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Triangulations, visibility graph and reflex vertices of a simple polygon

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Abstract

In this paper tight lower and upper bounds for the number of triangulations of a simple polygon are obtained as a function of the number of reflex vertices, thus relating these two shape descriptors. Tight bounds for the size of the visibility graph of a polygon are obtained too, with the same parameter. The former bounds are also studied from an asymptotical point of view.

Keywords: Simple polygon; Triangulation; Decomposition; Visibility; Visibility graph

1. Introduction

The number of reflex vertices of a simple polygon is a shape complexity measure describing how far it is from being a convex polygon. This number is a poor descriptor when considered alone, as pointed out by Toussaint [11]: given any polygon, if we insert a new vertex R in some side P_iP_{i+1} and pull it an infinitesimal amount towards the interior of the polygon, the basic shape will remain unchanged. In fact, only the visibility between P_i and P_{i+1} has been altered. But if R enters progressively into the interior of the polygon, the visibility between many pairs of vertices can disappear and R will become really significant.

The number of ways a polygon can be triangulated is again a shape descriptor. If the polygon has many arms and it is very twisted, the number of triangulations will be relatively low, and this number will increase if there are important "convex bags", because many internal diagonals are then available.

An internal diagonal is a visibility trajectory and could have been destroyed by a reflex vertex; it is then reasonable to expect a relation between the two numbers considered above. Let n and k be the number of sides and reflex vertices of a polygon, respectively. Hertel and Mehlhorn [6] described an algorithm for triangulating a simple polygon that performs better the fewer reflex vertices it has

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(the running time is $O(n + k \log k)$). This is natural because in general, as we show in this paper, the fewer the number of reflex vertices the higher the number of possible triangulations.

The proof is based on decomposing the polygon into convex pieces, a subject widely studied, although the main objective is usually to minimize the number of pieces. Chazelle and Dobkin [1–3] obtained a running time upper bound $O(n+k^3)$ with Steiner points allowed. The algorithms by Greene [5] and Keil [8] use only vertices from the original set and have a worst case complexity $O(n^2k^2)$ and $O(k^2 n \log n)$, respectively. The survey [9] by Keil and Sack provides a panorama on the subject. In our paper we seek for a fixed number of non-overlapping convex parts, not necessarily covering the whole polygon, but providing a total size large enough for visibility purposes.

The size of the visibility graph of a polygon [10] is naturally related to the number of reflex vertices, because adjacencies correspond to sides and internal diagonals, thus it is not surprising the former decomposition provides bounds for that size too.

The paper is organized as follows. Section 2 establishes some combinatorial lemmas and a procedure for breaking a polygon into a suitable number of convex pieces. After that, the results are combined together in a final theorem on the number of triangulations. In Section 3 these results are used in relation with the visibility graph. Finally, in Section 4 we study from an asymptotical point of view the lower and upper bounds obtained for the number of triangulations.

As we only consider simple polygons, the adjective will be omitted hereafter. The vertices will be numbered counterclockwise with indices modulus the number of sides.

2. The number of triangulations of a polygon

2.1. Triangulations and Catalan numbers

Let t_n and C_n be the number of triangulations of a convex n-polygon and the nth Catalan number, respectively. It is a well known fact (see for example [4]) that these sequences are the same up to a shift:

$$t_n = C_{n-2} = \frac{1}{n-1}\binom{2n-4}{n-2} \quad (n \geqslant 3).$$

We will accept as a convention that a segment is a convex polygon with two sides and $t_2 = C_0 = 1$. Given a polygon P, we denote $t(P)$ the number of triangulations of P.

Theorem 1. *Let P be an n-polygon. Then $1 \leqslant t(P) \leqslant t_n$, and these bounds are tight.*

Proof. Every polygon has at least one triangulation, hence $t(P) \geqslant 1$, and this bound is tight as we can see in Fig. 1, where $n-3$ consecutive reflex vertices force the triangulation to be unique. For the upper bound we simply observe that if we label the vertices of a convex n-polygon Q with the ordered labels of the vertices of P, then every triangulation of P translates into a triangulation of Q, and this polygon realizes the bound. □

The presence of reflex vertices diminishes the internal visibility (at least between the vertices adjacent to the reflex ones), implying that some triangulations are lost with regard to the convex model. But not

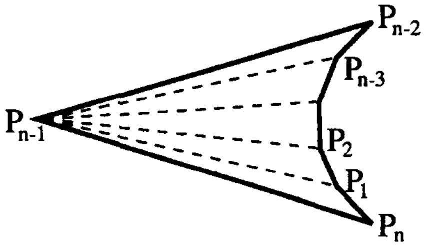

Fig. 1. An n-polygon that can be triangulated in a unique way.

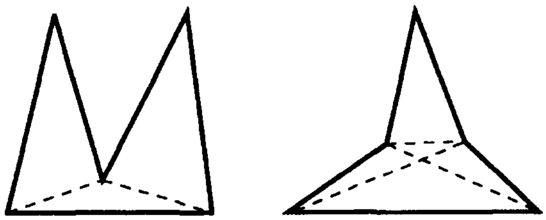

Fig. 2. The pentagon on the right has more reflex vertices and more triangulations that the pentagon on the left.

necessarily the number of triangulations decreases monotonically with the number of reflex vertices (Fig. 2).

The main theorem in this section provides tight bounds for the number of triangulations of a polygon as a function of the number of reflex vertices. To establish that result we need some preliminary lemmas. We use the Catalan numbers C_n for the purely combinatorial results; they can all be rephrased in terms of the t_n.

Lemma 2. *If $\alpha \geqslant 0$ and $\beta \geqslant 1$ then*

$$C_\alpha C_\beta - C_{\alpha+1}C_{\beta-1} = \frac{6(\beta-1-\alpha)}{(\alpha+2)(\beta+1)} C_\alpha C_{\beta-1}.$$

Proof. It suffices to develop the C_s in binomial form. □

We obtain now an immediate consequence and two subsequent results.

Lemma 3. *If $0 \leqslant \alpha \leqslant \beta - 2$ then $C_\alpha C_\beta > C_{\alpha+1}C_{\beta-1}$.*

Lemma 4. *Let $\alpha_1, \ldots, \alpha_m$ be m given nonnegative integers. Then*

$$C_{\alpha_1} C_{\alpha_2} \cdots C_{\alpha_m} \geqslant \left(C_{\lceil(\alpha_1+\cdots+\alpha_m)/m\rceil}\right)^s \left(C_{\lfloor(\alpha_1+\cdots+\alpha_m)/m\rfloor}\right)^{m-s},$$

where s is the residue of the division of $\alpha_1 + \cdots + \alpha_m$ by m. Moreover the right value in the inequality is the minimum when we allow the variation of $\alpha_1, \ldots, \alpha_m$ while maintaining constant its sum.

Proof. If the difference between some α_i and α_j is 2 or more, Lemma 3 says we can reduce the product if one unity is added to the least one and subtracted to the greatest one; that way the sum of the indices remains constant. The process goes on until the operation is no longer possible. Then

$$C_{\alpha_1}\cdots C_{\alpha_m} \geqslant \underbrace{C_x\cdots C_x}_{p}\underbrace{C_{x+1}\cdots C_{x+1}}_{q},$$

where $\alpha_1+\cdots+\alpha_m=(p+q)x+q=mx+q$, $0\leqslant q<m$, and we get the result. Now the second part of the statement is immediate because

$$s\left\lceil\frac{\alpha_1+\cdots+\alpha_m}{m}\right\rceil+(m-s)\left\lfloor\frac{\alpha_1+\cdots+\alpha_m}{m}\right\rfloor=\alpha_1+\cdots+\alpha_m. \qquad \square$$

Lemma 5. *Let $\alpha_1,\ldots,\alpha_m$, w be nonnegative integers with $\alpha_1+\cdots+\alpha_m\geqslant w$. Then*

$$C_{\alpha_1}\cdots C_{\alpha_m}\geqslant \left(C_{\lceil w/m\rceil}\right)^t\left(C_{\lfloor w/m\rfloor}\right)^{m-t},$$

where t is the residue of the division of w by m.

Proof. Since the sequence $\{C_n\}$ is strictly increasing, it suffices to replace the α_i by integers β_i in such a way that $0\leqslant\beta_i\leqslant\alpha_i$ and $\beta_1+\cdots+\beta_m=w$. Then $C_{\alpha_1}\cdots C_{\alpha_m}\geqslant C_{\beta_1}\cdots C_{\beta_m}$ and we apply Lemma 4. $\square$

2.2. Breaking a polygon

If a polygon P admits a collection of non-overlapping convex subpolygons, the product of the corresponding number of triangulations is certainly a lower bound for $t(P)$ and the former lemmas show that it is smallest when the pieces have balanced sizes. The following lemmas provide us a relation between the number of subpolygons and the sum of the sizes.

Definitions. Let P be a polygon with vertices $P_0,\ldots,P_{n-1}$ and let P_i be a reflex vertex of P. We say a triangle $P_sP_tP_i$ *breaks* the angle at P_i when its interior is contained in P and neither of the angles $P_{i+1}P_iP_s$, $P_sP_iP_t$, $P_tP_iP_{i-1}$ is reflex (Fig. 3). An internal diagonal P_iP_k *breaks* the angle at P_i if the angles $P_{i-1}P_iP_k$ and $P_kP_iP_{i+1}$ are not reflex. We use the term "to break" because we intend to work with the resultant pieces. The *central zone* of P_i is the part of the plane to the right of the ray $P_{i+1}P_i$ and to the left of the ray $P_{i-1}P_i$ (Fig. 4).

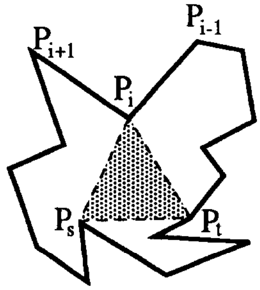

Fig. 3. A triangle breaking the angle at P_i.

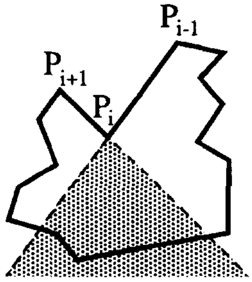

Fig. 4. Central zone (darkened) of the reflex angle at P_i.

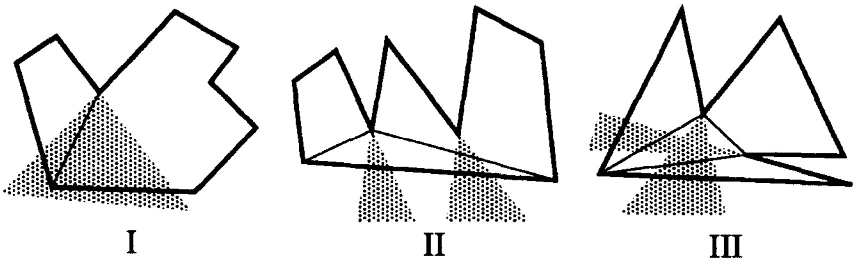

Fig. 5. The three possible situations for Lemma 6.

Lemma 6. *Every nonconvex polygon P admits at least one of the following configurations (refer to Fig. 5):*

I. *A reflex vertex seeing a vertex in its central zone;*

II. *A reflex vertex that can be broken by a triangle consisting of two diagonals and one side of P;*

III. *Two reflex vertices that can be simultaneously broken by a triangle constituted by three diagonals.*

Proof. By induction on the number k of reflex vertices.

If $k = 1$ and the central zone of the unique reflex vertex P_i is empty, then there is a triangle $P_sP_iP_{s+1}$ in situation II (see Fig. 6).

Let us suppose now that $k > 1$ and P does not have any reflex vertex in situations I or II, and consider any reflex vertex P_i. Let P_rP_{r+1} be the first edge intersected by the bisector of the central zone of P_i and let X be the point of intersection. Anchor segment P_iX at P_i and move X traversing the boundary. Let P_s be the first vertex obtained in clockwise order and X_s the corresponding position of X; P_t and X_t are defined in the same manner for the counter-clockwise order.

It is impossible to have both $P_s = P_r$ and $P_t = P_{r+1}$, as triangle $P_iP_rP_{r+1}$ would be in situation II. Assume then, without loss of generality, that $P_t \neq P_{r+1}$ and as a consequence that P_t is reflex. We observe that P_{t-1} and P_{t+1} lie in the same halfplane as P_{i-1} with respect to the line P_iP_t by construction, and that P_s lies in the opposite halfplane. As the diagonal P_sP_t cannot break the angle at P_t, the vertices P_{t-1} and P_{t+1} must lie in opposite halfplanes with respect to the line P_sP_t. We consider now the following possibilities.

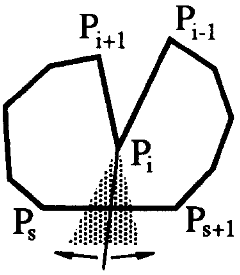

Fig. 6. Searching for situation II when the unique reflex vertex in not in situation I.

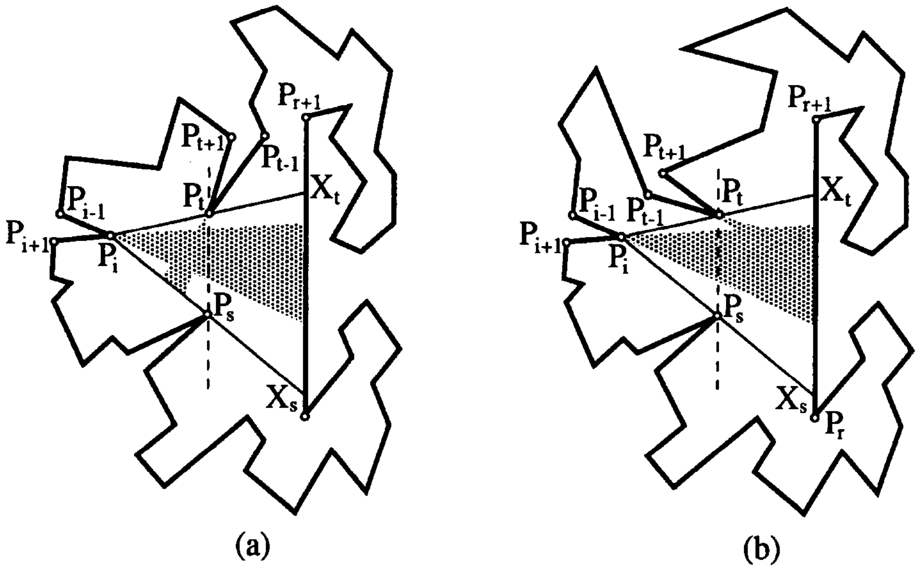

Fig. 7. Searching for situation III when situations I and II are not present.

Case A. P_{t-1} and P_{t+1} lie in opposite halfplane than P_i with respect to the line P_sP_t (Fig. 7a). In this case the triangle $T = P_sP_iP_t$ is in situation III, because its sides are internal diagonals and T breaks the angle at P_i and P_t.

Case B. P_{t-1} and P_{t+1} lie in the same halfplane than P_i with respect to the line P_sP_t (Fig. 7b). In this case we apply induction to the polygon $Q = P_tP_iP_sP_{s+1} \dots P_{t-1}$, in which the angle at P_t is still reflex and P_i is no longer reflex.

P_t is not in situation I in Q because its central zone is contained in the empty triangle $P_iX_sX_t$. As a consequence any triangle $P_uP_tP_v$ in situation II or III breaking the angle at P_t in Q would have diagonals P_uP_t and P_vP_t in opposite sides of the central zone in Q and would also break the angle P_t in P. In particular P_t cannot be in situation II in Q.

We split now the discussion depending on the position of P_s.

*Case B*1. $P_s = P_r$. Then P_s is a convex vertex of Q. If P_j is a reflex vertex of Q, $P_j \neq P_t$, then P_j is not in situation I in P. Similarly P_j cannot be in situation II for a triangle $P_jP_vP_{v+1}$ with an edge P_vP_{v+1} that was common to P and Q. As the triangle $P_iP_rX_t$ is empty in Q, the edges P_iP_r and P_iP_t cannot contribute to any triangle in situation II.

Hence we do not have situations I or II in Q, and by induction some triangle T is in situation III in Q breaking at least two angles at reflex vertices P_u and P_v. Let us see that T is also in situation III in P. If $P_u = P_t$ or $P_v = P_t$, the angle at P_t is also broken by T in P as shown above. Otherwise the claim is obvious.

Case B2. $P_s \neq P_r$ (implying that P_s is reflex in P). If P_{s-1} and P_{s+1} lie in opposite halfplane than P_i with respect to the line P_tP_s we proceed as in Case A. Otherwise we apply induction to polygon $Q = P_tP_iP_sP_{s+1}\dots P_{t-1}$ as before, where the angles at P_s and P_t are still reflex but P_i is not. Now P_s behaves exactly as P_t and the proof goes along the same lines: no reflex vertex of Q is either in situation I or II and by induction there is a triangle in situation III in Q, hence also in P. □

The preceding lemma will allow us later to work with the convex polygons obtained by breaking any given polygon in pieces by a diagonal or a triangle, suitable for diminishing the total number of reflex vertices, and then iterating the process until all the reflex vertices are destroyed. We now present the breaking procedure; the complete description and a proof of correctness will be given in Lemma 7. The steps of the procedure are illustrated in Fig. 8.

Procedure $break(P_1,\dots,P_n)$ *{$(P_1,\dots,P_n)$ is the list of vertices of a polygon in counterclockwise order}*

Step 0. If all the vertices are convex, then return $(P_1,\dots,P_n)$.

Step 1. If some reflex vertex is in situation I then select a reflex vertex P_i and a diagonal P_iP_j breaking the angle at P_i, such that P_j is either convex or reflex but not broken by P_iP_j. Return $break(P_i,P_{i+1},\dots,P_{j-1},P_j) \cup break(P_j,P_{j+1},\dots,P_{i-1},P_i)$.

Step 2. If some triangle $T = P_iP_jP_{j+1}$ is in situation II breaking the angle at P_i then return $break(P_i,P_{i+1},\dots,P_{j-1},P_j) \cup break(P_{j+1},P_{j+2},\dots,P_{i-1},P_i)$.

Step 3a. If some triangle $T = P_iP_jP_k$ is in situation III breaking the angles at P_i and P_j but not at P_k, then return

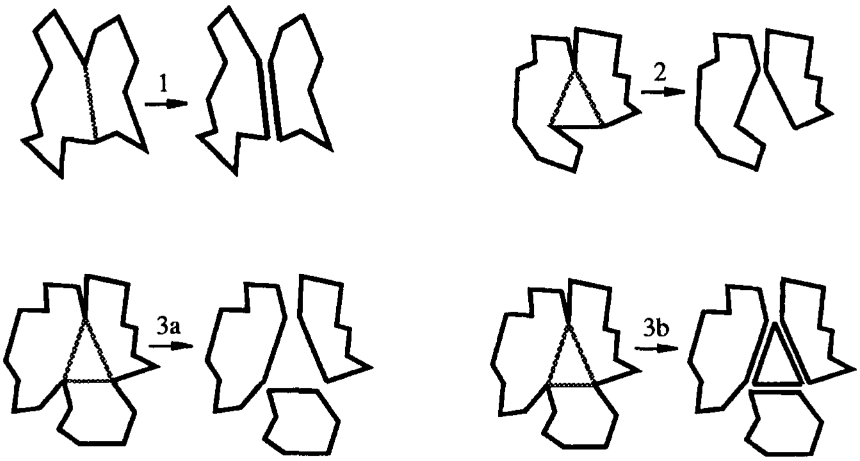

Fig. 8. Breaking recursively until the pieces are convex.

$$break(P_i, P_{i+1}, \ldots, P_{j-1}, P_j) \cup break(P_j, P_{j+1}, \ldots, P_{k-1}, P_k) \cup break(P_k, P_{k+1}, \ldots, P_{i-1}, P_i).$$

Step 3b. If some triangle $T = P_i P_j P_k$ is in situation III breaking the angles at P_i, P_j and P_k, then return

$$break(P_i, P_{i+1}, \ldots, P_{j-1}, P_j) \cup break(P_j, P_{j+1}, \ldots, P_{k-1}, P_k) \cup break(P_k, P_{k+1}, \ldots, P_{i-1}, P_i) \cup (P_i, P_j, P_k).$$

Lemma 7. *Let $P = (P_1, P_2, \ldots, P_n)$ be an n-polygon in counterclockwise order with k reflex vertices. Then procedure* break *applied to P returns $k+1$ convex polygons $C_1, \ldots, C_{k+1}$ such that*
(a) *Every vertex of C_i is a vertex of P, $i = 1, \ldots, k+1$;*
(b) $\sum_{i=1}^{k+1} |vertices(C_i)| \geqslant n + k$;
(c) *If $i \neq j$ then $C_i \cap C_j$ is empty, a common vertex or a common edge*;
(d) *Points interior to C_i are interior to P, $i = 1, \ldots, k+1$.*

Proof. Conditions (c) and (d) are obvious, as it is that the resulting pieces are convex polygons since every reflex vertex is broken and no new reflex vertices are created.

Step 1. If we are in situation I there is a diagonal $P_s P_t$ breaking the angle at P_s. If P_t is convex or reflex and not broken by $P_s P_t$ then Step 1 can be performed. Otherwise we look for a diagonal breaking a reflex angle at one end but not at the other end as follows. Assume without loss of generality that angle$(P_{s+1} P_s P_t) \leqslant$ angle$(P_{t+1} P_t P_s)$ and let us see that there is a diagonal breaking the angle at P_s but not at the other end (refer to Fig. 9). Let $P_t, P_{t+1}, \ldots, P_a$ be the longest sequence of consecutive reflex vertices visible from P_s and starting at P_t (observe that P_a might be equal to P_t). If $P_s P_a P_{a+1}$ is a right turn then we choose diagonal $P_s P_a$. If $P_s P_a P_{a+1}$ is a left turn and P_{a+1} is visible from P_s, then P_{a+1} is convex and we take diagonal $P_s P_{a+1}$. Finally if $P_s P_a P_{a+1}$ is a left turn and P_{a+1} is not

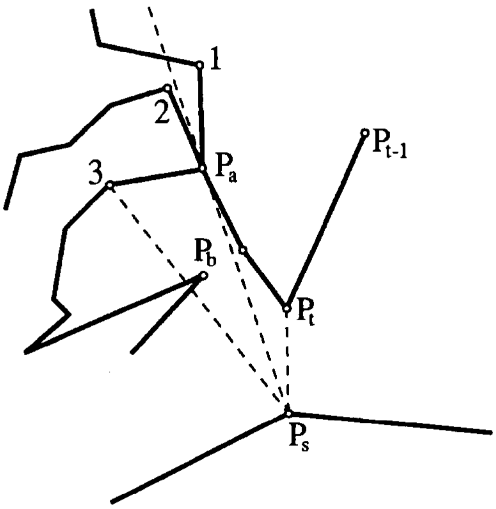

Fig. 9. The three possibilities for P_{a+1} in Step 1 of Lemma 7. (1) Right turn; (2) Left turn and P_{a+1} visible; (3) Left turn and P_{a+1} not visible.

visible from P_s, let $P_a, P_b, \ldots, P_{a+1}$ be the polar ordering around P_s of the vertices of P contained in the triangle $P_sP_aP_{a+1}$. Now we take diagonal P_sP_b.

Hence in situation I it is always possible to break a single reflex vertex by means of a diagonal P_iP_j and Step 1 is correct. Observe that P_i and P_j will contribute twice to the sum of sizes of the resulting pieces.

Step 2. Observe that we only arrive at this step when Step 1 does not apply, hence situation I is not present. Let $P_iP_jP_{j+1}$ be a triangle breaking the angle at P_i. If P_j were reflex and broken by diagonal P_iP_j then P_i would be in the central zone of P_j, and the same applies to P_{j+1}. Hence a single reflex vertex is broken at Step 2. Observe that P_i will contribute twice to the sum of sizes of the resulting pieces and that triangle $P_iP_jP_{j+1}$ is discarded.

Step 3a. Exactly two reflex vertices are broken. Vertices P_i, P_j and P_k will contribute twice to the sum of sizes of the resulting pieces and triangle $P_iP_jP_k$ is discarded.

Step 3b. Three reflex vertices are broken. Vertices P_i, P_j and P_k will contribute thrice to the sum of sizes of the resulting pieces and triangle $P_iP_jP_k$ is also included as a piece.

Number of pieces. As in every step the resulting number of pieces exceeds by one the number of broken reflex vertices and there are k of them, the final number of pieces will be $k+1$.

Total number of vertices. Let $n_1, \ldots, n_{k+1}$ be the sizes of the resulting convex pieces. If Steps 1, 2, 3a and 3b are applied α, β, γ_1 and γ_2 times, respectively, then $k = \alpha + \beta + 2\gamma_1 + 3\gamma_2$ and

$$n_1 + \cdots + n_{k+1} = n + 2\alpha + \beta + 3\gamma_1 + 6\gamma_2 = n + k + \alpha + \gamma_1 + 3\gamma_2 \geqslant n + k. \qquad \square$$

2.3. *Bounds for* $t(P)$

Definitions. A *reflex chain* of a polygon P is a set $\{P_{i+1}, \ldots, P_{i+t}\}$ of reflex vertices of P, consecutive on its boundary, followed and preceded by convex vertices P_i and P_{i+t+1}. In that situation the chain $\{P_i, P_{i+1}, \ldots, P_{i+t}, P_{i+t+1}\}$ will be called an *augmented reflex chain.* In both cases the *length* of a chain will be the number of vertices in the chain. A polygon will be called an *almost-convex polygon* if the only vertices not seeing each other are not consecutive vertices belonging to the same augmented reflex chain. An almost-convex polygon is shown in Fig. 10.

Almost-convex polygons are widely studied in [7]; the main results we recall here are the following two lemmas.

Lemma 8. *If P is an almost-convex polygon with n vertices and m reflex chains with lengths $\alpha_1, \ldots, \alpha_m$, then*

$$t(P) = \sum_{i=0}^{m} (-1)^i \left\| \begin{matrix} \alpha_1, \ldots, \alpha_m \\ i \end{matrix} \right\| t_{n-i},$$

where

$$\left\| \begin{matrix} \alpha \\ i \end{matrix} \right\| = \binom{a+i-1}{i} \quad \text{and} \quad \left\| \begin{matrix} \alpha_1, \ldots, \alpha_m \\ i \end{matrix} \right\| = \sum_{j=0}^{i} \left\| \begin{matrix} \alpha_1 \\ j \end{matrix} \right\| \cdot \left\| \begin{matrix} \alpha_2, \ldots, \alpha_m \\ i-j \end{matrix} \right\|.$$

Sketch of the proof. Let $P_1, \ldots, P_n$ be the vertices of P. We consider a convex polygon $Q = \{Q_1, \ldots, Q_n\}$, and the ordered correspondence $P_i \leftrightarrow Q_i$. If P_j is reflex in P, we say that $Q_{j-1}Q_{j+1}$

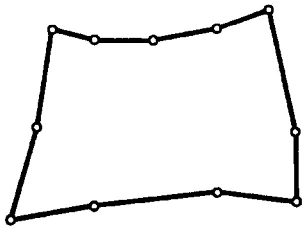

Fig. 10. An almost-convex polygon.

is a *reflex diagonal* of Q. This way, the triangulations of P correspond exactly to the triangulations of Q not using any reflex diagonal, and this number can be expressed by inclusion–exclusion as

$$t(P) = \sum_{i=0}^{m} (-1)^i \left\| \begin{matrix} \alpha_1, \ldots, \alpha_m \\ i \end{matrix} \right\| t_{n-i},$$

where $\left\| \begin{smallmatrix} \alpha_1, \ldots, \alpha_m \\ i \end{smallmatrix} \right\|$ is the number of ways of selecting i diagonals, any two noncrossing, from the set of reflex diagonals. These numbers can be directly computed when $m = 1$ and recursively otherwise.

Lemma 9. *Among all almost-convex polygons with n vertices, k of them being reflex, the maximum number of triangulations is reached by the* balanced *one: the k reflex vertices are distributed in $n - k$ reflex chains of length $\lfloor k/(n-k) \rfloor$ or $\lceil k/(n-k) \rceil$ between any two convex vertices.*

Sketch of the proof. Let P be an almost-convex polygon and let $K_1 = \{A, V_1, \ldots, V_s, B\}$, $K_2 = \{B, W_1, \ldots, W_{s+p}, C\}$ be consecutive augmented chains of P with $p \geqslant 2$. Let us call S the ordered set of the other vertices of P. We construct a new almost-convex polygon P' by inserting a new vertex in K_1, that becomes $K_1' = \{A, V_1, \ldots, V_s, V_{s+1}, B\}$, and deleting a vertex of K_2, that becomes $K_2' = \{B, W_1, \ldots, W_{s+p-1}, C\}$. Then $t(P) < t(P')$. This is proved by induction on p, by taking every triangle with basis $W_{s+p}C$ in P and the corresponding triangle with basis $W_{s+p-1}C$ in P'. These triangles dissect P and P' in pieces suitable for induction. The initial case $p = 2$ is treated in a similar way.

Now we have all the ingredients for our main result.

Theorem 10. *Let P be a n-polygon with k reflex vertices. Then*

$$\left(t_{\lceil (n+k)/(k+1) \rceil}\right)^s \left(t_{\lfloor (n+k)/(k+1) \rfloor}\right)^{k+1-s} \leqslant t(P) \leqslant \sum_{i=0}^{k} (-1)^i \left\| \begin{matrix} \gamma, \overset{u}{\ldots}, \gamma, \ \gamma+1, \overset{v}{\ldots}, \gamma+1 \\ i \end{matrix} \right\| t_{n-i},$$

where s is the residue of the division of $n + k$ by $k + 1$, $\gamma = \lfloor k/(n-k) \rfloor$, v is the residue of the division of k by $n - k$, $u = n - k - v$. These bounds are tight, and for $k \leqslant n/2$ the upper bound adopts the simpler form

$$t(P) \leqslant t_n - \binom{k}{1} t_{n-1} + \cdots + (-1)^k \binom{k}{k} t_{n-k}.$$

Proof. For the lower bound we use procedure *break* to obtain $k+1$ non-overlapping convex subpolygons of P, with sizes $n_1, \dots, n_{k+1}$ vertices and $n_1 + \cdots + n_{k+1} \geqslant n + k$. Then

$$t(P) \geqslant t_{n_1} t_{n_2} \cdots t_{n_{k+1}},$$

and Lemma 5 gives us the desired bound.

To show the lower bound is tight, we consider the polygon in Fig. 11(a). If the number s of convex vertices is less than $k+3$ then the lower bound is simply 1: vertices $v_s, v_{s+1}, \dots, v_{k+3}$ merge to a single vertex v_s and the sequence of segments is replaced by a single segment (Fig. 11(b)). Otherwise convex vertices $v_{k+4}, v_{k+5}, \dots$ are distributed in the "pockets" in a balanced way (Fig. 11(c)). Like this we determine the $k+1$ convex subpolygons that provide the bound.

For the upper bound we use the vertices of P as ordered labels for the vertices of a regular n-polygon R. For every reflex chain C of P we pull the corresponding vertices of R towards the center in such a way that the only diagonals passing in this operation from the interior to the exterior are between nonconsecutive vertices of C (Fig. 12); this way we obtain a polygon Q. It is evident that every triangulation of P translates into a triangulation of Q, hence $t(P) \leqslant t(Q)$. But Q is an almost-convex polygon with n vertices, k of them being reflex, and Lemmas 8 and 9 apply providing us the desired (tight) bound, that corresponds to a balanced almost-convex polygon. Let us observe that if $k \leqslant n/2$ then two reflex vertices of a balanced almost-convex polygon Q^* cannot be consecutive. In this case the Principle of Inclusion and Exclusion can be applied directly to obtain the simpler

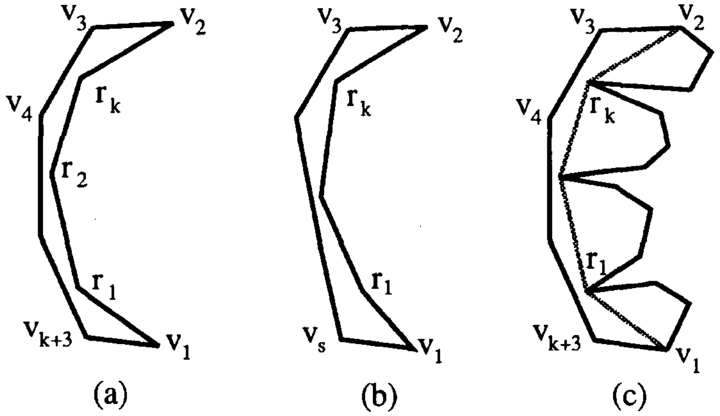

Fig. 11. The construction for the lower bound.

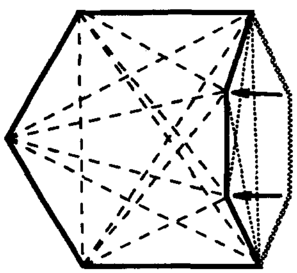

Fig. 12. Two vertices are pulled producing an almost-convex polygon.

expression given in the last part of the statement, since $t(Q^*)$ is the number of triangulations of a convex n-polygon not using a fixed set of k disjoint diagonals of the form P_iP_{i+2}. $\square$

The precedent results have some simple consequences that are worth noticing.

Observation 1. The lower bound in Theorem 10 is 1 when $2 \leqslant (n+k)/(k+1) \leqslant 3$ or, equivalently, $k \geqslant (n-3)/2$. So, if $k < (n-3)/2$ then P admits at least two triangulations.

Observation 2. An n-polygon with k reflex vertices admits always a convex $\lceil (n+k)/(k+1) \rceil$-subpolygon, and this value is tight. In other words, in order to guarantee a convex p-subpolygon we must have $n \geqslant kp + p - 2k$.

3. The size of the visibility graph

The *visibility graph* of a simple polygon P, denoted by $G_P = (V_P, E_P)$, is the graph with a node for each vertex of P, and an edge connecting pairs of nodes if and only if the corresponding vertices can see each other inside P. Many parameters of G_P are naturally related to the number of reflex vertices, because internal diagonals—adjacencies in G_P—can be destroyed by reflex vertices. For example, Observation 2 can be rephrased in the following way: if an n-polygon has k reflex vertices, then $\text{clique}(G_P) \geqslant \lceil (n+k)/(k+1) \rceil$. In this section we are concerned with the size $|E_P|$ of G_P.

For the sake of clarity, we begin with the statement for the general situation. The tightness is immediately provided by the same constructions given in Theorem 1.

Theorem 11. *Let P be an n-polygon and $G_P = (V_P, E_P)$ its visibility graph. Then $2n-3 \leqslant |E_P| \leqslant \binom{n}{2}$, and these bounds are tight.*

As we have $\binom{m}{2} + \binom{n}{2} > \binom{m+1}{2} + \binom{n-1}{2}$ when $n - m \geqslant 2$, the argumentation in Lemma 4 can be repeated and we obtain the following lemma.

Lemma 12. *Let $n_1, \ldots, n_m$ be m given nonnegative integers. Then*

$$\binom{n_1}{2} + \cdots + \binom{n_m}{2} \geqslant s\binom{\lceil \sum n_i/m \rceil}{2} + (m-s)\binom{\lfloor \sum n_i/m \rfloor}{2},$$

where s is the residue of the division of $\sum n_i$ by m. Moreover the right value in the inequality is the minimum when we allow the variation of $n_1, \ldots, n_m$ while maintaining constant its sum.

Now the natural step is a result similar to Theorem 10.

Theorem 13. *Let P be an n-polygon with k reflex vertices, and let $G_P = (V_P, E_P)$ its visibility graph. Then*

$$k + s\binom{\lceil \frac{n+k}{k+1} \rceil}{2} + (k+1-s)\binom{\lfloor \frac{n+k}{k+1} \rfloor}{2} \leqslant |E_P|,$$

$$|E_P| \leqslant \binom{n}{2} - k - u\binom{\lceil \frac{k}{n-k} \rceil}{2} - (n-k-u)\binom{\lfloor \frac{k}{n-k} \rfloor}{2},$$

where s is the residue of the division of $n+k$ by $k+1$ and u is the residue of the division of k by $n-k$. These bounds are tight, and for $k \leqslant n/2$ the right inequality adopts the simpler form

$$|E_P| \leqslant \binom{n}{2} - k.$$

Proof. For the lower bound, we first apply the procedure *break* to the polygon P. Let $n_1, \ldots, n_{k+1}$ be the sizes of the resulting convex pieces. If Steps 1, 2, 3a and 3b are applied α, β, γ_1 and γ_2 times, respectively, then we see that

$$n_1 + \cdots + n_{k+1} = n + k + \alpha + \gamma_1 + 2\gamma_2,$$
$$\alpha + \beta + 2\gamma_1 + 3\gamma_2 = k.$$

In the following inequalities, we bound $|E_P|$ from below by the sum of the sizes of the visibility graphs of the pieces, discounting edges counted twice and recovering discarded edges. We also use the notation $d = \alpha + \gamma_1 + 2\gamma_2$ and the fact that $\binom{n_i}{2} \geqslant \binom{n_i - 1}{2} + 2$ when $n_i \geqslant 3$.

$$\begin{aligned}
|E_P| &\geqslant \binom{n_1}{2} + \cdots + \binom{n_{k+1}}{2} - \alpha + \beta - 3\gamma_2 \\
&\geqslant \left[\binom{n_1 - 1}{2} + \cdots + \binom{n_d - 1}{2} + \binom{n_{d+1}}{2} + \cdots + \binom{n_{k+1}}{2} + 2d\right] - \alpha + \beta - 3\gamma_2 \\
&= \binom{n_1 - 1}{2} + \cdots + \binom{n_d - 1}{2} + \binom{n_{d+1}}{2} + \cdots + \binom{n_{k+1}}{2} + \alpha + \beta + 2\gamma_1 + 3\gamma_2 \\
&= \binom{n_1 - 1}{2} + \cdots + \binom{n_d - 1}{2} + \binom{n_{d+1}}{2} + \cdots + \binom{n_{k+1}}{2} + k \\
&\geqslant s\binom{\lceil \frac{n+k}{k+1} \rceil}{2} + (k+1-s)\binom{\lfloor \frac{n+k}{k+1} \rfloor}{2} + k,
\end{aligned}$$

where Lemma 12 has been applied in the last inequality and s is the residue of the division of $(n_1 - 1) + \cdots + (n_d - 1) + n_{d+1} + n_{k+1} = n + k$ by $k+1$. The tightness is obtained with the same construction we gave for the lower bound in Theorem 10.

For the upper bound, let us consider the $n-k$ convex vertices of P, ordered as they appear around P. Let $\gamma_1, \ldots, \gamma_{n-k}$ be the lengths of the corresponding reflex chains between any two of them (some γ_i can be 0); in any case $\sum \gamma_i = k$. The number of lost diagonals will be at least

$$k' = \sum_{i=1}^{n-k} \left[\binom{\gamma_i + 2}{2} - (\gamma_i + 1)\right],$$

a value that is exactly reached if P is an almost-convex polygon. We can express that result as

$$k' = k + \sum_{i=1}^{n-k} \binom{\gamma_i}{2}.$$

Now Lemma 12 applies, and we see that the minimum k' is obtained when the $\gamma_1, \ldots, \gamma_{n-k}$ are balanced, providing us the claimed (tight) bound. For $k \leqslant n/2$ we have $0 \leqslant \lfloor (n+k)/(k+1) \rfloor \leqslant \lceil (n+k)/(k+1) \rceil \leqslant 1$, and the upper bound becomes simply $\binom{n}{2} - k$. $\square$

4. Asymptotic analysis

We now proceed to compute asymptotic estimates for the bounds presented in Theorem 10, for fixed k, $n \to \infty$. In order to simplify matters, we will content ourselves with the following expressions for the lower and upper bounds, which do not alter its asymptotic behaviour:

$$L_k(n) = \left(C_{\lfloor n/(k+1)\rfloor}\right)^{k+1},$$
$$U_k(n) = C_n - \binom{k}{1}C_{n-1} + \binom{k}{2}C_{n-2} + \cdots + (-1)^k\binom{k}{k}C_{n-k},$$

where, as before, $C_n = (1/(n+1))\binom{2n}{n}$ is the nth Catalan number. The lower bound is easily dealt with thanks to Stirling's estimate $C_n \sim K'4^n n^{-3/2}$ ($K' = \pi^{-1/2}$) which immediately gives

$$L_k(n) \sim K4^n n^{-3(k+1)/2}$$

for a certain constant K depending on k. Thus the main asymptotic term 4^n remains the same as in the convex case ($k = 0$) but the degree of the polynomial in n has been decreased by $3k/2$.

As for the upper bound we use a generating function approach based on the well-known ordinary generating function for the Catalan numbers,

$$\sum C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$$

From this it follows easily that

$$\sum_n U_k(n) z^n = (1-z)^k \frac{1-\sqrt{1-4z}}{2z}.$$

Now Darboux's lemma [12] tells us that the main contribution to the $U_k(n)$ comes from the algebraic singularity $\sqrt{1-4z}$, the function $(1-z)^k$ being analytic on the complex domain. If we make the change of variable $w = 4z$, the function becomes

$$\left(1-\frac{w}{4}\right)^k \frac{2(1-\sqrt{1-w})}{w},$$

and Taylor development's about $w = 1$

$$\left(1-\frac{w}{4}\right)^k = (3/4)^k + \cdots$$

gives the first approximation to the asymptotic behaviour of the $U_k(n)$:

$$U_k(n) \sim (3/4)^k \pi^{-1/2} 4^n n^{-3/2}.$$

This result can be rephrased in a simple way as

$$\frac{U_k(n)}{C_n} \sim (3/4)^k,$$

that is, every time we add a reflex vertex, the maximum number of triangulations of a polygon is decreased (asymptotically) by a factor of 3/4.

We summarize these results in the following theorem.

Theorem 14. *Let* $L_k(n)$ *and* $U_k(n)$ *be as above. Then*

$$L_k(n) \sim K 4^n n^{-3(k+1)/2} \quad \textit{for a certain constant } K = K(k);$$

$$U_k(n) \sim \left(\frac{3}{4}\right)^k \pi^{-1/2} 4^n n^{-3/2}.$$

Observation 3. As an additional remark, we note that if we make the number of reflex vertices proportional to n, say $k = n/\alpha$, then

$$L_k(n) = \left((C_\alpha)^{1/\alpha}\right)^n.$$

The sequence $(C_\alpha)^{1/\alpha}$ is increasing with limit equal to 4 (this follows from the fact that $C_{\alpha+1}/C_\alpha = 2(2\alpha+1)/(\alpha+2)$ has limit 4 as α goes to infinity). This means that we are always asymptotically under the main term 4^n coming from the convex case, but we approach this limit as the number of reflex vertices becomes relatively scarce.

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