Structure of the set of bounded solutions for a class of nonautonomous second-order differential equations

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Abstract
Some precise descriptions of the structure and behavior of bounded solutions for a class of nonautonomous second-order differential equations are obtained. The nonautonomous differential equations here are in the form \( x'' = f(\theta, x, x') \), where the driving system \( \theta \) is defined in a compact base space \( H \) and the vector field \( f(h, x, p) \) is strictly increasing in state variable \( x \) on some interval \((a, b)\). The stability of the structure of bounded solutions with respect to parameter perturbations in \( f \) is also addressed.

1. Introduction
Nonautonomous differential equations arise naturally as mathematical models for complex evolutionary systems in engineering and sciences [8,13,15,18]. These equations have coefficients depending on time explicitly. We discuss bounded solutions for a class of such nonautonomous second-order differential equations. Specifically, we investigate the structure and stability of the set of all bounded solutions.
Let \((H, d)\) be a compact metric space and \(\theta = \{\theta_t\}_{t \in \mathbb{R}}\) be a dynamical system on \(H\). Namely, \(\theta\) is a family of mappings on \(H\) that satisfies the following group property

\[ \theta_0 = \text{id}, \quad \theta_{t+s} = \theta_t \circ \theta_s \quad (\forall t, s \in \mathbb{R}) \]

with \(\theta_th\) being continuous in \((t, h)\). We assume that \(H\) is minimal with respect to \(\theta\), that is, \(\theta\) has no proper closed invariant subset in \(H\).

Let \(f \in C(H \times (a, b) \times \mathbb{R})\). Consider the following nonautonomous differential equation

\[ x'' = f(\theta_th, x, x') \quad (1.1) \]

for \(h \in H\). This model covers the special cases when the vector filed \(f\) is periodic, quasiperiodic and almost periodic in time; see Section 6 below. We are interested in the structure and stability of the set of bounded solutions of this equation. This consideration is motivated by the works of Campos and Torres [3], Cieutat [6], Martínez-Amores and Torres [14] as well as our own previous works [9,10].

In [3,14], the authors have obtained a result about the dynamics of the following periodic Liénard type equation:

\[ x'' + g(x)x' + f(x) = h(t) \quad (1.2) \]

when \(g(x) \geq 0, f(x)\) is strictly decreasing on some interval \((a, b)\), and \(h(t)\) is periodic. It is shown that if the equation has a bounded solution on \(\mathbb{R}^+ := [0, \infty)\), then it has a unique periodic solution, and the set of bounded solutions on \(\mathbb{R}^+\) is homeomorphic to the graph of a continuous decreasing function \(\Phi(x)\) defined on a nonempty open interval \(I \subset (a, b)\). Moreover, each bounded solution \(x(t)\) on \(\mathbb{R}^+\) tends to the periodic orbit as \(t \to +\infty\). This result has been extended to almost periodic case in a recent paper of Cieutat [6]. However, since the approaches in these papers rely heavily on the special structure of the equation, we find it difficult to apply them to deal with other types of equations such as the one in Example 6.1, Section 6, in the present paper. We also remark that the approach in [3,14], which makes use of some topological methods such as free homeomorphisms [1] and the Massera’s convergence theorem due to Smith [16], also depends heavily on the periodicity of the equation.

In this present paper we establish some results on structure of the set of bounded solutions for (1.1), by developing some techniques inspired by our earlier works [9,10] and [13], combined with the basic theory of pullback attractors for cocycle dynamical systems (or skew-product flows). We also establish some stability results concerning the structure of bounded solutions.

Now let us give a detailed description of our results. Throughout the paper we assume that \(f\) satisfies the following structure conditions:

(F0) **Lipschitz condition:** For any compact subset \(V \subset (a,b) \times \mathbb{R}\), there exists an \(L > 0\) such that

\[ |f(h, x, p) - f(h, y, q)| \leq L(|x - y| + |p - q|), \quad \forall (x, p), (y, q) \in V, \ h \in H. \]

(F1) **Monotonicity condition:** \(f(h, x, p)\) is strictly increasing in \(x\).

(F2) **Growth condition:** For any compact interval \(I \subset (a,b)\), there exists \(c_0 > 0\) such that

\[ |f(h, x, p)| \leq c_0(1 + |p|^2), \quad \forall (h, x, p) \in H \times I \times \mathbb{R}. \quad (1.3) \]

For \(h \in H\) and \((x_0, x_1) \in (a,b) \times \mathbb{R}\) given, we denote by \(\psi_h(t; x_0, x_1)\) the unique solution \(x(t)\) of (1.1) with initial values \(x(0) = x_0\) and \(x'(0) = x_1\). By the basic theory of ordinary differential equations (ODEs) with parameters we know that for any \(t\) fixed, \(\psi_h(t; x_0, x_1)\) is continuous in \((h, x_0, x_1)\).

We denote by \([0, T^+]\) the right maximal existence interval for any solution \(\psi_h(t; x_0, x_1)\). Let \(J\) be an interval. By a **bounded solution** on \(J\) we mean a solution \(x(t)\) of (1.1) satisfying

\[ a < c \leq x(t) \leq d < b, \quad \forall t \in J. \]
We will first prove the following result.

**Theorem 1.1 (Structure theorem).** Assume (F0)–(F2). Suppose that for some \( h = h_0 \in H \) Eq. (1.1) has a bounded solution \( w \) on \( \mathbb{R}^+ \). Then:

1. There exists a \( \Gamma \in C(H) \) such that for each \( h \in H \), \( \gamma_h(t) := \Gamma(\theta_t h) \) is the unique bounded solution of (1.1) on \( \mathbb{R}^+ \).
2. For each \( h \in H \), there exists a continuous decreasing function \( \Phi_h \) defined on a maximal nonempty open interval \( D(\Phi_h) \subset (a, b) \), such that for any \( x \in D(\Phi_h) \), \( x(t) := \psi_h(t; \theta, \Phi_h(x)) \) is the unique bounded solution of (1.1) on \( \mathbb{R}^+ \) that satisfies \( x(0) = x \).
3. For any compact interval \( I \subset D(\Phi_h) \),

\[
\lim_{t \to +\infty} \left( |\psi_h(t; x, \Phi_h(x)) - \gamma_h(t)| + |\psi_h'(t; x, \Phi_h(x)) - \gamma_h'(t)| \right) = 0
\]

uniformly with respect to \( x \in I \).

**Remark 1.2.** In the case \( (a, b) = \mathbb{R} \), we have \( D(\Phi_h) = \mathbb{R} \); see Section 3.

It is also interesting to note that by the representation \( \gamma_h(t) = \Gamma(\theta_t h) \), the unique bounded solution \( \gamma_h \) of (1.1) synchronize with the motion \( \theta_t h \) of the driving system \( \theta \). In particular, if \( \theta_t h \) is translation compact (resp. almost periodic, quasiperiodic, periodic), then so is \( \gamma_h \).

For convenience we will call the function \( \Gamma \) (resp. \( \Phi_h \)) in Theorem 1.1 the structure function of bounded solutions of (1.1) on \( \mathbb{R} \) (resp. \( \mathbb{R}^+ \)).

As in autonomous case, a major issue in the theory of nonautonomous dynamical systems concerns the stability or persistence of certain dynamical properties under various kinds of perturbations. Here we are interested in the stability of the structure of bounded solutions of (1.1) with respect to parameter perturbations in \( f \). So we consider the following system with parameter \( \lambda \in \Lambda \):

\[
\dot{x}'' = f_\lambda(\theta_t h, x, x'),
\]

where \( \Lambda \) is a metric space with metric \( \rho(\cdot, \cdot) \). We make the following assumptions:

(H1) For each \( \lambda \) fixed, \( f := f_\lambda \) satisfies (F0) and (F1).
(H2) For any compact interval \( I \subset (a, b) \), there exists a \( c_0 > 0 \) independent of \( \lambda \) in \( \Lambda \) such that (1.3) holds for all \( f := f_\lambda \).
(H3) \( f_\lambda \) satisfies the continuity property at \( \lambda_0 \): for any compact set \( V \subset (a, b) \times \mathbb{R} \) and \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[
|f_\lambda(h, x, p) - f_{\lambda_0}(h, x, p)| < \varepsilon, \quad \forall (h, x, p) \in H \times V, \quad \rho(\lambda, \lambda_0) < \delta.
\]

We denote by \( \Gamma^\lambda \) and \( \Phi_\lambda^h \) the corresponding structure functions of bounded solutions of (1.4)\(\lambda\). Then we have the following result.

**Theorem 1.3 (Stability theorem).** Assume that (H1)–(H3) hold. Suppose that for some \( h = h_0 \in H \), Eq. (1.4)\(\lambda_0\) has a bounded solution \( w \) on \( \mathbb{R}^+ \). Then there exists a \( \eta > 0 \) such that when \( \rho(\lambda, \lambda_0) < \eta \), (1.4)\(\lambda\) has a bounded solution. Moreover, we have

1. \( \lim_{\lambda \to \lambda_0} \|\Gamma^\lambda - \Gamma^{\lambda_0}\|_{C(H)} = 0 \);
2. for each \( h \in H \) and compact interval \( I \subset D(\Phi_\lambda^{\lambda_0}) \),

\[
\lim_{\lambda \to \lambda_0} \|\Phi_\lambda^h - \Phi_\lambda^{\lambda_0}\|_{C(I)} = 0.
\]
Remark 1.4. As we will see in Section 6, there are situations where one can find an $h_0 \in H$ such that
\[
\{ \theta h_0 \mid t \in \mathbb{R} \} = H,
\]
while the base space $H$ may fail to be minimal. When this occurs, if instead of assuming the existence of a bounded solution on $\mathbb{R}^+$ we suppose that for $h = h_0$ Eq. (1.1) (resp. (1.4)$_{\lambda_0}$) has a bounded solution $w$ on $\mathbb{R}$, then Theorems 1.1 and 1.3 remain valid. Indeed, under this assumption one easily sees that the conclusion in Lemma 3.5 still holds, where the minimality of $H$ was used.

We allow $(a, b) \neq \mathbb{R}$. Therefore the results can be applied to describe local dynamics of nonautonomous differential equations, as we will see in Example 6.1. It can also be applied to describe the dynamical behavior of some types of equations with singular nonlinearities as
\[
x'' + cx' + 1/x^\alpha = h(t),
\]
where the nonlinearity $1/x^\alpha$ ($\alpha > 0$) relates to electrostatic or gravitational forces. Some investigations on the existence of periodic solutions of this equation can be found in [8,11].

This paper is organized as follows. In Section 2, we give some auxiliary results. In Sections 3 and 4 we prove the main results in case $(a, b) = \mathbb{R}$. Section 5 is devoted to the proof of the results in the general case. Section 6 consists of some applications.

2. Some auxiliary results

This section is concerned with some auxiliary results.

**Lemma 2.1.** Let $I = (\tau_0, \tau_1)$, where $-\infty \leq \tau_0 < \tau_1 \leq +\infty$. Suppose that $x \in C^2(I)$ satisfies $\|x\|_{C(I)} \leq M_0 < \infty$ and that
\[
|x'(t)| \leq c_0 (1 + |x'(t)|^2), \quad t \in I.
\]
Then there exists $M_1 > 0$, depending only on $c_0, M_0$ and the lower bounds of $\tau_1 - \tau_0$, such that $\|x'\|_{C(I)} \leq M_1$.

**Proof.** It is a particular case of Lemma 5.1 in [7, Chapter XII].

**Lemma 2.2.** Let $x(t)$ be a solution of (1.1) on $I = (\tau_0, \tau_1)$. Assume that
\[
a < c \leq x(t) \leq d < b, \quad \forall t \in I.
\]
Then there exists an $M > 0$ that depends only on $c, d$ and the lower bounds of $\tau_1 - \tau_0$ such that $\|x\|_{C^2(I)} \leq M$.

**Proof.** Assume (2.1) holds. Then by (F2) we see that there exists a $c_0 > 0$ such that
\[
|x''(t)| = |f(\theta_{\lambda}h, x(t), x'(t))| \leq c_0 (1 + |x'(t)|^2), \quad \forall t \in I.
\]
Now the conclusion follows immediately from Lemma 2.1.

**Lemma 2.3.** Assume that $\varepsilon > 0$, $M > 0$, and $T > 4M/\varepsilon + 2$. Then for any monotone function $x \in C^2([0, T])$ with $\|x\|_{C^2([0, T])} \leq M$, there is a $t \in [0, T]$ such that
\[
|x'(t)| < \varepsilon, \quad |x''(t)| < \varepsilon.
\]
Proof. We may assume that \( x \) is nonincreasing, and hence \( x'(t) \leq 0 \) for all \( t \in [0, T] \).
Let \( \Omega = \{ t \in [0, T] \mid x'(t) > -\varepsilon \} \). By \( \| x \|_{C([0,T])} \leq M \) one easily checks that
\[
|\Omega| > T - 2M/\varepsilon > 2M/\varepsilon + 2,
\]
where \( |\Omega| \) denotes the Lebesgue measure of \( \Omega \). Since \( \Omega \) is open in \([0, T]\), it consists of at most countably many disjoint intervals \( I_n \) (1 \( \leq n \leq N \leq +\infty \)), i.e., \( \Omega = \bigcup_{1 \leq n \leq N} I_n \).

Case 1. \( N \geq 3 \). In this case one can find at least one open interval \( I_n := (a_n, b_n) \) with \( 0 \leq a_n < b_n < T \). Note that we necessarily have \( x'(a_n) = x'(b_n) = -\varepsilon \). Thus there exists a \( t \in (a_n, b_n) \) such that \( x''(t) = 0 \), and (2.3) follows.

Case 2. \( N \leq 2 \). When this occurs there is at least an interval \( I_n := (a_n, b_n) \) such that
\[
b_n - a_n \geq \frac{1}{2} |\Omega| > M/\varepsilon + 1.
\]
If \( x''(t) \) changes sign on \( I_n \), then we can pick a \( t \in I_n \) such that \( x''(t) = 0 \) and complete the proof of (2.3). So we assume \( x''(t) \) does not change sign on \( I_n \).

(i) \( x''(t) \geq 0 \) on \( I_n \). In such a case if \( x''(t) \geq \varepsilon \) for all \( t \in I_n \), then
\[
0 \geq x'(b_n) = x'(a_n) + \int_{a_n}^{b_n} x''(t) \, dt \geq -\varepsilon + \varepsilon (b_n - a_n) > 0,
\]
a contradiction! Therefore there is at least a \( t \in I_n \) such that \( x''(t) < \varepsilon \).

(ii) \( x''(t) \leq 0 \) on \( I_n \). We recall that \( |x'(t)| \leq M \) for all \( t \in [0, T] \). As in (i), if \( x''(t) \leq -\varepsilon \) for all \( t \in I_n \), then
\[
-M \leq x'(b_n) = x'(a_n) + \int_{a_n}^{b_n} x''(t) \, dt < -\varepsilon (b_n - a_n) \leq -\varepsilon (M/\varepsilon + 1) < -M,
\]
which is again a contradiction! Thus there is at least a \( t \in I_n \) such that \( x''(t) > -\varepsilon \).

The proof is complete. \( \square \)

Lemma 2.4. Assume that \( x \in C^2(\mathbb{R}^+) \) and satisfies:
\[
\lim_{t \to +\infty} x(t) = 0; \quad \| x \|_{C^2(\mathbb{R}^+)} < +\infty.
\]
Then there exists a sequence \( t_n \to +\infty \) such that \( |x'(t_n)| + |x''(t_n)| \to 0 \) as \( n \to +\infty \).

Proof. For each positive integer \( n \) define a function \( x_n \) on \([0, 1]\) as
\[
x_n(t) = x(n + t), \quad \text{for } t \in [0, 1].
\]
Clearly the sequence \( x_n \) is bounded in \( C^2([0, 1]) \). Thus there exists a subsequence (still denoted by \( x_n \)) that converges in \( C^1([0, 1]) \). Since \( \lim_{t \to +\infty} x(t) = 0 \), we necessarily have \( \| x_n \|_{C^1([0,1])} \to 0 \) as \( n \to +\infty \).

Now as in Lemma 2.3 one easily picks a sequence \( s_n \in [0, 1] \) such that \( x_n''(s_n) \to 0 \), according to whether \( x_n'' \) changes its sign on \([0, 1]\). Let \( t_n = n + s_n \). Then the sequence \( t_n \) has the desired property. \( \square \)
3. Proof of Theorem 1.1 in the case \((a, b) = \mathbb{R}\)

In this section we give a detailed proof for Theorem 1.1 in case \((a, b) = \mathbb{R}\). We split the argument into several lemmas and remarks.

**Lemma 3.1.** For \(h \in H\) given, let \(u, v \in C^2(\mathbb{R}^+)\) be a lower and upper solution of (1.1) on \(\mathbb{R}^+\), respectively, namely,

\[ u'' \geq f(\theta_t h, u, u'), \quad v'' \leq f(\theta_t h, v, v'), \quad t \geq 0. \]

Assume that \(\|u\|_{C^1(\mathbb{R}^+)}\), \(\|v\|_{C^1(\mathbb{R}^+)} < +\infty\). Let \(x\) be a solution of (1.1) with the right maximal existence interval \([0, T^+)\). Suppose that there exists a \(t_0 \in [0, T^+)\) such that

\[ x(t_0) \geq v(t_0), \quad x'(t_0) > v'(t_0) \quad \text{(resp.} x(t_0) \leq u(t_0), \ x'(t_0) < u'(t_0)\text{)}. \]

Then

\[ \lim_{t \to T^+} x(t) = +\infty \quad \text{(resp.} \lim_{t \to T^+} x(t) = -\infty\text{)}. \quad (3.1) \]

**Proof.** We only consider the case where \(x(t_0) \geq v(t_0)\) and \(x'(t_0) > v'(t_0)\).

First by maximum principle one easily checks that \(x(t) - v(t)\) is strictly increasing on \([t_0, T^+)\). So to prove (3.1), by boundedness of \(v\) it suffices to verify \(\limsup_{t \to T^+} x(t) = +\infty\).

We argue by contradiction and suppose that \(\limsup_{t \to T^+} x(t) = x^* < +\infty\). Then \(x(t)\) is bounded on \([0, T^+)\). It follows by Lemma 2.2 that \(\|x\|_{C^1([0, T^+])} < +\infty\). The classical Extension Theorem then implies that \(T^* = +\infty\). Now by monotonicity of \(x(t) - v(t)\) on \([t_0, +\infty)\), we see that \(\lim_{t \to +\infty} (x(t) - v(t)) = \delta > 0\) exists. Applying Lemma 2.4 to \(x(t) - v(t) - \delta\), one deduces that there exists a sequence \(t_n \to +\infty\) such that

\[ |x'(t_n) - v'(t_n)| + |x''(t_n) - v''(t_n)| \to 0, \quad \text{as} \ n \to +\infty. \]

Extracting a subsequence if necessary, it can be assumed that

\[ (v(t_n), v'(t_n), v''(t_n)) \to (r, p, q), \quad \text{as} \ n \to +\infty. \]

Consequently

\[ (x(t_n), x'(t_n), x''(t_n)) \to (r + \delta, p, q), \quad \text{as} \ n \to +\infty. \]

One can also assume that \(\theta_{t_n} h \to \hat{h}\). Observe that

\[ x''(t_n) - v''(t_n) \geq f(\theta_{t_n} h, x(t_n), x'(t_n)) - f(\theta_{t_n} h, v(t_n), v'(t_n)). \]

Passing to the limit one finds that

\[ 0 \geq f(\hat{h}, r + \delta, p) - f(\hat{h}, r, p), \]

which clearly contradicts to (F1). \(\square\)

**Lemma 3.2.** Let \(h \in H\) and \(u, v \in C^2(\mathbb{R})\) be as in Lemma 3.1, \(x\) be a bounded solution of (1.1) on \(\mathbb{R}^+\). Then:

1. If \(x(0) \leq v(0)\), then \(x(t) \leq v(t)\) for all \(t \geq 0\).
2. If \(x(0) \geq u(0)\), then \(x(t) \geq u(t)\) for all \(t \geq 0\).
Proof. In case \(x(0) \leq v(0)\), if there is a \(t > 0\) such that \(x(t) > v(t)\), then one can find a \(t_0 \geq 0\) such that \(x(t_0) \geq v(t_0)\) and \(x'(t_0) > v'(t_0)\). It then follows by Lemma 3.1 that \(\lim_{t \to +\infty} x(t) = +\infty\). This contradicts the boundedness of \(x\) and thus proves (1).

A parallel argument applies to prove the validity of (2). \(\square\)

Remark 3.3. As a direct consequence of Lemma 3.2, we conclude that for any \(x_0 \in \mathbb{R}\) and \(h \in H\), there is at most one \(x_1 \in \mathbb{R}\) such that \(\psi_h(t; x_0, x_1)\) is bounded on \(\mathbb{R}^+\).

Remark 3.4. Let \(x\) and \(y\) be bounded solutions of (1.1) on \(\mathbb{R}^+\) with \(x(0) \leq y(0)\). Then \(y(t) - x(t) \geq 0\) and is decreasing on \(\mathbb{R}^+\).

Indeed, by Lemma 3.2 we see that \(x(t) \leq y(t)\) for all \(t \geq 0\). Now if there is a \(t > 0\) such that \(y'(t) > x'(t)\), then by Lemma 3.1, one deduces that \(\lim_{t \to +\infty} y(t) = +\infty\), which yields a contradiction.

Lemma 3.5. Let \(w\) be the bounded solution of (1.1) given in Theorem 1.1. Assume that

\[-\infty < a_1 \leq w(t) \leq b_1 < +\infty, \quad t \geq 0.\]

Then for any \(h \in H\), (1.1) has a bounded solution \(w_h\) on \(\mathbb{R}^+\) with

\[a_1 \leq w_h(t) \leq b_1, \quad t \geq 0.\]  \(\text{(3.2)}\)

Proof. Let \(h \in H\). By minimality of \(H\) we deduce that \(\omega(h_0) = H\), where \(\omega(h_0)\) is the \(\omega\)-limit set of \(h_0\) under the system \(\theta\). Hence there exists a sequence \(t_n \in \mathbb{R}^+\) such that \(t_n h_0 \to h\). Let \(h_n = t_n h_0\). For each \(n\) define \(w_n \in C^2(\mathbb{R}^+)\) as \(w_n(t) = w(t_n + t)\) \((t \geq 0)\). Then \(w_n\) satisfies

\[w''_n = f(\theta t h_n, w_n, w'_n), \quad t \geq 0.\]  \(\text{(3.3)}\)

We also infer from Lemma 2.2 that \(w'(t)\) and \(w''(t)\) are bounded on \(\mathbb{R}^+\). Noting that \(\|w_n\|_{C^2(\mathbb{R}^+)} \leq \|w\|_{C^2(\mathbb{R}^+)}\) for all \(n\), by some standard argument it is easy to show that (1.1) has a bounded solution \(w_h\) on \(\mathbb{R}^+\), which can be obtained by passing to the limit in (3.3) with respect to some appropriate subsequence \(n_k\). \(w_h\) naturally satisfies (3.2). \(\square\)

Lemma 3.6. For any \(x_0 \in \mathbb{R}\) and \(h \in H\), there exists a unique \(x_1 \in \mathbb{R}\) such that the solution \(\psi_h(t; x_0, x_1)\) is a bounded one of (1.1) on \(\mathbb{R}^+\).

Proof. Let \(x_0 \in \mathbb{R}\) and \(h \in H\). Define

\[D^\pm(x_0) = \left\{ y \in \mathbb{R} : \lim_{t \to T^\pm} \psi_h(t; x_0, y) = \pm \infty \right\}.\]

We prove that \(D^\pm(x_0)\) are nonempty open subsets of \(\mathbb{R}\). We only consider \(D^+(x_0)\). We first prove that \(D^+(x_0)\) is open.

Let \(y_1 \in D^+(x_0)\). We claim that there exists a \(t_0 \in [0, T^+)\) such that

\[\psi_h(t_0; x_0, y_1) > w_h(t_0), \quad \psi'_h(t_0; x_0, y_1) > w'_h(t_0),\]

where \(w_h\) is the bounded solution of (1.1) given in Lemma 3.5. Indeed, if this is not the case, then we have \(\psi'_h(t; x_0, y_1) \leq w'_h(t)\) whenever \(\psi_h(t; x_0, y_1) > w_h(t)\), by which and the boundedness of \(w_h\), one concludes that \(\psi_h(t; x_0, y_1)\) is bounded from above, which leads to a contradiction and proves the claim.
Now we infer from the continuity of solutions with respect to initial data that there exists a \( \delta > 0 \) such that when \( |y - y_1| < \delta \), \( \psi_h(t; x_0, y) \) exists on \([0, t_0]\), moreover,

\[
\psi_h(t_0; x_0, y) > w_h(t_0), \quad \psi'_h(t_0; x_0, y) > w'_h(t_0).
\]

It then follows from Lemma 3.1 that \( \limsup_{t \to \tau^+} \psi_h(t; x_0, y) = +\infty \); hence \( y \in D^+(x_0) \).

In the sequel we show that \( D^+(x_0) \) is nonempty. Let

\[
M = \|w_h\|_{C^2(\mathbb{R}^+)}.
\]

Take a \( y_0 \) be such that \( y_0 > \max(x_0, M) \). Fix an \( x_1 \in \mathbb{R} \) and let \( u(t) = \psi_h(t; x_0, x_1) \). Take a \( \tau > 0 \) sufficiently small so that \( u(t) \) exists on \([0, \tau]\) with \( u(t) \leq y_0 \) for \( t \in [0, \tau] \) and

\[
(y_0 - x_0)/\tau > M. \quad (3.4)
\]

Define a cutoff function \( \zeta : [0, \tau] \times \mathbb{R} \to \mathbb{R} \) as

\[
\zeta(t, x) = \begin{cases} 
  y_0, & x > y_0; \\
  x, & u(t) \leq x \leq y_0; \\
  u(t), & x < u(t).
\end{cases}
\]

Let

\[
g(t, x, p) = f(\partial th, \zeta(t, x), p) + (x - \zeta(t, x)), \quad (t, x, p) \in [0, \tau] \times \mathbb{R}^2.
\]

Then \( g \) is strictly increasing in \( x \). Clearly \( g(t, x, 0) \to \pm \infty \) as \( x \to \pm \infty \) uniformly with respect to \( t \in [0, \tau] \). Consider the boundary value problem on \([0, \tau] \):

\[
x'' = g(t, x, x'); \quad x(0) = x_0, \quad x(\tau) = y_0. \quad (3.5)
\]

By very standard argument using the well-known upper and lower solutions method (see, e.g., [12,13, 15]), one can easily prove that (3.5) has a solution \( x \in C^2([0, \tau]) \). Noting that \( u(0) = x_0 \) and \( u(\tau) \leq y_0 \), by comparison we find that

\[
x(t) \geq u(t), \quad \forall t \in [0, \tau].
\]

Let

\[
s = \max\{\tau' \in [0, \tau] : x(t) \leq y_0 \text{ for } t \in [0, \tau']\}.
\]

Since \( x(0) = x_0 < y_0 \), it is clear that \( s > 0 \). By the definition of \( \zeta \) we know that \( x \) is a solution of (1.1) on \([0, s]\).

If \( x_0 \leq w_h(0) \), then since \( x(s) = y_0 > M \geq w_h(s) \), it can be easily seen that there exists a \( t_0 \in [0, s] \) such that

\[
x(t_0) \geq w_h(t_0), \quad x'(t_0) > w'_h(t_0). \quad (3.6)
\]

Assume that \( x_0 > w_h(0) \). Then since \( x(s) > w_h(s) \), by comparison we find that \( x(t) \geq w_h(t) \) for \( t \in [0, s] \). On the other hand, invoking of the classical mean value theorem, we deduce that there exists a \( t_0 \in (0, s) \) such that

\[
x'(t_0) = (y_0 - x_0)/s \geq (y_0 - x_0)/\tau > (\text{by (3.4)}) > M \geq w'_h(t_0).
\]

Thus \( x \) satisfies (3.6) at the point \( t_0 \).
Now we consider the solution \( \psi_h(t; x_0, x'(0)) \) of (1.1). By uniqueness for initial value problems, we have
\[
\psi_h(t; x_0, x'(0)) = x(t), \quad \text{for } t \in [0, s].
\]

Thanks to (3.6) and Lemma 3.1, we see that
\[
\lim_{t \to T^+} \psi_h(t; x_0, x'(0)) = +\infty.
\]

Therefore \( x'(0) \in D^+(x_0) \).

An analogous argument applies to prove that \( D^-(x_0) \) is a nonempty open subset of \( \mathbb{R} \).

Let us now complete the proof of the lemma. Set
\[
DB(x_0) = \{ x_1 \in \mathbb{R} : \psi_h(t; x_0, x_1) \text{ is a bounded solution of (1.1) on } \mathbb{R}^+ \}.
\]

Then
\[
D^+(x_0) \cup D^-(x_0) \cup DB(x_0) = \mathbb{R}. \tag{3.7}
\]

Since \( D^+(x_0) \cap D^-(x_0) = \emptyset \), as a topological consequence one concludes immediately that \( DB(x_0) \) is a nonempty closed subset of \( \mathbb{R} \).

We also infer from Remark 3.3 that \( DB(x_0) \) is a singleton. The proof is complete. \( \square \)

**Lemma 3.7.** For \( h \in H \) we denote by \( x_h \) any bounded solution of (1.1) on \( \mathbb{R}^+ \). Let \( I \) be a compact interval of \( \mathbb{R} \). Then \( \forall \varepsilon > 0 \), there exists a \( T > 0 \) such that
\[
|x(t) - w(t)| + |x'(t) - w'(t)| < \varepsilon, \quad t > T, \tag{3.8}
\]
for any \( h \in H \) and \( x_h \) with \( x_h(0) \in I \), where \( w_h \) is the bounded solution of (1.1) on \( \mathbb{R}^+ \) given in Lemma 3.5.

**Proof.** We only need to consider the case \( x_h(0) \geq w_h(0) \).

By Lemma 3.2 and Remark 3.4, we know that \( x_h(t) \geq w_h(t) \) for all \( t \geq 0 \); moreover, \( x_h(t) - w_h(t) \geq 0 \) and is decreasing on \( \mathbb{R}^+ \). Noting that \( \{ w_h \mid h \in H \} \) is uniformly bounded on \( \mathbb{R}^+ \) (see (3.2)), we deduce that the family
\[
B := \{ x_h \mid x_h(0) \in I, \ x_h(0) \geq w_h(0), \ h \in H \}
\]
is also uniformly bounded on \( \mathbb{R}^+ \). Further we infer from Lemma 2.2 that there is an \( M > 0 \) such that
\[
\|x_h\|_{C^2(\mathbb{R}^+)} \leq M, \quad \|w_h\|_{C^2(\mathbb{R}^+)} \leq M \tag{3.9}
\]
for all \( x_h \in B \) and all \( w_h \).

Let \( \varepsilon > 0 \) be given arbitrary. We first show that there exists a \( T > 0 \) such that
\[
|x(t) - w(t)| < \varepsilon, \quad t > T, \tag{3.10}
\]
for any \( h \in H \) and \( x_h \) in \( B \).

Suppose the contrary. Then there exist an \( \varepsilon_0 > 0 \), a sequence \( h_n \in H \) and a sequence \( t_n \to +\infty \) such that \( \text{recall that } x(t) - w(t) \geq 0 \)
\[
x_h(t_n) - w_h(t_n) \geq \varepsilon_0
\]
for all \( n \). On the other hand, by Lemma 2.3 and the monotonicity of \( x_h - w_h \), for each positive integer \( k \) one can find a \( T_k > 0 \) sufficiently large such that for any \( h \in H \) and \( x_h \), there is a \( t \in [0, T_k] \) such that

\[
|\phi'_h(t) - w'_h(t)| < 1/k, \quad |\phi''_h(t) - w''_h(t)| < 1/k.
\]

For each \( k \), we now choose an \( n_k \) so that \( t_{n_k} \geq T_k \). Then we deduce that there exists an \( s_k \leq t_{n_k} \leq t_{n_k} \) such that

\[
|\phi'_k(s_k) - w'_k(s_k)| < 1/k, \quad |\phi''_k(s_k) - w''_k(s_k)| < 1/k,
\]

where we have rewritten \( x_{h_{n_k}} \) and \( w_{h_{n_k}} \) as \( x_k \) and \( w_k \), respectively. Note that

\[
x_k(s_k) - w_k(s_k) \geq x_k(t_{n_k}) - w_k(t_{n_k}) \geq \varepsilon_0
\]

for all \( k \). We may assume that \( \theta_{s_k} h_{n_k} \to \hat{h} \), and that

\[
\lim_{k \to \infty} w_k(s_k) = r, \quad \lim_{k \to \infty} w'_k(s_k) = p.
\]

Then

\[
\lim_{k \to \infty} x_k(s_k) = r + \delta, \quad \lim_{k \to \infty} x'_k(s_k) = p,
\]

where \( \delta \geq \varepsilon_0 > 0 \). Now passing to the limit in the following equation

\[
x'_k(s_k) - w'_k(s_k) = f (\theta_{s_k} h_{n_k} x_k(s_k), x'_k(s_k)) - f (\theta_{s_k} h_{n_k} w_k(s_k), w'_k(s_k)),
\]

one finds

\[
0 = f (\hat{h}, r + \delta, p) - f (\hat{h}, r, p),
\]

which contradicts to the strict monotonicity of \( f (h, x, p) \) in \( x \). Hence (3.10) holds true.

Since \( x_h(t) - w_h(t) \) is decreasing on \( \mathbb{R}^+ \), we have \( x'_h(t) - w'_h(t) \leq 0 \) for \( t \geq 0 \). Now using (3.10) and the equi-continuity of the family \( \{ x'_h - w'_h \mid h \in H, x_h \in \mathcal{B} \} \) on \( \mathbb{R}^+ \) (see (3.9)), it can be easily shown that for any \( \varepsilon > 0 \), there is a \( T' > 0 \) such that

\[
|x'_h(t) - w'_h(t)| < \varepsilon, \quad t > T',
\]

for any \( h \in H \) and \( x_h \in \mathcal{B} \).

The proof of the lemma is complete. \( \Box \)

By Lemma 3.6, we can define a function \( \Phi_h(x) \) on \( \mathbb{R} \) as follows: for each \( x \in \mathbb{R} \), \( \Phi_h(x) \) denotes the unique \( y \) such that \( \psi_h(t; x, y) \) is a bounded solution of (1.1) on \( \mathbb{R}^+ \). We have

**Lemma 3.8.** \( \Phi_h(x) \) is continuous in \((h, x)\) and decreasing in \( x \).

**Proof.** We first prove that \( \Phi_h(x) \) is decreasing. Suppose not. Then there would exist \( x_1, x_2 \in \mathbb{R} \) such that

\[
x_1 < x_2, \quad \Phi_h(x_1) < \Phi_h(x_2).
\]


Let \( u(t) = \psi_h(t; x_1, \Phi_h(x_1)) \), \( v(t) = \psi_h(t; x_2, \Phi_h(x_2)) \). Since \( \lim_{t \to +\infty} (v(t) - u(t)) = 0 \), in view of (3.12), one easily sees that \( v(t) - u(t) \) has a maximum point \( s > 0 \), at which

\[
v(s) > u(s), \quad v'(s) = u'(s), \quad v''(s) \leq u''(s).
\]

Subtracting the respective equations for \( u \) and \( v \) at \( s \), it yields immediately a contradiction!

Now we examine the continuity property of \( \Phi_h(x) \). Let \( h \in H \) and \( x \in \mathbb{R} \). If \( \Phi_h(x) \) is not continuous at \((h, x)\), then there exist an \( \varepsilon_0 > 0 \) as well as sequences \( h_n \to h \) and \( x_n \to x \) such that

\[
|\Phi_{h_n}(x_n) - \Phi_h(x)| \geq \varepsilon_0, \quad n = 1, 2, \ldots
\]

Let \( x_{h_n}(t) = \psi_{h_n}(t; x_n, \Phi_{h_n}(x_n)) \). As in (3.9) we conclude that there is an \( M > 0 \) such that \( \|x_{h_n}\|_{C^2(\mathbb{R}^+)} \leq M \) for all \( n \). Further by standard argument one deduces that there is a subsequence of \( x_{h_n} \) (still denoted by \( x_{h_n} \)) such that for any \( T > 0, x_{h_n} \) converges in \( C^2([0, T]) \) to a bounded solution \( x_h \in C^2(\mathbb{R}^+) \) of (1.1) on \( \mathbb{R}^+ \). It is clear that

\[
x_h(0) = \lim_{n \to \infty} x_{h_n} = x, \quad x_h'(0) = \lim_{n \to \infty} x_{h_n}'(0) = \lim_{n \to \infty} \Phi_{h_n}(x_n).
\]

On the other hand by uniqueness of bounded solutions and \( x_h(0) = x \) we should also have \( x_h'(0) = \Phi_h(x) \), and thus \( \lim_{n \to \infty} \Phi_{h_n}(x_n) = \Phi_h(x) \). This contradicts to (3.13). \( \square \)

By far we have proved the second conclusion in Theorem 1.1. To prove the first one we define a nonautonomous dynamical system \( \Psi \) on \( \mathbb{R} \) in terms of a cocycle mapping \( \Psi : \mathbb{R}^+ \times H \times \mathbb{R} \to \mathbb{R} \) with driving system \( \theta \) and base space \( H \) as follows:

\[
\Psi(t, h, x) = \psi_h(t; x, \Phi_h(x)), \quad (t, h, x) \in \mathbb{R}^+ \times H \times \mathbb{R}.
\]

By uniqueness of bounded solutions on \( \mathbb{R}^+ \), one easily checks that \( \Psi \) is well defined and satisfies:

1. \( \Psi(0, h, x) = x \) for all \( h \in H \) and \( x \in \mathbb{R} \);
2. \( \Psi(s + t, h, x) = \Psi(s, \theta_t h, \Psi(t, h, x)) \) for all \( s, t \geq 0, h \in H \) and \( x \in \mathbb{R} \);
3. \( \Psi(t, h, x) \) is continuous in \( (t, h, x) \).

For any \( A, B \subset \mathbb{R} \), we denote by \( d_\mathbb{R}(A, B) \) the Hausdorff semi-distance of \( A \) and \( B \),

\[
d_\mathbb{R}(A, B) = \sup_{x \in A} \inf_{y \in B} |x - y|.
\]

Thanks to Lemma 3.7, we see that the interval \( U = [a_1, b_1] \), where \( a_1, b_1 \) are the constants in (3.2), is a uniformly attracting set for \( \Psi \), namely, for any compact interval \( I \),

\[
\lim_{t \to +\infty} \sup_{h \in H} d_\mathbb{R}(\Psi(t, h, I), U) = 0.
\]

Invoking existence results on pullback attractors (see [2,4,5] and [17], etc.), we deduce that \( \Psi \) has a pullback attractor, i.e., a family of nonempty compact sets \( \mathcal{A} = \{A_h\}_{h \in H} \) satisfying:

(A1) \( \Psi(t, h, A_h) = A_{h_0} \) for all \( t \geq 0 \) and \( h \in H \) (nonautonomous invariance);
(A2) \( \lim_{t \to +\infty} d_\mathbb{R}(\Psi(t, \theta_{-t} h, I), A_h) = 0 \) for any compact subset \( I \) of \( \mathbb{R} \) and \( h \in H \) (pullback attraction).

One can also write the nonautonomous invariance property (A1) as

\[
\Psi(t, \theta_{-t} h, A_{h-t}) = A_h, \quad \text{for all } t \geq 0, \ h \in H.
\]
from which it can be easily seen that for any $h \in H$ and $x \in A_h$, there is a bounded solution $x_h$ of (1.1) on $\mathbb{R}$ with $x_h(0) = x$ that lies in $\mathcal{A}$.

We also infer from [2,4] that $A_h$ is upper semi-continuous in $h$, that is,

$$\lim_{h \to h_0} d_R(A_h, A_{h_0}) = 0 \quad (3.14)$$

for any $h_0 \in H$. In what follows we show that for each $h \in H$, $A_h$ is in fact a singleton, i.e.,

$$A_h = \{a_h\}$$

for some $a_h \in \mathbb{R}$. Hence (3.14) reduces to the continuity of $a_h$ in $h$.

Let $x, y \in A_h$. Then there exist two bounded solutions $x_h$ and $y_h$ of (1.1) on $\mathbb{R}$ with $x_h(0) = x$ and $y_h(0) = y$; moreover, both $x_h$ and $y_h$ lie in $\mathcal{A}$. Since $\mathcal{A}$ is contained in $U = [-c, c]$, by Lemma 3.7 we find that

$$|x - y| = |x_h(0) - y_h(0)|$$

$$= |\psi_{\theta, h}(t; x_h(-t), \Phi_{\theta, h}(x_h(-t))) - \psi_{\theta, h}(t; y_h(-t), \Phi_{\theta, h}(y_h(-t)))|$$

$$\leq |\psi_{\theta, h}(t; x_h(-t), \Phi_{\theta, h}(x_h(-t))) - w_{\theta, h}(t)|$$

$$+ |w_{\theta, h}(t) - \psi_{\theta, h}(t; y_h(-t), \Phi_{\theta, h}(y_h(-t)))| \to 0 \quad (\text{as } t \to +\infty).$$

Hence $x = y$.

Now define $\Gamma : H \to \mathbb{R}$ as

$$\Gamma(h) = a_h, \quad \forall h \in H.$$

Then $\Gamma$ is continuous. For each $h \in H$, set

$$\gamma_h(t) = \Gamma(\theta_t h), \quad t \in \mathbb{R}.$$

By invariance property of $\mathcal{A}$ one trivially checks that $\gamma_h$ is precisely the unique bounded solution of (1.1) on $\mathbb{R}$.

Replacing $w_h$ in Lemma 3.7 with $\gamma_h$, we immediately derive the validity of the third conclusion (3) in Theorem 1.1.

The proof of Theorem 1.1 in case $(a, b) = \mathbb{R}$ is complete.

Remark 3.9. We remark that

$$\Gamma(H) \subset \bigcup_{h \in H} A_h \subset U = [a_1, b_1].$$

4. Proof of Theorem 1.3 in the case $(a, b) = \mathbb{R}$

In this section we prove Theorem 1.3 in case $(a, b) = \mathbb{R}$.

Proof. Let $\Gamma^\lambda$ and $\Phi^\lambda_\theta$ be the structure functions for bounded solutions of (1.4).

By virtue of Theorem 1.1 we know that for each $h \in H$ Eq. (1.4)$_{\theta_0}$ has a bounded solution $\gamma_h(t) = \Gamma^\lambda(\theta_t h)$; moreover we infer from Lemma 2.2 that there is an $M > 0$ that only depends on $\|\Gamma^\lambda\|_{C(H)}$ and the structure condition of $f_h$ such that

$$\|\gamma_h\|_{C^2(\mathbb{R})} \leq M, \quad \forall h \in H.$$
Let \( \varepsilon > 0 \) be given arbitrary. Then by (H1) we have

\[
 f_{\lambda_0}(h, x - \varepsilon, p) < f_{\lambda_0}(h, x, p) < f_{\lambda_0}(h, x + \varepsilon, p), \quad \forall h \in H, \quad |x|, |p| \leq M.
\]

Further by continuity, we can find an \( r > 0 \) sufficiently small so that

\[
 f_{\lambda_0}(h, x - \varepsilon, p) + r \leq f_{\lambda_0}(h, x, p) \leq f_{\lambda_0}(h, x + \varepsilon, p) - r, \quad \forall h \in H, \quad |x|, |p| \leq M.
\]

In view of the continuity assumption (H3), we see that there exists an \( \eta > 0 \) such that

\[
 f_{\lambda}(h, x - \varepsilon, p) + \frac{r}{2} \leq f_{\lambda_0}(h, x, p) \leq f_{\lambda}(h, x + \varepsilon, p) - \frac{r}{2}, \quad \forall h \in H, \quad |x|, |p| \leq M.
\]

provided \( \rho(\lambda, \lambda_0) < \eta \).

Assume that \( \rho(\lambda, \lambda_0) < \eta \). Then (4.2) implies that for any \( h \in H \), \( \gamma_h(t) + \varepsilon \) and \( \gamma_h(t) - \varepsilon \) are a bounded upper solution and lower solution of (1.4)\( \lambda \) on \( \mathbb{R} \), respectively. By very standard argument via upper and lower solutions method (see, e.g., [12,13]), it can be easily shown that (1.4)\( \lambda \) has a bounded solution \( \gamma_h^\lambda \) with

\[
 \gamma_h(t) - \varepsilon \leq \gamma_h^\lambda(t) \leq \gamma_h(t) + \varepsilon, \quad \forall t \in \mathbb{R}.
\]

Since \( h \in H \) is arbitrary, we have

\[
 |\Gamma^\lambda(h) - \Gamma^{\lambda_0}(h)| = |\gamma_h^\lambda(0) - \gamma_h(0)| \leq \varepsilon, \quad \forall h \in H.
\]

This proves the first conclusion (1).

Now we proceed to prove the second conclusion (2).

Let \( h \in H \) be fixed, and let \( I = [c, d] \subset \mathbb{R} \) be a compact interval.

We first show that there exist a \( 0 < \delta \leq \eta \) and \( M_0 > 0 \) such that when \( \rho(\lambda, \lambda_0) < \delta \),

\[
 |\psi_h^\lambda(t; x, \Phi_h^\lambda(x))| \leq M_0, \quad \forall t \geq 0, \quad x \in I,
\]

where \( \psi_h^\lambda(t; x, \Phi_h^\lambda(x)) \) denotes the unique bounded solution of (1.4)\( \lambda \) on \( \mathbb{R}^+ \) with initial value \( (x, \Phi_h^\lambda(x)) \). For simplicity we will rewrite here \( \psi_h^\lambda(t; x, \Phi_h^\lambda(x)) \) as \( \psi_h^\lambda(t; x) \) when there is no confusion. By Lemma 3.2 we know that if \( x \leq y \), then \( \psi_h^\lambda(t, x) \leq \psi_h^\lambda(t, y) \) for all \( t \geq 0 \).

Assume that

\[-\infty < \alpha \leq \psi_0^\lambda(t; c) \leq \psi_0^\lambda(t; d) \leq \beta < +\infty, \quad t \geq 0.\]

Then by Lemma 2.2 there exists an \( M_1 > 0 \) such that

\[
 \left|\left(\psi_0^\lambda\right)'(t; c)\right|, \left|\left(\psi_0^\lambda\right)'(t; d)\right| \leq M_1, \quad \forall t \geq 0.
\]

Take a \( \mu > 0 \). We infer from the first conclusion (1) that there exists \( 0 < \delta_1 \leq \eta \) such that

\[
 |\gamma_h^\lambda(t) - \gamma_h(t)| < \mu, \quad \forall t \in \mathbb{R},
\]

provided \( \rho(\lambda, \lambda_0) < \delta_1 \). As in (4.2) we have for some \( r > 0 \) that

\[
 f_{\lambda_0}(h', x + \mu, p) - f_{\lambda_0}(h', x, p) \geq r, \quad \forall h' \in H, \quad x \in [\alpha, \beta], \quad |p| \leq M_1.
\]
Further by continuity we find that for some $\delta_2 > 0$,
\begin{equation}
 f_\lambda(h', x + \mu, p) - f_{\lambda_0}(h', x, p) \geq r/2, \quad \forall h' \in H, \ x \in [\alpha, \beta], \ |p| \leq M_1,
\end{equation}
provided $\rho(\lambda, \lambda_0) < \delta_2$. Similarly there also exists a $\delta_3 > 0$ such that
\begin{equation}
 f_\lambda(h', x - \mu, p) - f_{\lambda_0}(h', x, p) \leq -r/2, \quad \forall h' \in H, \ x \in [\alpha, \beta], \ |p| \leq M_1,
\end{equation}
provided $\rho(\lambda, \lambda_0) < \delta_3$. Set $\delta = \min(\delta_1, \delta_2, \delta_3)$. We claim that when $\rho(\lambda, \lambda_0) < \delta$,
\begin{equation}
 \alpha - \mu \leq \psi^\lambda(t; c) \leq \psi^\lambda(t; d) \leq \beta + \mu, \quad \forall t \geq 0.
\end{equation}

We argue by contradiction and suppose, say, that $\psi^\lambda(t; d) > \beta + \mu$ for some $t > 0$. Let $x(t) = \psi^\lambda(t; d)$, $y(t) = \psi^\lambda_0(t; d)$. Then $x(0) - y(0) = 0$,
\begin{equation*}
 x(t) - y(t) = \psi^\lambda(t; d) - \psi^\lambda_0(t; d) \geq \psi^\lambda(t; d) - \beta > 0.
\end{equation*}

Noting that
\begin{equation*}
 \limsup_{t \to +\infty} |x(t) - y(t)| \leq \limsup_{t \to +\infty} \left( |x(t) - y^\lambda(t)| + |y(t) - y_h(t)| + |y^\lambda(t) - y_h(t)| \right) \\
 \leq (\text{by Theorem 1.1 and (4.5)}) \leq \mu,
\end{equation*}
we deduce that $x(t) - y(t)$ has a maximum point $s \in (0, +\infty)$, at which we have
\begin{equation*}
 x(s) > y(s) + \mu, \quad x'(s) = y'(s) := p, \quad x''(s) \leq y''(s).
\end{equation*}

Now we obtain that
\begin{equation*}
 0 \leq y''(s) - x''(s) = f_{\lambda_0}(\theta_3 h, y(s), p) - f_{\lambda}(\theta_3 h, x(s), p) \\
 < f_{\lambda_0}(\theta_3 h, y(s), p) - f_{\lambda}(\theta_3 h, y(s) + \mu, p) \\
 \leq (\text{by (4.6)}) \leq -r/2,
\end{equation*}
which is a contradiction.

Since $\psi^\lambda(t; c) \leq \psi^\lambda(t; x) \leq \psi^\lambda(t; d)$ for all $t \geq 0$ and $x \in [c, d]$, the validity of (4.4) follows immediately from (4.8).

Thanks to Lemma 2.2, making use of (4.4) and the structure condition (H2), one easily concludes that there exists an $M_2 > 0$ such that
\begin{equation}
 |(\psi^\lambda)'(t; x)| + |(\psi^\lambda)''(t; x)| \leq M_2, \quad \forall t \geq 0, \ x \in I,
\end{equation}
for all $\lambda \in \Lambda$ with $\rho(\lambda, \lambda_0) < \delta$.

We continue to prove the second conclusion (2). Suppose that it fails to be true. Then there would exist sequences $\lambda_n \to \lambda_0$ and $x_n \in I$ such that
\begin{equation}
 |\Phi^\lambda_n(h_0) - \Phi^\lambda_0(h_0) | \geq \varepsilon_0 > 0, \quad \text{for all } n \geq 1.
\end{equation}

Extracting a subsequence if necessary, we can assume that $x_n \to x_0 \in I$. Observing that
\begin{equation*}
 |\Phi^\lambda_n(h_0) | = |(\psi^\lambda_n)'(0; x_n)| \leq (\text{by (4.9)}) \leq M_2.
\end{equation*}
it can also be assumed that \( \lim_{n \to \infty} \Phi_h^{\lambda_n}(x_n) = y_0 \). Note that (4.10) implies

\[
|y_0 - \Phi_h^{\lambda_n}(x_0)| \geq \varepsilon_0. \tag{4.11}
\]

Let \( u_n(t) = \psi^{\lambda_n}(t; x_n) \). Due to (4.4) and (4.9), we can find a subsequence of \( u_n \) (still denoted by \( u_n \)) and a bounded function \( u \in C^1(\mathbb{R}^+) \), such that for any \( T > 0 \), \( \|u_n - u\|_{C^1([0,T])} \to 0 \). We observe that

\[
u' = \lim_{n \to +\infty} \psi^{\lambda_n}'(0; x_n) = \lim_{n \to +\infty} \Phi_h^{\lambda_n}(x_n) = y_0. \tag{4.12}
\]

By standard argument via the equations on \( u_n \) one can also verify that \( u \in C^2(\mathbb{R}^+) \) and solves the equation \( u'' = f_{\lambda_0}(\theta_h t, u, u') \) on \( \mathbb{R}^+ \). Now since \( u(0) = \lim_{n \to +\infty} u_n(0) = \lim_{n \to +\infty} x_n = x_0 \), by uniqueness of bounded solutions we necessarily have \( u'(0) = \Phi_h^{\lambda_0}(x_0) \). However, this contradicts to (4.11) and (4.12).

The proof of the theorem is complete. \( \square \)

5. Proofs of Theorems 1.1 and 1.3 in the general case

In this section we extend the argument above to the general case where \((a, b)\) is a proper subset or bounded subset of \( \mathbb{R} \).

**Proof of Theorem 1.1.** Since \( w \) is a bounded solution on \( \mathbb{R}^+ \), by definition there exist two constants \( a_1 \) and \( b_1 \) with \( a < a_1 \leq b_1 < b \) such that

\[
a_1 \leq w(t) \leq b_1. \quad \forall t > 0.
\]

Fix an \( r > 0 \) sufficiently small so that \( a < a_1 - r \) and \( b_1 + r < b \). Let \( c, d \in \mathbb{R} \) satisfy

\[
a < c \leq a_1 - r, \quad b_1 + r \leq d < b. \tag{5.1}
\]

Take a strictly increasing function \( \zeta \in C^1(\mathbb{R}) \) which satisfies:

\[
\zeta(x) = x, \quad \forall x \in [c, d]; \tag{5.2}
\]

\[
\lim_{x \to -\infty} \zeta(x) = a, \quad \lim_{x \to +\infty} \zeta(x) = b. \tag{5.3}
\]

For \( h \in H \) consider the modified equation:

\[
x'' = f(\theta_h t, \zeta(x), x'). \tag{5.4}
\]

Clearly for \( h = h_0, w \) is a bounded solution of (5.4). By what we have proved, there exists a \( \Gamma \in C(H) \) such that for each \( h \in H \), \( \gamma_h(t) = \Gamma(\theta_h t) \) \( t \in \mathbb{R} \) is a bounded solution of (5.4) on \( \mathbb{R} \). We also infer from Remark 3.9 that \( \Gamma(H) \subset [a_1, b_1] \). Hence \( \gamma_h \) is in fact a bounded solution of (1.1) on \( \mathbb{R} \).

Let \( h \in H \) be fixed. Set \( u(t) = \gamma_h(t) - r \) and \( v(t) = \gamma_h(t) + r \). Then

\[
c \leq u(t) < v(t) \leq d, \quad t \in \mathbb{R}.
\]
By monotonicity property of \( f \) it is trivial to examine that \( u \) and \( v \) are a lower and upper solution of (5.4) on \( \mathbb{R} \), respectively. Let

\[ x \in [u(0), v(0)] = [\gamma_h(0) - r, \gamma_h(0) + r]. \]

By Lemma 3.2 we see that the bounded solution \( x(t) \) of (5.4) on \( \mathbb{R}^+ \) with \( x(0) = x \) necessarily satisfies \( u(t) \leq x(t) \leq v(t) \) on \( \mathbb{R}^+ \), and hence is a bounded solution of (1.1) on \( \mathbb{R}^+ \).

We claim that for any \( x \in (a, b) \), (1.1) has at most one bounded solution \( x(t) \) with \( x(0) = x \); moreover, the bounded solution \( x(t) \), if exists, satisfies

\[ \lim_{t \to +\infty} \left( |x(t) - \gamma_h(t)| + |x'(t) - \gamma'_h(t)| \right) = 0. \tag{5.5} \]

Indeed, if \( x(t) \) and \( y(t) \) are both bounded solutions of (1.1), then we can take \( a < c \leq d < b \) satisfying (5.1) such that

\[ c \leq x(t), \quad y(t) \leq d, \quad \forall t \geq 0. \]

Take a function \( \zeta \in C^1(\mathbb{R}) \) satisfying (5.2)–(5.3) and consider the modified equation (5.4). We see that \( x(t) \) and \( y(t) \) are both bounded solutions of (5.4) on \( \mathbb{R}^+ \). Thus if \( x(0) = y(0) = x \), then by uniqueness of bounded solutions for (5.4) one necessarily has \( x(t) \equiv y(t) \) on \( \mathbb{R}^+ \). (5.5) also follows naturally from what we have proved for (1.1) in case \( (a, b) = \mathbb{R} \).

Now consider the family \( \mathcal{F} \) of open intervals \( I \subset (a, b) \) which is defined as follows:

\( I \in \mathcal{F} \) if and only if \( \gamma_h(0) \in I \), moreover, there exists a continuous decreasing function \( \Phi^I_h(x) \) on \( I \) such that for each \( x \in I \), \( x(t) := \psi_h(t; x, \Phi^I_h(x)) \) is the unique bounded solution of (1.1) on \( \mathbb{R}^+ \) with \( x(0) = x \).

Clearly \( (\gamma_h(0) - r, \gamma_h(0) + r) \in \mathcal{F} \), hence \( \mathcal{F} \) is nonvoid. Set \( D(\Phi_h) = \bigcup_{I \in \mathcal{F}} I \). Then \( D(\Phi_h) \) is an open interval of \( (a, b) \). Define \( \Phi_h(x) \) on \( D(\Phi_h) \) as: on each \( I \in \mathcal{F} \),

\[ \Phi_h(x) = \Phi^I_h(x), \quad x \in I. \]

Note that if \( x \in I \cap J \), then by uniqueness of bounded solutions we have \( \Phi^I_h(x) = \Phi^J_h(x) \). Therefore \( \Phi_h \) is well defined. As for each \( I \), \( \Phi^I_h \) is continuous and decreasing on \( I \), it is clear that \( \Phi_h \) is a continuous and decreasing function on \( D(\Phi_h) \).

To complete the proof of the theorem, there remains to check that if \( x_0 \notin D(\Phi_h) \), then there is no bounded solution \( y \) satisfying \( y(0) = x_0 \). We may assume that \( x_0 > \gamma_h(0) \). We argue by contradiction and assume that (1.1) has a bounded solution \( y(t) \) on \( \mathbb{R}^+ \) with \( y(0) = x_0 \). Take two constants \( c < d \) satisfying (5.1) and an \( \varepsilon > 0 \) such that \( c \leq y(t) + \varepsilon \leq d \). Pick a function \( \zeta \in C^1(\mathbb{R}) \) satisfying (5.2)–(5.3) and consider the modified equation (5.4). Then \( y(t) \) is a bounded solution of (5.4) on \( \mathbb{R}^+ \). Since \( \gamma_h(0) < y(0) \), we also have \( \gamma_h(t) \leq y(t) \) \( (t \geq 0) \).

Let \( u(t) = \gamma_h(t) - r, v(t) = y(t) + \varepsilon \). Noting that \( u, v \) are a lower and upper solution of (5.4) on \( \mathbb{R}^+ \), respectively, one deduces that for any \( u(0) < x < v(0) \), the bounded solution \( x(t) \) of (5.4) with \( x(0) = x \) satisfies

\[ c \leq u(t) \leq x(t) \leq v(t) \leq d, \quad t \in \mathbb{R}^+. \]

Hence \( x(t) \) is a bounded solution of (1.1). Now by Theorem 1.1 in the case \( (a, b) = \mathbb{R} \), we deduce that

\[ (u(0), v(0)) = (\gamma_h(0) - r, x_0 + \varepsilon) \in \mathcal{F}. \]

Therefore \( x_0 \in D(\Phi_h) \). A contradiction! \( \Box \)
Proof of Theorem 1.3. Let \( h \in H \), and \( I = [\alpha, \beta] \subset D(\Phi_h^{\lambda_0}) \) be a compact interval. Set

\[
\tilde{u}(t) = \psi_h^{\lambda_0}(t; \alpha, \Phi_h^{\lambda_0}(\alpha)), \quad \tilde{v}(t) = \psi_h^{\lambda_0}(t; \beta, \Phi_h^{\lambda_0}(\beta)),
\]

where \( \psi_h^{\lambda_0}(t; x_0, x_1) \) denotes the solution of (1.4) with initial value \((x_0, x_1)\). Then

\[
a_1 \leq \tilde{u}(t) \leq \tilde{v}(t) \leq b_1, \quad t \geq 0,
\]

for some \( a_1, b_1 \in (a, b) \). Fix an \( r > 0 \) sufficiently small so that \( a < a_1 - r < b_1 + r < b \).

By virtue of Lemma 2.2, we know that \(|\tilde{u}'(t)|, |\tilde{v}'(t)| \leq M < \infty \) for all \( t \geq 0 \). Using (H1) and (H3) it is easy to deduce that there exists an \( \eta > 0 \) such that when \( \rho(\lambda, \lambda_0) < \eta \),

\[
f(\lambda, x - r, p) \leq f_0(\lambda, x, p) \leq f(\lambda, x + r, p), \quad \forall \lambda \in H, \ x \in [a_1, b_1], \ |p| \leq M. \tag{5.6}
\]

Let \( \lambda \in \Lambda \) be such that \( \rho(\lambda, \lambda_0) < \eta \). Set \( u = \tilde{u}(t) - r, \ v = \tilde{v}(t) + r \). By (5.6) we find that \( u \) and \( v \) are a lower and upper solution of (1.4) on \( R^+ \), respectively. Let \( c = a_1 - r, \ d = b_1 + r \). Choose a \( \xi \in C^1(R) \) satisfying (5.2)–(5.3) and consider the modified equation:

\[
x'' = f_\lambda(\theta_t h, \xi(x), x'). \tag{5.7}
\]

Clearly \( u \) and \( v \) are also a lower and upper solution of (5.7) on \( R^+ \), respectively. We denote by \( \tilde{\Gamma}_h^\lambda \) and \( \tilde{\Phi}_h^\lambda \) the corresponding structure functions for bounded solutions of (5.7).

Let \( x \in I \subset [u(0), v(0)] \). Then Eq. (5.7) has a unique bounded solution \( x(t) \) on \( R^+ \) with \( x(0) = x \).

Since

\[
c \leq u(t) \leq x(t) \leq v(t) \leq d, \quad t \geq 0,
\]

we see that \( x(t) \) is a bounded solution of (1.4) on \( I \), which indicates that \( \tilde{\Phi}_h^\lambda = \Phi_h^\lambda \) on \( I \).

We also infer from Remark 3.9 that \( \tilde{\Gamma}_h^\lambda(H) \subset [c, d] \), and thus \( \tilde{\Gamma}_h^\lambda = \Gamma_h^\lambda \).

Now the conclusion follows from what we have proved in case \( (a, b) = R \).

The proof is complete. \( \square \)

6. Applications

In this section we demonstrate how the abstract results in previous sections can be applied to differential equations.

Let \( (\mathcal{M}, d) \) be a complete metric space and \( C(R, \mathcal{M}) \) be the set of continuous functions from \( R \) to \( \mathcal{M} \). We denote by \( \theta \) the translation group on \( C(R, \mathcal{M}) \), i.e.,

\[
\theta_{\tau} h = h(\tau + \cdot), \quad \forall h \in C(R, \mathcal{M}), \ \tau \in R.
\]

Let \( g \in C(\mathcal{M} \times (a, b) \times R) \), and \( h_0 \in C(R, \mathcal{M}) \). Consider the nonautonomous equation:

\[
x'' = g(h_0(t), x, x'). \tag{6.1}
\]

We always assume that

(G1) \( g(z, x, p) \) is locally Lipschitz in \( (x, p) \) in a uniform manner with respect to \( z \) in any compact subset of \( \mathcal{M} \);
(G2) \( g(z, x, p) \) is strictly increasing in \( x \).
Define $f : C(\mathbb{R}, \mathcal{M}) \times (a, b) \times \mathbb{R} \to \mathbb{R}$ as

$$f(h, x, p) = g(h(0), x, p), \quad \forall (h, x, p) \in C(\mathbb{R}, \mathcal{M}) \times (a, b) \times \mathbb{R}.$$ 

Then we can rewrite (6.1) as

$$x'' = f(\theta th_0, x, x').$$

(6.2)

### 6.1. Translation compact case

Let the space $C(\mathbb{R}, \mathcal{M})$ be equipped with the local uniform convergence topology on any compact interval of $\mathbb{R}$ (also called the compact-open topology). It is well known that this topology is metrizable and $C(\mathbb{R}, \mathcal{M})$ is a complete metric space.

A function $h \in C(\mathbb{R}, \mathcal{M})$ is said to be *translation compact* [5], if the set $\{\theta th \mid t \in \mathbb{R}\}$ is precompact in $C(\mathbb{R}, \mathcal{M})$.

Assume that the function $h_0$ in (6.1) is translation compact. Define the hull of $h_0$ in $C(\mathbb{R}, \mathcal{M})$ as

$$\mathcal{H}(h_0) = \text{the closure of } \{\theta th_0 \mid t \in \mathbb{R}\} \text{ in } C(\mathbb{R}, \mathcal{M}).$$

We know that each $h \in \mathcal{H}(h_0)$ is translation compact, moreover, $\theta th_0 \in \mathcal{H}(h_0)$ (see [5]). Let $H = \mathcal{H}(h_0)$. We associate with (6.1) (or (6.2)) the following system:

$$x' = f(\theta th, x, x'), \quad h \in H.$$ 

(6.3)

Then by virtue of Remark 1.4 one can immediately apply Theorem 1.1 to derive some interesting results. In particular, we have

**Theorem 6.1.** Assume that (6.1) has a bounded solution $\gamma$ on $\mathbb{R}$. Then:

1. There is a $\Gamma \in C(H)$ such that $\gamma(t) = \Gamma(\theta_0 h_0)$.
2. There exist a maximal nonempty open interval $D(\Phi) \subset (a, b)$ and a continuous decreasing function $\Phi$ on $D(\Phi)$ such that for any $x \in D(\Phi)$, the solution $x(t)$ of (6.1) with initial value $x(0) = x, x'(0) = \Phi(x)$ is the unique bounded one on $\mathbb{R}^+.
3. For any $x \in D(\Phi)$ and bounded solution $x(t)$ of (6.1) with $x(0) = x$,

$$\lim_{t \to +\infty} \left( |x(t) - \gamma(t)| + |x'(t) - \gamma'(t)| \right) = 0.$$ 

It is known that if $h_0$ is uniformly continuous with $h_0(\mathbb{R})$ being precompact in $\mathcal{M}$, then it is translation compact [5]. In particular, if $h_0 \in C(\mathbb{R}, \mathcal{M})$ satisfies that $\lim_{t \to \pm\infty} h_0(t) = \alpha^\pm$ exist, then $h_0$ is translation compact with

$$\mathcal{H}(h_0) = \{\theta th_0 \mid t \in \mathbb{R}\} \cup \{h_0^\pm \mid h_0^\pm (t) = \alpha^\pm\};$$

see [5]. In such a case we necessarily have $\lim_{t \to \pm\infty} \theta_0 h_0 = \alpha^\pm$. Thus if (6.1) has a bounded solution $\gamma$ on $\mathbb{R}$, then by Theorem 6.1 we see that $\lim_{t \to \pm\infty} \gamma(t) = \Gamma(\alpha^\pm)$ both exist. One can also deduce by Theorem 1.1 that $\alpha^\pm$ satisfy

$$g(\alpha^\pm, \Gamma(\alpha^\pm), 0) = 0.$$
6.2. Almost periodic and quasiperiodic case

Denote by \( C_b(\mathbb{R}, \mathcal{M}) \) the set of bounded continuous functions from \( \mathbb{R} \) to \( \mathcal{M} \). Equip with \( C_b(\mathbb{R}, \mathcal{M}) \) the uniform convergence topology generated by the metric:

\[
\rho(h_1, h_2) = \sup_{t \in \mathbb{R}} d(h_1(t), h_2(t)), \quad \forall h_1, h_2 \in C_b(\mathbb{R}, \mathcal{M}).
\]

Then \( C_b(\mathbb{R}, \mathcal{M}) \) is a complete metric space.

A function \( h \in C_b(\mathbb{R}, \mathcal{M}) \) is said to be almost periodic, if for any \( \varepsilon > 0 \) there exists a number \( l = l(\varepsilon) \) such that for any \( \alpha \in \mathbb{R} \), one can find a \( \tau \in [\alpha, \alpha + l] \) with \( \rho(\theta_{\tau} h, h) < \varepsilon \).

Now assume that the function \( h_0 \) in (6.1) is almost periodic. Then the hull

\[
\mathcal{H}(h_0) = \text{the closure of } \{\theta_{\tau}h_0 | \tau \in \mathbb{R}\} \text{ in } C_b(\mathbb{R}, \mathcal{M})
\]

is compact and minimal [5] with each \( h \in \mathcal{H}(h_0) \) being almost periodic. It is trivial to check that \( \theta_{\tau} h \) is an almost periodic function from \( \mathbb{R} \) to \( \mathcal{H}(h_0) \).

Suppose that (6.1) has a bounded solution \( w \) on \( \mathbb{R}^+ \). Then we have similar results as in Theorem 6.1. In particular, we know that Eq. (6.1) has an almost periodic solution \( \gamma(t) = \Gamma(\theta_{\tau} h_0) \), where \( \Gamma \) is the structure function of bounded solution for the associated system (6.3).

A special case for almost periodic function \( h_0 \) is that \( h_0 \) is quasiperiodic, i.e., there is a function \( \phi \in C(\mathbb{R}^n, \mathcal{M}) \) with \( \phi(s_1, \ldots, s_n) \) being \( 2\pi \)-periodic in each argument \( s_k \) such that

\[
h_0(t) = \phi(\alpha_1 t, \ldots, \alpha_n t)
\]

for some rationally independent real numbers \( \alpha_1, \ldots, \alpha_n \). One can also easily verify that \( \theta_{\tau} h_0 \) is quasiperiodic. Further by the representation \( \gamma(t) = \Gamma(\theta_{\tau} h_0) \) we conclude that the bounded solution \( \gamma(t) \) of (6.1) on \( \mathbb{R} \) is quasiperiodic as well.

6.3. Periodic case

Consider the periodic differential equation:

\[
x'' = g(t, x, x'),
\]

where \( g(t, x, p) \) is \( 2\pi \)-periodic in \( t \). Let \( S^1 = \mathbb{R} \mod 2\pi \). Define \( h_0 : \mathbb{R} \to S^1 \) as

\[
h : t \mapsto t \mod 2\pi.
\]

Then (6.4) can be reformulated as (6.1) with \( \mathcal{M} = S^1 \). Since \( h_0 \) is continuous and \( 2\pi \)-periodic, we see that the above argument applies. Then we have similar results as in Theorem 6.1, in particular, we know that the bounded solution of the equation on \( \mathbb{R} \) is \( 2\pi \)-periodic.

6.4. An example

Example 6.1. Consider the following equation:

\[
x'' + ((|x'| - a)x' + x^3 - x = \lambda h(t),
\]

where \( a, \lambda \) are constants, \( h \in C_b(\mathbb{R}) \) with \( \|h\|_{C(\mathbb{R})} = 1 \). Note that the function \( g(x) = x^3 - x \) is strictly decreasing on the interval \( J = (-\sqrt{3}/3, \sqrt{3}/3) \) with \( g(\mp \sqrt{3}/3) = \pm 2\sqrt{3}/9 \). Therefore if \( |\lambda| < 2\sqrt{3}/9 \), then by standard argument as in [13] it can be easily shown that (6.5) has a unique bounded solution \( \gamma_\lambda \) on \( \mathbb{R} \) with \( \|\gamma_\lambda\|_{C(\mathbb{R})} < 2\sqrt{3}/9 \), which enables us to apply the general results in Theorems 1.1 and 1.3.
When $h$ is translation compact (resp. almost periodic, quasiperiodic, periodic), we know that $\gamma_\lambda$ is translation compact (resp. almost periodic, quasiperiodic, periodic). In addition, there exist a maximal nonempty open interval $D(\Phi_\lambda) \subset J$ and a continuous decreasing function $\Phi_\lambda(x)$ on $D(\Phi_\lambda)$ such that for each $x \in D(\Phi_\lambda)$, the solution $x(t)$ with initial data $x(0) = x, x'(0) = \Phi_\lambda(x)$ is the unique bounded solution of the equation on $\mathbb{R}^+$; moreover,

$$\lim_{t \to +\infty} \left( ||x(t) - \gamma_\lambda(t)|| + ||x'(t) - \gamma'_\lambda(t)|| \right) = 0.$$  

We also infer from Theorem 1.3 that $\gamma_\lambda$ and $\Phi_\lambda$ are stable with respect to variation of the parameter $\lambda$.

**Remark 6.2.** We remark that even in periodic case, the above results for Eq. (6.5) cannot be deduced from those in [3,6] and [14], for example. Similarly, one can discuss dynamical behavior of Eq. (1.5) with $h(t)$ being translation compact, a case which is also not covered in the literature mentioned above.

**References**


