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The simplest proof of Burnside's theorem on matrix algebras

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To Heydar Radjavi, in honor of his 70th birthday, with deep gratitude for his friendship and his mathematics

Abstract

A very simple, short and self-contained proof is presented of Burnside's Theorem that every proper algebra of matrices over an algebraically closed field has a non-trivial invariant subspace.

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Burnside's very well known and very useful theorem states that the only irreducible algebra of linear transformations on a vector space of finite dimension greater than 1 over an algebraically closed field is the algebra of all linear transformations on the space.

There are many known proofs of Burnside's Theorem; the simplest published so far is apparently that of [1]. The proof in [1], however, depends on a little trick that seems artificial and is therefore not so easily remembered. In this note we show that trick can be avoided by using induction on the dimension of the space. The resulting proof is extremely easy and entirely straightforward. This proof is, of course,

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motivated by that of [1], which was itself suggested by a result of Heydar Radjavi's [3, Lemma 1].

Recall that a collection of linear transformations on a vector space is said to be *irreducible* if the only linear subspaces that are invariant under all the operators in the collection are $\{0\}$ and the entire space.

Burnside's Theorem. *The only irreducible algebra of linear transformations on a vector space of finite dimension greater than* 1 *over an algebraically closed field is the algebra of all linear transformations on the vector space.*

Proof. Say that an algebra $\mathscr{A}$ of linear transformations on a vector space $\mathscr{V}$ is *transitive* if $\{Ax : A \in \mathscr{A}\}$ is $\mathscr{V}$ for every vector x other than 0. For every x, $\{Ax : A \in \mathscr{A}\}$ is an invariant subspace for $\mathscr{A}$, so an algebra of linear transformations on a space of dimension greater than 1 is transitive if and only if it is irreducible. (On a one-dimensional space, the algebra $\{0\}$ is irreducible but not transitive.)

We now prove the following by induction on the dimension of $\mathscr{V}$: If $\mathscr{A}$ is a transitive algebra of linear transformations on $\mathscr{V}$, then $\mathscr{A}$ is the algebra of all linear transformations on $\mathscr{V}$.

Let n denote the dimension of $\mathscr{V}$. The above assertion is obviously true for $n = 1$, so we assume it holds on all spaces of dimension from 1 to $n - 1$, and suppose that $\mathscr{A}$ is a transitive algebra on a space of dimension n.

First, note that $\mathscr{A}$ contains a non-invertible operator F other than 0. To see this, let T be any operator in $\mathscr{A}$ that is not a scalar multiple of the identity. If T is not invertible, let $F = T$. If T is invertible, let λ be any eigenvalue of T, and then let $F = \lambda T - T^2$.

Since F is not invertible, the dimension of its range, $F\mathscr{V}$, is less than n. The set of all restrictions to $F\mathscr{V}$ of $\{FA : \mathscr{A} \in \mathscr{A}\}$ is a transitive algebra on $F\mathscr{V}$, so the inductive hypothesis implies that this algebra contains all linear transformations on $F\mathscr{V}$. In particular, there is an $A_0 \in \mathscr{A}$ such that the restriction of FA_0 to $F\mathscr{V}$ has rank 1. Then FA_0F is a transformation of rank 1 in $\mathscr{A}$.

Once it is known that a transitive algebra contains a transformation of rank 1, it is very easy to show (as in [1]) that it contains all transformations. For suppose that $\mathscr{A}$ contains the transformation $y_0 \otimes f_0$, with y_0 in the space and f_0 in the dual space (defined by $(y_0 \otimes f_0)(x) = f_0(x)y_0$ for all $x \in \mathscr{V}$). For $A \in \mathscr{A}$, $A(y_0 \otimes f_0) = Ay_0 \otimes f_0$, so the transitivity of $\mathscr{A}$ implies that $y \otimes f_0$ is in $\mathscr{A}$ for every y. It is clear that the transitivity of $\mathscr{A}$ implies transitivity of the set of all operators on the dual space of $\mathscr{V}$ that are duals of operators in $\mathscr{A}$. Moreover,

$$(y \otimes f_0)A = y \otimes A' f_0,$$

where A' is the dual of A.

Hence $y \otimes f$ is in $\mathscr{A}$ for all y in $\mathscr{V}$ and f in the dual space; i.e., $\mathscr{A}$ contains all transformations on $\mathscr{V}$ of rank 1. Since every linear transformation is a sum of transformations of rank 1, $\mathscr{A}$ is the algebra of all linear transformations on $\mathscr{V}$. $\square$

Other interesting and quite different proofs of Burnside's Theorem are presented in [2] and [4].

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