Topology and its Applications 156 (2009) 1463–1481

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Hyperbolic $(1,2)$-knots in S^3 with crosscap number two and tunnel number one

Enrique Ramírez-Losada [a], Luis G. Valdez-Sánchez [b,*]

[a] Centro de Investigación en Matemáticas, A.C., Guanajuato, Gto. 36240, México
[b] Department of Mathematical Sciences, University of Texas at El Paso, 500 West University Ave., El Paso, TX 79968, USA

ARTICLE INFO

Article history:
Received 30 April 2008
Accepted 9 December 2008

Keywords:
Once-punctured Klein bottles
Seifert Klein bottles
$(1,1)$-knot
$(1,2)$-knot
Toroidal boundary slope
Klein bottle boundary slope

ABSTRACT

A knot in S^3 is said to have crosscap number two if it bounds a once-punctured Klein bottle but not a Moebius band. In this paper we give a method of constructing crosscap number two hyperbolic $(1,2)$-knots with tunnel number one which are neither 2-bridge nor $(1,1)$-knots. An explicit infinite family of such knots is discussed in detail.

1. Introduction

Let K be a knot in the 3-sphere S^3 with exterior $X_K = S^3 \setminus \operatorname{int} N(K)$ (where $N(\cdot)$ denotes regular neighborhood). For r a slope in ∂X_K, we let $K(r) = X_K \cup_\partial S^1 \times D^2$ denote the manifold obtained by performing surgery on K along the slope r, so that r bounds a disk in $S^1 \times D^2$. A *Seifert Klein bottle* for K is a once-punctured Klein bottle P properly embedded in X_K which has integral boundary slope; such a surface P is *unknotted* if $\operatorname{cl}(X_K \setminus N(P))$ is a genus two handlebody. We say that K has *crosscap number two* if K has a Seifert Klein bottle and its exterior contains no properly embedded Moebius band. The knot K has *tunnel number one* if there is a properly embedded arc τ in X_K such that $\operatorname{cl}(X_K \setminus N(\tau))$ is a (genus two) handlebody. We also say that the knot K *admits a* (g,n) *decomposition*, or that K is a (g,n)-*knot*, if there is a Heegaard splitting surface S in S^3 of genus g which intersects K transversely and bounds handlebodies H, H', such that both $K \cap H \subset H$ and $K \cap H' \subset H'$ are trivial n-string arc systems. Finally, we will use the notation $S^2(a,b,c)$ to represent any small Seifert fibered space over a 2-sphere with three singular fibers of indices a, b, c.

Any $(1,1)$-knot has tunnel number one; the converse, however, does not hold in general. It is therefore remarkable that for genus one hyperbolic knots the properties of having tunnel number one, admitting a $(1,1)$ decomposition, or being 2-bridge are all mutually equivalent; this is the content of the Goda–Teragaito conjecture, which is the main result of [13]. Since a genus one knot bounds a once punctured torus, which is the orientable homotopy equivalent of a once punctured Klein bottle, one might expect crosscap number two hyperbolic knots to exhibit similar behavior, i.e. with having tunnel number one, admitting a $(1,1)$ decomposition, and being 2-bridge are all equivalent conditions. That this is not the case follows from [11, Theorem 1.1], which shows that crosscap number two hyperbolic $(1,1)$-knots are in general not 2-bridge.

* Corresponding author.
E-mail addresses: kikis@cimat.mx (E. Ramírez-Losada), lvsanchez@utep.edu (L.G. Valdez-Sánchez).

0166-8641/$ – see front matter
doi:10.1016/j.topol.2008.12.031

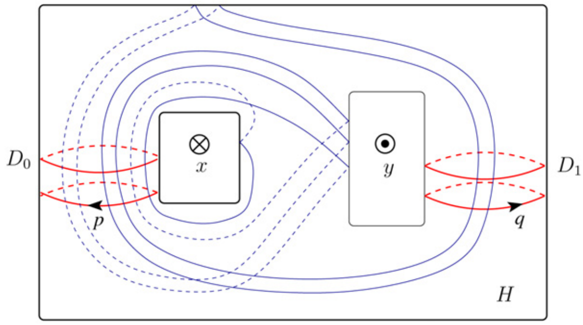

Fig. 1. The knot $K(0,0) \subset \partial H$.

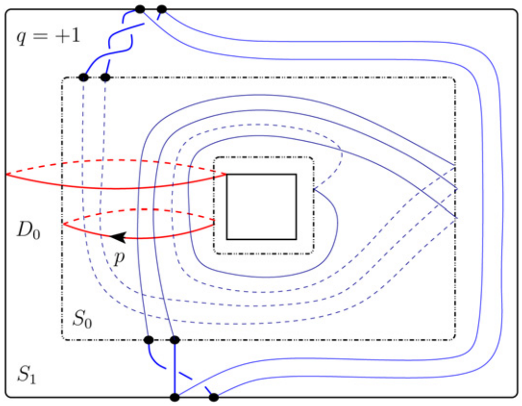

Fig. 2. A $(1,2)$ decomposition of the knot $K(0,1)$.

One may still ask if, for crosscap number two hyperbolic knots, having tunnel number one is equivalent to admit a $(1,1)$ decomposition. In this paper we show this is not the case by constructing explicit examples of crosscap number two hyperbolic knots that have tunnel number one but do not admit $(1,1)$ decompositions; moreover, all such knots admit a $(1,2)$ decomposition. The construction of such examples is based on the classification of the nontrivial crosscap number two $(1,1)$-knots in S^3 given in [11]. Explicit examples of general tunnel number one knots without $(1,1)$ decompositions were first given by Morimoto, Sakuma and Yokota [10], and more recently by Eudave-Muñoz [4] (see also [6]).

Our family of examples is constructed starting from the trivial knot $K(0,0)$ shown in Fig. 1, which lies in the boundary ∂H of an unknotted genus two handlebody H standardly embedded in S^3. We obtain a two-parameter infinite family of knots $K(p,q) \subset \partial H$, for any integers p,q, by Dehn-twisting $K(0,0)$ p-times along ∂D_0 and q-times along ∂D_1, where D_0, D_1 is the complete meridian system for H shown in the figure; the Dehn-twists are carried out by cutting ∂H along ∂D_i, obtaining a 4-punctured 2-sphere F_0, and then twisting only the two 'bottom' boundary circles of F_0 indicated by the arrows of the figure the required number of times (in the direction of the arrows for $p,q > 0$, see e.g. the knot $K(-1,1)$ in Fig. 11).

It is not hard to see that each knot $K(p,q)$ is a $(1,2)$-knot. Fig. 2 shows a pair of parallel unknotted tori S_0, S_1 in S^3, with the region between S_0 and S_1 a product of the form $S_0 \times I$; in the figure, the knot $K(0,1)$ is represented as the union of 4 pairs of arcs: one pair on each S_0, S_1 and two pairs of arcs in $S_0 \times I$, with each arc in the latter pairs intersecting each level torus $S_0 \times \{t\}$, $t \in I$, transversely in one point. It is well known (and easy to prove) that under such conditions the given representation is in fact a $(1,2)$ decomposition of $K(0,1)$ relative to either S_0 or S_1. A $(1,2)$ decomposition for the knot $K(p,q)$ can be obtained by Dehn twisting the knot $K(0,1)$ p-times along the disk D_0 in Fig. 2 (following our previous convention) and varying the number q of full twists on the top pair of strands that run between S_0 and S_1 (with $q > 0$ corresponding to q positive full twists on the strands).

On the other hand, it is not easy to see that most of the knots $K(p,q)$ are not $(1,1)$-knots. Our main result is the following:

Theorem 1.1. *The $(1,2)$-knot $K(p,q)$ is trivial iff $(p,q) = (0,0), (0,1)$, and a torus knot iff $p = 0$ and $q \neq -1, 0$ (with $K(0,q) = T(2, 2q-1)$) or $(p,q) = (-1,1)$ (with $K(-1,1) = T(5,8)$); in all other cases,*

(a) *$K(p,q)$ is a hyperbolic tunnel number one knot which is not 2-bridge;*
(b) *$K(p,q)$ bounds an unknotted Seifert Klein bottle $P(p,q)$ of boundary slope $r = 4q - 36p$;*

(c) *$K(p,q)$ is a $(1,1)$-knot iff (p,q) is a pair of the form $(p,1)$, $(p,2)$, $(1,q)$, or $(-1,0)$;*
(d) *with the exception of $K(-1,2)(r) = S^2(2,2,3)$, $K(-2,1)(r) = S^2(2,2,7)$, and $K(p,0)(r) = S^2(2,2,|6p-1|)$, the manifold $K(p,q)(r)$ is irreducible and toroidal.*

In particular, there are infinitely many hyperbolic $(1,2)$-knots with crosscap number two and tunnel number one which admit no $(1,1)$ decompositions.

Proof. We have already seen that each $K(p,q)$ is a $(1,2)$-knot; all other claims follow from Lemmas 4.3, 4.4, and 4.6. □

We remark that the family of knots $K(p,q)$ is one of the simplest families that can be obtained following our method of construction, which is quite general. The paper is organized as follows. In Section 2 we provide definitions and background results, and develop several specific properties of circles embedded in the boundary of a genus two handlebody, in both algebraic and topological versions, which will be needed in later sections. Section 3 contains, among many other miscellaneous results, the criteria used to determine if a crosscap number two knot with an unknotted Seifert Klein bottle has tunnel number one, and if so whether or not it admits a $(1,1)$ decomposition; such criteria are given in decidable algebraic terms involving primitive or power words in a rank two free group. Finally, in Section 4 we apply these criteria to prove Lemmas 4.3, 4.4 and 4.6, which establish the properties of the family of knots $K(p,q)$ given in Theorem 1.1.

2. Preliminaries

2.1. Once-punctured Klein bottles

Let P denote a once-punctured Klein bottle. Any circle embedded in P is, up to isotopy, of one of the following types (cf. [12, §2]):

(i) a *meridian circle* m: this is an orientation preserving circle which cuts P into a pair of pants;
(ii) a *center* c: this is an orientation reversing circle whose regular neighborhood in P is a Moebius band;
(iii) a *longitude* ℓ: this is an orientation preserving circle which separates P into two components, one of which is a Moebius band.

The meridian circle of P is unique, while there are infinitely many isotopy classes of center and longitude circles (cf. [16, Lemma 3.1]). This contrasts with the situation in a *closed* Klein bottle, where up to isotopy there is only one longitude circle and two center circles.

Denote by $P \tilde{\times} I$ the orientable twisted I-bundle over P, where $I = [0,1]$. $P \tilde{\times} I$ is a genus two handlebody; the pair $(P \tilde{\times} I, \partial P)$ can be seen in Fig. 4, up to homeomorphism. In particular, if $N(P)$ is the regular neighborhood of a once-punctured Klein bottle which is properly embedded in an orientable 3-manifold with boundary, then $(N(P), P) \approx (P \tilde{\times} I, P \tilde{\times} \frac{1}{2})$, where $\approx$ denotes homeomorphism.

2.2. Lifts of meridian, center, and longitude circles

Let T_P be the twice punctured torus $\partial N(P) \setminus \operatorname{int} N(\partial P) \subset \partial N(P)$. For any meridian circle m or longitude circle ℓ of P, the restriction of the I-bundle $N(P)$ to m, ℓ is a fibered annulus $A(m), A(\ell)$, respectively, properly embedded in $N(P)$, which intersects P transversely in m, ℓ, respectively; if c is a center circle of P, the restriction of $N(P)$ to c is a fibered Moebius band $B(c)$ which intersects P transversely in c. Notice that $A(m), A(\ell)$, and $B(c)$ are all unique up to isotopy in $N(P)$ (i.e., any annulus $(A, \partial A) \subset (N(P), T_P)$ intersecting P in m is isotopic to $A(m)$, etc.), and that the boundary circles $\partial A(m), \partial A(\ell), \partial B(c)$ may all be assumed to lie in T_P. We call the circles $\partial A(m), \partial A(\ell), \partial B(c)$, respectively, the *lifts* of m, ℓ, c to T_P; thus m has two distinct nonparallel lifts $m_0 \sqcup m_1 = \partial A(m)$, while c has a unique lift. Each longitude ℓ also has a unique lift, of which $\partial A(\ell)$ gives two parallel copies. In fact, if ℓ splits off a Moebius band from P with center c_ℓ, then the lifts of c_ℓ and ℓ are isotopic in T_P: for $A(\ell)$ is isotopic to the frontier of the regular neighborhood in $N(P)$ of the Moebius band $B(c_\ell)$. Thus, the set of lifts of centers of P coincides with the set of lifts of longitudes of P.

2.3. Seifert Klein bottles

Let $P \subset X_K$ be a Seifert Klein bottle for a knot $K \subset S^3$, and let $N(P) \approx P \tilde{\times} I$ be a small regular neighborhood of P in X_K. We define the *exterior of P in S^3* as the manifold $X(P) = S^3 \setminus \operatorname{int} N(P)$; we thus have

$$S^3 = N(P) \cup_\partial X(P) \tag{2.1}$$

with $\partial P \subset \partial N(P) = \partial X(P)$. We will identify the twice punctured torus $T_P \subset \partial N(P)$ with the frontier in X_K of $N(P)$, so that $T_P \subset N(P) \cap \partial X(P)$.

Given that K and ∂P are isotopic in S^3, the translation of properties of $K \subset S^3$ or $P \subset X_K$ into properties involving the decomposition given in (2.1) can be easily carried out. For instance, it is easy to see that P is unknotted in S^3 iff $X(P)$ is a handlebody.

2.4. Companion annuli and multiplicity

Let $\mathcal{M}$ be an orientable, irreducible, and geometrically atoroidal 3-manifold with connected boundary, and let γ be a circle embedded in $\partial\mathcal{M}$ which is nontrivial (i.e., it does not bound a disk) in $\mathcal{M}$.

Let A be an annular regular neighborhood of γ in $\partial\mathcal{M}$, and A' a properly embedded separating annulus in $\mathcal{M}$ with $\partial A' = \partial A$. We say that A' is a *companion annulus for* γ *in* $\mathcal{M}$ if A' is not parallel into $\partial\mathcal{M}$. It follows from [16, Lemma 5.1] that the region cobounded in $\mathcal{M}$ by A' and the annular neighborhood A of γ is a solid torus, the *companion solid torus of* γ *in* M, and that a companion annulus and a companion solid torus for γ are unique up to isotopy. We define the *multiplicity* $\mu(\gamma)$ *of* γ in $\mathcal{M}$ to be 1 if γ has no companion annuli, and as the number of times γ runs around its companion solid torus when γ has a companion annulus. Thus γ has a companion annulus in $\mathcal{M}$ iff $\mu(\gamma) \geqslant 2$.

Multiplicities of circles in the case where $\mathcal{M}$ is a genus two handlebody will be of particular interest in later developments. So let H be a genus two handlebody; we shall see that the fact that $\pi_1(H)$ (rel some base point) is a free group on two generators allows for a simple interpretation of multiplicities of circles in ∂H in purely algebraic terms. We will need the following general definitions.

Let F_2 denote the free group on the two generators; if free generators (i.e., a basis) x, y for F_2 are given, so that $F_2 = \langle x, y \mid - \rangle$, we may refer to the elements of F_2 as words in x and y. For convenience, we will denote the inverse u^{-1} of u by $\bar{u}$, and by $[u, v]$ the commutator $uv\bar{u}\bar{v}$ of any two elements u, v of F_2. A word $u \in F_2$ is *primitive* if there is $v \in F_2$ such that $\{u, v\}$ is a basis of F_2, and that u is a *power* if there is a nontrivial element $w \in F_2$ and an integer $n \geqslant 2$ such that $u = w^n$. We write $u \equiv v$ for $u, v \in F_2$ if $u = \bar{c}v^{\varepsilon}c$ for some $c \in F_2$ and $\varepsilon \in \{1, -1\}$. A word $u \in \langle x, y \mid - \rangle$ is said to be *cyclically reduced* if, for $\varepsilon = \pm 1$ and $z \in \{x, y\}$, the pair of symbols $z^{\varepsilon}, z^{-\varepsilon}$ do not occur consecutively in u nor u simultaneously begins with z^{ε} and ends with $z^{-\varepsilon}$; notice $u \equiv v$ whenever v is a cyclic reduction of u or $\bar{u}$.

We will see in Lemma 2.3 below that multiplicities of circles in ∂H can be characterized in terms of primitive or power elements in $\pi_1(H)$. A complete characterization of primitive words in $F_2 = \langle x, y \mid - \rangle$ can be found in [5]; for our purposes, the following partial characterization of such primitive words (originally given in [3]), which easily extends to words that are powers of primitive elements, will suffice.

Lemma 2.1. *([3,5].) If an element u in the free group $F_2 = \langle x, y \mid - \rangle$ is primitive or a power of a primitive then there is a basis $\{a, b\} \subset \{x, \bar{x}, y, \bar{y}\}$ of F_2 and an integer $n \geqslant 1$ such that either $u \equiv ab^{n-1}$, $u \equiv a^{n+1}$, or $u \equiv ab^{m_1} \cdots ab^{m_k}$, $k \geqslant 2$, for some integers $\{m_1, \ldots, m_k\} = \{n, n+1\}$.*

The next result establishes some useful equivalences in F_2.

Lemma 2.2. *Let $\{u, v\}$ and $\{a, b\}$ be any two bases of F_2 and $m, n \geqslant 1$ any two integers.*

(a) *The identities $[\overline{w}, w'] \equiv [w, w'] \equiv [w', w]$ hold for any $w, w' \in F_2$.*
(b) *If $w \in F_2$, then $\{u, w\}$ is a basis for F_2 iff $w = u^k v^{\varepsilon} u^l$ for some integers k, l and $\varepsilon \in \{-1, 1\}$.*
(c) *If $u^m \equiv a^m$ then $[u^m, v] \equiv [a^m, b]$.*
(d) *If $u^m \equiv a^m$ and $v^n \equiv b^n$ then $[u^m, v^n] \equiv [a^m, b^n]$.*

Proof. Part (a) follows by direct computation, while the result in (b) is well known (cf. [9, §3.5, Problem 3]).

For part (c) we have that $u^m = \bar{c}a^{m\varepsilon}c = (\bar{c}a^{\varepsilon}c)^m$ for some $c \in F_2$ and $\varepsilon \in \{-1, 1\}$, and hence that $u = \bar{c}a^{\varepsilon}c$ (cf. [9, §1.4]). It follows that $cu\bar{c} = a^{\varepsilon}$ and $cv\bar{c}$ form a basis for F_2, and hence by (b) that $cv\bar{c} = a^k b^{\varepsilon'} a^l$ for some integers k, l and $\varepsilon' \in \{-1, 1\}$. Therefore

$$[u^m, v] = [\bar{c}a^{m\varepsilon}c, \bar{c}a^k b^{\varepsilon'} a^l c] \equiv [a^{m\varepsilon}, b^{\varepsilon'}] \equiv [a^m, b].$$

Similarly, for part (d) we continue to have $u^m = \bar{c}a^{m\varepsilon}c = (\bar{c}a^{\varepsilon}c)^m$ and also have $v^n = \bar{d}b^{n\delta}d = (\bar{d}b^{\delta}d)^n$ for some $d \in F_2$ and $\delta \in \{-1, 1\}$, whence $u = \bar{c}a^{\varepsilon}c$ and $v = \bar{d}b^{\delta}d$ hold in F_2. It follows that $cu\bar{c} = a^{\varepsilon}$ and $cv\bar{c} = \bar{e}b^{\delta}e$ form a basis of F_2, where $e = d\bar{c}$. By (b) we must have $\bar{e}b^{\delta}e = \bar{a}^k b^{\delta} a^k$, and so $[u^m, v^n] = [\bar{c}a^{m\varepsilon}c, \bar{c}\bar{a}^k b^{\delta} a^k c] \equiv [a^{m\varepsilon}, b^{\delta}] \equiv [a^m, b^n]$ holds. □

The following sequence of lemmas will establish several fundamental facts about circles in ∂H which may represent primitive/power elements in $\pi_1(H)$. For any loop $\alpha \subset H$, denote by $[\alpha]$ the element of $\pi_1(H)$ representing α (rel some base point). We will call any disk properly embedded in H which separates H into two solid torus components a *waist disk* of H. Also, for a 3-manifold $\mathcal{M}$, we will say that the pair $(\mathcal{M}, \partial\mathcal{M})$ is *irreducible* if $\mathcal{M}$ is irreducible and $\partial\mathcal{M}$ is incompressible in $\mathcal{M}$.

Lemma 2.3. *Let H be a handlebody of genus two and γ, γ' be disjoint circles embedded in ∂H which are nontrivial in H.*

(a) *$\partial H \setminus \gamma$ compresses in H iff $[\gamma]$ is primitive or a power in $\pi_1(H)$, in which case there is a waist disk of H which is disjoint from γ. In particular, if $[\gamma]$ is a power in $\pi_1(H)$ then $[\gamma]$ is a power of a primitive element.*
(b) *γ has multiplicity $n \geqslant 2$ in H iff $[\gamma]$ is the nth power of some primitive element in $\pi_1(H)$; moreover, if $[\gamma] = \lambda^n$ for some integer $n \geqslant 1$ and some primitive element $\lambda \in \pi_1(H)$ then $n = \mu([\gamma])$.*
(c) *Suppose γ and γ' are not parallel in ∂H. If $[\gamma]$ is primitive or a power in $\pi_1(H)$ and $[\gamma']$ is conjugate to $[\gamma]$ in $\pi_1(H)$, then γ and γ' cobound a nonseparating annulus in H.*
(d) *Suppose γ' does not cobound an annulus with γ in H and that, in $\pi_1(H)$, either $[\gamma]$ is primitive or a power while $[\gamma']$ is a power. Then there is a waist disk D of H which separates γ and γ'.*

Proof. Parts (a) and (b) follow from the argument used in the proof of [2, Theorem 4.1], which deals with roots in the fundamental group of a compression body; we prove them here in the context of handlebodies of genus two for the convenience of the reader.

Suppose $\partial H \setminus \gamma$ compresses in H along a disk D. If D is nonseparating then there is an embedded circle $\alpha \subset \partial H \setminus \gamma$ which intersects ∂D transversely in a single point, and hence the frontier of a small regular neighborhood in H of $D \cup \alpha$ is a waist disk in H which compresses $\partial H \setminus \gamma$; we may thus assume D is a waist disk of H. Cutting H along D produces two solid torus components, one of which, say V, contains γ in its boundary; given that γ is nontrivial in H, it follows that γ must be a nonseparating circle in ∂H. By Van Kampen's theorem, if β is a core of V then $[\beta]$ is primitive in $\pi_1(H)$ and $[\gamma] = [\beta]^k$ for some integer $k \neq 0$. Thus $[\gamma]$ is either primitive or a power of a primitive in $\pi_1(H)$; moreover, it is not hard to see that if $|k| \geqslant 2$ then V is the companion solid torus of γ in H, so $\mu(\gamma) = |k|$. This proves one direction of part (a).

Conversely, let M be the 3-manifold obtained by adding a 2-handle to H along γ. If, in $\pi_1(H)$, $[\gamma]$ is primitive then $\pi_1(M) = \mathbb{Z}$ and so M is a solid torus, while if $[\gamma]$ is a power then $\pi_1(M)$ has nontrivial torsion by [9, Theorems N3 and 4.12] and hence M is reducible (cf. [8, Theorem 9.8]). Therefore the pair $(M, \partial M)$ is not irreducible, so the surface $\partial H \setminus \gamma$ compresses in H by the 2-handle addition theorem (cf. [2]), and hence by the above argument $[\gamma]$ is a primitive or a power of a primitive in $\pi_1(H)$. Thus (a) holds.

For part (b), assume γ has multiplicity $n \geqslant 2$; that is, for $A \subset \partial H$ an annular neighborhood of γ and A' a companion annulus of γ with $\partial A' = \partial A$, A and A' cobound a solid torus $V \subset H$ such that γ runs n times around V. Since γ is nontrivial in H, and A' separates H, it follows that A' boundary compresses in H into a nontrivial separating compression disk of $\partial H \setminus \gamma$; hence, by the first part of the argument for (a), γ is the nth power of a primitive element of $\pi_1(H)$. Conversely, suppose $[\gamma] = \lambda^n$ for some integer $n \geqslant 1$ and some primitive element $\lambda \in \pi_1(H)$. By the first part of the argument for (a), there is a loop β in H with $[\beta]$ primitive in $\pi_1(H)$ and $[\gamma] = [\beta]^{\mu(\gamma)}$. Therefore the abelianization of $\pi_1(H)/\langle[\gamma]\rangle$ is isomorphic to both $\mathbb{Z} \oplus \mathbb{Z}_n$ and $\mathbb{Z} \oplus \mathbb{Z}_{\mu(\gamma)}$, so $n = \mu(\gamma)$. Thus (b) follows.

In parts (c) and (d), let $F = \partial H \setminus \gamma$, so that $\gamma' \subset F$; observe that F compresses in H by (a), given that $[\gamma]$ is either primitive or a power in $\pi_1(H)$. Let M be the manifold obtained by attaching a 2-handle to H along γ', so that $\gamma \subset \partial M$. In part (d), $[\gamma']$ is a power in $\pi_1(H)$ and so M is reducible by the argument used in part (a); in part (c), since $[\gamma]$ is conjugate to $[\gamma']$ in $\pi_1(H)$ but γ and γ' are not parallel in ∂H, and $\gamma \subset \partial M$, $[\gamma]$ must be trivial in $\pi_1(M)$ but nontrivial in ∂M, and hence γ bounds a nonseparating disk in M. Either way the pair $(M, \partial M \setminus \gamma)$ is not irreducible, so by the 2-handle addition theorem the surface $F \setminus \gamma' = \partial H \setminus (\gamma \cup \gamma')$ compresses in H along some disk $D \subset H$. In part (c) the disk D must be nonseparating, so γ, γ' lie in the boundary of the solid torus $H \setminus \operatorname{int} N(D)$ and hence cobound a nonseparating annulus in H; similarly, in part (d) the disk D must be a waist disk of H which separates γ and γ'. □

For a Seifert Klein bottle P in a knot exterior X_K, recall that T_P is the twice punctured torus obtained from the frontier of $N(P)$ in the knot exterior X_K, so $T_P \subset \partial X(P) \setminus \partial P$ and $\operatorname{int} T_P$, $X(P) \setminus \partial P$ are homeomorphic surfaces. In this context, Lemma 2.3(a) has the following immediate consequence.

Corollary 2.4. *An unknotted Seifert Klein bottle P for a knot K is π_1-injective in X_K iff $[\partial P]$ is neither primitive nor a power of a primitive in $\pi_1(X(P))$. Specifically, T_P is boundary compressible in X_K iff $[\partial P]$ is primitive in $\pi_1(X(P))$.*

Proof. Since $N(P)$ is an I-bundle over P, we have that P is π_1-injective in $N(P)$ and T_P is incompressible in $N(P)$; hence, by the Dehn's lemma-loop theorem [8], P is π_1-injective in X_K iff T_P is geometrically incompressible in $X(P)$. Thus the first part follows from Lemma 2.3(a).

Now, given the relationship between the surfaces T_P and $X(P) \setminus \partial P$, it is not hard to see that the boundary compressibility of T_P in $X(P)$ is equivalent to the existence of a properly embedded disk D in $X(P)$ which intersects the circle $\partial P \subset \partial X(P)$ transversely in one point; as the latter condition is equivalent to $[\partial P]$ being primitive in $\pi_1(X(P))$, the second part of the claim follows. □

The following result gives a simple algebraic way of determining if the manifold $T \times I$, T a torus, is obtained by attaching a 2-handle to a genus two handlebody H; though the result is well known, we sketch its proof as preparation for the argument used in its generalization given in Lemma 2.6, which deals with the case of attaching a 2-handle to a genus two sub-handlebody of H.

Lemma 2.5. *Let H be a genus two handlebody and $\mathcal{T}$ be a closed torus. Let γ be a circle embedded in ∂H and $M = H \cup_\gamma N(D)$ be the manifold obtained by attaching a 2-handle $N(D)$ to H along γ. Then $M \approx \mathcal{T} \times I$ iff $[\gamma] \equiv [a, b]$ for some (and hence any) basis $\{a, b\}$ of $\pi_1(H)$.*

Proof. Suppose that $\pi_1(M)/\langle[\gamma]\rangle = \mathbb{Z} \oplus \mathbb{Z}$, so that γ is nontrivial in H. If $\partial H \setminus \gamma$ compresses in H then by Lemma 2.3(a) there is a waist disk in H disjoint from γ and so M is a manifold of the form $S^1 \times D^2 \# L$ for $L = S^3$ or a lens space; but then $\pi_1(M)/\langle[\gamma]\rangle \neq \mathbb{Z} \oplus \mathbb{Z}$, contradicting our hypothesis. Thus $\partial H \setminus \gamma$ is incompressible in H and so the pair $(M, \partial M)$ is irreducible by the 2-handle addition theorem, hence by [8, Theorem 12.10] the condition $M \approx \mathcal{T} \times I$ is equivalent to the condition $\pi_1(H)/\langle[\gamma]\rangle = \mathbb{Z} \oplus \mathbb{Z}$. The lemma follows now from the fact (due to Nielsen, cf. [9, §4.4]) that, for any word w in $\langle x, y \mid -\rangle$, $\langle x, y \mid -\rangle/\langle w\rangle = \mathbb{Z} \oplus \mathbb{Z}$ iff $w \equiv [x, y]$. □

Lemma 2.6. *Let H be a handlebody of genus two and $\gamma_0, \gamma_1, \gamma_2$ be disjoint circles embedded in ∂H which are nontrivial in H; let $\mathcal{T}$ denote a closed torus.*

(a) *If $A_0 \subset H$ is a companion annulus for γ_0 with core α_0 and corresponding companion solid torus $V_0 \subset H$, then $H' = \mathrm{cl}(H \setminus V_0)$ is a genus two handlebody, and there is a common waist disk D of H and H' such that*
   - (i) *$H' = W_0 \cup_D W_1$ for some solid tori W_0, W_1 in H' with $D = \partial W_0 \cap \partial W_1 = W_0 \cap W_1$ and $A_0 \subset \partial W_0 \setminus D$,*
   - (ii) *if β_1 is a core of W_1 then $\{w_0 = [\alpha_0], w_1 = [\beta_1]\}$ is a basis for $\pi_1(H')$,*
   - (iii) *if β_0 is a core of V_0 then $\{u = [\beta_0], v = [\beta_1]\}$ is a basis for $\pi_1(H)$, and the inclusion map $i\colon H' \subset H$ induces an injection $\pi_1(H') \xrightarrow{i_*} \pi_1(H)$ given by $w_0 \mapsto u^{\mu(\gamma_0)} \equiv [\gamma_0]$ and $w_1 \mapsto v$,*
   - (iv) *if $H' \cup N(D(\gamma_2))$ is the manifold obtained by attaching a 2-handle $N(D(\gamma_2))$ along γ_2, then $H' \cup N(D(\gamma_2)) \approx \mathcal{T} \times I$ iff $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b]$ for some (and hence any) basis $\{a, b\}$ of $\pi_1(H)$ such that $[\gamma_0] \equiv a^{\mu(\gamma_0)}$.*

(b) *Suppose $A_0, A_1 \subset H$ are disjoint companion annuli for γ_0, γ_1, respectively, with corresponding cores α_0, α_1 and companion solid tori $V_0, V_1 \subset H$. Then $H' = \mathrm{cl}(H \setminus (V_0 \sqcup V_1))$ is a genus two handlebody, and there is a common waist disk D of H and H' such that*
   - (i) *$H' = W_0 \cup_D W_1$ for some solid tori W_0, W_1 in H' with $D = \partial W_0 \cap \partial W_1 = W_0 \cap W_1$, $A_0 \subset \partial W_0 \setminus D$, and $A_1 \subset \partial W_1 \setminus D$,*
   - (ii) *$\{w_0 = [\alpha_0], w_1 = [\alpha_1]\}$ is a basis for $\pi_1(H')$,*
   - (iii) *if β_0, β_1 are cores of V_0, V_1, respectively, then $\{u = [\beta_0], v = [\beta_1]\}$ is a basis for $\pi_1(H)$ and the inclusion map $i\colon H' \subset H$ induces an injection $\pi_1(H') \xrightarrow{i_*} \pi_1(H)$ given by $w_0 \mapsto u^{\mu(\gamma_0)} \equiv [\gamma_0]$ and $w_1 \mapsto v^{\mu(\gamma_1)} \equiv [\gamma_1]$,*
   - (iv) *if $H' \cup N(D(\gamma_2))$ is the manifold obtained by attaching a 2-handle $N(D(\gamma_2))$ along γ_2, then $H' \cup N(D(\gamma_2)) \approx \mathcal{T} \times I$ iff $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b^{\mu(\gamma_1)}]$ for some (and hence any) basis $\{a, b\}$ of $\pi_1(H)$ such that $[\gamma_0] \equiv a^{\mu(\gamma_0)}$ and $[\gamma_1] \equiv b^{\mu(\gamma_1)}$.*

Proof. For part (a), by Lemma 2.3(a), (b), there is a waist disk D for H such that $H = U_0 \cup_D W_1$ for some solid tori U_0, W_1 with $\gamma_0 \subset \partial U_0 \setminus D$. After a slight isotopy we may also assume that $A_0 \subset U_0$, and then we may write $U_0 = W_0 \cup_{A_0} V_0$ for some solid torus $W_0 \subset U_0$. Thus $H' = \mathrm{cl}(H \setminus V_0) = W_0 \cup_D W_1$ is a genus two handlebody and (i) holds.

As the circles $\gamma, \alpha \subset \partial V_0$ run $\mu(\gamma_0) \geqslant 2$ around V_0, and $U_0 = W_0 \cup_{A_0} V_0$ is a solid torus, it follows that $\alpha_0 \subset \partial W_0$ must run once around W_0 and hence that α_0 is isotopic to a core of W_0; therefore, that (ii) and (iii) hold follows by Van Kampen's theorem and the fact that α_0 and γ_0 are isotopic in V_0.

For part (a)(iv), we assume as we may that ∂A_0 and γ_2 are disjoint in ∂H, whence $\gamma_2 \subset \partial H'$; we write $[\gamma_2]', [\gamma_2]$ for the elements in $\pi_1(H'), \pi_1(H)$ represented by γ_2, respectively, so that $i_*([\gamma_2]') = [\gamma_2]$. Recall by Lemma 2.5 that $H' \cup N(D(\gamma_2)) \approx \mathcal{T} \times I$ iff $[\gamma_2]' \equiv [x, y]$ for some and in fact any basis $\{x, y\}$ of $\pi_1(H')$. Thus, if $H' \cup N(D(\gamma_2)) \approx \mathcal{T} \times I$ then $[\gamma_2]' \equiv [w_0, w_1]$ in $\pi_1(H')$ and hence $[\gamma_2] = i_*([\gamma_2]') \equiv [u^{\mu(\gamma_0)}, v]$ in $\pi_1(H)$; that $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b]$ holds for any basis $\{a, b\}$ of $\pi_1(H)$ with $[\gamma_0] \equiv a^{\mu(\gamma_0)}$ now follows from Lemma 2.2(c).

Suppose now $\{a, b\}$ is any basis of $\pi_1(H)$ such that $[\gamma_0] \equiv a^{\mu(\gamma_0)}$ and $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b]$ hold in $\pi_1(H)$; by Lemma 2.2(c), we then also have that $[\gamma_2] \equiv [u^{\mu(\gamma_0)}, v]$. Observe that $[u^{\mu(\gamma_0)}, v]$ is a cyclically reduced word in $\pi_1(H) = \langle u, v \mid -\rangle$; for definiteness, we will assume, that $[u^{\mu(\gamma_0)}, v]$ is a cyclic reduction of $[\gamma_2]$ in $\pi_1(H) = \langle u, v \mid -\rangle$.

Let $W(w_0, w_1)$ be a cyclic reduction of $[\gamma_2]'$ in $\pi_1(H') = \langle w_0, w_1 \mid -\rangle$. Then, in $\pi_1(H) = \langle u, v \mid -\rangle$, $i_*(W(w_0, w_1)) = W(i_*(w_0), i_*(w_1)) = W(u^{\mu(\gamma_0)}, v)$ is also a cyclically reduced word, which must then be a cyclic reduction of $[\gamma_2] = i_*([\gamma_2]')$. Thus the words $i_*(W(w_0, w_1))$ and $[u^{\mu(\gamma_0)}, v]$ are identical except for the cyclic order of their factors. Given that $i_*(w_0) = u^{\mu(\gamma_0)}$ and $i_*(w_1) = v$, it is not hard to see that changing the cyclic order of the w_0, w_1 factors in $W(w_0, w_1)$ will produce the identity $i_*(W(w_0, w_1)) = [u^{\mu(\gamma_0)}, v]$, so we may assume that such identity holds. As $i_*([w_0, w_1]) = [u^{\mu(\gamma_0)}, v]$ holds too, so that $i_*([w_0, w_1]) = i_*(W(w_0, w_1))$, we must have $[w_0, w_1] = W(w_0, w_1) \equiv [\gamma_2]'$ in $\pi_1(H')$. Thus (a)(iv) holds.

The proof for part (b) is similar: by Lemma 2.3(d), there is a waist disk D for H such that $H = U_0 \cup_D U_1$ for some solid tori U_0, U_1 with $\gamma_0 \subset \partial U_0 \setminus D$ and $\gamma_1 \subset \partial U_1 \setminus D$, and we may also assume that $A_0 \subset U_0$, $A_1 \subset U_1$. Thus $U_0 = W_0 \cup_{A_0} V_0$ and $U_1 = W_1 \cup_{A_1} V_1$ for some solid tori $W_0 \subset U_0$, $W_1 \subset U_1$, so $H' = \mathrm{cl}(H \setminus (V_0 \cup V_1)) = W_0 \cup_D W_1$ is a genus two handlebody and (i) holds. As before, α_0, α_1 are isotopic to cores of W_0, W_1, respectively, so that (ii) and (iii) hold follows by Van Kampen's theorem. The proof of (b)(iv) follows from the same argument as that of (a)(iv), using Lemma 2.2(d) instead of Lemma 2.2(c) to deduce $[\gamma_2] \equiv [u^{\mu(\gamma_0)}, v^{\mu(\gamma_1)}]$ from $[\gamma_2] \equiv [a^{\mu(\gamma_0)}, b^{\mu(\gamma_1)}]$. □

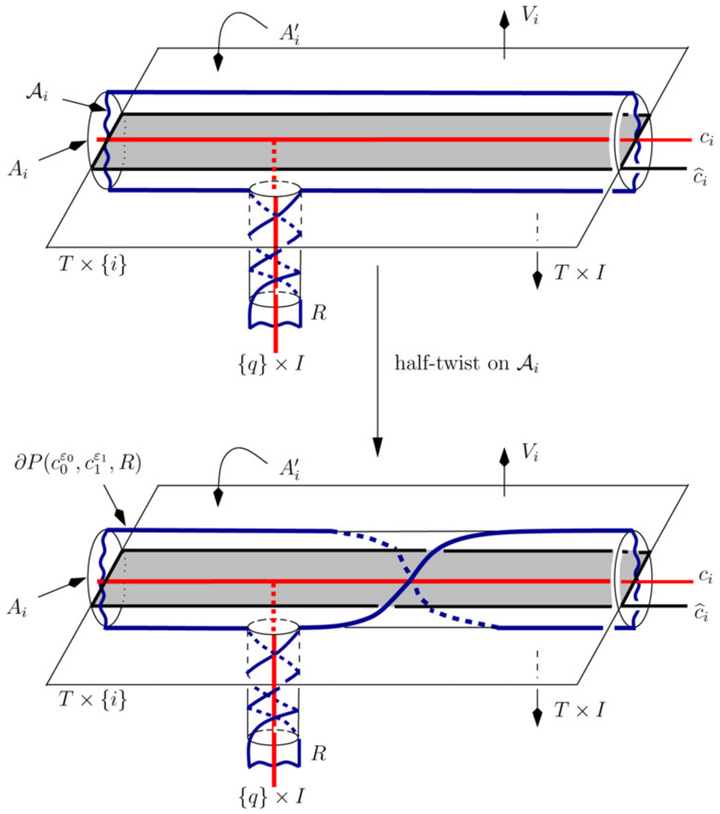

Fig. 3. Construction of the Seifert Klein bottle $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$.

3. Crosscap number two knots

In this section we establish necessary and sufficient conditions for a crosscap number two hyperbolic knot to admit a $(1,1)$ decomposition. We also establish miscellaneous results that can be used to detect when a crosscap number two knot with an unknotted Seifert Klein bottle has tunnel number one or is hyperbolic, as well as means of identifying and constructing the lifts of meridians, centers, and longitudes of a Seifert Klein bottle.

3.1. $(1,1)$ *decompositions*

The following construction of a special family of Seifert Klein bottles $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ in S^3 with boundary a $(1,1)$-knot is taken from [11, §1]. Let T be an unknotted (i.e., Heegaard) torus embedded in S^3; we identify a small regular neighborhood of T in S^3 with a product $T \times I$, where $I = [0,1]$. Thus, there are unknotted solid tori $V_0, V_1 \subset S^3$ such that

$$S^3 = V_0 \cup_{\partial V_0 = T\times\{0\}} T \times I \cup_{\partial V_1 = T\times\{1\}} V_1. \tag{3.1}$$

We say that an arc γ embedded in $T \times I$ is *monotone* if the natural projection map $T \times I \to I$ is monotone on γ. We may further assume that $T \times I$ lies within a slightly larger embedding of the form $T \times [-\delta, 1+\delta]$, for some small $\delta > 0$.

For $i = 0, 1$, let c_i be a circle nontrivially embedded in $T \times \{i\}$. Let R be a rectangle properly embedded in $T \times I$ with one boundary side along c_0 and the opposite side along c_1, such that $R \cap (T \times [0,\delta] \cup T \times [1-\delta, 1]) \subset c_0 \times [0,\delta] \cup c_1 \times [1-\delta, 1]$ and some core $\beta \subset T \times I$ of R is monotone. The union of R with the annuli $\mathcal{A}_i = c_i \times [i-\delta, i+\delta]$, $i = 0,1$, is then a pair of pants; giving one half-twist relative to $T \times \{i\}$ to each annulus piece $\mathcal{A}_i$, away from R, produces a once punctured Klein bottle $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$, where $\varepsilon_i \in \{+,-\}$ and the notation $c_i^{\varepsilon_i}$ stands for one of the two possible half-twists that can be performed on the annulus $\mathcal{A}_i$ (see Fig. 3). We remark that in [11] the knot $\partial P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ is denoted by $K(c_0^*, c_1^*, R)$.

We say that $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ is in *vertical position* if some monotone core $\beta \subset R$ is a fiber $\{q\} \times I$ of $T \times I$; in such case, $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ can be isotoped so as to properly embed in some regular neighborhood of $c_0 \cup (\{q\} \times I) \cup c_1$ in S^3. The next result states that vertical position is always attainable for any surface of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$.

Lemma 3.1. *Any once-punctured Klein bottle of the form* $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ *can be isotoped into vertical position.*

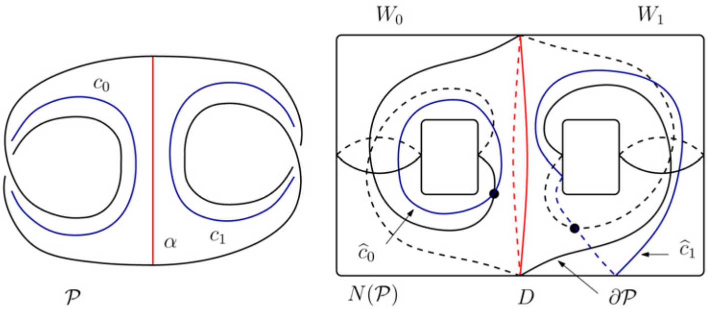

Fig. 4. A pair of disjoint centers c_0, c_1 in $\mathcal{P}$ and some twisted lifts $\hat{c}_0, \hat{c}_1 \subset T_{\mathcal{P}}$.

Proof. Consider an arbitrary once-punctured Klein bottle of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ with β a monotone core of R. We claim that the arc β is isotopic in $T \times I$ to some (and hence any) fiber $\{q\} \times I$, $q \in T$; in such case, the isotopy that moves β onto some fiber $\{p\} \times I$ can be extended to an isotopy of $c_0 \cup \beta \cup c_1$ in $T \times I$, and then further extended to an isotopy of S^3 that puts $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ in vertical position.

Given $0 \leqslant x \leqslant y \leqslant 1$ and any monotone arc $\gamma \subset T \times I$, denote the point $\gamma \cap (T \times \{x\})$ by γ_x and the arc $\gamma \cap (T \times [x, y])$ by $\gamma_{[x,y]}$. Since β is monotone in $T \times I$, there is a sufficiently large integer $n > 0$ such that, for each integer $1 \leqslant k \leqslant n$, the arc $\beta_{[(k-1)/n,k/n]}$ is isotopic, rel $\beta_{k/n}$ (i.e., fixing the set $\{\beta_{k/n}\}$), to the arc $\{\beta_{k/n}\} \times [(k-1)/n, k/n]$ in $T \times [(k-1)/n, k/n]$.

Isotope the arc $\beta_{[0,1/n]}$ onto the arc $\{\beta_{1/n}\} \times [0, 1/n]$ rel $\beta_{1/n}$, and let $\beta^{(1)}$ be the union of the arcs $\{\beta_{1/n}\} \times [0, 1/n]$ and $\beta_{[1/n,1]}$; clearly, $\beta^{(1)}$ and β are isotopic in $T \times I$ rel $\beta_{[1/n,1]}$. Now isotope the arc $\beta^{(1)}_{[1/n,2/n]}$ $(= \beta_{[1/n,2/n]})$ onto the arc $\{\beta_{2/n}\} \times [1/n, 2/n]$ in $T \times [1/n, 2/n]$ rel $\beta_{2/n}$; this isotopy easily extends to an isotopy of the arcs $\beta^{(1)}_{[0,2/n]}$ and $\{\beta^{(1)}_{2/n}\} \times [0, 2/n]$ in $T \times [0, 2/n]$ rel $\beta^{(1)}_{2/n}$, and produces the arc $\beta^{(2)} = \{\beta^{(1)}_{2/n}\} \times [0, 2/n] \cup \beta^{(1)}_{[2/n,1]}$ in $T \times I$, isotopic to $\beta^{(1)}$ rel $\beta^{(1)}_{[2/n,1]} = \beta_{[2/n,1]}$. Continuing the process in this fashion, the claim follows by induction, with β isotopic to $\beta^{(n)} = \{\beta_1\} \times [0, 1]$ in $T \times I$ rel β_1. □

Assumption 3.2. In light of Lemma 3.1, any Seifert Klein bottle of the form $\mathcal{P} = P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ constructed relatively to an unknotted $T \times I \subset S^3$ will be assumed to be in vertical position relatively to $T \times I$. In particular, we may always assume that $N(\mathcal{P}) = N(c_0 \cup \{q\} \times I \cup c_1)$ for some point $q \in T$ and that $T \times I \setminus \operatorname{int} N(\mathcal{P})$ is isotopic to $T_0 \times I$ for the once punctured torus $T_0 = T \setminus \operatorname{int} N(q) \subset T$ (see Fig. 3).

Now let K be a knot in S^3 spanning a once-punctured Klein bottle $\mathcal{P}$. If c_0, c_1 are two disjoint center circles of $\mathcal{P}$, we say that K *admits a* $\{\mathcal{P}, c_0, c_1\}$*-structure* if $\mathcal{P}$ is isotopic to some once-punctured Klein bottle of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$. In this context, [11, Theorem 1.1] can be restated, in the case of hyperbolic non 2-bridge knots, as follows:

Theorem 3.3. *([11].) Let K be a hyperbolic knot in S^3 which is not 2-bridge and bounds a Seifert Klein bottle $\mathcal{P}$. Then K has a $(1, 1)$ decomposition iff K admits a $\{\mathcal{P}, c_0, c_1\}$-structure for some pair of disjoint centers c_0, c_1 of $\mathcal{P}$.*

3.2. Twisted lifts of centers

Let $\mathcal{P} \subset X_K$ be any Seifert Klein bottle for a knot $K \subset S^3$, and let $N(\mathcal{P}) \subset X_K$ be its regular neighborhood. If c_0, c_1 are disjoint centers of $\mathcal{P}$, then, up to isotopy, there is a unique arc α properly embedded in $\mathcal{P}$ which separates c_0 from c_1, and which gives rise (via the I-bundle structure of $N(\mathcal{P})$) to a waist disk $D \subset N(\mathcal{P})$ with $D \cap \mathcal{P} = \alpha$, which cuts $N(\mathcal{P})$ into two solid tori W_0, W_1 with $c_i \subset W_i$ (see Fig. 4). Notice that such a waist disk is also unique up to isotopy, and that $B_i = \mathcal{P} \cap W_i$ is a Moebius band for $i = 1, 2$. Performing one half-twist to B_i in W_i produces an annulus in W_i; there are two ways of half-twisting B_i, and each way produces an annulus, say with boundary slope $\hat{c}_i$ or $\hat{c}'_i \subset \partial W_i \setminus D$, respectively. Each of the circles $\hat{c}_i, \hat{c}'_i$ runs once around W_i and intersects $\partial\mathcal{P}$ transversely in one point. We call the circles $\hat{c}_i, \hat{c}'_i$ the *twisted lifts* of the center $c_i \subset \mathcal{P}$ to $\partial N(\mathcal{P})$.

In the particular case where $\mathcal{P} = P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ for some disjoint centers $c_0, c_1 \subset \mathcal{P}$, relative to some unknotted $T \times I \subset S^3$, so that $N(\mathcal{P}) = N(c_0 \cup (\{q\} \times I) \cup c_1)$, it follows that any circle of intersection $\hat{c}_i$ between $\partial N(c_0 \cup (\{q\} \times I) \cup c_1)$ and $T \times \{i\}$ is a twisted lift of c_i (see Fig. 3), which we call the *induced twisted lift of* c_i.

Back in the general case, assume further that $\mathcal{P}$ has atoroidal exterior $X(\mathcal{P}) \subset S^3$; as $X(\mathcal{P})$ is irreducible, companion solid tori and multiplicities of circles on $\partial X(\mathcal{P}) \subset X(\mathcal{P})$ are thus defined. So, for any pair c_0, c_1 of disjoint centers of $\mathcal{P}$ with twisted lifts $\hat{c}_0, \hat{c}_1 \subset T_{\mathcal{P}} \subset \partial X(\mathcal{P})$, define $V(\hat{c}_i) \subset X(\mathcal{P})$ as the companion solid torus of $\hat{c}_i$ if $\mu(\hat{c}_i) \geqslant 2$ in $X(P)$, with $A(\hat{c}_i) \subset \partial W_i$ the annular neighborhood of $\hat{c}_i \subset \partial W_i$ such that $V(\hat{c}_i) \cap N(\mathcal{P}) = A(\hat{c}_i)$, and otherwise set $V(\hat{c}_i) = \emptyset = A(\hat{c}_i)$; we now construct the manifold

$$M(\mathcal{P}, \hat{c}_0, \hat{c}_1) = \operatorname{cl}\big(X(\mathcal{P}) \setminus \big(V(\hat{c}_0) \cup V(\hat{c}_0)\big)\big) \subset S^3. \tag{3.2}$$

Let D be a waist disk of $N(\mathcal{P})$ that separates $\hat{c}_0$ and $\hat{c}_1$ (see Fig. 4); D is unique up to isotopy, and can always be chosen such that

$$\partial D \subset \partial M(\mathcal{P}, \hat{c}_0, \hat{c}_1). \tag{3.3}$$

Let $\mathrm{cl}(N(\mathcal{P}) \setminus N(D)) = W_0 \sqcup W_1$, where W_0, W_1 are solid tori with $\hat{c}_0 \subset \partial W_0$ and $\hat{c}_1 \subset \partial W_1$. Then $U_0 = W_0 \cup_{A(\hat{c}_0)} V(\hat{c}_0)$ and $U_1 = W_1 \cup_{A(\hat{c}_1)} V(\hat{c}_1)$ are solid tori since, whenever present, each annulus $A(\hat{c}_i)$ runs once around W_i (see Fig. 4); since, in light of (2.1), we have

$$S^3 = U_0 \cup_\partial \big(M(\mathcal{P}, \hat{c}_0, \hat{c}_1) \cup N(D)\big) \cup_\partial U_1, \tag{3.4}$$

it follows that $M(\mathcal{P}, \hat{c}_0, \hat{c}_1) \cup N(D)$ is the exterior in S^3 of the link formed by cores of the solid tori U_0 and U_1.

The manifolds $M(\mathcal{P}, \hat{c}_0, \hat{c}_1)$ and $M(\mathcal{P}, \hat{c}_0, \hat{c}_1) \cup N(D)$ can be readily identified whenever $\mathcal{P}$ is of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$.

Lemma 3.4. *Suppose $\mathcal{P} = P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ for some disjoint centers $c_0, c_1 \subset \mathcal{P}$, relative to some unknotted $T \times I \subset S^3$; let $\hat{c}_0, \hat{c}_1$ be the induced twisted lifts, respectively, and let $D \subset N(\mathcal{P})$ be the unique waist disk that separates $\hat{c}_0$ and $\hat{c}_1$. If $X(\mathcal{P})$ is atoroidal then there is a once punctured torus $T_0 \subset T$ such that, in S^3, $M(\mathcal{P}, \hat{c}_0, \hat{c}_1)$ is isotopic to $T_0 \times I$ and $M(\mathcal{P}, \hat{c}_0, \hat{c}_1) \cup N(D)$ is isotopic to $T \times I$, with the isotopy carrying the circle $\partial D \subset \partial M(\mathcal{P}, \hat{c}_0, \hat{c}_1)$ to a circle in $\partial(T_0 \times I)$ isotopic to $(\partial T_0) \times \{0\}$.*

Proof. Recall from (3.1) that $S^3 = V_0 \cup T \times I \cup V_1$; also, by Assumption 3.2, we may assume that $N(\mathcal{P}) = N(c_0 \cup \{q\} \times I \cup c_1)$ for some $q \in T$, so that $T \times I \setminus \operatorname{int} N(\mathcal{P})$ is isotopic to $T_0 \times I$ for the once punctured torus $T_0 = T \setminus \operatorname{int} N(q) \subset T$.

For $i = 0, 1$, consider the annuli $A_i \subset \partial X(\mathcal{P})$ and $A'_i \subset T \times \{i\}$ indicated in Fig. 3, with $\partial A_i = \partial A'_i$. We then have

$$X(\mathcal{P}) = V'_0 \cup_{A'_0} T_0 \times I \cup_{A'_1} V'_1,$$

where $V'_i = V_i \setminus \operatorname{int} N(\mathcal{P}) \subset V_i$ is a solid torus. Now, for $i = 0, 1$, the circle $\hat{c}_i$ is isotopic to a core of the annulus $A_i \subset \partial X(\mathcal{P})$; moreover, if $\mu(\hat{c}_i) \geqslant 2$ in $X(\mathcal{P})$ then the annulus A'_i is a companion annulus for a core of A_i, so we can take $V(\hat{c}_i) = V'_i$, while if $\mu(\hat{c}_i) = 1$ in $X(\mathcal{P})$ then the annuli A_i and A'_i are isotopic in $V'_i \subset X(\mathcal{P})$. It is not hard to see now that the manifold $M(\mathcal{P}, \hat{c}_0, \hat{c}_1)$ may always be isotoped in S^3 onto the manifold $T_0 \times I$ in such a way that $\partial D \subset \partial M(\mathcal{P}, \hat{c}_0, \hat{c}_1)$ isotopes to a circle in $\partial(T_0 \times I)$ isotopic to $(\partial T_0) \times \{0\}$, and that such an isotopy can be extended to an isotopy that maps $M(\mathcal{P}, \hat{c}_0, \hat{c}_1) \cup N(D)$ onto $T \times I$. □

We are now ready to give necessary and sufficient conditions for a Seifert Klein bottle $\mathcal{P}$ to be of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$.

Lemma 3.5. *Let $K \subset S^3$ be a knot, P any Seifert Klein bottle in X_K with atoroidal exterior $X(P)$, c_0, c_1 any two disjoint centers of P, $\hat{c}_0, \hat{c}_1 \subset T_P$ any two twisted lifts of c_0, c_1, respectively, and D the unique waist disk of $N(P)$ that separates $\hat{c}_0$ and $\hat{c}_1$. Then, K admits a $\{P, c_0, c_1\}$-structure with induced twisted lifts $\hat{c}_0, \hat{c}_1$ iff the following conditions hold*:

(a) *the manifold $M(P, \hat{c}_0, \hat{c}_1)$ is a genus two handlebody, and*
(b) *$M(P, \hat{c}_0, \hat{c}_1) \cup N(D) \approx \mathcal{T} \times I$, where $\mathcal{T}$ is a closed torus.*

Proof. If such a $\{c_0, c_1\}$-structure exists, that (a) and (b) hold follows from Lemma 3.4.

Conversely, suppose (a) and (b) hold. By (b), $M(P, \hat{c}_0, \hat{c}_1) \cup N(D) = T \times I$ for some closed torus $T = T \times \{0\} \subset S^3$; the identity in (3.4) implies that T is unknotted in S^3 and hence that $M(P, \hat{c}_0, \hat{c}_1) \cup N(D)$ is the exterior of the Hopf link in S^3. We will assume that $\hat{c}_i \subset T \times \{i\}$ for $i = 0, 1$, and that $T \times I$ lies inside a slightly larger product of the form $T \times [-\delta, 1 + \delta]$ for some sufficiently small $\delta > 0$.

Let $N(D)$ be a small regular neighborhood of D in $N(P)$ which is disjoint from c_i and $\hat{c}_i$ for $i = 0, 1$, and such that $R = N(D) \cap P$ is a rectangle properly embedded in $N(D)$ whose core $\alpha \subset R$ is also a cocore of the 2-handle $N(D)$.

Recall from the construction of $M(P, \hat{c}_0, \hat{c}_1)$ that $\mathrm{cl}(N(P) \setminus N(D)) = W_0 \sqcup W_1$, where W_0, W_1 are solid tori in $N(P)$ with $\hat{c}_0 \subset \partial W_0$ and $\hat{c}_1 \subset \partial W_1$, so that for $i = 0, 1$, we have

$$A_i = W_i \cap \big(M(P, \hat{c}_0, \hat{c}_1) \cup N(D)\big) = W_i \cap (T \times I) = (\partial W_i) \cap \partial(T \times I)$$

is an annulus with core $\hat{c}'_i \subset \operatorname{int} A_i$ isotopic to $\hat{c}_i$ in ∂W_i and $\partial(T \times I)$. We may further assume, after an isotopy of $W_0 \cup (T \times I) \cup W_1$ which leaves $T \times I$ fixed, that $W_0 = A_0 \times [-\delta, 0]$, and $W_1 = A_1 \times [1, 1 + \delta]$ in $T \times [-\delta, 1 + \delta]$.

Consider now the Moebius bands $B_i = P \cap W_i$, $i = 0, 1$; the rectangle R has one boundary side on $\partial B_0 \cap T \times \{0\}$ and its opposite side on $\partial B_1 \cap T \times \{1\}$, with $P = B_0 \cup R \cup B_1$. Since $\hat{c}_0$ is a twisted lift of c_0 in $N(P)$, giving one half-twist to B_0 in W_0 away from $R \subset N(D)$ produces an annulus properly embedded in W_0 with the same boundary slope as the core $\hat{c}'_0$ of A_0; this annulus can be isotoped within W_0 onto the annulus $\hat{c}'_0 \times [-\delta, 0]$, with $c_0 \subset B_0$ corresponding to the circle $\hat{c}'_0 \times \{-\delta/2\}$. In a similar way, one half twist on $B_1 \subset W_1$ (away from R) may be assumed to produce the annulus $\hat{c}'_1 \times [1, 1 + \delta]$, with $c_1 \subset B_1$ corresponding to the circle $\hat{c}'_1 \times \{1 + \delta/2\}$. Finally, the rectangle $R \subset T \times I$ may be slightly isotoped within $T \times I$ so that its side $R \cap T \times \{0\}$ lies on $\hat{c}'_0$ and its side $R \cap T \times \{1\}$ on $\hat{c}'_1$, in such a way that P may be recovered from the pair of pants $Q = \hat{c}_0' \times [-\delta, 0] \cup R \cup \hat{c}_1' \times [1, 1+\delta] \subset N(P)$ by performing the corresponding reverse one half twists on the annuli $\hat{c}_0' \times [-\delta, 0] \subset W_0$ and $\hat{c}_1' \times [1, 1+\delta] \subset W_1$.

Now, the endpoints $q_0 = \alpha \cap T \times \{0\}$, $q_1 = \alpha \cap T \times \{1\}$ of α lie on $\hat{c}_0', \hat{c}_1'$, respectively. Therefore, if α' is the arc $\{q_0\} \times [-\delta/2, 0] \cup \alpha \cup \{q_1\} \times [1, 1+\delta/2] \subset Q$ then α' is an arc properly embedded in the product $T \times [-\delta/2, 1+\delta/2]$ with endpoints on $c_0 \sqcup c_1$ such that $\mathrm{cl}(T \times [-\delta/2, 1+\delta/2] \setminus N(\alpha'))$ is homeomorphic to $\mathrm{cl}(T \times [0,1] \setminus N(\alpha))$.

By (a), the manifold $\mathrm{cl}(T \times [0,1] \setminus N(\alpha)) \approx \mathrm{cl}(T \times [0,1] \setminus N(D)) = M(P, \hat{c}_0, \hat{c}_1)$ is a genus two handlebody. Since $T \times [-\delta/2, 1+\delta/2]$ and $T \times [0,1]$ are isotopic in S^3, and $T \times [0,1] \subset S^3$ is the exterior of the Hopf link, it follows that α' is a tunnel for the Hopf link exterior $T \times [-\delta/2, 1+\delta/2] \subset S^3$. As the Hopf link is a 2-braid link, by [1, Theorem 2.1] such a tunnel arc α' is unique up to isotopy, hence α' can be isotoped into a fiber $\{*\} \times [-\delta/2, 1+\delta/2]$ of $T \times [-\delta/2, 1+\delta/2]$. Therefore P is isotopic to a surface of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ and so K admits a $\{P, c_0, c_1\}$-structure. □

With the aid of Lemma 2.6, the conditions in Lemma 3.5 can now be expressed entirely in simple algebraic terms whenever the Seifert Klein bottle surface P is unknotted; this is the main content of the next result.

Lemma 3.6. *Suppose P is an unknotted Seifert Klein bottle for a knot $K \subset S^3$. Let c_0, c_1 be disjoint centers of P with corresponding twisted lifts $\hat{c}_0, \hat{c}_1 \subset \partial X(P)$ and D the unique waist disk of $N(P)$ that separates $\hat{c}_0$ and $\hat{c}_1$.*

(a) *The knot K admits a $\{P, c_0, c_1\}$-structure with induced twisted lifts $\hat{c}_0, \hat{c}_1$ iff one of the following set of conditions holds:*
  (i) *$\mu(\hat{c}_0) = 1 = \mu(\hat{c}_1)$ and $[\partial D] \equiv [u, v]$ in $\pi_1(X(P))$ for some (and hence any) basis $\{u, v\}$ of $\pi_1(X(P))$;*
  (ii) *for some $\{i, j\} = \{0, 1\}$: $\mu(\hat{c}_i) = 1$, $\mu(\hat{c}_j) \geqslant 2$, and $\partial D \equiv [u, v^{\mu(\hat{c}_j)}]$ in $\pi_1(X(P))$ for some (and hence any) basis $\{u, v\}$ of $\pi_1(X(P))$ such that $[\hat{c}_j] \equiv v^{\mu(\hat{c}_j)}$;*
  (iii) *$\mu(\hat{c}_0) \geqslant 2$, $\mu(\hat{c}_1) \geqslant 2$, and $\partial D \equiv [u^{\mu(\hat{c}_0)}, v^{\mu(\hat{c}_1)}]$ in $\pi_1(X(P))$ for some (and hence any) basis $\{u, v\}$ of $\pi_1(X(P))$ such that $[\hat{c}_0] \equiv u^{\mu(\hat{c}_0)}$ and $[\hat{c}_1] \equiv v^{\mu(\hat{c}_1)}$.*

(b) *Suppose K admits a $\{P, c_0, c_1\}$-structure with induced twisted lifts $\hat{c}_0, \hat{c}_1$, and let $\{i, j\} = \{0, 1\}$. If $\mu(\hat{c}_i) = 1$ in $X(P)$ then, in $\pi_1(X(P))$, either $[\hat{c}_i], [\hat{c}_j]$ are both primitive or $[\hat{c}_j]$ is a power of a primitive.*

Proof. Since $X(P)$ is a genus two handlebody, part (a) follows from a direct application of Lemma 2.6 to Lemma 3.5; in particular, observe that, by Lemma 2.6, a basis $\{u, v\}$ of $\pi_1(X(P))$ satisfying the condition $[\hat{c}_j] \equiv v^{\mu(\hat{c}_j)}$ in (ii) or the conditions $[\hat{c}_0] \equiv u^{\mu(\hat{c}_0)}$ and $[\hat{c}_1] \equiv v^{\mu(\hat{c}_1)}$ in (iii) always exists.

For part (b) we have that P is a surface of the form $P(c_0^{\varepsilon_0}, c_1^{\varepsilon_1}, R)$ with induced twisted lifts $\hat{c}_0, \hat{c}_1$; we assume for definiteness that $\mu(\hat{c}_0) = 1$ in $X(P)$. If $\mu(\hat{c}_1) = 1$ too then $X(P) = M(P, \hat{c}_0, \hat{c}_1)$; since, by Lemma 3.4, $M(P, \hat{c}_0, \hat{c}_1) = T_0 \times I$ for some once punctured torus $T_0 \subset S^3$ such that $\hat{c}_0 \subset T_0 \times \{0\}$ and $\hat{c}_1 \subset T_0 \times \{1\}$, it follows that $[\hat{c}_0], [\hat{c}_1]$ are primitive in $\pi_1(X(P)) = \pi_1(T_0 \times I) = \pi_1(T_0)$. Otherwise, $\mu(\hat{c}_1) \geqslant 2$ and so $[\hat{c}_1]$ is a power of a primitive in $\pi_1(X(P))$ by Lemma 2.3(b). □

3.3. A tunnel number one criterion

The next result gives a condition for a knot with an unknotted Seifert Klein bottle to have tunnel number one.

Lemma 3.7. *Let $K \subset S^3$ be a nontrivial knot with an unknotted Seifert Klein bottle P. Then K has tunnel number one if some center circle of P has a lift to T_P which is primitive in $X(P)$.*

Proof. Since P is unknotted, it follows from (2.1) that $N(P) \cup X(P)$ is a genus two Heegaard decomposition of S^3. Let c_0 be any center circle of P, and let c_1 be the only other center circle of P which is disjoint from c_0. Then $N(P) = W_0 \cup_D W_1$, where $W_0, W_1 \subset N(P)$ are solid tori and D is the (unique) waist disk D of $N(P)$ which separates c_0, c_1. Finally, let $B(c_0)$ be the Moebius band in $N(P)$ constructed in Section 2.1, so that $\partial B(c_0) \subset T_P$ is the lift of c, and let $V(c_0)$ be a regular neighborhood of $B(c_0)$ in $N(P)$ disjoint from ∂P; $V(c)$ is a solid torus.

If $\partial B(c_0)$ is primitive in $X(P)$ then $H_1 = X(P) \cup V(c_0)$ is a genus two handlebody, while clearly $H_2 = \mathrm{cl}(N(P) \setminus V(c_0)) \subset S^3$ is also a genus two handlebody; thus $H_1 \cup H_2$ is a Heegaard decomposition for S^3. From the representation of $N(P), D, \partial P$ shown in Fig. 4 it is not hard to see that $[\partial P]$ is primitive in $\pi_1(H_2)$, which implies that there is a properly embedded disk D_0 in H_2 which intersects ∂P transversely in one point. Therefore the knot ∂P, and hence K, has tunnel number one. □

3.4. Hyperbolicity

Lemma 3.8. *Let $k \subset S^3$ be a knot which is a (possibly trivial) torus knot or a 2-bridge knot. Let P be a Seifert Klein bottle for k with meridian circle m, and let T_P be the frontier of $N(P)$ in X_k.*

(a) *If k is a trivial or 2-bridge knot then m is a trivial knot in S^3.*

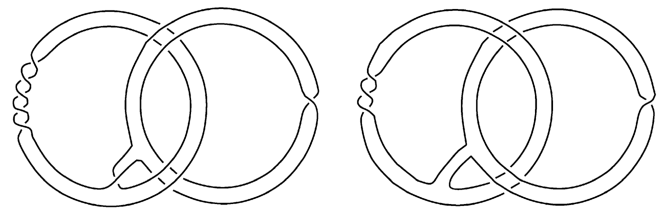

Fig. 5. The torus knots $T(3,7)$ and $T(3,5)$.

(b) *If k is nontrivial then P is unknotted; moreover, if k is a torus knot then*
 (i) *T_P is not π_1-injective in X_k,*
 (ii) *T_P is boundary compressible in X_k iff $k = T(3,5)$, $T(3,7)$, or $T(2,n)$ for some odd integer $n \neq \pm 1$, in which case*

$$m = \begin{cases} \text{trivial knot}, & k \neq T(3,7), \\ T(2,3), & k = T(3,7). \end{cases}$$

Proof. If k is a 2-bridge knot then it follows from [11, Theorem 1.2(a)] that k is a plumbing of an annulus A and a Moebius band B, which are unknotted and unlinked in S^3. Moreover, k bounds a unique Seifert Klein bottle P by [7], namely the one obtained as the plumbing of A and B, so P is unknotted. This proves part (a) for k a 2-bridge knot; since the meridian circle m of P is the core of A, it also follows that m is a trivial knot in S^3.

The only nontrivial torus knots that bound a Moebius band are those of the form $T(2,n)$ for some odd integer $n \neq \pm 1$. Hence, any nontrivial torus knot k which is not of the form $T(2,n)$ and bounds a Seifert Klein bottle has crosscap number two, so it follows from [16, Corollary 1.6] that P is unique in X_k up to isotopy and that P is unknotted and not π_1-injective in X_k; moreover, by the argument used in the proof of [16, Corollary 1.6], T_P is boundary compressible in X_k iff k is a torus knot of the form $T(5,3)$ or $T(3,7)$. Projections of the knots $T(3,5)$ and $T(3,7)$ spanning their unique Seifert Klein bottle P are shown in Fig. 5; it can be seen that indeed the meridian circle m of P is the trivial knot and the trefoil $T(2,3)$, respectively ($T(3,5)$ can also be drawn as the pretzel $(-2,3,5)$, again showing the meridian circle of its Seifert Klein bottle is trivial).

We now deal with the cases where k is either a nontrivial torus knot of the form $T(2,n)$, or k is the trivial knot with a Seifert Klein bottle whose boundary slope, relative to a standard meridian-longitude pair $\mu, \lambda \subset \partial X_k$, is not 0/1. These cases have in common that any Seifert Klein bottle P in X_k, and hence the associated surface T_P, boundary compress in X_k: this follows from [12, Lemma 4.2] if $k = T(2,n)$, while if k is trivial then X_k is a solid torus and hence the surface P must boundary compress in X_k. In either case, P boundary compresses into a Moebius band $B \subset X_k$ such that $\Delta(\partial P, \partial B) = 2$. If $k = T(2,n)$ then B is unique and so, for a fixed boundary slope, P is obtained by adding a band (rectangle) in ∂X_k to the Moebius band B, and hence P is unique up to isotopy in X_k; if k is the trivial knot we will see below that though B may not be unique, the surfaces P are unique relative to their boundary slopes.

Using a standard meridian-longitude pair μ, λ in ∂X_k, we may assume that $\partial P = a\mu + \lambda$ and

$$\partial B = \begin{cases} 2\mu + b\lambda \ (b = \text{odd}), & k = \text{trivial}, \\ 2n\mu + \lambda, & k = T(2,n). \end{cases}$$

Since $\Delta(\partial P, \partial B) = 2$, we must then have

$$a = \begin{cases} \pm 4 \text{ and } b = \mp 1, & k = \text{trivial}, \\ 2n \pm 2, & k = T(2,n). \end{cases}$$

The knot $k = T(2,n)$ can be represented as the pretzel knots $(n-2, 1, -2)$ or $(n+2, -1, 2)$, and in these projections k spans a Seifert Klein bottle with boundary slope $2n-2, 2n+2$, respectively, all having meridian circle a trivial knot. Since the Seifert Klein bottle for k is unique for each of these slopes, it follows that the meridian circle of any Seifert Klein bottle bounded by $k = T(2,n)$ is a trivial knot.

If k is the trivial knot then it follows from the above computation that k bounds two Seifert Klein bottles with boundary slopes distinct from 0/1: P_1 with boundary slope $+4$, which boundary compresses to the unique Moebius band $B_1 \subset X_k$ such that $\partial B_1 = 2\mu - \lambda$, and P_2 with boundary slope -4, which boundary compresses to the unique Moebius band $B_2 \subset X_k$ such that $\partial B_2 = 2\mu + \lambda$. Therefore P_1 and P_2 are unique relative to their boundary slope; the trivial knot k can be represented as the pretzel knots $(0,1,1)$ or $(0,-1,-1)$ and in these projections k spans the unique Seifert Klein bottles with boundary slopes $4, -4$, respectively, both of which have the trivial knots as meridian circle.

Therefore part (b) holds, and for part (a) only the case when k is trivial and P has boundary slope 0/1 remains. But in this last case, if D is the meridian disk of the solid torus X_k then ∂D and ∂P both have the same boundary slope and so P and D can be isotoped in X_k so that ∂D and ∂P are disjoint, and hence so that $P \cap D$ intersect transversely and minimally in circles only. Necessarily, $P \cap D$ is nonempty and consists of orientation preserving circles in P, i.e. of meridians or longitudes of P (cf. Section 2.1). As an innermost disk D' of $P \cap D \subset D$ compresses P along $\partial D'$, the circle ∂D must be a meridian of P, for otherwise ∂D would be a longitude of P that bounds some Moebius band $B \subset P$, and $B \cup_\partial D'$ would be a closed projective plane in S^3, an impossibility. Hence the meridian circle of P is a trivial knot in S^3. □

The above lemma can be used to give conditions under which a knot bounding a Seifert Klein bottle P is hyperbolic in terms of the lifts of the meridians and the longitudes of P.

Lemma 3.9. *Let K be a nontrivial knot with Seifert Klein bottle P such that $X(P)$ is atoroidal. Let m_1, m_2 be the lifts of the meridian circle m of P and $\mathcal{L}$ the collection of lifts of longitudes of P. If, in $X(P)$, $\mu(\ell') = 1$ for each $\ell' \in \mathcal{L}$ and either $\mu(m_1) = 1$ or $\mu(m_2) = 1$, then X_K is atoroidal, and in such case,*

(a) *if P is π_1-injective in X_K then K is hyperbolic,*
(b) *if the frontier T_P of $N(P)$ is boundary compressible in X_K and the meridian circle m of P is a nontrivial knot in S^3 then either K is hyperbolic and not 2-bridge, or $K = T(3, 7)$ and $m = T(2, 3)$.*

Proof. By Lemma 2.3(b), no element of $\mathcal{L}$, nor say m_2, has a companion annulus in $X(P)$.

Suppose T is an essential torus in X_K. Since, by [12, Lemma 4.2], the fact that K is a nontrivial knot implies that the surface P is incompressible in X_K, and both $N(P)$ and $X(P)$ are atoroidal, we may assume T intersects P transversely and minimally with $P \cap T$ a nonempty collection of circles which are nontrivial in both P and T. Necessarily, $P \cap T \subset T$ is a family of mutually parallel nontrivial circles, so we may assume that $T \cap N(P)$ is a collection of annuli which are fibered under the I-bundle structure of $N(P) = P \tilde{\times} I$, i.e. $T \cap N(P) = (T \cap P) \tilde{\times} I \subset N(P)$, and that each component of $T \cap X(P)$ is an annulus. Moreover, each circle in $P \cap T \subset P$ is either a meridian circle, a longitude circle, or a circle parallel to ∂P. It is not hard to see that if all circles $P \cap T \subset P$ are parallel to ∂P then T must be parallel to ∂X_K, which is not the case. Therefore there is a component A of $T \cap X(P)$ such that at least one of its boundary components $\partial_1 A, \partial_2 A$ is not parallel to ∂P in P. By [16, Lemma 2.3] we then have that, in $\partial X(P)$, the circles $\partial_1 A, \partial_2 A$ are either both parallel to m_1, both parallel to m_2, or both parallel to some element of $\mathcal{L}$. Given our hypothesis on $\mathcal{L}$ and m_1, m_2, since $|P \cap T|$ is minimal, this implies necessarily that A is a companion annulus in $X(P)$ for m_1. By minimality of $|P \cap T|$, there are two annular components A', A'' of $T \cap N(P)$ with, say, $\partial_1 A = \partial_1 A'$, $\partial_2 A = \partial_1 A''$, and with $\partial_2 A, \partial_2 A'' \subset \partial N(P) = \partial X(P)$ parallel to m_2. Applying [16, Lemma 2.3] again, it follows that some component of $T \cap X(P)$ must be a companion annulus of m_2 in $X(P)$, which by hypothesis is not the case.

Therefore X_K is atoroidal and so by [15] the knot K is either a hyperbolic or torus knot, hence part (a) holds by Lemma 3.8(b)(i). Suppose now T_P is boundary compressible in X_K. If m is a nontrivial knot then K is not a 2-bridge knot by Lemma 3.8(a), and if K is a torus knot then by Lemma 3.8(b)(ii) K must be the $T(3, 7)$ torus knot and $m = T(2, 3)$; hence (b) holds. □

3.5. Meridians, centers, longitudes, and their lifts

Our first result gives a criterion to identify the lifts of a meridian or longitude circle of a once punctured Klein bottle$P$; along with Lemma 2.3(c), this result will provide a simple algebraic way of identifying the lifts of the meridian.

Lemma 3.10. *Let P be a once punctured Klein bottle and $H = P \tilde{\times} I$. If A is an incompressible annulus properly embedded in H with $\partial A \subset \partial H \setminus \partial P$, and A is not parallel into ∂H, then A can be isotoped in H so that $A \cap P$ is either one meridian (if A is nonseparating) or one longitude (if A is separating) circle of P; in particular, ∂A are the lifts of the circle $A \cap P$ to ∂H.*

Proof. Let A_P be an annular regular neighborhood of ∂P in ∂H, T be the twice punctured torus $\partial H \setminus \operatorname{int} A_P$, and M be the manifold obtained by cutting H along P, so that $M = T \times I$ with T corresponding to $T \times 0$.

Isotope A and P in H so as to intersect transversely and minimally. If $A \cap P = \emptyset$ then A lies in $M = T \times I$ with $\partial A \subset T \times 0$; but then A is parallel into $T \times 0$ in M (cf. [17, Corollary 3.2]), and hence A is parallel into ∂H in H, contradicting our hypothesis; thus $A \cap P \neq \emptyset$. Since A is orientable, $A \cap P$ consists of a collection of circles which preserve orientation in P, i.e., of longitudes of P only or meridians of P only. If α, β are distinct components of $A \cap P$ which cobound an annulus $A' \subset A$ with $P \cap \operatorname{int} A' = \emptyset$, then the annulus $A' \cap M$ has both of its boundary components in $T \times 1$, and hence A' is parallel into $T \times 1$ in $T \times I$. It follows that A' is parallel into P in H, contradicting the fact that $A \cap P$ is minimal. Therefore, $A \cap P$ consists of a single circle γ, a meridian or longitude of P, and so ∂A are the lifts of γ to ∂H. □

Now let m be the meridian circle of P, and let c, ℓ be a center and a longitude circle of P, respectively. If $c(n), \ell(n) \subset P$ denote the circles obtained by Dehn-twisting n-times the circles c, ℓ along m, it follows from [16, Lemma 3.1] that the collections $\{c(n) \mid n \in \mathbb{Z}\}$ and $\{\ell(n) \mid n \in \mathbb{Z}\}$ consist of all center and longitude circles of P up to isotopy.

This fact generalizes into the following result, which describes the construction of the twisted lifts of all centers of P and of all lifts of longitudes; its proof follows easily from the fact that Dehn-twisting $P \tilde{\times} I$ along the fibered annulus $A(m) \subset P \tilde{\times} I$ defined in Section 2.1 is an automorphism of H that fixes m and maps $P \subset H$ into itself.

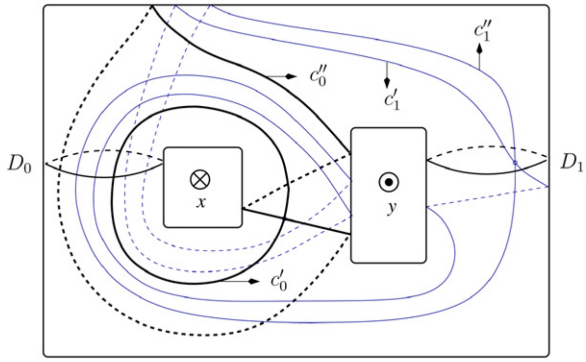

Fig. 6.

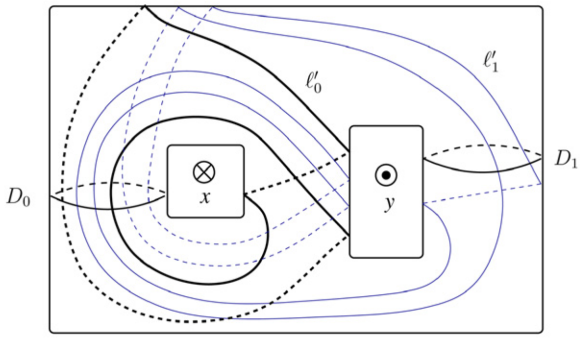

Fig. 7.

Lemma 3.11. *Let P be a once punctured Klein with meridian circle $m \in P$ and fibered annulus $A(m) \subset H = P \tilde{\times} I$. Let c_0, c_1 be a pair of disjoint center circles of P with c_0', c_0'' and c_1', c_1'' their twisted lifts to ∂H, respectively, and let $\ell' \subset \partial H$ be the lift of any longitude of P.*

Let $c_0(n), c_1(n) \subset P$ and $c_0'(n), c_0''(n), c_1'(n), c_1''(n), \ell'(n) \subset \partial H$ be the circles obtained from $c_0, c_1, c_0', c_0'', c_1', c_1'', \ell'$ after Dehn twisting H n-times along $A(m)$. Then,

(a) *the collection $\{(c_0(n), c_1(n)) \mid n \in \mathbb{Z}\}$ consists of all pairs of disjoint centers of P, and the twisted lifts of each $c_i(n)$ are the circles $c_i'(n), c_i''(n)$;*
(b) *the collection $\{\ell'(n) \mid n \in \mathbb{Z}\}$ consists of all lifts of longitudes of P.*

4. The knots $K(p,q)$

We begin this section by providing a detailed construction of the family of knots $K(p,q)$ given in Section 1. Let H be a genus two handlebody standardly embedded in S^3 with a complete system of meridian disks D_0, D_1, as shown in Fig. 6, and let c_0', c_0'' and c_1', c_1'' be the four circles embedded in the boundary ∂H shown in the same figure; notice that any pair formed by one circle in c_0', c_0'' and one in c_1', c_1'' give rise to a basis of $\pi_1(H)$. The homological sums ℓ_0' and ℓ_1' of the pairs of curves c_0', c_0'' and c_1', c_1'', respectively, indicated in Fig. 7, bound disjoint Moebius bands B_0, B_1 in H, respectively.

A waist disk D_2 of H separating B_1 and B_2 can be constructed from the meridian disk D_0 of H in Fig. 9, which is disjoint from B_1 and intersects B_0 in a single essential arc, by taking the frontier of a regular neighborhood of $D_0 \cup c_1'$ (or of $D_0 \cup c_1''$) in H; such a waist disk of H is unique up to isotopy.

We now connect the Moebius bands B_0, B_1 with the rectangle R shown in Fig. 8 and produce a properly embedded Seifert Klein bottle $P(0,0)$ in H whose boundary $\partial P(0,0)$ is isotopic to the knot $K(0,0)$ shown in Fig. 1; notice the cores c_0, c_1 of B_0, B_1, respectively, are disjoint centers of $P(0,0)$. The fact that R intersects D_2 transversely in a single arc implies that $H = P(0,0) \tilde{\times} I$, whence $H' = X(P(0,0))$, by Section 3.2 that the twisted lifts of the centers c_0, c_1 are the pairs c_0', c_0'' and c_1', c_1'', respectively, and by Section 2.1 that $\ell_0' = \partial B_0$ and $\ell_1' = \partial B_1$ are the lifts of the longitudes of P corresponding to the centers c_0, c_1, respectively. In particular, $P(0,0)$ is unknotted.

Finally, consider the circles m_0, m_1 shown in Fig. 10, which are disjoint from $K(0,0)$. Relative to the base of $\pi_1(H)$ dual to the meridian disks D_0, D_1, m_0, m_1 give rise to conjugate primitive words and so m_0, m_1 cobound a nonseparating annulus in $A(0,0)$ in H by Lemma 2.3(c). By Lemma 3.10, it follows that m_0, m_1 are the lifts of the meridian m of $P(0,0)$, so $A(0,0)$ can be isotoped so as to intersect $P(0,0)$ in m; thus $A(0,0)$ is the fibered annulus generated by m in $H = P \tilde{\times} I$.

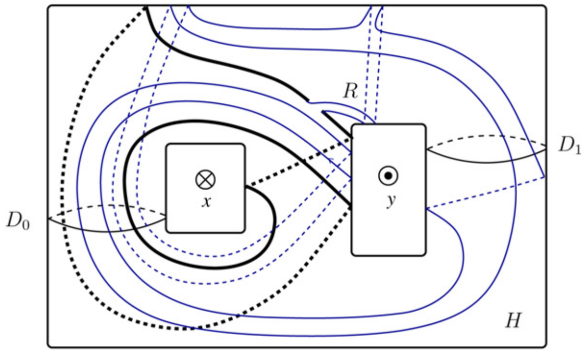

Fig. 8. The Seifert Klein bottle bounded by $K(0,0)$ in H.

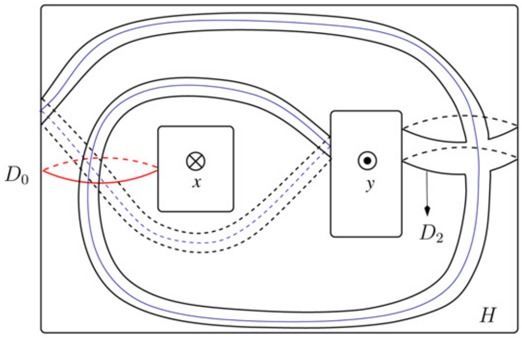

Fig. 9.

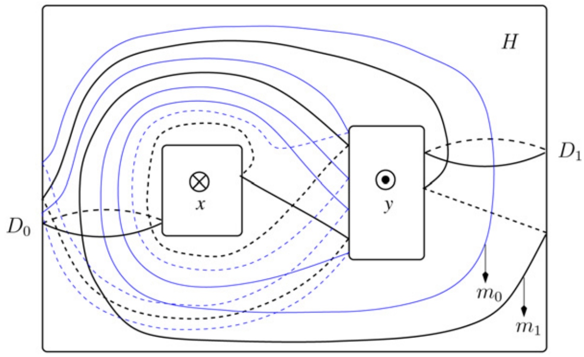

Fig. 10.

By Lemma 3.11, Dehn twisting H n times along $A(0,0)$ for $n \in \mathbb{Z}$ gives rise to the collections $\{(c'_0(n), c''_0(n)) \mid n \in \mathbb{Z}\}$ and $\{(c'_1(n), c''_1(n)) \mid n \in \mathbb{Z}\}$ of twisted lifts of disjoint pairs of centers of P, as well as to the collection $\{\ell'(n) \mid n \in \mathbb{Z}\}$ of all lifts of longitudes of P (with c'_0 giving rise to $c'_0(n)$, ℓ'_0 to $\ell'_0(n)$, etc.). At the same time the waist disk $D_2 \subset H$ gives rise to a waist disk $D_2(n)$ separating the twisted lifts $c'_0(n) \cup c''_0(n)$ and $c'_1(n) \cup c''_1(n)$.

Remark 4.1. Any once punctured Klein bottle with H as regular neighborhood can be constructed following the procedure outlined above for $P(0,0)$. That is, once the pairs of circles c'_0, c''_0 and c'_1, c''_1 (such that any pair formed by one circle in c'_0, c''_0 and one in c'_1, c''_1 is a basis for $\pi_1(H)$), and the disk D_2 separating them are given, any rectangle $R \subset H$ that intersects D_2 in one arc may be used to join the Moebius bands B_0, B_1; the latter condition on R guarantees that $H = P \tilde{\times} I$. In general, different rectangles R will give rise in H to nonisotopic once punctured Klein bottles whose meridians induce nonisotopic fibered annuli.

For any integers p, q, the Seifert Klein bottle $P(p,q)$ and the knot $K(p,q) = \partial P(p,q)$ are then obtained by Dehn twisting the pair $(P(0,0), K(0,0))$ p and q times along the disks D_0, D_1 of Fig. 1, respectively, as explained in Section 1. We will denote the pairs of twisted lifts of centers of $P(p,q)$ by $c'_0(n,p,q), c''_0(n,p,q)$ and $c'_1(n,p,q), c''_1(n,p,q)$, by $D_2(n,p,q) \subset H$

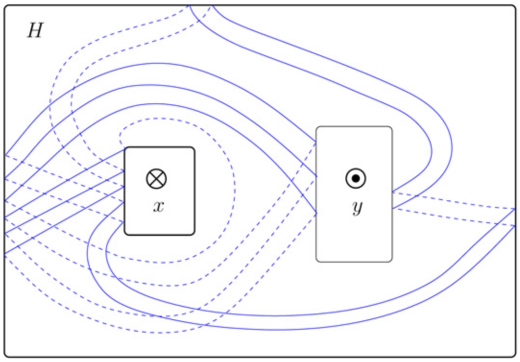

Fig. 11. The knot $K(-1,1) \subset \partial H$.

the waist disk that separates these pairs, by $\ell'(n,p,q)$ the lifts of the longitudes of $P(p,q)$, and by $A(p,q) \subset H$ the fibered annulus in $H = P(p,q) \tilde{\times} I$ induced by the meridian circle m of $P(p,q)$ (with $c_0'(n)$ giving rise to $c_0'(n,p,q)$ after the Dehn twists along D_0, D_1, etc.).

By construction we have $H = N(P(p,q)) = P(p,q) \tilde{\times} I$ and $H' = X(P(p,q))$, so $P(p,q)$ is unknotted. Let $\pi_1(H') = \langle x, y \rangle$, where x, y is the base dual to the complete disk system of H' indicated in Fig. 1 (with the disks, and hence the circles x, y, oriented by the arrow head and arrow tail shown in the same figure). It is not hard to see that the words in $\pi_1(H') = \langle x, y \mid - \rangle$ corresponding to the circles constructed above, up to equivalence, are the ones given in the next lemma; for convenience, we may denote the elements $[c_0'(n,p,q)], [c_0''(n,p,q)]$, etc., of $\pi_1(H')$ simply by $c_0'(n,p,q), c_0''(n,p,q)$, etc.

Lemma 4.2. *In $\pi_1(H') = \langle x, y \mid - \rangle$,*

(i) $c_0'(n,p,q) \equiv x^p(x^{p+1}yx^py^{q+1}x^py)^n$;
(ii) $c_0''(n,p,q) \equiv x^pyxy(\bar{y}x^pyx^py^qx^pyx^p\bar{y}\bar{x}^p)^n$;
(iii) $c_1'(n,p,q) \equiv y\bar{x}^p\bar{y}\bar{x}^p\bar{y}^q(y^{q+1}x^pyx^{p+1}yx^p)^n$;
(iv) $c_1''(n,p,q) \equiv \bar{y}^q\bar{x}^p\bar{y}\bar{x}^p(x^pyx^py^qx^pyx^p\bar{y}\bar{x}^p\bar{y})^n$;
(v) $\ell_0'(n,p,q) \equiv (x^{p+1}yx^py^{q+1}x^py)^nx^{p+1}y(\bar{y}x^pyx^py^qx^pyx^p\bar{y}\bar{x}^p)^nx^py$;
(vi) $\ell_1'(n,p,q) \equiv (y^{q+1}x^pyx^{p+1}yx^p)^ny\bar{x}^p\bar{y}\bar{x}^p\bar{y}^q\bar{x}^p\bar{y}\bar{x}^p(x^pyx^py^qx^pyx^p\bar{y}\bar{x}^p\bar{y})^n\bar{y}^q$;
(vii) $$\partial D_2(n,p,q) \equiv \left(x^pyx^{p+1}yx^py^{q+1}\right)^n\left(\bar{y}^q\bar{x}^p\bar{y}\bar{x}^pyx^py\bar{x}^p\bar{y}\bar{x}^p\right)^n\left(yx^py\bar{x}^p\bar{y}\bar{x}^p\bar{y}^q\bar{x}^p\bar{y}\bar{x}^p\right)^nx^pyx^p\left(y^qx^pyx^p\bar{y}\bar{x}^p\bar{y}x^pyx^p\right)^n\bar{y} \cdot \left(\bar{x}^p\bar{y}\bar{x}^{p+1}\bar{y}\bar{x}^p\bar{y}^{q+1}\right)^n\bar{x}^p\bar{y}\bar{x}^p\left(x^pyx^py^qx^pyx^p\bar{y}\bar{x}^p\bar{y}\right)^ny;$$
(viii) $m_0(p,q) \equiv x^pyx^p\bar{y}\bar{x}^p\bar{y}x^pyx^py^q$;
(ix) $m_1(p,q) \equiv x^pyx^{p+1}yx^py^{q+1}$;
(x) $\partial P(p,q) \equiv x^pyx^{2p+1}yx^py^qx^pyx^py^q$.

Lemma 4.3. *The knot $K(p,q)$ is trivial if $(p,q) = (0,0), (0,1)$, and a torus knot if $p = 0$ and $q \neq -1, 0$ (with $K(0,q) = T(2, 2q-1)$) or $(p,q) = (-1,1)$ (with $K(-1,1) = T(5,8)$). In all other cases, $K(p,q)$ is a hyperbolic tunnel number one knot which is not 2-bridge, and its Seifert Klein bottle $P(p,q)$ is π_1-injective in the knot exterior iff (p,q) is not a pair of the form $(-1,2), (-2,1)$, or $(p,0)$.*

Proof. It is easy to see that $K(0,0)$ and $K(0,1)$ are trivial knots, $K(-1,1) = T(5,8)$ (see Fig. 11), and $K(0,q) = T(2,2q-1)$ for all q. From now on we assume that (p,q) is not a pair of the form $(0,0), (0,1), (-1,1)$, or $(0,q)$.

By Lemma 3.7, since $\ell_0'(0,p,q) \equiv x(x^py)^2$ is primitive in $\pi_1(H')$, all the knots $K(p,q)$ have tunnel number one.

By Corollary 2.4, the Seifert Klein bottle $P(p,q)$ is π_1-injective iff $[\partial P(p,q)]$ is not primitive nor a power in $\pi_1(H')$. By Lemma 4.2(x), $\partial P(p,q) \equiv x^pyx^{2p+1}yx^py^qx^pyx^py^q$, so $\partial P(p,0) \equiv x(x^{2p}y)^3$ is a primitive word. Suppose now $q \neq 0$. If $p \neq -1$ then by Lemma 2.1 $\partial P(p,q)$ is primitive/power iff $q = 1$; since $\partial P(p,1) \equiv x^{p+1}(x^py)^5$, it follows that $\partial P(p,1)$ is primitive/power iff $|p+1| \leqslant 1$ iff $p = -2$ (as $p \neq 0, -1$), in which case $\partial P(-2,1)$ is primitive. If $p = -1$ then $\partial P(-1,q) = \bar{x}y\bar{x}y\bar{x}y^q\bar{x}y\bar{x}y^q$, so by Lemma 2.1, as $q \neq 0$, $\partial P(-1,q)$ is a power iff $q = 1$ (which corresponds to the torus knot $K(-1,1) = T(5,8)$) and primitive iff $q = 2$. Therefore, $P(p,q)$ is π_1-injective iff (p,q) is not one of the pairs $(-1,2), (-2,1)$, or $(p,0)$.

Now, for $p \neq 0$, the circle $m_1(p,q)$ is a positive or negative braid on $s = 3$ strings with $c = 6|p|$ crossings (see Fig. 10), whence its genus is $g = (c-s+1)/2 = 3|p| - 1 \geqslant 2$ (cf. [14]), and so the meridian circle of $P(p,q)$ is a nontrivial knot which is distinct from the genus one knot $T(2,3)$. Since $\partial P(-1,2), \partial P(-2,1), \partial P(p,0)$ are all primitive in $\pi_1(H')$ by the argument above, the hyperbolicity of $K(p,q)$ for $(p,q) \neq (0,0), (0,1), (-1,1), (0,q)$ can be established via Lemma 3.9 by checking that $\mu(\ell'_0(n,p,q)) = 1 = \mu(\ell'_1(n,p,q))$ and either $\mu(m_0(p,q)) = 1$ or $\mu(m_1(p,q)) = 1$ hold in $H' = X(P(p,q))$ for all n and the allowed values of (p,q).

By Lemma 4.2, $m_0(p,q) \equiv x^p y x^p \bar{y}\bar{x}^p \bar{y} x^p y x^p y^q \in \pi_1(H')$. As $p \neq 0$, if $q \neq 0$ then the word for $m_0(p,q)$ is cyclically reduced, while the cyclic reduction of $m_0(p,0)$ is $x^{2p} y x^p \bar{y}\bar{x}^p \bar{y} x^p y$; in either case the word for $m_0(p,q)$ contains both y and $\bar{y}$ factors and so it is neither primitive nor a power by Lemma 2.1; thus $\mu(m_0(p,q)) = 1$ for all p, q by Lemma 2.3(b).

Consider now $\ell'_0(n,p,q) \equiv (x^{p+1} y x^p y^{q+1} x^p y)^n x^{p+1} y (\bar{y} x^p y x^p y^q x^p y x^p \bar{y}\bar{x}^p)^n x^p y \in \pi_1(H')$. Clearly $\ell'_0(0,p,q) \equiv x(x^p y)^2$ is primitive. If $|n| \geqslant 2$ then, since $p \neq 0$, it is not hard to see that the cyclic reduction of the word $\ell'_0(n,p,q)$ contains both y and $\bar{y}$ factors, while for $n = \pm 1$ the word $\ell'_0(n,p,q)$ contains both y^n and y^{3n} factors. Hence in all cases $\mu(\ell'_0(n,p,q)) = 1$ by Lemmas 2.1 and 2.3(b).

Finally, $\ell'_1(0,p,q) \equiv \bar{x}^p \bar{y}\bar{x}^p \bar{y}^q \bar{x}^p \bar{y}\bar{x}^p \bar{y}^{q-1} \in \pi_1(H')$. For $n = 0$, if $|p| \geqslant 2$ then $\ell'_1(0,p,q)$ contains at least three factors $\bar{x}^p$; if $|p| = 1$ then $\ell'_1(0,p,0)$ contains both y and $\bar{y}$ factors, $\ell'_1(0,p,1) = \bar{x}^p(\bar{x}^p y)^3$ is primitive, and for any $q \neq 0, 1$ the word $\ell'_1(0,p,q)$ contains three different powers $\bar{y}, \bar{y}^q, \bar{y}^{q-1}$. Finally, for $|n| = 1$ we have $\ell'_1(n,p,q) \equiv (x(x^p y)^2)^{|n|}$, a primitive word, and for $|n| \geqslant 2$ the word $\ell'_1(n,p,q)$ contains both y and $\bar{y}$ factors. Thus again $\mu(\ell'_1(n,p,q)) = 1$ in all cases. □

Lemma 4.4.

(a) *The boundary slope of the surface $P(p,q)$ is $r = 4q - 36p$.*
(b) *If (p,q) is not a pair of the form $(0,0), (0,1), (-1,1), (0,q)$ then, except for $K(-1,2)(r) = S^2(2,2,3)$, $K(-2,1)(r) = S^2(2,2,7)$, and $K(p,0)(r) = S^2(2,2,|6p-1|)$, the manifold $K(p,q)(r)$ is irreducible and toroidal.*

Proof. Since $\partial P(p,q)$ lies in ∂H, the boundary slope of $\partial P(p,q)$ coincides with the linking number between $\partial P(p,q)$ and a parallel copy in ∂H. From Fig. 1 we can thus see that the boundary slope of $\partial P(p,q)$ is $r = 0$, and that the above linking number decreases by $6^2 = 36$ with each positive Dehn twist along D_0, and increases by $2^2 = 4$ with each positive Dehn twist along D_1. Thus the boundary slope of $P(p,q)$ is $r = 4q - 36p$ and (a) holds.

For part (b) suppose $(p,q) \neq (0,0), (0,1), (-1,1), (0,q)$, and denote $P(p,q)$ by P for simplicity. Let $\tilde{X}$ and $\tilde{N}$ be the manifolds obtained by attaching 2-handles to $H' = X(P)$ and $N(P)$ along ∂P, respectively; thus, if r is the boundary slope of P then $K(p,q)(r) = \tilde{X} \cup_\partial \tilde{N}$. We will consider $N(P)$ as a Seifert fibered space over a disk with two singular fibers of indices $2, 2$; in particular, the pair $(\tilde{N}, \partial\tilde{N})$ is irreducible, and the circle $\ell'_0(0,p,q) \subset \partial\tilde{N}$ is a fiber of $\tilde{N}$ disjoint from $\partial P(p,q)$.

By Lemma 4.3, if $(p,q) \neq (-1,2), (-2,1), (p,0)$ then the surface P is π_1-injective in the exterior of $K(p,q)$, which implies that the surface T_P is incompressible in $H' = X(P)$ and hence, by the 2-handle addition theorem, that the pair $(\tilde{X}, \partial\tilde{X})$ is irreducible. Therefore $K(r) = \tilde{X} \cup_\partial \tilde{N}$ is irreducible and $\partial\tilde{X} = \partial\tilde{N}$ is an incompressible torus in $K(r)$.

Now, by Lemma 4.2, $[\partial P(p,0)] \equiv x(x^{2p} y)^3$ is primitive in $\pi_1(H') = \pi_1(X(P))$ and so $\tilde{X}$ is a solid torus; thus the Seifert fibration of $\tilde{N}$ extends to a Seifert fibration of $K(p,0)(r) = \tilde{X} \cup_\partial \tilde{N}$ over the 2-sphere with fibers of indices $2, 2, n$, where $n = |6p-1|$ is the order of the cyclic group

$$\pi_1(\tilde{X})/\langle[\ell'_0(0,p,0)]\rangle = \pi_1(H')/\langle[\partial P(p,0)], [\ell'_0(0,p,0)]\rangle = \mathbb{Z}_{|6p-1|}.$$

Therefore $K(p,0)(r) = S^2(2,2,|6p-1|)$. The identities $K(-1,2)(r) = S^2(2,2,3)$ and $K(-2,1)(r) = S^2(2,2,7)$ follow in a similar way. □

We now classify the words $[c'_i(n,p,q)]$ and $[c''_i(n,p,q)]$ which are primitive or a power in $\pi_1(H')$; we will need this information to discern which knots $K(p,q)$ admit a $(1,1)$ decomposition.

Lemma 4.5. *In $\pi_1(H')$, for $p \neq 0$,*

(a) • *$c'_0(n,p,q)$ is primitive iff $(n,p,q) = (0,\pm 1,q)$, $(1,-2,0)$, $(1,-1,1)$, $(1,p,-1)$, $(-1,2,0)$, $(-1,1,1)$, $(2,-1,0)$, $(-2,1,0)$, or $(n,p,q) = (n,1,-1)$ for all n;*
 • *$c'_0(n,p,q)$ is a power iff $(n,p,q) = (0,p,q)$ for $|p| \geqslant 2$, or $(n,p,q) = (1,-1,0)$, $(-1,1,0)$;*
(b) • *$c''_0(n,p,q)$ is primitive iff $(n,p,q) = (0,2,q)$, $(1,p,0)$, $(1,-1,2)$, $(1,-2,1)$;*
 • *$c''_0(n,p,q)$ is a power iff $(n,p,q) = (0,1,q)$, $(1,-1,1)$;*
(c) • *$c'_1(n,p,q)$ is primitive iff $(n,p,q) = (0,\pm 1,1)$, $(0,\pm 1,3)$, $(-1,2,0)$, $(-1,-1,2)$, or $(-1,p,1)$;*
 • *$c'_1(n,p,q)$ is a power iff $(n,p,q) = (-1,1,0)$ or $(0,p,2)$;*
(d) • *$c''_1(n,p,q)$ is primitive iff $(n,p,q) = (0,\pm 1,2)$ or $(0,p,0)$;*
 • *$c''_1(n,p,q)$ is a power iff $(n,p,q) = (0,p,1)$.*

Proof. The words for $c'_i(n,p,q)$ and $c''_i(n,p,q)$ in $\pi_1(H') = \langle x, y \mid - \rangle$ are given in Lemma 4.2. It is easy to see that in the given cases the words $c'_i(n,p,q)$ and $c''_i(n,p,q)$ are indeed primitive or powers as claimed. In order to establish the converse statements we apply Lemma 2.1; the fact that a word of the form $u^s v^t$ in the free group $\langle u, v \mid - \rangle$ is primitive iff $|s| = 1$ or $|t| = 1$, and a power iff $s = 0$, $|t| \geqslant 2$ or $|s| \geqslant 2$, $t = 0$, will also be of use.

So suppose some word $c_i'(n,p,q)$ or $c_i''(n,p,q)$ is primitive or a power in $\pi_1(H')$. We consider the case of the word $c_0'(n,p,q) \equiv x^p(x^{p+1}yx^py^{q+1}x^py)^n$ in part (a) in full detail; the other parts of the lemma follow along entirely similar lines, so their proof will be omitted.

(1) For $n=0$, $c_0'(0,p,q) \equiv x^p$. Thus $c_0'(0,p,q)$ is primitive iff $p=\pm 1$ and a power iff $|p| \geqslant 2$.

(2) For $n=1$, $c_0'(1,p,q) \equiv x^{2p+1}yx^py^{q+1}x^py$.

- If $q \neq -1$ then $x^{2p+1}yx^py^{q+1}x^py$ is cyclically reduced, so by Lemma 2.1 either all exponents of x are 1 or all -1, or all exponents of y are 1 or all -1; that is, either $p=-1$ or $q=0$. By Lemma 2.1, the word $c_0'(1,-1,q) \equiv \bar{x}y\bar{x}y^{q+1}\bar{x}y$ is primitive iff $q=1$ and a power iff $q=0$, while the word $c_0'(1,p,0) \equiv x^{2p+1}yx^pyx^py = x^{p+1}(x^py)^3$ is primitive iff $p+1=\pm 1$ iff $p=-2$ (as $p \neq 0$) and a power iff $p=-1$.
- If $q=-1$ then $c_0'(1,p,-1) \equiv x^{2p+1}yx^{2p}y$ is primitive for all p.

Remark. For the rest of the argument in this proof, we will implement the strategy used in (2) and will not explicitly indicate the use of Lemma 2.1 for the sake of brevity.

(3) For $n=-1$, $c_0'(-1,p,q) \equiv xyx^py^{q+1}x^py$.

- If $q \neq -1$ then $xyx^py^{q+1}x^py$ is cyclically reduced, hence either $p=1$ or $q=0$. The word $c_0'(-1,1,q) \equiv xyxy^{q+1}xy$ is primitive iff $q=1$ and a power iff $q=0$, while the word $c_0'(-1,p,0) \equiv xyx^pyx^py$ is primitive iff $p=2$ and a power iff $p=1$.
- If $q=-1$ then the word $c_0'(-1,p,-1) \equiv xyx^{2p}y \equiv x^{2p-1}(xy)^2$ is primitive iff $p=1$ (as $p \neq 0$) and never a power.

(4) For $n \geqslant 2$, $c_0'(n,p,q) \equiv x^p(x^{p+1}yx^py^{q+1}x^py)^n$.

- If $p \neq -1$ and $q \neq -1$ then $c_0'(n,p,q) \equiv x^p(x^{p+1}yx^py^{q+1}x^py)^n$ is cyclically reduced and x appears with the three distinct exponents $2p+1, p+1, p$, so $c_0'(n,p,q)$ is never primitive nor a power.
- If $p=-1$, $c_0'(n,-1,q) \equiv \bar{x}(y\bar{x}y^{q+1}\bar{x}y)^n$ is cyclically reduced and y appears with exponents $1, 2, q+1$, hence we must have $q+1=0,1,2$, i.e. $q=-1,0,1$.
  - If $q=-1$ then $c_0'(n,-1,-1) \equiv \bar{x}(y\bar{x}^2y)^n = \bar{x}y\bar{x}^2y^2\bar{x}^2y(y\bar{x}^2y)^{n-2}$ is cyclically reduced and both $\bar{x}$ and y appear with exponents 1 and 2, so $c_0'(n,-1,-1)$ is never primitive nor a power.
  - If $q=0$ then $c_0'(n,-1,0) \equiv \bar{x}(y\bar{x}y\bar{x}y)^n \equiv u^{n-1}v^3$ for the basis $u=y\bar{x}y\bar{x}y$ and $v=y\bar{x}$ of $\pi_1(H')$, hence $c_0'(n,-1,0)$ is primitive iff $n=2$ and never a power.
  - If $q=1$ then $c_0'(n,-1,1) \equiv \bar{x}(y\bar{x}y^2\bar{x}y)^n \equiv u^{2n-1}v^2$ for the basis $u=y\bar{x}y$ and $v=y\bar{x}$ of $\pi_1(H')$, hence $c_0'(n,-1,1)$ is neither primitive nor a power.
- For $q=-1$ and $p \neq -1$ the word

$$c_0'(n,p,-1) \equiv x^p\left(x^{p+1}yx^{2p}y\right)^n = x^{2p+1}yx^{2p}yx^{p+1}yx^{2p}y\left(x^{p+1}yx^{2p}y\right)^{n-2}$$

  is cyclically reduced and x appears with exponents $2p+1, 2p, p+1$, which are mutually distinct for $p \neq 1$, in which case $c_0'(n,p,-1)$ is neither primitive nor a power, while for $p=1$ the word $c_0'(n,1,-1) \equiv x(x^2y)^{2n}$ is always primitive.

(5) For $n \leqslant -2$,

$$c_0'(n,p,q) \equiv \bar{x}^p\left(x^{p+1}yx^py^{q+1}x^py\right)^{|n|} = xyx^py^{q+1}x^pyx^{p+1}yx^py^{q+1}x^py\left(x^{p+1}yx^py^{q+1}x^py\right)^{|n|-2}.$$

- If $q \neq -1$ then the word for $c_0'(n,p,q)$ is cyclically reduced and x appears with exponents $p+1, p, 1$ while y appears with exponents $q+1, 1$, so we must have $p=1$ and $q=0$; thus $c_0'(n,1,0) \equiv \bar{x}(x^2yxyxy)^{|n|} \equiv u^{|n|-1}v^3$ for the basis $u=x^2yxyxy$ and $v=xy$ of $\pi_1(H')$, so $c_0'(n,1,0)$ is primitive iff $n=-2$ and never a power.
- If $q=-1$ then $c_0'(n,p,q) \equiv xyx^{2p}yx^{p+1}yx^{2p}y(x^{p+1}yx^{2p}y)^{|n|-2}$ and x appears with exponents $2p, p+1, 1$, so $p=1$ again; clearly $c_0'(n,1,-1) \equiv \bar{x}(x^2y)^{2|n|}$ is always primitive.

Therefore part (a) holds. □

The following lemma is the last result needed in the proof of Theorem 1.1.

Lemma 4.6. *If $K(p,q)$ is a hyperbolic knot then $K(p,q)$ is a $(1,1)$-knot iff (p,q) is a pair of the form $(-1,0)$, $(1,q)$, or $(p,1)$, $(p,2)$ for $p \neq 0$.*

Proof. By Lemma 4.3, (p,q) is not a pair of the form $(0,0)$, $(0,1)$, $(-1,1)$, or $(0,q)$. That $K(p,q)$ admits a $(1,1)$ decomposition for $(p,q)=(p,1)$, $(p,2)$, $(1,q)$, and $(-1,0)$ follows directly from Lemma 3.6(a) using the following choices for $\hat{c}_0, \hat{c}_1$ and the fact that $H' = X(P(p,q))$:

(1) $\hat{c}_0 = c_0'(0,p,1)$ and $\hat{c}_1 = c_1''(0,p,1)$; then $[\hat{c}_0] \equiv x^p$, $[\hat{c}_1] \equiv (yx^p)^2$, whence $\mu(\hat{c}_0) = |p|$ and $\mu(\hat{c}_1) = 2$, and $[\partial D_2(0,p,1)] \equiv (yx^p)^2(\bar{y}\bar{x}^p)^2 \equiv [x^{|p|}, v^2]$ for the basis $\{x, v = yx^p\}$ of $\pi_1(H')$;
(2) $\hat{c}_0 = c_0'(0,p,2)$ and $\hat{c}_1 = c_1'(0,p,2)$; then $[\hat{c}_0] \equiv x^p$, $[\hat{c}_1] \equiv (yx^p)^2$, so again $\mu(\hat{c}_0) = |p|$ and $\mu(\hat{c}_1) = 2$ while $[\partial D_2(0,p,2)] \equiv (yx^p)^2(\bar{y}\bar{x}^p)^2 \equiv [x^{|p|}, v^2]$ for the basis $\{x, v = yx^p\}$ of $\pi_1(H')$;
(3) $\hat{c}_0 = c_0''(0,1,q)$ for $q \neq 1,2$ (the cases $q = 1,2$ follow from (1) and (2) above); then $[c_0''(0,1,q)] = (yx)^2$, so $\mu(\hat{c}_0) = 2$, while $\mu(\hat{c}_1) = 1$ by Lemma 4.5(c) and $[\partial D_2(0,1,q)] \equiv [u^2, y]$ for the basis $\{u = yx, y\}$ of $\pi_1(H')$;
(4) $\hat{c}_0 = c_0'(1,-1,0)$; then $[\hat{c}_0] \equiv (y\bar{x})^3$, so $\mu(\hat{c}_0) = 3$, and $\mu(\hat{c}_1) = 1$ by Lemma 4.5(c) while $[\partial D_2(1,-1,0)] \equiv (y\bar{x})^2 y(x\bar{y})^3\bar{x} \equiv [u^3, y]$ for the basis $\{u = y\bar{x}, y\}$ of $\pi_1(H')$.

Conversely, assume that $K(p,q)$ admits a $(1,1)$ decomposition and the pair (p,q) is not of the form $(-1,0)$, $(1,q)$, or $(p,1)$, $(p,2)$ for $p \neq 0$ (recall also that (p,q) is not a pair of the form $(0,0)$, $(0,1)$, $(-1,1)$, or $(0,q)$); we show this situation contradicts Lemma 3.6(a). The following fact will be useful in the sequel.

Claim 4.7. *If $\mu(\hat{c}_0) = 1 = \mu(\hat{c}_1)$ then $n \neq -1, 0$.*

Proof. For $n = -1, 0$ the word for $[\partial D_2(n,p,q)]$ in $\pi_1(H')$ has the following cyclic reductions:

$$\begin{aligned}
&\left[\partial D_2(0,p,q)\right] \equiv \left(yx^p\right)^2\left(\bar{y}\bar{x}^p\right)^2,\\
&\left[\partial D_2(-1,p,q)\right] \equiv x^p yx^p y^q x^p yx^p \bar{y}\bar{x}^p \bar{y}\bar{x}\bar{y}\bar{x}^p \bar{y}^q\bar{x}^p \bar{y}\bar{x}^p \bar{y}^q\bar{x}^p \bar{y}\bar{x}^p yx^p y^q yxyx^p y^q, \quad q \neq 0,\\
&\left[\partial D_2(-1,p,0)\right] \equiv x^{2p} yx^{2p} yx^p \bar{y}\bar{x}^p \bar{y}\bar{x}\bar{y}\bar{x}^{2p} \bar{y}\bar{x}^{2p} \bar{y}\bar{x}^p yx^p yxy.
\end{aligned}$$

Thus $[\partial D_2(n,p,q)] \not\equiv [x,y]$ for $n = -1, 0$, so the claim follows by Lemma 3.6(a)(i). □

We now consider two cases:

Case 1. $\mu(\hat{c}_0) = 1$.

By Lemmas 2.3(b) and 3.6(b), in $\pi_1(H')$, $[\hat{c}_0]$ is not a power and $[\hat{c}_1]$ is either primitive with $\mu(\hat{c}_1) = 1$ or a power with $\mu(\hat{c}_1) \geqslant 2$. We thus consider the following subcases.

Subcase 1.1. $[\hat{c}_1]$ is primitive, $\mu(\hat{c}_1) = 1$.

As $\hat{c}_1 = c_1'(n,p,q)$ or $c_1''(n,p,q)$ gives rise to a primitive word in $\pi_1(H')$ then $n = -1, 0$ by Lemma 4.5, contradicting Claim 4.7.

Subcase 1.2. $[\hat{c}_1]$ is a power, $\mu(\hat{c}_1) \geqslant 2$.

By Lemma 4.5, $\hat{c}_1$ must be one of $c_1'(-1,1,0)$, $c_1'(0,p,2)$, or $c_1''(0,p,1)$, contradicting our hypothesis that (p,q) is not of the form $(1,q)$, $(p,1)$, $(p,2)$.

Case 2. $\mu(\hat{c}_0) \geqslant 2$.

Then $[\hat{c}_0]$ is a power in $\pi_1(H')$; by Lemma 4.5, with the given restrictions on the pair (p,q), we must have $\hat{c}_0 = c_0'(0,p,q)$ for $|p| \geqslant 2$ and $q \neq 1,2$, and hence either $[\hat{c}_1] = [c_1'(0,p,q)] \equiv y^{q-1}x^p yx^p$ or $[\hat{c}_1] = [c_1''(0,p,q)] \equiv y^q x^p yx^p$.

Now, by Lemma 2.1, for any $p \neq 0$, $y^{q-1}x^p yx^p$ is a power iff $q = 2$ while $y^q x^p yx^p$ is a power iff $q = 1$; thus $\mu(\hat{c}_1) = 1$ in all cases. However, $\mu(\hat{c}_0) = |p| \geqslant 2$ since $[\hat{c}_0] = [c_0'(0,p,q)] \equiv x^p$, while $[\partial D_2(0,p,q)] \equiv (yx^p)^2(\bar{y}\bar{x}^p)^2 \not\equiv [x^p, y]$ for the basis $\{x, y\}$ of $\pi_1(H')$, contradicting Lemma 3.6(a)(ii). □

References

[1] C.C. Adams, A.W. Reid, Unknotting tunnels in two-bridge knot and link complements, Comment. Math. Helv. 71 (4) (1996) 617–627.
[2] A.J. Casson, C.M. Gordon, Reducing Heegaard splittings, Topology Appl. 27 (3) (1987) 275–283.
[3] M. Cohen, W. Metzler, A. Zimmermann, What does a basis of $F(a,b)$ look like?, Math. Ann. 257 (4) (1981) 435–445.
[4] M. Eudave-Muñoz, Incompressible surfaces and (1,2)-knots, Geom. Topol. Monogr. 12 (2007) 35–88.
[5] F. González-Acuña, A. Ramírez, A composition formula in the rank two free group, Proc. Amer. Math. Soc. 127 (9) (1999) 2779–2782.

[6] C.M. Gordon, Dehn surgery and satellite knots, Trans. Amer. Math. Soc. 275 (1983) 687–708.
[7] A. Hatcher, W. Thurston, Incompressible surfaces in 2-bridge knot complements, Invent. Math. 79 (2) (1985) 225–246.
[8] J. Hempel, 3-Manifolds, Ann. of Math. Stud., vol. 86, Princeton Univ. Press, Princeton, NJ, 1976.
[9] W. Magnus, A. Karrass, D. Solitar, Combinatorial Group Theory, revised ed., Dover Publications Inc., New York, 1976.
[10] K. Morimoto, M. Sakuma, Y. Yokota, Identifying tunnel number one knots, J. Math. Soc. Japan 48 (4) (1996) 667–688.
[11] E. Ramírez-Losada, L.G. Valdez-Sánchez, Crosscap number two knots in S^3 with $(1,1)$ decompositions, Bol. Soc. Mat. Mexicana (3) 10 (Special Issue) (2004) 451–465.
[12] E. Ramírez-Losada, L.G. Valdez-Sánchez, Once-punctured Klein bottles in knot complements, Topology Appl. 146/147 (2005) 159–188.
[13] M. Scharlemann, There are no unexpected tunnel number one knots of genus one, Trans. Amer. Math. Soc. 356 (2004) 1385–1442.
[14] J.R. Stallings, Constructions of fibred knots and links, in: Algebraic and Geometric Topology, Proc. Sympos. Pure Math., Part 2, Stanford Univ., Stanford, CA, 1976, in: Proc. Sympos. Pure Math., vol. XXXII, Amer. Math. Soc., Providence, RI, 1978, pp. 55–60.
[15] W.P. Thurston, Three-dimensional manifolds, Kleinian groups and hyperbolic geometry, Bull. Amer. Math. Soc. (N.S.) 6 (3) (1982) 357–381.
[16] L.G. Valdez-Sánchez, Seifert Klein bottles for knots with common boundary slopes, in: Proceedings of the Casson Fest, in: Geom. Topol. Monogr., vol. 7, Geom. Topol. Publ., Coventry, 2004, pp. 27–68 (electronic).
[17] F. Waldhausen, On irreducible 3-manifolds which are sufficiently large, Ann. of Math. (2) 87 (1968) 56–88.