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Overinterpolation ☆

Dan Coman, Evgeny A. Poletsky *

Department of Mathematics, 215 Carnegie Hall, Syracuse University, Syracuse, NY 13244-1150, USA

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Abstract

In this paper we study the consequences of overinterpolation, i.e., the situation when a function can be interpolated by polynomial, or rational, or algebraic functions in more points than normally expected. We show that in many cases such function has specific forms.

Keywords: Overinterpolation; Interpolation; Overconvergence

1. Introduction

Let $\mathcal{P}_n$ be the space of all polynomials on the complex plane $\mathbb{C}$ whose degree is at most n. Let $\mathcal{R}_{nm}$ be the space of rational functions $R_{nm} = P_n/Q_m$ where $P_n \in \mathcal{P}_n$ and $Q_m \in \mathcal{P}_m$.

If f is a function on a compact set $K \subset \mathbb{C}$, then we denote by $N_K(n)$ and $N_K(n,m)$ the maximal number of zeros on K of the functions $f - p$, where $p \in \mathcal{P}_n$, respectively $p \in \mathcal{R}_{nm}$. Since functions in $\mathcal{P}_n$ or $\mathcal{R}_{nm}$ have $n+1$ or, respectively, $n+m+2$ coefficients, $N_K(n) \geqslant n+1$ and $N_K(n,m) \geqslant n+1$.

In this paper we consider the situations when for a fixed function f we have either polynomial or rational *overinterpolation*. This means that

$$\lim_{n\to\infty} \frac{N_K(n)}{n} = \infty \quad \text{or} \quad \lim_{n\to\infty} \frac{N_K(n,m)}{n} = \infty.$$

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* Corresponding author.
E-mail addresses: dcoman@syr.edu (D. Coman), eapolets@syr.edu (E.A. Poletsky).

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One can expect that in the case of overinterpolation, the function f must be either polynomial or rational. We prove two theorems of this kind. Before we state them, let us introduce some notation.

Let $\Delta_r \subset \mathbb{C}$ be the open disk of radius r centered at the origin, and $\Delta \subset \mathbb{C}$ be the open unit disk. We denote by $O(\Delta_r)$ and $O(\overline{\Delta}_r)$ the set of holomorphic functions on Δ_r, respectively on neighborhoods of $\overline{\Delta}_r$.

The first theorem proved in Section 2 states that for an analytic function f overinterpolation by polynomials implies that f is a polynomial.

Theorem 1.1. *Let $f \in O(\Delta)$ and $K = \overline{\Delta}_r$, where $r < 1$. If $\lim_{n\to\infty} N_K(n)/n = \infty$ then f is a polynomial.*

For a function f as above, the second theorem states that overinterpolation in $\mathcal{R}_{n1}$ implies that either f is entire or it belongs to $\mathcal{R}_{n1}$.

Theorem 1.2. *Let $f \in O(\Delta)$ and $K = \overline{\Delta}_r$, where $r < 1$. If $\lim_{n\to\infty} N_K(n,1)/n = \infty$ then either f is entire or $f = P/Q$, where P, Q are polynomials, $\deg Q = 1$ and Q does not divide P.*

This theorem is proved in Section 3, where we also consider the case of Padé interpolation, i.e. when $K = \{0\}$. For any germ of an analytic function f at 0 and any fixed $m \in \mathbb{N}$, we show that $N_K(n,m) \leqslant n + m + 1$ for infinitely many n, unless f is the germ of a rational function.

The expected rate of rational approximation of continuous or analytic functions is at most geometric, but in some cases functions can be approximated faster. This phenomenon is called *overconvergence*. In [6] and [1] Gonchar and Chirka have shown that in this case the functions have specific forms. In Section 4 we prove that overinterpolation implies overconvergence on some circle and, therefore, overinterpolated functions have the same specific forms as in the results of Gonchar and Chirka.

For the entire function $f(z) = \sum 2^{-n!} z^n$, its Taylor series is overconvergent but by Theorem 1.1 f cannot be overinterpolated by polynomials. Hence overconvergence does not imply overinterpolation.

In the last section we consider the interpolation of a general set S in $\mathbb{C}^2$ by algebraic functions, i.e., we are looking for the maximal number $N_S^a(n)$ of zeros on S of a polynomial of degree n which does not vanish on S. The desirable estimate is $N_S^a(n) \leqslant An^\alpha$, where A and α are some constants. We show that either S is finite, or $\alpha = 1$ and S is contained in an irreducible algebraic curve, or $\alpha \geqslant 2$.

It should be noted that in [2] and [4] we proved for a large class of meromorphic functions f on $\mathbb{C}$ with finitely many poles, including the Riemann ζ-function, that if S is the graph of f over Δ_r, then $N_S^a(n) \leqslant An^2 \log r$.

2. Overinterpolation by polynomials

If $f \in O(\overline{\Delta}_R)$ we set

$$M(r, f) = \max\{|f(z)|: |z| = r\}, \quad r \leqslant R.$$

We will need the following lemmas:

Lemma 2.1. *Let $f \in O(\overline{\Delta}_R)$ and $L_n f$ denote the Lagrange interpolating polynomial of f at the (not necessarily distinct) points $z_0, \dots, z_n \in \overline{\Delta}_r$, where $r < R$. If $0 < s < R$ then*

$$M(s, f - L_n f) \leqslant M(R, f)\frac{R}{R-s}\left(\frac{s+r}{R-r}\right)^{n+1}.$$

Proof. Let $\omega(z) = (z - z_0)\dots(z - z_n)$. By [5, p. 59, (1.4)] we have

$$f(z) - L_n f(z) = \frac{1}{2\pi i}\int\limits_{|t|=R} \frac{\omega(z) f(t)}{\omega(t)(t-z)}\,dt.$$

The lemma follows since $|\omega(t)| \geqslant (R-r)^{n+1}$ for $|t| = R$, and since $M(s, \omega) \leqslant (s+r)^{n+1}$. □

For $R > 0$ let

$$R_+ = \max\{R, 1\}.$$

We have the following estimate of Taylor coefficients.

Lemma 2.2. *Let $f \in O(\Delta)$, $f(z) = \sum_{k\geqslant 0} f_k z^k$. Suppose that $0 < r < 1$ and the function $f - P_n$ has N zeros in $\overline{\Delta}_r$, where P_n is a polynomial of degree at most n. There exist positive constants $A \geqslant 1$, $a < 1$ and δ, depending only on r, with the following property: If $N \geqslant A(n+1)$, then*

$$|f_k| \leqslant \frac{M(R, f)}{R_+^{n+1}} a^N,$$

for $n < k \leqslant \delta N$ and every $R \geqslant (r+2)/3$ such that $f \in O(\overline{\Delta}_R)$.

Proof. Let $s = (2r+1)/3$ and fix R as in the statement. Let $z_0, \dots, z_{N-1}$ be zeros of $f - P_n$ in $\overline{\Delta}_r$. Since $N \geqslant n+1$ the polynomial $P_n = L_n f$ is the Lagrange interpolating polynomial of f at $z_0, \dots, z_n$. Since $f - P_n$ has N zeros in $\overline{\Delta}_r$, we have by [3, Theorem 2.2] (see the formula on p. 578)

$$M(r, f - P_n) \leqslant M(s, f - P_n)\left(\frac{2rs}{r^2 + s^2}\right)^N.$$

Hence by Lemma 2.1

$$\begin{aligned} M(r, f - P_n) &\leqslant M(R, f)\frac{R}{R-s}\left(\frac{s+r}{R-r}\right)^{n+1}\left(\frac{2rs}{r^2+s^2}\right)^N \\ &= \frac{M(R, f)}{R^{n+1}}\frac{1}{1 - s/R}\left(\frac{s+r}{1-r/R}\right)^{n+1}\left(\frac{2rs}{r^2+s^2}\right)^N. \end{aligned}$$

Notice that

$$\frac{1}{1-s/R} < \frac{3}{1-r}, \qquad \frac{s+r}{1-r/R} < \frac{3}{1-r},$$

and

$$a_1 := \frac{2rs}{r^2+s^2} < 1.$$

Since $R > 2/3$ we obtain

$$M(r, f - P_n) \leqslant \frac{M(R, f)}{R^{n+1}}\left(\frac{3}{1-r}\right)^{n+2} a_1^N$$
$$< \frac{M(R, f)}{R_+^{n+1}}\left(\frac{3}{2}\right)^{n+1}\left(\frac{3}{1-r}\right)^{n+2} a_1^N$$
$$< \frac{M(R, f)}{R_+^{n+1}}\left(\frac{5}{1-r}\right)^{n+2} a_1^N. \tag{1}$$

Let

$$A = \max\left\{-4\frac{\log 5 - \log(1-r)}{\log a_1}, 1\right\}.$$

As $N \geqslant A(n+1)$ we obtain

$$M(r, f - P_n) \leqslant \frac{M(R, f)}{R_+^{n+1}} a_2^N,$$

where $a_2 = a_1^{1/2}$. Since $k > \deg P_n$ it follows by Cauchy's inequalities that

$$|f_k| \leqslant \frac{M(r, f - P_n)}{r^k} \leqslant \frac{M(R, f)}{R_+^{n+1}} a_2^N r^{-k}.$$

We define $a = a_2^{1/2}$ and δ by $r^\delta = a$. If $k \leqslant \delta N$ then

$$|f_k| \leqslant \frac{M(R, f)}{R_+^{n+1}} a^N. \qquad \square$$

Proof of Theorem 1.1. Let $f(z) = \sum_{n\geqslant 0} f_n z^n$. We can find an increasing sequence of integers $N(n) \leqslant N_K(n)$ such that $N(n)/n \to \infty$ and the function $f - P_n$ has at least $N(n)$ zeros in $\overline{\Delta}_r$, where $P_n \in \mathcal{P}_n$.

Fix $R \geqslant (r+2)/3$ so that $f \in O(\overline{\Delta}_R)$. Let $a < 1 \leqslant A$ be the constants from Lemma 2.2 and n_0 be so that $N(n) \geqslant A(n+1)$ if $n \geqslant n_0$. Lemma 2.2 implies that for $n \geqslant n_0$

$$|f_{n+1}| \leqslant \frac{M(R, f)}{R_+^{n+1}} a^{N(n)}. \tag{2}$$

Therefore $|f_n|^{1/n} \to 0$, so f is entire, hence (2) holds for any $R \geqslant 1$. By Cauchy's inequalities we have $|f_n| \leqslant M(R, f)/R^n$ for $n \leqslant n_0$. Using these estimates of the coefficients, we obtain the following bound for $M(2R, f)$, $R \geqslant 1$:

$$M(2R, f) \leqslant \sum_{n\geqslant 0} |f_n|(2R)^n \leqslant CM(R, f), \tag{3}$$

where

$$C = \sum_{n=0}^{n_0} 2^n + \sum_{n=n_0}^{\infty} 2^{n+1} a^{N(n)}$$

is independent on R. Note that

$$2^n a^{N(n)} = \left(2a^{N(n)/n}\right)^n \leqslant 2^{-n},$$

provided that n is sufficiently large, thus C is finite.

Applying the doubling inequality (3) successively we obtain

$$M\big(2^j, f\big) \leqslant C^j M(1, f),$$

for any $j > 0$. Hence

$$|f_n| \leqslant \frac{C^j M(1, f)}{2^{nj}} \to 0 \quad \text{as } j \to \infty,$$

provided that $2^n > C$. We conclude that f is a polynomial of degree at most $\log C/\log 2$. □

Theorem 1.1 has the following immediate corollary:

Corollary 2.3. *Let $\{n_k\}_{k\geqslant 0}$ be an increasing sequence of natural numbers such that $n_{k+1}/n_k \leqslant C$ for some constant C. Let $f \in O(\Delta)$ and $K = \overline{\Delta}_r$, where $r < 1$. If $\lim_{k\to\infty} N_K(n_k)/n_k = \infty$ then f is a polynomial.*

Proof. Let $N(n) = N_K(n)$. If $n_k \leqslant n < n_{k+1}$ then

$$\frac{N(n)}{n} > \frac{N(n_k)}{n_{k+1}} \geqslant \frac{N(n_k)}{Cn_k}. \qquad \square$$

3. Overinterpolation by rational functions

We prove here Theorem 1.2. We can find an increasing sequence of integers $N(n) \leqslant N_K(n, 1)$ such that $N(n)/n \to \infty$ and the function $Q_n f - P_n$ has at least $N(n)$ zeros in $\overline{\Delta}_r$, where $P_n \in \mathcal{P}_n$, $Q_n \in \mathcal{P}_1$ and $Q_n \neq 0$.

Let us write

$$f(z) = \sum_{k\geqslant 0} f_k z^k, \qquad Q_n(z) = \alpha_n z - \beta_n.$$

Let

$$\rho = \frac{1}{\limsup |f_k|^{1/k}} \geqslant 1$$

be the radius of convergence of the power series of f at the origin.

By considering functions $c(Q_n f - P_n)$, where $c \in \mathbb{C} \setminus \{0\}$, we can identify Q_n with the point $[\alpha_n : \beta_n] \in \mathbb{P}^1$. Thus

$$Q_n(z) = \alpha_n z - 1, \quad \alpha_n \in \mathbb{C} \quad \text{or} \quad Q_n(z) = z.$$

The latter case corresponds to $\alpha_n = \infty$ in the extended complex plane.

We begin with a few lemmas.

Lemma 3.1. *There exist constants $a < 1$, $\delta < 1$ and an integer n_0, depending only on r, with the following property: If $n \geqslant n_0$, then one of the inequalities*

$$|\alpha_n f_{k-1} - f_k| \leqslant \frac{M(R, f)}{R_+^{n+1}}\big(|\alpha_n| R_+ + 1\big) a^{N(n)}, \qquad |f_{k-1}| \leqslant \frac{M(R, f)}{R_+^n} a^{N(n)},$$

holds for every k, $n < k \leqslant \delta N(n)$, and every $R \geqslant (r + 2)/3$ such that $f \in O(\overline{\Delta}_R)$.

Proof. Let $a < 1 \leqslant A$, $\delta > 0$, be the constants from Lemma 2.2, and let $n_0 = n_0(r)$ be an integer such that $N(n) \geqslant A(n+1)$ for $n \geqslant n_0$. We fix such an n, and apply Lemma 2.2 to the function $Q_n f$ and the polynomial P_n. If $Q_n(z) = \alpha_n z - 1$ then

$$Q_n(z)f(z) = -f_0 + \sum_{k\geqslant 1}(\alpha_n f_{k-1} - f_k)z^k,$$

and $M(R, Q_n f) \leqslant M(R, f)(|\alpha_n|R_+ + 1)$. This yields the first inequality of the lemma. The second one is obtained in a similar way, in the case when $Q_n(z) = z$ (or by letting $\alpha_n \to \infty$). □

Lemma 3.2. *If $f \in O(\Delta_s)$, $1 \leqslant s \leqslant \infty$, and if $\liminf_{n\to\infty} |\alpha_n| > 1/s$, then f is a polynomial.*

Proof. There exist $n_1 \geqslant n_0$ and $\epsilon > 0$ such that $|\alpha_n| > 1/s + \epsilon$, for $n \geqslant n_1$. Let

$$c = \left(\frac{1}{s} + \epsilon\right)^{-1}, \qquad d = 1 + c.$$

By Lemma 3.1 with $k = n+1$ we have

$$|f_n| \leqslant \frac{|f_{n+1}|}{|\alpha_n|} + \frac{M(R, f)}{R_+^n}\left(1 + \frac{1}{R_+|\alpha_n|}\right)a^{N(n)} \leqslant c|f_{n+1}| + \frac{dM(R, f)}{R_+^n}a^{N(n)},$$

for every $R \geqslant (r+2)/3$ such that $f \in O(\overline{\Delta}_R)$. Note that this estimate obviously holds in the case $\alpha_n = \infty$. Applying it successively we obtain

$$|f_n| \leqslant c^k|f_{n+k}| + \frac{dM(R, f)}{R_+^n}\sum_{j=0}^{k-1}\frac{c^j}{R_+^j}a^{N(n+j)}, \tag{4}$$

for every $k \geqslant 1$. Fix $s_1 \geqslant (r+2)/3$ such that $c < s_1 < s$. Since $f \in O(\Delta_s)$ we have

$$|f_{n+k}| \leqslant \left(\frac{1}{s} + \frac{\epsilon}{2}\right)^{n+k},$$

for k sufficiently large. Since $N(n)$ is increasing, and if $R \geqslant s_1$, we obtain by (4)

$$|f_n| \leqslant c^k|f_{n+k}| + \frac{dM(R, f)}{R_+^n}a^{N(n)}\sum_{j=0}^{\infty}\frac{c^j}{s_1^j} \leqslant \frac{(\frac{1}{s} + \frac{\epsilon}{2})^{n+k}}{(\frac{1}{s} + \epsilon)^k} + \frac{ds_1 M(R, f)}{(s_1 - c)R_+^n}a^{N(n)}.$$

Letting $k \to \infty$ we conclude that

$$|f_n| \leqslant \frac{ds_1 M(R, f)}{(s_1 - c)R_+^n}a^{N(n)}$$

holds for all $n \geqslant n_1$ and $R \geqslant s_1$ such that $f \in O(\overline{\Delta}_R)$. This is a similar estimate to (2) from the proof of Theorem 1.1. Therefore, by the same argument as in the proof of Theorem 1.1, it follows that f is entire and $M(2R, f) \leqslant CM(R, f)$ for every $R \geqslant s_1$, where

$$C = \sum_{n=0}^{n_1-1} 2^n + \frac{ds_1}{s_1 - c}\sum_{n=n_1}^{\infty} 2^n a^{N(n)}$$

is independent on R. Hence f is a polynomial. □

Lemma 3.3. $\limsup_{n\to\infty} |\alpha_n| \geqslant 1/\rho$.

Proof. We assume for a contradiction that there exist $n_1 \geqslant n_0$ and $0 < \epsilon < 1/\rho$ such that

$$|\alpha_n| < c := \rho^{-1} - \epsilon, \quad n \geqslant n_1.$$

As $c < 1$, we obtain by Lemma 3.1, applied with $k = n+1$ and $R = (r+2)/3 < 1$, that

$$|f_{n+1}| \leqslant |\alpha_n f_n| + M\big(|\alpha_n| + 1\big)a^{N(n)} \leqslant c|f_n| + 2Ma^{N(n)},$$

where $M = M(R, f)$ and $n \geqslant n_1$. Hence

$$|f_{n+k}| \leqslant c^k |f_n| + 2M \sum_{j=0}^{k-1} c^j a^{N(n+k-1-j)} \leqslant c^k |f_n| + 2Ma^{N(n)} \sum_{j=0}^{\infty} c^j,$$

for all $k \geqslant 1$.

Let $C = 2M/(1-c)$. Then for $n \geqslant n_1$ we have

$$|f_{2n}| \leqslant c^n |f_n| + Ca^{N(n)}, \qquad |f_{2n+1}| \leqslant c^{n+1}|f_n| + Ca^{N(n)}.$$

Since, for n large, $|f_n| \leqslant (\rho^{-1} + \epsilon)^n$, it follows that

$$|f_{2n}|^{1/(2n)} \leqslant c^{1/2}\big(\rho^{-1} + \epsilon\big)^{1/2} + C^{1/(2n)} a^{N(n)/(2n)},$$
$$|f_{2n+1}|^{1/(2n+1)} \leqslant c^{(n+1)/(2n+1)}\big(\rho^{-1} + \epsilon\big)^{n/(2n+1)} + C^{1/(2n+1)} a^{N(n)/(2n+1)}.$$

Note that $a^{N(n)/n} \to 0$. Therefore

$$\rho^{-1} = \limsup_{j\to\infty} |f_j|^{1/j} \leqslant \big(\rho^{-1} - \epsilon\big)^{1/2}\big(\rho^{-1} + \epsilon\big)^{1/2},$$

a contradiction. □

We continue with the proof of Theorem 1.2 in several steps. We can assume $\rho < \infty$, otherwise f is entire. The radius of convergence of the power series of $f(\rho z)$ at the origin is 1, and the function $Q_n(\rho z) f(\rho z) - P_n(\rho z)$ has $N(n)$ zeros in the disk $\overline{\Delta}_{r/\rho}$. Therefore we may assume that $\rho = 1$.

Let $R = (r+2)/3$ and $M = M(R, f)$. By Lemma 3.1, one of the estimates

$$|\alpha_n f_{k-1} - f_k| \leqslant M\big(|\alpha_n| + 1\big)a^{N(n)}, \qquad |f_{k-1}| \leqslant Ma^{N(n)}, \tag{5}$$

holds for $n < k \leqslant \delta N(n)$, provided that $n \geqslant n_0$. By Lemma 3.3, $|\alpha_n| > 1/3$ or $\alpha_n = \infty$ for infinitely many n.

Step 1. We show that there exists a sequence $m_j \to \infty$ such that

$$\alpha_{m_j} \in \mathbb{C}, \qquad |\alpha_{m_j}| > 1/3, \qquad |f_{m_j+1}| > 2^{-m_j-2}.$$

Fix any n large with $|\alpha_n| > 1/3$ or $\alpha_n = \infty$. Let $k \geqslant n$ be the smallest integer such that $|f_k| > 2^{-k}$. Such k exists since $\rho = 1$. If $k > n$ then α_{k-1} is finite. Otherwise by (5)

$$2^{-k} < |f_k| \leqslant Ma^{N(k-1)},$$

which is impossible as n is large. By the definition of k, $|f_{k-1}| \leqslant 2^{-k+1}$. We claim that $|\alpha_{k-1}| > 1/3$. If not, then using (5)

$$2^{-k} < |f_k| \leqslant |\alpha_{k-1} f_{k-1}| + 2Ma^{N(k-1)} \leqslant \frac{2^{-k+1}}{3} + 2Ma^{N(k-1)},$$

so $2^{-k} < 6Ma^{N(k-1)}$. This is a contradiction since n is large.

If $k=n$, then $|f_n| > 2^{-n} > Ma^{N(n)}$ shows that $\alpha_n \in \mathbb{C}$, so $|\alpha_n| > 1/3$. We have by (5)

$$|f_{n+1}| \geqslant |\alpha_n f_n| - 4M|\alpha_n|a^{N(n)} > |\alpha_n|\big(2^{-n} - 4Ma^{N(n)}\big)$$
$$> \frac{2^{-n}}{3}\big(1 - 2^{n+2}Ma^{N(n)}\big) \geqslant 2^{-n-2}.$$

This establishes the existence of the desired sequence m_j and concludes Step 1.

Since $N(n)/n \to \infty$ and $\rho = 1$, we can find $n_1 \geqslant n_0$ with the property that

$$2^{3n+6k+16}Ma^{N(n+k)} < 1, \qquad |f_n| < 2^n, \tag{6}$$

hold for every $n \geqslant n_1$ and for every $k \geqslant 0$. Then we fix $n \geqslant n_1$ such that $\alpha_n \in \mathbb{C}$, $|\alpha_n| > 1/3$ and $|f_{n+1}| > 2^{-n-2}$. We have using (5) that

$$2^{-n-2}|\alpha_n| < |\alpha_n||f_{n+1}| \leqslant |f_{n+2}| + 4M|\alpha_n|a^{N(n)} \leqslant 2^{n+2} + 4M|\alpha_n|a^{N(n)},$$

so by (6)

$$|\alpha_n| \leqslant \frac{2^{2n+4}}{1 - 2^{n+4}Ma^{N(n)}} \leqslant 2^{2n+5}.$$

Step 2. We will show by induction that for every $k \geqslant 0$

$$|f_{n+k+1}| > 2^{-n-2}6^{-k}, \qquad \frac{1}{6} + 6^{-k-1} < |\alpha_{n+k}| < 2^{2n+6} - 6^{-k-1}. \tag{7}$$

Evidently, these inequalities hold for $k = 0$. Suppose that they are true for some $k \geqslant 0$. Then using (5)

$$|f_{n+k+2}| \geqslant |\alpha_{n+k} f_{n+k+1}| - 7M|\alpha_{n+k}|a^{N(n+k)}$$
$$\geqslant 2^{-n-2}6^{-k-1} + 2^{-n-2}6^{-2k-1} - 2^{2n+9}Ma^{N(n+k)}.$$

By (6)

$$2^{-n-2}6^{-2k-1} - 2^{2n+9}Ma^{N(n+k)} > 0,$$

so we see that $|f_{n+k+2}| > 2^{-n-2}6^{-k-1}$.

Since $|f_{n+k+1}| > 2^{-n-2}6^{-k}$, we have in view of (5) and (6) that $\alpha_{n+k+1} \in \mathbb{C}$. Therefore by (5)

$$|\alpha_{n+k} f_{n+k+1} - f_{n+k+2}| \leqslant M\big(|\alpha_{n+k}| + 1\big)a^{N(n+k)},$$
$$|\alpha_{n+k+1} f_{n+k+1} - f_{n+k+2}| \leqslant M\big(|\alpha_{n+k+1}| + 1\big)a^{N(n+k+1)}.$$

As $|\alpha_{n+k}| > 1/6$ and $N(n)$ is increasing, it follows that

$$|f_{n+k+1}||\alpha_{n+k} - \alpha_{n+k+1}| \leqslant M\big(13|\alpha_{n+k}| + |\alpha_{n+k+1}|\big)a^{N(n+k)}.$$

Hence

$$|\alpha_{n+k+1}|\left(1 - \frac{Ma^{N(n+k)}}{|f_{n+k+1}|}\right) \leqslant |\alpha_{n+k}|\left(1 + 13\frac{Ma^{N(n+k)}}{|f_{n+k+1}|}\right).$$

So, by (6) and (7), $|\alpha_{n+k+1}| < 4|\alpha_{n+k}|$.

Thus

$$|\alpha_{n+k} - \alpha_{n+k+1}| \leqslant 17M2^{2n+6}a^{N(n+k)}2^{n+2}6^k \leqslant 2^{3n+3k+13}Ma^{N(n+k)}, \tag{8}$$

and by (6)

$$|\alpha_{n+k} - \alpha_{n+k+1}| \leqslant 2^{-3k-3} < 6^{-k-1} - 6^{-k-2}.$$

Using the bounds for $|\alpha_{n+k}|$, this yields the desired estimates for $|\alpha_{n+k+1}|$. The inductive proof of the inequalities (7) and Step 2 are now concluded.

Note that we have shown in (8) that

$$|\alpha_m - \alpha_{m+1}| \leqslant 2^{3m+13} M a^{N(m)},$$

for all $m \geqslant n$. This implies that $\alpha_m \to \alpha \in \mathbb{C}$, and for $m \geqslant n$

$$|\alpha - \alpha_m| \leqslant 2^{13} M \sum_{j=m}^{\infty} 2^{3j} a^{N(j)} \leqslant 2^{13} M a^{N(m)/2} \sum_{j=m}^{\infty} 2^{3j} a^{N(j)/2}.$$

Hence

$$|\alpha - \alpha_m| \leqslant B a^{N(m)/2}, \quad B = 2^{13} M \sum_{j=0}^{\infty} 2^{3j} a^{N(j)/2}. \tag{9}$$

Let $Q(z) = \alpha z - 1$. Lemma 3.3 implies that $|\alpha| \geqslant 1$. If $|\alpha| > 1$ then by Lemma 3.2 f is a polynomial, which is in contradiction to $\rho = 1$. Thus $|\alpha| = 1$.

Step 3. We let

$$P(z) = Q(z) f(z) = \sum_{k \geqslant 0} c_k z^k,$$

and show that P is a polynomial to conclude the proof of the theorem.

Note that

$$P(z) - P_m(z) = Q_m(z) f(z) - P_m(z) + (\alpha - \alpha_m) z f(z).$$

It follows, using (5)–(7) and (9), that

$$\begin{aligned} |c_{m+1}| &\leqslant |\alpha_m f_m - f_{m+1}| + |\alpha - \alpha_m||f_m| \\ &\leqslant M\big(2^{2n+6} + 1\big) a^{N(m)} + 2^m B a^{N(m)/2}, \end{aligned}$$

for all $m \geqslant n$. This implies that $|c_m|^{1/m} \to 0$, hence P is an entire function.

Observe that $Q_m(z)P(z) - (\alpha z - 1)P_m(z)$ has $N(m)$ zeros in $\overline{\Delta}_r$. Since P is entire, it follows by Lemma 3.2 that P is in fact a polynomial. So $f = P/Q$, and Q does not divide P since f is not entire. This completes the proof.

Theorem 1.2 has the following corollary, which is proved exactly as Corollary 2.3.

Corollary 3.4. *Let $\{n_k\}_{k \geqslant 0}$ be an increasing sequence of natural numbers such that $n_{k+1}/n_k \leqslant C$ for some constant C. Let $f \in O(\Delta)$ and $K = \overline{\Delta}_r$, where $r < 1$. If $\lim_{k \to \infty} N_K(n_k, 1)/n_k = \infty$, then either f is entire or $f = P/Q$, where P, Q are polynomials,* $\deg Q = 1$ *and Q does not divide P.*

We conclude this section with a remark about Padé overinterpolation. Let f be a germ of a holomorphic function at the origin. A rational function $R \in \mathcal{R}_{nm}$ is called a Padé interpolator (or Padé approximant) of type (m,n) of f if $f - R$ has a zero of the highest possible order at the origin, i.e. of order $N_K(n,m)$, where $K = \{0\}$. We prove the following simple fact about overinterpolation in the mth row of the Padé table.

Proposition 3.5. *Let f be a holomorphic germ at the origin and $m \in \mathbb{N}$. If, for all $n \geqslant k$, there exist functions $R_n \in \mathcal{R}_{nm}$ so that $f - R_n$ vanishes to order at least $n+m+2$ at the origin, then $f \in \mathcal{R}_{km}$.*

Proof. Let us write $R_n = P_n/Q_n$, where $P_n \in \mathcal{P}_n$ and $Q_n \in \mathcal{P}_m$, $Q_n \neq 0$. For $n \geqslant k$ the function $R_n - R_{n+1}$ vanishes to order at least $n+m+2$ at the origin. Since $\deg(P_n Q_{n+1} - P_{n+1} Q_n) \leqslant n+m+1$, this implies $R_n = R_{n+1} = R \in \mathcal{R}_{km}$, for $n \geqslant k$. It follows that $f = R$. $\square$

4. Overinterpolation and overconvergence

Throughout this section we assume that $f \in O(\overline{\Delta})$ and that $0 < r < 1$ is fixed. For a compact set $E \subset \mathbb{C}$ and a continuous complex-valued function g on E, we denote by $\|g\|_E$ the uniform norm of g on E.

The following theorem shows that, in the presence of overinterpolation, the functions R_{nm} quickly approximate f on some circle $S_t = \{z \in \mathbb{C}\colon |z| = t\}$.

Theorem 4.1. *Let $m(n) \in \mathbb{N}$, and $d_n > 0$ be so that $\sum d_n$ converges. Suppose that for every n there are polynomials $P_n \in \mathcal{P}_n$ and $Q_{m(n)} \in \mathcal{P}_{m(n)}$, $Q_{m(n)} \neq 0$, so that the function $Q_{m(n)} f - P_n$ has $N(n)$ zeros in $\overline{\Delta}_r$. There exist positive constants $b < 1$, c, depending only on r, and $t \in [r, (1+r)/2]$, such that*

$$\|f - R_n\|_{S_t} \leqslant M\left(\frac{c}{d_n}\right)^{m(n)} b^{N(n)}$$

holds for all n sufficiently large, where $R_n = P_n/Q_{m(n)}$ and $M = M(1, f)$.

Proof. We may assume that $M(r/2, Q_{m(n)}) = 1$. Following [3], we define the nth diameter of a set $G \subset \mathbb{C}$ by

$$\operatorname{diam}_n(G) = \inf\left\{r_1 + \cdots + r_k\colon k \leqslant n,\ G \subset \bigcup_{j=1}^{k} C_j(r_j)\right\},$$

where $C_j(r_j)$ are closed disks of radii $r_j > 0$. If $H_n(z) = Q_{m(n)}(rz/2)$ then by Lemma 3.3 from [3], for every $0 < h \leqslant 1/(8e)$, the nth diameter of the set

$$G' = \left\{z \in \mathbb{C}\colon \left|H_n(z)\right| \leqslant \left(\frac{hr^2|z|}{(1+r)^2}\right)^{m(n)},\ 2 \leqslant |z| \leqslant \frac{1+r}{r}\right\}$$

does not exceed $36eh$. Hence the nth diameter of the set

$$G = \left\{z \in \mathbb{C}\colon \left|Q_{m(n)}(z)\right| \leqslant \left(\frac{2hr|z|}{(1+r)^2}\right)^{m(n)},\ r \leqslant |z| \leqslant \frac{1+r}{2}\right\}$$

does not exceed $18ehr$. This means that the measure of the set

$$F_n = \left\{ t \in \left[r, \frac{1+r}{2} \right] \colon \left| Q_{m(n)}(z) \right| \geqslant \left(\frac{2hr|z|}{(1+r)^2} \right)^{m(n)}, \ \forall z \in S_t \right\}$$

is at least $(1-r)/2 - 36ehr$.

Since $M(r/2, Q_{m(n)}) = 1$, the classical Bernstein–Walsh inequality implies that

$$M(1, Q_{m(n)}) \leqslant \left(\frac{2}{r} \right)^{m(n)}.$$

If $t \in F_n$ then by (1) we have

$$M(t, Q_{m(n)} f - P_n) \leqslant M \left(\frac{2}{r} \right)^{m(n)} \left(\frac{3}{1-t} \right)^{m(n)+2} a_1^{N(n)},$$

where

$$a_1 = a_1(t) = \frac{12t^2 + 6t}{13t^2 + 4t + 1} < 1.$$

The function $a_1(t)$ is increasing on [0, 1] and, therefore, it does not exceed

$$b = a_1 \left(\frac{1+r}{2} \right)$$

on F_n. Hence for $t \in F_n$ we have

$$\| f - R_n \|_{S_t} \leqslant \frac{9M}{(1-t)^2} \left(\frac{3(1+r)^2}{hr^2 t(1-t)} \right)^{m(n)} b^{N(n)} \leqslant M \left(\frac{c_1}{h} \right)^{m(n)} b^{N(n)},$$

where c_1 is a constant depending only on r.

If we let $h = d_n/(36er)$ then the measure of F_n is at least $(1-r)/2 - d_n$ and for $t \in F_n$ we have

$$\| f - R_n \|_{S_t} \leqslant M \left(\frac{c}{d_n} \right)^{m(n)} b^{N(n)},$$

where $c = 36erc_1$. Since $\sum d_n < \infty$ there is n_0 such that the set $F = \bigcap_{n=n_0}^{\infty} F_n$ is not empty. If $t \in F$ then the conclusion of the theorem holds for t and for all $n \geqslant n_0$. □

If g is a continuous function on a compact set $E \subset \mathbb{C}$, we let

$$\rho(n, m) = \inf \| g - R \|_E,$$

where the infimum is taken over all $R \in \mathcal{R}_{nm}$. We say that rational functions *overconverge* to g on E if

$$\lim_{n \to \infty} \rho\big(n, m(n)\big)^{1/n} = 0,$$

for some sequence $m(n) \in \mathbb{N}$.

The following corollary shows that, under suitable conditions, overinterpolation implies overconvergence.

Corollary 4.2. *Under the assumptions of Theorem* 4.1*, suppose that there is a sequence* $\{a_n\}$ *of positive numbers converging to* 0 *such that*

$$\sum_{n=1}^{\infty} \frac{b^{N(n)/m(n)}}{a_n^{n/m(n)}} < \infty.$$

Then there exists $t \in [r, (1+r)/2]$ *for which*

$$\lim_{n\to\infty} \|f - R_n\|_{S_t}^{1/n} = 0.$$

Proof. For the proof, take $d_n = cb^{N(n)/m(n)}/a_n^{n/m(n)}$. □

The fact that overinterpolation implies overconvergence allows us to use results of Gonchar and Chirka to prove other results about overinterpolation. Let us first recall some definitions from [1]. The class $\mathcal{R}_{n,(m)}$ consists of all rational functions of degree at most n and with at most m geometrically distinct poles. The class $\mathcal{A}_m^0$ consists of all functions meromorphic on $\mathbb{P}^1$ except for at most m singularities of finite order. This means that for every singular point a there is a number p such that $|f(z)| < \exp(1/|z-a|^p)$ near a.

Theorem 4.3. *Let* $m \geqslant 0$ *be an integer. If for every* n *there are functions* $R_n \in \mathcal{R}_{nm}$ *such that* $f - R_n$ *has* $N(n)$ *zeros in* $\overline{\Delta}_r$*, where* $N(n)/n \to \infty$*, then* f *extends to a meromorphic function on* $\mathbb{C}$ *with at most* m *poles.*

If the functions $R_n \in \mathcal{R}_{n,(m)}$ *and*

$$\liminf_{n\to\infty} \frac{N(n)}{n\log n} > -\frac{1}{\log b},$$

where b *is the constant from Theorem* 4.1*, then* f *has an extension in* $\mathcal{A}_m^0$.

Proof. To prove the first statement, we take a number α such that $b < \alpha < 1$ and let $a_n = \alpha^{N(n)/n}$. By Corollary 4.2, there is $t \in [r, (1+r)/2]$ for which

$$\lim_{n\to\infty} \|f - R_n\|_{S_t}^{1/n} = 0.$$

By Theorem 1 from [6], f extends to a meromorphic function to $\mathbb{C}$ with at most m poles.

A result of Chirka and Gonchar (see [1, Theorem 1]) states that if f is analytic in a neighborhood of a compact set $E \subset \mathbb{C}$ of positive capacity then f has an extension in $\mathcal{A}_m^0$ if and only if there are a sequence of rational functions $R_n \in \mathcal{R}_{n,(m)}$ and a number $\lambda > 0$ such that

$$\|f - R_n\|_E^{1/n} < \frac{1}{n^\lambda}$$

for all n sufficiently large. (The theorem is stated for $E = \overline{\Delta}_s$, but see the note after the statement.)

Take numbers α and λ such that

$$\liminf_{n\to\infty} \frac{N(n)}{n\log n} > \alpha > -\frac{1}{\log b}, \quad 0 < \lambda < -1 - \alpha\log b.$$

Let

$$d_n = cb^{\alpha\log n} n^\lambda = cn^{\alpha\log b + \lambda},$$

where c is the constant from Theorem 4.1. Then $\sum d_n < \infty$ and by Theorem 4.1 there is $t \in [r, (1+r)/2]$ such that

$$\|f - R_n\|_{S_t}^{1/n} \leqslant M^{1/n} \frac{c}{d_n} b^{N(n)/n} = M^{1/n} \frac{b^{N(n)/n - \alpha \log n}}{n^\lambda} < \frac{1}{n^\lambda},$$

for all n sufficiently large. Now the second statement of the theorem follows from the result of Chirka and Gonchar mentioned above. □

5. Interpolation by algebraic functions

Proposition 5.1. *Let S be an infinite set in $\mathbb{C}^2$ with the following property: There exist positive constants $A \geqslant 1$ and $\alpha < 2$ such that*

$$|S \cap X| \leqslant A(\deg X)^\alpha,$$

for any algebraic curve $X \subset \mathbb{C}^2$ not containing S. Then $\alpha \geqslant 1$ and S is contained in an irreducible algebraic curve of degree at most $(2A)^{1/(2-\alpha)}$. Moreover,

$$|S \cap X| \leqslant (2A)^{1/(2-\alpha)} \deg X,$$

for any algebraic curve $X \subset \mathbb{C}^2$ not containing S.

Proof. Suppose $\alpha < 1$. Assume that $\{z_1, \dots, z_n\} \subseteq S$, where $n \geqslant 2$, and let L_j, $1 \leqslant j < n$, be a complex line passing through z_j and not containing z_n. If $X = L_1 \cup \dots \cup L_{n-1}$, then X does not contain S, hence

$$n - 1 \leqslant |S \cap X| \leqslant A(n-1)^\alpha.$$

Thus $|S| \leqslant 1 + A^{1/(1-\alpha)}$, which is a contradiction.

Let k denote the greatest integer in $x = (2A)^{1/(2-\alpha)}$. Then

$$2Ak^\alpha = x^{2-\alpha} k^\alpha < (k+1)^2 \leqslant k^2 + 3k.$$

Note that the dimension of the space of polynomials in $\mathbb{C}^2$ of degree at most n is $(n+1)(n+2)/2$. Therefore there exists a curve C of degree at most k so that

$$|S \cap C| \geqslant \big(k^2 + 3k\big)/2 > Ak^\alpha. \tag{10}$$

It follows that $S \subseteq C$. Assume that $C = C_1 \cup \dots \cup C_m$, where C_j is an irreducible algebraic curve of degree k_j, $k_1 + \dots + k_m \leqslant k$. If no curve C_j contains S then, since $\alpha \geqslant 1$,

$$|S \cap C| \leqslant \sum_{j=1}^m |S \cap C_j| \leqslant A \sum_{j=1}^m k_j^\alpha \leqslant Ak^\alpha,$$

which contradicts (10). We conclude that S is contained in an irreducible curve Γ of degree at most k. Hence by Bezout's theorem,

$$|S \cap X| \leqslant |\Gamma \cap X| \leqslant (2A)^{1/(2-\alpha)} \deg X,$$

for any algebraic curve $X \subset \mathbb{C}^2$ not containing S. □

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