COVERING A SET BY SUBSETS

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We give formulae for determining the number of ways of writing a finite set as the union of a given number of subsets, in such a way that none of the subsets may be omitted. In particular, we consider the case in which the elements of the set are identical.

1. Introduction

Let \( X \) be a set with \( n \) elements (an \( n \)-set) and let \( k \) be a positive integer. A \( k \)-cover of \( X \) is a collection of \( k \) not necessarily distinct subsets of \( X \) whose union is \( X \). A \( k \)-cover of \( X \) is minimal if none of its proper subsets covers \( X \). We wish to enumerate the \( k \)-covers and the minimal \( k \)-covers of an \( n \)-set. There are essentially four cases to consider. The elements of \( X \) may be taken to be identical or distinguishable. We shall describe \( X \) as unlabelled or labelled respectively. The order of the sets \( Y_1, \ldots, Y_k \) comprising the \( k \)-cover of \( X \) may or may not be material, and we shall describe the cover as ordered or disordered respectively. Hearne and Wagner [1] enumerate the minimal disordered covers of a labelled set. For completeness we include a derivation of their formula in Section 2.

We note that their formula has been generalized in a different direction by Wagner [2].

Throughout the following, \( X \) denotes a labelled \( n \)-set and \( Y_1, \ldots, Y_k \) a cover of \( X \). \( \mathcal{F} \) denotes the set of all ordered \( k \)-covers of \( X \). We adopt a systematic notation for the number of covers of a particular kind. Thus \( \text{tol}(n, k) \) denotes the cardinality of \( \mathcal{F} \), i.e. the number of ordered \( k \)-covers of a labelled \( n \)-set, while \( \text{mol}(n, k) \) denotes the number of minimal such covers. The symmetric group \( S_n \) acts naturally on \( \mathcal{F} \), and \( \text{tou}(n, k) \) denotes the number of orbits under this action, i.e. the number of ordered \( k \)-covers of an unlabelled \( n \)-set. The symmetric group \( S_k \) acts naturally on \( \mathcal{F} \), and \( \text{tdl}(n, k) \) denotes the number of orbits under this action, i.e. the number of disordered \( k \)-covers of a labelled \( n \)-set. Finally, the group \( S_n \times S_k \) acts naturally on \( \mathcal{F} \), and \( \text{mdl}(n, k) \) denotes the number of orbits under this action, i.e. the number of disordered \( k \)-covers of an unlabelled \( n \)-set. We will discuss this action more fully in Section 3. The functions \( \text{mou} \), \( \text{mdu} \) and \( \text{mdl} \) are defined in a similar way to \( \text{mol} \).

2. Labelled or ordered covers

The following well known results are inserted here for completeness.

**Theorem 1.** For all natural numbers \( n \) and \( k \),

(i) \( \text{Tol}(n, k) = (2^k - 1)^n. \)

(ii) \( \text{Mol}(n, k) = \sum_{i=0}^{k} (-1)^i \binom{k}{i} (2^k - i - 1)^n. \)

(iii) \( \text{Tou}(n, k) = \binom{2^k + n - 2}{n}. \)

(iv) \( \text{Tdl}(n, k) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{j} (2^i + k - 1)^n. \)

(v) \( \text{Mdl}(n, k) = \text{Mol}(n, k)/k!. \)

**Proof.** (i) Since each element of \( X \) may be placed in the subsets \( Y_1, \ldots, Y_k \) in \( 2^k - 1 \) ways, the result follows.

(ii) Let a cover in \( \mathcal{C} \) have property \( i \) if the set \( Y_i \) is redundant. Then the number of covers in \( \mathcal{C} \) with properties \( 1, 2, \ldots, i \) is clearly \( (2^k - i - 1)^n. \) The result now follows from the inclusion–exclusion principle.

(iii) Each element of \( X \) may be placed in the subsets \( Y_1, \ldots, Y_k \) in \( 2^k - 1 \) ways. Hence \( \text{Tou}(n, k) \) is the number of ways of placing \( n \) identical objects into \( 2^k - 1 \) labelled boxes, which gives the result.

(iv) Let \( \mathcal{S} \) be the set of all selections of \( k \) subsets of the set \( \{1, 2, \ldots, n\} \). The cardinality of \( \mathcal{S} \) is the number of ways of placing \( k \) identical objects in \( 2^n \) labelled boxes, which is \( (2^n + k - 1). \) Now let an element of \( \mathcal{S} \) have property \( i \) if the element \( i \) lies in none of the selected subsets. Then \( \text{Tdl}(n, k) \) is the number of elements of \( \mathcal{S} \) which have none of the properties \( 1, 2, \ldots, n. \) The required result now follows by applying the inclusion–exclusion principle.

(v) This follows immediately from the observation that all of the sets in a minimal cover are unequal. \( \Box \)

We note that formula (v) is given in [1, page 250].

3. Covers of an unlabelled set

**Lemma 1.** For all \( n \geq k \),

(i) \( \text{Mou}(n, k) = \text{Tou}(n - k, k) \)

(ii) \( \text{Mdu}(n, k) = \text{Tdu}(n - k, k). \)
Proof. Let \( Y_1, \ldots, Y_k \) be a minimal \( k \)-cover of \( X \). We define an element \( y \) of \( Y_i \) to be \emph{loyal} to \( Y_j \) if \( y \) does not lie in \( Y_j \) for any \( j \neq i \). Clearly, each set of a minimal cover must contain at least one loyal element. Now delete from each set \( Y_i \) one of its loyal elements. We obtain a \( k \)-cover of an \((n-k)\)-set. Conversely, if to each of the sets of any \( k \)-cover of an \((n-k)\)-set we adjoin a new element, adjoining \( k \) new elements in all, we obtain a minimal \( k \)-cover of an \( n \)-set. This proves the result.

We observe that the above result does not hold for covers of labelled sets.

Using these results, we can calculate each of the functions \( T_{ol}, M_{ol}, T_{dl}, M_{dl}, T_{ou} \) and \( M_{ou} \) for all values of their arguments.

We shall now obtain expressions for \( T_{du} \) and \( M_{du} \). These expressions involve a double summation over all partitions of \( n \) and \( k \). This dependence on partitions is to be expected, as we are considering here partitions with "overlapping parts". To illustrate this point, we show \( M_{du}(5,3) \) in Fig. 1.

For the rest of this section, we denote by \( S_k \) the symmetric group on \( k \) letters and by \( \mathcal{F}_k \) the set of all partitions \( \lambda = (\lambda_1, \ldots, \lambda_k) \) of \( k \). To be more precise,

\[
\mathcal{F}_k = \{ (\lambda_1, \ldots, \lambda_k) \mid \lambda_1 \geq \cdots \geq \lambda_k \geq 0, \lambda_1 + \cdots + \lambda_k = k \}.
\]

The \emph{length} \( l(\lambda) \) of a partition \( \lambda \) is the number of non zero parts in \( \lambda \). The \emph{spectrum} of \( \lambda \) is the vector \( \mu = (\mu_1, \ldots, \mu_k) \) such that \( \lambda \) has \( \mu_i \) parts of size \( i \). The number of permutations of \( k \) with cycle type \( \lambda \) is denoted by \( [\lambda] \) and is given by the formula

\[
[k] = \frac{k!}{\mu_1! \cdots \mu_k! 2^{\mu_2} \cdots k^{\mu_k}}.
\]

Let \( U(n, k) \) denote the number of ways in which \( k \) unordered sets may be

![Fig. 1. \( M_{du}(5,3) = 9 \).](image)
chosen from an unlabelled $n$-set. Then clearly
\[ Tdu(n, k) = U(n, k) - U(n - 1, k). \] (1)

We calculate $U(n, k)$.

Let $X = \{1, 2, \ldots, n\}$ and let $\mathcal{U} = \{(Y_1, \ldots, Y_k) \mid Y_1, \ldots, Y_k \subseteq X\}$. The groups $S_n$ and $S_k$ have natural actions on $X$ and on $\mathcal{U}$ respectively. Let $G = S_n \times S_k$. Let $\sigma = \sigma' \sigma'' \in G$, where $\sigma' \in S_n$ and $\sigma'' \in S_k$. Define the action of $G$ on $\mathcal{U}$ by
\[ (Y_1, \ldots, Y_k) \sigma = (Z_1, \ldots, Z_k), \]
where
\[ Z_i\sigma = (Y_i)\sigma' = \{x\sigma' \mid x \in Y_i\}. \]

Then $U(n, k)$ is the number of orbits of $\mathcal{U}$ under $G$.

Let $\sigma'$ and $\sigma''$ have cycle types $\lambda' = (\lambda'_1)$ and $\lambda'' = (\lambda''_1)$ respectively. Consider cycles $(1, 2, \ldots, u)$ of $\sigma'$ and $(1, 2, \ldots, v)$ of $\sigma''$, and consider for the moment the natural action of $\sigma = \sigma' \sigma''$ on the set $\{(a, b) \mid a = 1, \ldots, u, b = 1, \ldots, v\}$. If $\sigma$ is the orbit of $(1, 1)$,
\[ (a, b) \in \sigma \text{ if and only if } a = i(u), b = i(v) \text{ for some } i \]
if and only if $a = ru + i, b = sv + i$ for some $i, r, s$
if and only if $a - b = ru - sv$ for some $r, s$
if and only if $d \mid a - b$, where $d = (u, v)$.

Here $(u, v)$ denotes as usual the g.c.d. of $u$ and $v$. Returning to our proof, let $\sigma$ fix $(Y_1, \ldots, Y_k) \in U$. Then
\[ i \in Y_j \text{ if and only if } i\sigma' \in Y_{j'} \]
for all $i, j$. We count the number $\chi(\sigma)$ of such elements of $\mathcal{U}$. Let
\[ Y'_i = Y_j \cap \{1, \ldots, u\}. \]

Then from the above remarks we see that
(i) $Y'_2, \ldots, Y'_u$ are completely determined by $Y'_1$;
(ii) if $i \in Y'_j$ then $j \in Y'_i$ for $j = i \mod d$.
Hence we may construct $Y = (Y'_1, \ldots, Y'_u)$ as follows.
(i) Choose a subset $S = S(Y)$ of $\{1, \ldots, d\}$.
(ii) Let $Y'_i = \{i \mid 1 \leq i \leq u, i = j \mod d \text{ for some } j \in S\}$.
(iii) Let $Y'_i = \{i + l - 1 \mod u \mid i \in Y'_i\}$.
Hence the number of ways of choosing $Y$ is $2^d = 2^{(u, v)}$. We therefore have
\[ \chi(\sigma) = \prod_{i,j} 2^{(\lambda'_i, \lambda''_j)} \]
and
\[ U(n, k) = \frac{1}{n! \cdot k!} \sum_{\lambda' \in S_n} \left[ \begin{array}{c} n \\ \lambda' \end{array} \right] \left[ \begin{array}{c} k \\ \lambda'' \end{array} \right] \prod_{i,j} 2^{(\lambda'_i, \lambda''_j)}. \]
(2)
For example,

$$U(1, k) = 1/k! \sum_{\lambda \in P_k} \left[ \begin{array}{c} k \\ \lambda \end{array} \right] \prod_{j} 2^{l(\lambda)}$$

$$= 1/k! \sum_{\lambda \in P_k} \left[ \begin{array}{c} k \\ \lambda \end{array} \right] 2^{l(\lambda)},$$

as $2^{(1,v)} = 2^{(v,0)} = 1$.

Since it is obvious that $U(1, k) = k + 1$, we have the well known result that

$$\sum_{\sigma \in S_k} 2^{l(\sigma)} = (k + 1)!,$$

where $l(\sigma)$ denotes (by abuse of notation) the length of the permutation associated with $\sigma$.

Similarly, one obtains

$$U(2, k) = \frac{1}{2k!} \sum_{\lambda \in P_k} \left[ \begin{array}{c} k \\ \lambda \end{array} \right] \{2^{l(\lambda) + e(\lambda)} + 2^{2l(\lambda)}\},$$

where $e(\lambda)$ equals the number of even parts in $\lambda$.

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However, instead of using Eqs 1 and 2 to calculate $Tdu(n, k)$, we may calculate it directly as follows. In the proof of 2, replace $\mathcal{U}$ by $\mathcal{T}$, the set of ordered $k$-covers of $X$. Then $Tdu(n, k)$ is the number of orbits of $\mathcal{T}$ under $G$. Now in counting the number $\chi(\sigma)$ of elements $(Y_1, \ldots, Y_k)$ of $\mathcal{T}$ fixed by $\sigma \in G$, we must ensure that each element of $X$ occurs in at least one of $Y_1, \ldots, Y_k$. Using our previous notation, each element $1, \ldots, u$ must occur in one of the subsets $Y'_1, \ldots, Y'_u$ for some cycle $(1, \ldots, v)$ of $\sigma'$. Now if $Y'_i \neq \emptyset$, each of $1, \ldots, u$ occurs in one of the subsets $Y'_1, \ldots, Y'_u$. Hence, we need only ensure that $Y'_1 \neq \emptyset$ for some cycle $(1, \ldots, v)$ of $\sigma''$. Hence $S = S(Y'_1, \ldots, Y'_v)$ must be non empty, and we get

$$\chi(\sigma) = \prod \left( \left( \prod_{j} 2^{(\lambda'_j, \lambda'_v)} \right) - 1 \right)$$

We have proved the following result.
Theorem 2.

\[ T_{du}(n, k) = \frac{1}{n!k!} \sum_{\lambda \in \mathcal{P}_n} \binom{n}{\lambda} \binom{k}{\lambda^*} \prod_{i} \left( \prod_{j} 2^{(\lambda_i, \lambda_j)} - 1 \right). \]

Note that it appears far from obvious that this last formula yields the same result as Eqs 1 and 2. It can be shown that the equality of these two formulae leads to the following proposition.

**Proposition.** For any \( m \)-tuple of positive integers \((a_1, \ldots, a_m)\) and any positive integer \( n \),

\[ \sum_{\lambda \in \mathcal{P}_n} \binom{n}{\lambda} \prod_{i} \left( \prod_{j} 2^{(\lambda_i, a_j)} - 1 \right) = \sum_{\lambda \in \mathcal{P}_n} \binom{n}{\lambda} (1 - \mu_1(\lambda)/2^m) \prod_{i,j} 2^{(\lambda_i, a_j)}, \]

where \( \mu_1(\lambda) \) denotes the number of parts of \( \lambda \) of size 1.

Table 1 presents some values of \( T_{du}(n, k) \) calculated using Theorem 2.

References