Mean value property and a Berezin-type transform on the half-space

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Abstract

Let $B$ be the open unit ball in $\mathbb{R}^n$. Liu (2007) [6] has shown that if $f \in C(B)$ then the iterates of a Berezin-type transform $B_k f$ of $f$ converge to the Poisson extension of the boundary values of $f$, as $k \to \infty$. In this paper, we extend this to the half-space setting. First, we obtain the mean value property for harmonic functions on the half-space $H$. Based on this property, we define a Berezin-type transform $B_H$ and investigate the limit of the iterates of $B_H$.

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1. Introduction

For a fixed positive integer $n \geq 2$, let $H = \mathbb{R}^{n-1} \times \mathbb{R}_+$ be the upper half-space where $\mathbb{R}_+$ denotes the set of all positive real numbers. We write a point $z \in H$ as $z = (z', z_n)$ where $z' \in \mathbb{R}^{n-1}$ and $z_n \in \mathbb{R}_+$. Let $z = (z', -z_n)$ and $e = (0', 1) \in H$. We denote by $V$ the normalized Lebesgue volume measure on $\mathbb{R}^n$. Let $\mu$ denote the weighted measure on $H$ given by

$$d\mu(w) = \frac{4dV(w)}{|w + e|^{n+2}}$$

where the constant factor 4 is for the purpose of normalization.

Given $z \in H$, define

$$\phi_z(w) = (z_0w' + z', z_0w_n), \quad w \in H.$$

Note $\phi_z(e) = z$. A straightforward calculation shows that $f \circ \phi_z$ is harmonic for each $z \in H$ if $f$ is. Also, note that $f \circ \phi_z \in L^1(H, \mu)$ for all $z \in H$ if $f \in L^1(H, \mu)$. See Section 3 for details.

The harmonic functions in $L^1(H, \mu)$ satisfy the mean value property (Proposition 3.2): The equality

$$f(e) = \int_H f(w) d\mu(w)$$

holds on $H$ for harmonic functions $f \in L^1(H, \mu)$. Replacing $f$ by $f \circ \phi_z$, we have

$$f(z) = \int_H (f \circ \phi_z)(w) d\mu(w), \quad z \in H.$$ (1.1)

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1 The author was supported by the Basic Science Research Program through the NRF grant funded by the MEST (2010-0006518).
Motivated by this property, we define a Berezin-type transform which is the half-space version of that considered in [6] for the open unit ball in \( \mathbb{R}^n \). For \( f \in L^1(\mathbb{H}, \mu) \), the Berezin-type transform \( B_H f \) of \( f \) is defined by

\[
B_H f(z) := \int_H (f \circ \phi)(w) \, d\mu(w), \quad z \in H.
\]

The formula (1.1) implies that \( B_H \) fixes harmonic functions in \( L^1(\mathbb{H}, \mu) \).

We say \( f \in \Omega(\mathbb{H}) \) if \( f \in C(\mathbb{H}) \) and \( \lim_{|z| \to \infty} f(z)/|z|^{n-2} \) exists.

The following is our main result.

**Theorem 1.1.** If \( f \in \Omega(\mathbb{H}) \), then the iterates \( B_{H}^k f \) converge uniformly on \( \mathbb{H} \) to the Poisson extension of the boundary values of \( f \), as \( k \to \infty \).

Engliš [4] and Zhu [7] proved this result on the open unit disk in \( \mathbb{C} \). Afterwards, Arazy and Engliš [3] studied this in the higher dimensional case. Recently, Liu [6] showed Theorem 1.1 in the setting of the open unit ball in \( \mathbb{R}^n \).

Let \( B \) be the open unit ball in \( \mathbb{R}^n \). Theorem 1.1 is the extension of the result for \( B \) obtained in [6] to the case of \( \mathbb{H} \). To prove Theorem 1.1, we first introduce a modified Kelvin transform \( K \) which connects \( B \) and \( \mathbb{H} \), and then we establish direct connections via \( K \) between main objects under the study for both above mentioned cases. Then our main result follows from the known result in [6].

In Section 2 we collect some well-known facts to be used later. In Section 3 we prove the mean value property for harmonic functions on \( \mathbb{H} \) (Proposition 3.2) and show that the Berezin-type transform \( B_H \) commutes with the Kelvin transform in the sense of Proposition 3.3. In addition, we prove that the Kelvin transform \( K \) commutes with the Poisson integral transform in the sense of Proposition 3.4. Using these properties, we prove our main result Theorem 1.1 in Section 4.

2. Preliminaries

In this section we first recall a modified Kelvin transform and a Berezin-type transform on \( B \). At the end of this section, we introduce the result concerning the limit of iterates of a Berezin-type transform on \( B \).

2.1. Kelvin transform

We consider the map \( \Phi : \mathbb{R}^n \cup \{ \infty \} \to \mathbb{R}^n \cup \{ \infty \} \) given by

\[
\Phi(x) = 2(x + e)^* - e.
\]

Here, the map \( x \mapsto x^* \), where

\[
x^* = \begin{cases} x/|x|^2 & \text{if } x \neq 0, \infty, \\ 0 & \text{if } x = \infty, \\ \infty & \text{if } x = 0 \end{cases}
\]

is called the inversion of \( \mathbb{R}^n \cup \{ \infty \} \) relative to the unit sphere \( S = \partial B \). The map \( \Phi \) is a Möbius transformation taking \( B \) onto \( \mathbb{H} \) with \( \Phi(e) = 0 \) and \( \Phi(-e) = \infty \).

Let \( x \in \mathbb{R}^n \setminus \{-e\} \). The explicit form of \( \Phi \) is

\[
\Phi(x) = \frac{(2x^*, 1 - |x|^2)}{|x + e|^2}. \quad (2.1)
\]

In case \( n = 2 \), we have

\[
\Phi(z) = -i \frac{z - i}{z + i} = \frac{1 - iz}{z - i}
\]

for every \( z \in \mathbb{C} \setminus \{-i\} \) in the complex notation. Thus, the following identities are straightforward:

\[
|\Phi(x) + e| = \frac{2}{|x + e|} \quad (2.2)
\]

and

\[
|\Phi(x)| = \frac{|x - e|}{|x + e|}. \quad (2.3)
\]

Also, the Jacobian determinant \( J\Phi \) is easily computed as

\[
J\Phi(x) = \left( \frac{2}{|x + e|^2} \right)^n. \quad (2.4)
\]
Note that the map \( \Phi: \mathbb{R}^{n-1} \to S \) is the inverse of the well-known stereographic projection whose Jacobian is given by

\[
J \Phi|_{\mathbb{R}^{n-1}}(\zeta) = \left( \frac{2}{|\zeta + e|^2} \right)^{n-1}.
\]  

The next proposition summarizes the properties of \( \Phi \). See [2, Proposition 7.18] for the proof.

**Proposition 2.1.** The map \( \Phi \) has the following properties:

(a) \( \Phi \) is an involution, i.e., \( \Phi \circ \Phi \) is the identity map on \( \mathbb{R}^n \cup \{ \infty \} \);

(b) \( \Phi \) is a conformal, one-to-one map of \( \mathbb{R}^n \setminus \{-e\} \) onto \( \mathbb{R}^n \setminus \{-e\} \);

(c) \( \Phi \) maps \( B \) onto \( H \) and \( H \) onto \( B \);

(d) \( \Phi \) maps \( S \setminus \{-e\} \) onto \( \mathbb{R}^{n-1} \) and \( \mathbb{R}^{n-1} \) onto \( S \setminus \{-e\} \).

Moreover, \( \Phi \) satisfies the following identity (2.6) which will be useful in proving Proposition 3.3.

**Lemma 2.2.** The identity

\[
|\Phi(x) - \Phi(y)| = \frac{2|x - y|}{|x + e||y + e|}
\]

holds for \( x, y \in B \).

Here and throughout the paper, we use the notation

\[
[x, y] := \sqrt{1 - 2x \cdot y + |x|^2|y|^2}, \quad x, y \in B
\]

where \( x \cdot y \) is the dot product of \( x \) and \( y \) in \( \mathbb{R}^n \).

**Proof of Lemma 2.2.** We first establish the identity

\[
|\Phi(x) - \Phi(y)| = \frac{2|x - y|}{|x + e||y + e|}
\]

valid for all \( x, y \in \mathbb{R}^n \setminus \{-e\} \). Note that from (2.1)

\[
\Phi(x) \cdot \Phi(y) = \frac{4(x \cdot y - x_n y_n) + (1 - |x|^2)(1 - |y|^2)}{|x + e|^2|y + e|^2}.
\]

Meanwhile, (2.3) gives us that

\[
|\Phi(x)|^2 + |\Phi(y)|^2 = \frac{|x - e|^2}{|x + e|^2} + \frac{|y - e|^2}{|y + e|^2} = 2\left(1 + |x|^2\right)\left(1 + |y|^2\right) - 4x_n y_n
\]

\[
\frac{|x + e|^2|y + e|^2}{|x + e|^2|y + e|^2}.
\]

Accordingly, (2.8) and (2.9) yield

\[
|\Phi(x) - \Phi(y)|^2 = \frac{4|x - y|^2}{|x + e|^2|y + e|^2}
\]

so that (2.7) holds.

Now, assume \( x, y \in H \). Using (2.3), we have

\[
(1 - |\Phi(x)|^2)(1 - |\Phi(y)|^2) = \frac{16x_n y_n}{|x + e|^2|y + e|^2}.
\]

This, together with (2.7), implies

\[
|\Phi(x) \cdot \Phi(y)|^2 = |\Phi(x) - \Phi(y)|^2 + (1 - |\Phi(x)|^2)(1 - |\Phi(y)|^2)
\]

\[
= 4\frac{|x - y|^2 + 4x_n y_n}{|x + e|^2|y + e|^2}
\]

\[
= 4\frac{|x - y|^2}{|x + e|^2|y + e|^2}.
\]
Consequently, we obtain
\[ |x - y| = \frac{1}{2} [\Phi(x), \Phi(y)] |x + e||y + e| \]
for all \( x, y \in H \). Now, replacing \( x, y \) by \( \Phi(x) \) and \( \Phi(y) \), respectively, we conclude the lemma by Proposition 2.1 and (2.2).

Now, we introduce a modified Kelvin transform \( K \) with respect to the point \(-e\). Given a function \( f \) defined on a set \( E \subset \mathbb{R}^n \setminus \{-e\} \), let
\[ K[f](x) = 2^{(n-2)/2} \frac{f(\Phi(x))}{|x + e|^{n-2}}, \quad x \in \Phi(E). \]
(2.10)

Note that \( K[f](x) = f(\Phi(x)) \) when \( n = 2 \). The constant factor \( 2^{(n-2)/2} \) is included so that \( K \) will be its own inverse. Also, \( K \) preserves harmonicity. That is, \( f \) is harmonic on \( E \) if and only if \( K[f] \) is harmonic on \( \Phi(E) \). In particular, if \( f \) is harmonic on \( H \), then \( K[f] \) is harmonic on \( B \), and vice versa. See [2, Proposition 7.19] for details.

2.2. Möbius transformations on \( B \)

We recall Möbius transformations on \( B \). Let \( a \in B \). The canonical Möbius transformation \( \varphi_a \) that exchanges \( a \) and 0 is given by
\[ \varphi_a(x) = a + (1 - |a|^2)(a - x^*)^* \]
for \( x \in B \). Avoiding \( x^* \) notation, we have
\[ \varphi_a(x) = \frac{(1 - |a|^2)(a - x) + |a - x|^2a}{[x, a]^2}. \]

See [1] for the proof of the next lemma.

Lemma 2.3. For every \( a \in B \), \( \varphi_a \) has the following properties:

(a) The map \( \varphi_a \) is an involution of \( B \), i.e., \( \varphi_a^{-1} = \varphi_a \).
(b) The identities
\[ |\varphi_a(x)| = \frac{|x - a|}{|x, a|}; \]
\[ 1 - |\varphi_a(x)|^2 = \frac{(1 - |x|^2)(1 - |a|^2)}{|x, a|^2}; \]
\[ [\varphi_a(x), a] = \frac{1 - |a|^2}{|x, a|} \]
and
\[ J\varphi_a(x) = \left( \frac{1 - |a|^2}{|x, a|^2} \right)^n \]
(2.13)
hold for every \( x \in B \).

2.3. Berezin-type transform on \( B \)

Consider the differential operator
\[ \Delta_0 := (1 - |x|^2)^2 \Delta \]
where \( \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \). For \( u \in \mathcal{C}^2(B) \) and \( a \in B \), it is known that
\[ \Delta_0[(u \circ \varphi_a) j_a] = [(\Delta_0 u) \circ \varphi_a] j_a \]
where
\[ j_a(x) = [x, a]^{-2-n}, \quad x \in B. \]
As a consequence, we see that $u$ is harmonic on $B$ if and only if $(u \circ \varphi_0)_{ja}$ is harmonic on $B$. Also, note that $(u \circ \varphi_0)_{ja} \in L^1(B, V)$ if $u \in L^1(B, V)$. Thus, by the mean value property, we have

$$u(a) = u \circ \varphi_0(0)_{ja}(0) = \int_B (u \circ \varphi_0)_{ja} dV$$

for $a \in B$ and integrable harmonic function $u$.

Based on the observations in the preceding paragraph, Liu [6] introduced the following Berezin-type transform $B_B$:

$$B_B u(a) = \int_B (u \circ \varphi_0)_{ja} dV$$

for $a \in B$ and $u \in L^1(B, V)$. Then (2.14) implies that integrable harmonic functions are fixed by this transform. Without the Möbius transformation the above defining equation of $B_B$ can be expressed as

$$B_B u(a) = (1 - |a|^2)^2 \int_B \frac{u(x)}{|x, a|^{n+2}} dV(x),$$

which is easily seen by a change of variables, (2.12) and (2.13). Note that $B_B u$ is infinitely differentiable on $B$.

Let $\sigma$ denote the normalized surface area measure on $S$. For arbitrary $u \in L^1(S, \sigma)$, we define the Poisson integral transform of $u$ to be the function $P_B[u]$ on $B$ given by

$$P_B[u](x) = \int_S u(\xi) P_B(x, \xi) d\sigma(\xi)$$

where $P_B(x, \xi)$ is the Poisson kernel for $B$ defined by

$$P_B(x, \xi) = \frac{1 - |x|^2}{|x - \xi|^n}, \quad (x, \xi) \in B \times S.$$

Liu [6] proved the next theorem concerning the limit of iterates of $B_B$.

**Theorem 2.4.** (See [6].) If $u \in C(B)$, then $B_B^k u$ converges uniformly on $B$ to $P_B[u|_S]$ as $k \to \infty$.

### 3. Berezin-type transform on the half-spaces

This section is devoted to the proofs of the mean value property on $H$ (Proposition 3.2), Propositions 3.3 and 3.4. We first define a Berezin-type transform $B_H$ on $H$ and show that $B_H$ commutes with the Kelvin transform $K$ in the sense of Proposition 3.3. Also, we prove that the Kelvin transform $K$ commutes with the Poisson integral transform in the sense of Proposition 3.4.

Given $z \in H$, recall the mapping $\phi_z$ defined by

$$\phi_z(w) = (z_{n+w} + z_{n+w}), \quad w \in H.$$  

Note $\phi_z(w) = w$. The map $\phi_z : H \to H$ is a bijection and its inverse is given by

$$\phi_z^{-1}(w) = z_{n-1}(w' - z', w_{n}), \quad w \in H.$$  

The Jacobian $J_{\phi_z^{-1}}$ is $z_{n-1}^{-1}$. A simple computation shows that the transform $f \mapsto f \circ \phi_z$ preserves harmonicity. In fact, we have $\Delta(f \circ \phi_z) = z_{n-1}^2(\Delta f) \circ \phi_z$ for any $f \in C^2(H)$.

Recall the weighted measure $\mu$ given by

$$d\mu(w) = \frac{4 dV(w)}{|w + e|^{n+2}}, \quad w \in H.$$  

One can check $\mu(H) = 1$ by the next lemma which follows from [5, Lemma 2.2].

**Lemma 3.1.** Given $s + t > -1$ and $t < 0$, the equality

$$\int_H \frac{w^{n+t} dV(w)}{|w - z|^{n+s}} = \frac{\Gamma\left(\frac{n}{2} + 1\right) \Gamma\left(\frac{n}{2} + s + t + 1\right) \Gamma(-t) \Gamma(s + t + 1)}{\Gamma\left(\frac{n}{2} + s\right) \Gamma\left(\frac{n}{2} + 1\right) \Gamma(s + 1)} z^n$$

holds for every $z \in H$.  

Note \(|\phi_2^{-1}(w) + e| = \bar{z} \cdot |w - \bar{z}|.\) Thus, using a change of variables, one can check that \(f \circ \phi_2 \in L^1(\mathbb{H}, \mu)\) for each \(z \in \mathbb{H}\) if \(f \in L^1(\mathbb{H}, \mu)\).

Now, we can state the mean value property for harmonic functions on the half-space.

**Proposition 3.2.** The equality

\[
f(z) = \int_{\mathbb{H}} f \circ \phi_2(w) \, d\mu(w)
\]

holds on \(\mathbb{H}\) for harmonic functions \(f \in L^1(\mathbb{H}, \mu)\).

**Proof.** Let \(f \in L^1(\mathbb{H}, \mu)\). First note that a change of variables, the explicit Jacobian (2.4) and (2.2) yield

\[
2^{(2-n)/2} \int_{\mathbb{B}} Kf(x) \, dV(x) = 2^n \int_{\mathbb{H}} \frac{f(w)}{|w + e|^{2n}|\Phi(w) + e|^{n-2}} \, dV(w)
\]

\[
= 4 \int_{\mathbb{H}} \frac{f(w)}{|w + e|^{n+2}} \, dV(w).
\]  

(3.1)

This means that if \(f \in L^1(\mathbb{H}, \mu)\), then \(Kf \in L^1(\mathbb{B}, V)\) and vice versa.

Now, assume \(f \in L^1(\mathbb{H}, \mu)\) is harmonic. Then \(Kf \in L^1(\mathbb{B}, V)\) is also harmonic on \(\mathbb{B}\) by what we have proved above. Thus, the mean value property on \(\mathbb{B}\), (2.10) and (3.1) give us that

\[
f(e) = 2^{(2-n)/2} Kf(0) = 4 \int_{\mathbb{H}} \frac{f(w)}{|w + e|^{n+2}} \, dV(w).
\]

This proves the proposition for \(z = e\). As mentioned before the proposition, \(f \circ \phi_2 \in L^1(\mathbb{H}, \mu)\) is also harmonic for each \(z \in \mathbb{H}\). Accordingly, replacing \(f\) by \(f \circ \phi_2\) in the above, we have

\[
f(z) = (f \circ \phi_2)(e) = \int_{\mathbb{H}} (f \circ \phi_2)(w) \, d\mu(w)
\]  

(3.2)

as desired. \(\Box\)

Based on this proposition, we now introduce a Berezin-type transform on the half-space given by

\[
B_{\mathbb{H}}f(z) := \int_{\mathbb{H}} (f \circ \phi_2)(w) \, d\mu(w), \quad z \in \mathbb{H}
\]  

(3.3)

for functions \(f \in L^1(\mathbb{H}, \mu)\). By a change of variables, equality (3.3) becomes

\[
B_{\mathbb{H}}f(z) = 4\pi^n \int_{\mathbb{H}} \frac{f(w)}{4|w - \bar{z}|^{n+2}} \, dV(w).
\]  

(3.4)

Proposition 3.2 shows that harmonic functions in \(L^1(\mathbb{H}, \mu)\) are fixed by this transform. Moreover, \(B_{\mathbb{H}}\) commutes with the Kelvin transform \(K\) in the sense of the next proposition.

**Proposition 3.3.** \(KB_{\mathbb{H}}K = B_{\mathbb{H}}\) on \(L^1(\mathbb{B}, V)\). Also, \(KB_{\mathbb{H}}K = B_{\mathbb{H}}\) on \(L^1(\mathbb{H}, \mu)\).

**Proof.** Since \(K\) is its own inverse, we only need to prove \(KB_{\mathbb{H}} = B_{\mathbb{H}}K\) on \(L^1(\mathbb{H}, \mu)\). Let \(f \in L^1(\mathbb{H}, \mu)\) and \(x \in \mathbb{B}\). Using (2.10), (3.4) and (2.1), we have

\[
K[B_{\mathbb{H}}f](x) = 2^{(n-2)/2} \frac{B_{\mathbb{H}}f(\Phi(x))}{|x + e|^{n-2}}
\]

\[
= 4 \cdot 2^{(n-2)/2} \frac{(1 - |x|^2)^2}{|x + e|^{n+2}} \int_{\mathbb{H}} \frac{f(w)}{|w - \Phi(x)|^{n+2}} \, dV(w).
\]

Meanwhile, making the change of variables \(w = \Phi(y)\), we see that the integral above is equal to
The proof is complete. Combining these observations, we obtain

$$K[B_H f](x) = \left(1 - |x|^2\right)^2 \int_B K f(y) \frac{dV(y)}{|y|^n+2} = B_B[Kf](x)$$

where the last equality comes from (2.15). Since $f \in L^1(H, d\mu)$ and $x \in B$ are arbitrary, we obtain $KB_H = B_B K$, as required. The proof is complete. □

Let $P_H(z, \zeta)$ be the Poisson kernel for $H$ defined by

$$P_H(z, \zeta) = c_n \frac{z_n}{|z - \zeta|^n}, \quad (z, \zeta) \in H \times \mathbb{R}^{n-1}$$

where $c_n = 2/(n|B|)$ and $|B|$ is the volume of $B$. The two kernels $P_H$ and $P_B$ are related by the Möbius transformation $\Phi$:

$$P_H(z, \zeta) = c_n 2^{n-2} \frac{P_B(\Phi(z), \Phi(\zeta))}{|z + e|^{n-2}(1 + |\zeta|^2)^{n/2}}$$

(3.5)

for all $z \in H$ and $\zeta \in \mathbb{R}^{n-1}$. One may verify this by a direct calculation. Also, see [2, Theorem 7.23] for another proof.

Let $\nu = \nu_{n-1}$ be the weighted measure on $\mathbb{R}^{n-1}$ defined by

$$d\nu(\zeta) = c_n 2^{(n-2)/2} \frac{d\zeta}{(1 + |\zeta|^2)^{n/2}}.$$

The Poisson integral transformation of $f \in L^1(\mathbb{R}^{n-1}, \nu)$ is the function $P_H f$ on $H$ defined by

$$P_H f(z) = \int_{\mathbb{R}^{n-1}} f(\zeta) P_H(z, \zeta) d\zeta.$$

By (2.5) and (2.2), one may easily check that $K : L^1(\mathbb{R}^{n-1}, \nu) \to L^1(S, \sigma)$ is an isometric isomorphism. As a consequence, we see that $P_B K$ is defined on $L^1(\mathbb{R}^{n-1}, \nu)$ and that $P_H K$ is defined on $L^1(S, \sigma)$. The Kelvin transform $K$ turns out to commute with Poisson integral transforms in the sense of the next proposition.

**Proposition 3.4.** $K P_B K = P_B$ on $L^1(S, \sigma)$. Also, $K P_B K = P_H$ on $L^1(\mathbb{R}^{n-1}, \nu)$.

**Proof.** Since $K$ is its own inverse, we only need to prove $K P_B = P_B K$ on $L^1(\mathbb{R}^{n-1}, \nu)$. Let $f \in L^1(\mathbb{R}^{n-1}, \nu)$ and $x \in B$. We have by (3.5) and (2.2),

$$K[P_B f](x) = c_n 2^{(n-2)/2} \int_{\mathbb{R}^{n-1}} f(\zeta) P_B(\Phi(x), \zeta) d\zeta$$

$$= c_n 2^{3(n-2)/2} \int_{\mathbb{R}^{n-1}} \frac{f(\zeta) P_B(x, \Phi(\zeta))}{|\zeta + e|^{n-2}(1 + |\zeta|^2)^{n/2}} d\zeta$$

$$= c_n 2^{(n-2)/2} \int_{\mathbb{R}^{n-1}} \frac{f(\zeta) P_B(x, \Phi(\zeta))}{(1 + |\zeta|^2)^{n/2}} d\zeta.$$

Making the change of variables $\xi = \Phi(\zeta)$ and using (2.5) in the last integral above, we obtain

$$K[P_B f](x) = c_n 2^{(n-2)/2} \int_{S} \frac{f(\Phi(\xi)) P_B(x, \xi)}{|\xi + e|^{n-2}(1 + |\Phi(\xi)|^2)^{n/2}} d\xi$$

$$= c_n 2^{(n-2)/2} \int_{S} \frac{f(\Phi(\xi)) P_B(x, \xi)}{|\xi + e|^{n-2}} d\xi$$

where the first equality holds by Lemma 2.2. Combining these observations, we obtain

$$K[B_H f](x) = \left(1 - |x|^2\right)^2 \int_B K f(y) \frac{dV(y)}{|y|^n+2} = B_B[Kf](x)$$

where the last equality comes from (2.15). Since $f \in L^1(H, d\mu)$ and $x \in B$ are arbitrary, we obtain $KB_H = B_B K$, as required. The proof is complete. □
where the last equality holds by (2.3). Note \( 2^{-1}c_n d\xi = d\sigma(\xi) \). So, the right-hand side of the above is equal to

\[
\int_B K[f](\xi)P_B(x, \xi) d\sigma(\xi) = P_B[Kf](x).
\]

Since \( f \in L^1(\mathbb{R}^{n-1}, \nu) \) and \( x \in B \) are arbitrary, we obtain \( KP_B = P_BK \), as required. The proof is complete. \( \Box \)

4. Proof of the main result

In this section, we prove our main result. Recall that \( f \in \Omega(H) \) if \( f \in C(\bar{H}) \) and \( \lim_{|z| \to \infty} f(z)|z|^{n-2} \) exists. Note that a function in \( \Omega(H) \) is necessarily continuous at \( \infty \).

Lemma 4.1. \( K[\Omega(H)] = C(\bar{B}) \).

Proof. Assume \( f \in \Omega(H) \). Clearly, \( Kf \) is continuous on \( \bar{B} \setminus \{ -e \} \). Also, we have

\[
\lim_{x \to -e} Kf(x) = 2^{(2-n)/2} \lim_{x \to -e} f(\Phi(x))|\Phi(x)| + e|n-2
\]

\[
= 2^{(2-n)/2} \lim_{|z| \to \infty} f(z)|z + e|^{n-2}
\]

\[
= 2^{(2-n)/2} \lim_{|z| \to \infty} f(z)|z|^{n-2}
\]

where the first equality comes from (2.2). This means that \( Kf \in C(\bar{B}) \) and so \( K[\Omega(H)] \subset C(\bar{B}) \). Similarly, we have \( K[C(\bar{B})] \subset \Omega(H) \), or equivalently, \( C(\bar{B}) \subset K[\Omega(H)] \). This completes the proof. \( \Box \)

Now, we are ready to prove the main result.

Theorem 4.2. If \( f \in \Omega(H) \), then the iterates \( B^k_H f \) converge uniformly on \( H \) to \( P_H[f|_{\mathbb{R}^{n-1}}] \), as \( k \to \infty \).

Proof. Assume \( f \in \Omega(H) \). Let \( u = Kf \). Then we have \( u \in C(\bar{B}) \) by Lemma 4.1. Thus Theorem 2.4 implies that \( B^k_H u \) converges uniformly on \( B \) to \( P_B[u|_S] \), as \( k \to \infty \). Since \( K \) is its own inverse, it follows from Proposition 3.3,

\[
K[B^k_H f] = B^k_H[Kf] = B^k_H u \rightarrow P_B[u|_S]
\]

uniformly on \( B \) as \( k \to \infty \). Note that from (2.2), \( K : L^\infty(B) \to L^\infty(H) \) is norm decreasing. Consequently, applying \( K \) to both sides of (4.1), we see from Proposition 3.4 that

\[
B^k_H f \rightarrow K[P_B(u|_S)] = P_H[K(u|_S)] = P_H(f|_{\mathbb{R}^{n-1}})
\]

uniformly on \( H \) as \( k \to \infty \). The proof is complete. \( \Box \)

Acknowledgments

The author thanks Professor B.R. Choe and the referee for their valuable comments and suggestions.

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