particle equation for z. We have the three first integrals

$$x'^2 + \tfrac{1}{2}q^2 e^{2x} = \lambda_1^2, \qquad y'^2 + 8g^2 e^{2y} = \lambda_2^2,$$
$$z' = \lambda_3, \tag{A.5}$$

and the Hamiltonian constraint implies that the three constants of integration obey the relation

$$\lambda_2^2 = \lambda_1^2 + \lambda_3^2. \tag{A.6}$$

Note that λ_3 is related to the constant k in (2.18) by $k = 2\lambda_3$.

The general solution can be taken, without loss of generality, to be given by

$$e^{-x} = \frac{q}{\sqrt{2}\lambda_1}\cosh\lambda_1(\eta - \eta_1),$$
$$e^{-y} = \frac{2\sqrt{2}g}{\lambda_2}\cosh\lambda_2(\eta - \eta_2), \qquad z = \lambda_3\eta. \tag{A.7}$$

The metric and dilaton are therefore given by

$$d\hat{s}_6^2 = W^2\, dx^\mu\, dx_\mu + a^2 W^8\, d\eta^2 + a^2\, d\psi^2,$$
$$e^{\phi} = W^4 e^{2\lambda_3\eta}, \tag{A.8}$$

where W and a are given by

$$W^4 = \frac{q\lambda_2}{4g\lambda_1}\frac{\cosh\lambda_1(\eta - \eta_1)}{\cosh\lambda_2(\eta - \eta_2)},$$

$$a^{-4} = \frac{gq^3}{\lambda_1^3\lambda_2} e^{-2\lambda_3\eta}\cosh^3\lambda_1(\eta - \eta_1)\cosh\lambda_2(\eta - \eta_2). \tag{A.9}$$

The solutions in Section 2 that are regular at the origin correspond to taking $\lambda_3 = 0$, and hence $\lambda_1 = \lambda_2$. This solution, in the form (2.19), is obtained by setting

$$\lambda_1 = \lambda_2 = 1, \qquad r = r_1 e^{\eta - \eta_1}, \qquad e^{\eta_1 - \eta_2} = \frac{4g}{q}. \tag{A.10}$$

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